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Set-Membership Filtering for Nonlinear Dynamic Systems With Quadratic Inequality Constraints

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ABSTRACT The present study investigates the problem of set-membership filtering for nonlinear dynamic systems with general nonconvex inhomogeneous quadratic inequality constraints. The investigators propose an ellipsoidal state bounding estimation in the setting of unknown but bounded noise. In order to guarantee the on-line usage, the nonlinear function is linearized by Taylor expansion at each time step, where the bounding ellipsoid of the remainder is updated on-line based on the current state bounding ellipsoid. Furthermore, based on the remainder bounds and the constraints, both the state prediction and measurement update of the filtering can be transformed into a semidefinite programming problem that can be efficiently solved. In order to further reduce the computational complexity, a part-analytical formula of the shape matrix and the center of the bounding ellipsoid is derived using a decoupled technique, which is also helpful to clarify how these constraints affect the state estimation. Finally, typical numerical examples demonstrate the effectiveness of this filtering.

INDEX TERMS Set-membership filter, quadratic inequality constraints, nonlinear dynamic systems, ellipsoidal estimation.

I. INTRODUCTION

Filtering techniques are widely used in target tracking, signal processing, system identification, fault diagnosis, robotics, navigation, etc [1]–[5]. For linear dynamic systems, Kalman filter (KF) [6] is the minimum-variance linear state estimator for both Gaussian and non-Gaussian noise [7]. However, this is not possible for general nonlinear dynamic systems. Furthermore, estimation for nonlinear systems is quite extensive in practice. Nonlinearities are widely included in vehicle navigation, dialysis machines, and many other areas [8]. In the present case, a few modifications of KF, including the extended Kalman filter (EKF) [9], the unscented Kalman filter (UKF) [10], and the particle filter (PF) [11], were used to estimate the state.

Constrained dynamic systems frequently occur in practical applications [12]–[14]. The constraints may arise from physical laws or mathematical properties. For instance, civil aircrafts and land-based vehicles are constrained within a preset flight channel and a known road (straight line or curve) [15], [16], respectively. These constraints, which are determined by the state physical properties, can provide valuable information for the estimator designers. By taking full advantage of the constraint, the state estimation error can be effectively reduced. Constraints of dynamic systems come in many forms, including set constraint, equality constraint, inequality constraint, probability constraint, etc [12]. Various point estimation methods for the constrained state estimation have been proposed in literature [17]. For instance, Simon and Simon [3], [18] focused on equality and inequality constrained Kalman filtering. Julier and LaViola [2] presented a nonlinear equality constrained Kalman filtering. Ko and Bitmead [19] proposed the state estimation for state equality constrained linear systems. Teixeira *et al.* [20] discussed the unscented filtering for nonlinear systems with interval-constraints. Lan and Li [21] provided the state estimation for nonlinear inequality constrained systems *via* unscented transformation, etc.

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Point estimation filtering techniques require the probability information of the state and measurement noises, and provide a probabilistic state estimation [22]. In some applications, the probabilistic assumption of noises is not suitable (for example, the main perturbation can be deterministic). Hence, the assumptions that the state perturbations and measurement noise are unknown but bounded appears to be more natural [23]. This has motivated the set-membership filtering [24]–[26]. Different from point estimation, set-membership filtering can provide a state estimation based on unknown but bounded noises. The problem of set-membership filtering was first considered by Witsenhausen [27] in the late 1960s. The ideas of set-membership filtering have been extensively investigated [23]–[26], [28]–[33], and the references therein. The present studies do not consider set-membership filtering with constraints.

Set-membership filtering for nonlinear equality constrained linear dynamic systems was addressed in a study [34], and linear dynamic systems with state linear equality constraints and a specific positive semidefinite quadratic inequality constraint were also given concerned in a study [5]. Both of these focused on linear dynamic systems. Nevertheless, the filtering problem for nonlinear dynamic systems with general nonconvex inhomogeneous quadratic inequality constraints has not been considered under the set-membership filter framework. In real world applications, many physical systems can be described as nonlinear dynamic systems with state quadratic inequality constraints. For instance, in a vehicle tracking problem, when the vehicle is traveling on a known curve road, the geometric structure of some roads can be approximately formed as quadratic inequality constraints on the kinematic variables of the target vehicle [35], [36]. The kinematic constraint can be utilized when the target's trajectory satisfies a kinematic constraint, such as the quadratic parabolic inequality constraint [17], [37]–[39]. For real-time space applications, the quaternion-of-rotation is the preferred attitude representation. In order to represent a rotation, the quaternion obeys a unit-norm (i.e., quadratic) constraint [40], [41]. In addition, in order to guarantee the on-line usage, when the nonlinear function is linearized on the current estimate, consideration should be given on how to update the bound of the remainder on-line, but not give it before filtering.

In the present study, focus was given on the set-membership filtering problem for nonlinear dynamic systems with general nonconvex inhomogeneous quadratic inequality constraints. The main contributions of the present study are as follows:

- A constrained set-membership filtering method *via* the constraint information and the remainder bounding techniques is proposed, which can guarantee the on-line usage. In the set-membership prediction step, based on the remainder bounds, S-procedure and Schur complements, the state estimation problem with the quadratic constraints can be transformed into a semidefinite programming (SDP) problem. Through path-following interior point methods in convex programming, the set-membership filtering prediction step can be efficiently solved. The optimization problem of the measurement update step can be similarly derived.
- In order to further reduce the computational complexity, a part-analytical formula of the predicted state bounding ellipsoid and the state bounding ellipsoid is derived through a decoupled technique, and this is helpful to clarify how the constraints affect the state estimation.
- Set-membership filtering for special cases of the quadratic inequality constraint, including the unilateral quadratic inequality constraint and linear inequality constraint, is derived.

Consistent with the discussion about the role of constraints in the state estimation, the numerical example illustrates that the new method has better performance than the set-membership filtering without using constraints.

The present study is structured as follows. Section II describes the set-membership filtering problem for nonlinear dynamic systems with quadratic inequality constraints. The prediction step, the measurement update step, the part-analytical formula of the predicted and updated state bounding ellipsoid, and the summarized algorithm of the set-membership filter with quadratic inequality constraints reside in Section III. In Section IV, a typical numerical example is used to illustrate the effectiveness of this method. The conclusion is provided in the last section.

II. PROBLEM FORMULATION

Consider the following dynamic system:

$$x_{k+1} = f_k(x_k) + w_k, \tag{1}$$

$$y_k = h_k(x_k) + v_k, \tag{2}$$

$$d_k^1 \leq x_k^T \hat{G}_k x_k + \hat{\beta}_k^T x_k + \hat{\alpha}_k \leq d_k^2, \tag{3}$$

where k is the time step, $x_k \in \mathbb{R}^n$ is the system state, $f_k(x_k)$ and $h_k(x_k)$ are the nonlinear functions of x_k, and $y_k \in \mathbb{R}^m$ is the measurement output; $\hat{G}_k \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\hat{\beta}_k \in \mathbb{R}^n$ is a vector, $\hat{\alpha}_k \in \mathbb{R}$ is a scalar, and these are all known; $d_k^1 \in \mathbb{R}$ and $d_k^2 \in \mathbb{R}$ are the known scalars that satisfy $d_k^2 \geq d_k^1$. The constraint (3) is the general nonconvex inhomogeneous quadratic inequality constraint. If $\hat{G}_k$ is a positive semidefinite matrix, it is a convex constraint. If $\hat{\beta}_k = 0$ and $\hat{\alpha}_k = 0$, it is a homogeneous constraint. $w_k \in \mathbb{R}^n$ is the process noise and $v_k \in \mathbb{R}^m$ is the measurement noise, and these are assumed in ellipsoidal sets:

$$W_k = \{w_k : w_k^T Q_k^{-1} w_k \leq 1\}, \tag{4}$$

$$V_k = \{v_k : v_k^T R_k^{-1} v_k \leq 1\}, \tag{5}$$

where Q_k and R_k are known shape matrices of W_k and V_k with compatible dimensions, respectively. These are symmetric positive-definite matrices.

When nonlinear functions $f_k(x_k)$ and $h_k(x_k)$ are linearized about the center of the given state bounding ellipsoid, the remainder terms can be restricted in an ellipsoid *via* the remainder bounding techniques in [33]. Specifically, assume that the system state at time k is bounded in an ellipsoid, that is:

$$\begin{aligned} x_k &\in \mathcal{E}_k(\hat{x}_k, P_k) \\ &= \{x \in \mathbb{R}^n : (x-\hat{x}_k)^T P_k^{-1}(x-\hat{x}_k) \le 1\} \\ &= \{x \in \mathbb{R}^n : x=\hat{x}_k + E_k u_k, P_k = E_k E_k^T, \|u_k\| \le 1\}. \end{aligned} \tag{6}$$

By Taylor's Theorem, the nonlinear function $h_k(x_k)$ can be linearized to:

$$h_k(\hat{x}_k + E_{h_k}u_k) = h_k(\hat{x}_k) + J_{h_k}E_{h_k}u_k + \Delta h_k(u_k), \tag{7}$$

where J_{h_k} is the Jacobian matrix of $h_k(x_k)$, $\Delta h_k(u_k)$ is the high-order remainder, and for all $\|u_k\| \le 1$, $\Delta h_k(u_k)$ can be restricted in an ellipsoid, that is:

$$\begin{aligned} \Delta h_k(u_k) &\in \mathcal{E}_{h_k}(e_{h_k}, P_{h_k}) \\ &= \{x \in \mathbb{R}^n : (x-e_{h_k})^T P_{h_k}^{-1}(x-e_{h_k}) \le 1\} \\ &= \{x \in \mathbb{R}^n : x = e_{h_k} + B_{h_k}\Delta_{h_k}, P_{h_k} = B_{h_k} \\ &\quad B_{h_k}^T, \|\Delta_{h_k}\| \le 1\}, \end{aligned} \tag{8}$$

where P_{h_k} and e_{h_k} are the shape matrix of $\mathcal{E}_{h_k}$ and the center of it, respectively. Similarly, the bounding ellipsoid $\mathcal{E}_{f_k}(e_{f_k}, P_{f_k})$ of the high-order remainder of $f_k(x_k)$ can be obtained on-line.

Suppose that x_0 is the initial state, and x_0 is bounded in a given bounding ellipsoid:

$$\mathcal{E}_0(\hat{x}_0, P_0) = \{x \in \mathbb{R}^n : (x-\hat{x}_0)^T P_0^{-1}(x-\hat{x}_0) \le 1\}, \tag{9}$$

where P_0 and $\hat{x}_0$ are known shape matrix and center of $\mathcal{E}_0$, respectively.

The goal, at time $k+1$, is to determine P_{k+1} and $\hat{x}_{k+1}$ that satisfying:

$$(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \le 1. \tag{10}$$

The filtering problem mentioned above aims to determine the center and shape matrix of the state bounding ellipsoid in the setting of bounded noises, which is called the set-membership filtering problem for nonlinear dynamic systems with quadratic inequality constraints. The prediction step and update step of set-membership filtering has been extensively investigated [23], [29], [32], [33], [42]. In the present study, focus was given on the prediction step and update step of the filtering with quadratic inequality constraints, as follows.

- *Prediction step:* Determining a predicted bounding ellipsoid, such that $x_{k+1} \in \mathcal{E}_{k+1|k}(\hat{x}_{k+1|k}, P_{k+1|k})$ in the conditions of (i) $x_k \in \mathcal{E}_k(\hat{x}_k, P_k)$; (ii) w_k and v_k are bounded in W_k and V_k, respectively; (iii) $\Delta f_k(u_k) \in \mathcal{E}_{f_k}(e_{f_k}, P_{f_k})$ and $\Delta h_k(u_k) \in \mathcal{E}_{h_k}(e_{h_k}, P_{h_k})$; (iv) the state satisfies the quadratic inequality constraint at time k.
- *Update step:* Determining an ellipsoid, such that $x_{k+1} \in \mathcal{E}_{k+1}(\hat{x}_{k+1}, P_{k+1})$ in the conditions of (i) $x_{k+1} \in \mathcal{E}_{k+1|k}(\hat{x}_{k+1|k}, P_{k+1|k})$; (ii) v_{k+1} is bounded in V_{k+1}; (iii) $\Delta h_{k+1}(u_{k+1}) \in \mathcal{E}_{h_{k+1}}(e_{h_{k+1}}, P_{h_{k+1}})$; (iv) the state satisfies the quadratic inequality constraint at time $k+1$.

III. SET-MEMBERSHIP FILTER WITH QUADRATIC INEQUALITY CONSTRAINTS

The constraint (3) of the dynamic system is equivalent to:

$$-1 \le x_k^T G_k x_k + \beta_k^T x_k + \alpha_k \le 1, \tag{11}$$

where $G_k = \frac{1}{d_k}\hat{G}_k$, $\beta_k^T = \frac{1}{d_k}\hat{\beta}_k^T$, $\alpha_k = \frac{1}{d_k}(\hat{\alpha}_k - \frac{d_k^1+d_k^2}{2})$, and $d_k = \frac{d_k^2-d_k^1}{2}$.

The constraint (11) is equivalent to:

$$x_k^T G_k x_k + \beta_k^T x_k + \alpha_k + \Delta = 0, \tag{12}$$

where $\Delta \in \mathbb{R}$ and $\|\Delta\| \le 1$.

Next, the prediction and measurement update steps of the new method are introduced. The following propositions provide a method for designing the set-membership filtering for quadratic inequality constrained nonlinear dynamic systems.

A. PREDICTION STEP

Proposition 1: A predicted state bounding ellipsoid $\mathcal{E}_{k+1|k}(\hat{x}_{k+1|k}, P_{k+1|k})$ can be obtained by solving the following optimization problem:

$$\min f(P_{k+1|k}) \tag{13}$$

$$\text{subject to } \lambda_2^g \in \mathbb{R}, \quad \lambda^u \ge 0, \ \lambda^w \ge 0, \ \lambda_1^g \ge 0, \tag{14}$$

$$\lambda^v \ge 0, \quad \lambda^f \ge 0, \ \lambda^h \ge 0, \tag{15}$$

$$-P_{k+1|k} \prec 0, \tag{16}$$

$$\begin{bmatrix} -P_{k+1|k} & \Phi_{k+1|k}(\hat{x}_{k+1|k})(\Psi_{k+1|k}(y_k))^\perp \\ (\Phi_{k+1|k}(\hat{x}_{k+1|k})(\Psi_{k+1|k}(y_k))^\perp)^T & -((\Psi_{k+1|k}(y_k))^\perp)^T \Xi (\Psi_{k+1|k}(y_k))^\perp \end{bmatrix} \preceq 0, \tag{17}$$

where

$$\Phi_{k+1|k}(\hat{x}_{k+1|k}) = [f_k(\hat{x}_k) - \hat{x}_{k+1|k} + e_{f_k}, J_{f_k}E_k, I, 0, B_{f_k}, 0, 0], \tag{18}$$

$$\Psi_{k+1|k}(y_k) = [h_k(\hat{x}_k) + e_{h_k} - y_k, J_{h_k}E_k, 0, I, 0, B_{h_k}, 0], \tag{19}$$

$$\Gamma_1 = \begin{bmatrix} \hat{x}_k^T G_k \hat{x}_k + \beta_k^T \hat{x}_k + \alpha_k & \hat{x}_k^T G_k E_k + \frac{1}{2}\beta_k^T E_k \\ (\hat{x}_k^T G_k E_k + \frac{1}{2}\beta_k^T E_k)^T & E_k^T G_k E_k \end{bmatrix}, \tag{20}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{21}$$

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^T & 0 \end{bmatrix}, \tag{22}$$

$$P_k = E_k(E_k)^T, \tag{23}$$

$$\begin{aligned} \Xi &= diag(1 - \lambda^u - \lambda^w - \lambda_1^g - \lambda^v - \lambda^f - \lambda^h, \lambda^u I, \\ &\quad \lambda^w Q_k^{-1}, \lambda^v R_k^{-1}, \lambda^f I, \lambda^h I, \lambda_1^g I) + \lambda_2^g \Gamma, \end{aligned} \tag{24}$$

where zero matrices and identity matrices have compatible dimensions among the optimization problem, and $(\cdot)^\perp$ denotes the orthogonal complement.

Proof: Refer to Appendix A.

Remark 1: The optimization problem in Proposition 1 is an SDP problem if trace function is chosen as the objective function, and if logdet function is chosen, the optimization problem becomes a *MAXDET* problem. Both of these optimization problems can be solved *via* path-following interior point methods in the convex programming [30], [43].

In order to further reduce the computation complexity of the SDP problem in Proposition 1, the part-analytical formula of the optimization problem can be derived using a decoupled technique. Note that the appropriate orthogonal complement of $\Psi_{k+1|k}(y_k)$ can be chosen as:

$$(\Psi_{k+1|k}(y_k))^{\perp} = \begin{bmatrix} -1 & 0 \\ \Psi_1 & \Psi_2 \end{bmatrix}, \tag{25}$$

where

$$\Psi_1 = [0, 0, (h_k(\hat{x}_k) - y_k)^T, (B_{f_k}^{-1} e_{f_k})^T, (B_{h_k}^{-1} e_{h_k})^T, 0]^T, \tag{26}$$

$$\Psi_2 = \begin{bmatrix} E_k^{-1} & 0 & 0 \\ -J_{f_k} & I & 0 \\ -J_{h_k} & 0 & I \\ B_{f_k}^{-1} & -B_{f_k}^{-1} & 0 \\ 0 & 0 & -B_{h_k}^{-1} \\ 0 & 0 & I \end{bmatrix}. \tag{27}$$

Denote

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix}, \tag{28}$$

where

$$\begin{aligned} \Xi_{11} &= 1 - \lambda^u - \lambda^w - \lambda_1^g - \lambda^v - \lambda^f - \lambda^h \\ &\quad + \lambda_2^g(\hat{x}_k^T G_k \hat{x}_k + \beta_k^T \hat{x}_k + \alpha_k), \end{aligned} \tag{29}$$

$$\Xi_{12} = [\lambda_2^g(\hat{x}_k^T G_k E_k + \frac{1}{2}\beta_k^T E_k), 0, 0, 0, 0, \frac{1}{2}\lambda_2^g], \tag{30}$$

$$\begin{aligned} \Xi_{22} &= diag(\lambda^u I + \lambda_2^g(E_k^T G_k E_k), \lambda^w Q_k^{-1}, \\ &\quad \lambda^v R_k^{-1}, \lambda^f I, \lambda^h I, \lambda_1^g I). \end{aligned} \tag{31}$$

Proposition 2: Let all symbols be defined as those in Proposition 1, and if $f(P_{k+1|k})$ is either trace, or the logdet function of $P_{k+1|k}$, then the shape matrix and center of the optimal predicted ellipsoid in Proposition 1 can be decoupled. Specifically, the part-analytical formula of the optimization problem is given by:

$$\begin{aligned} P_{k+1|k}^{-1} &= \lambda^u P_k^{-1} + \lambda_2^g G_k + (J_{f_k} - I)^T (\frac{Q_k}{\lambda^w} + \frac{P_{f_k}}{\lambda^f})^{-1} \\ &\quad \times (J_{f_k} - I) + J_{h_k}^T (\frac{R_k}{\lambda^v} + (\lambda^h P_{h_k}^{-1} + \lambda_1^g I)^{-1})^{-1} J_{h_k}, \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{x}_{k+1|k} &= f_k(\hat{x}_k) + P_{k+1|k} X_1 + P_{k+1|k} C_1 X_2 \\ &\quad + P_{k+1|k} C_2 X_3, \end{aligned} \tag{33}$$

where

$$\begin{aligned} X_1 &= \lambda^f P_{f_k}^{-1} e_{f_k} - \lambda^v J_{h_k}^T R_k^{-1}(h_k(\hat{x}_k) - y_k) \\ &\quad - \lambda_2^g (G_k \hat{x}_k + \frac{1}{2}\beta_k), \end{aligned} \tag{34}$$

$$X_2 = -\lambda^f P_{f_k}^{-1} e_{f_k}, \tag{35}$$

$$X_3 = \lambda^v R_k^{-1}(h_k(\hat{x}_k) - y_k) - \lambda^h P_{h_k}^{-1} e_{h_k} - \frac{1}{2}\lambda_2^g I, \tag{36}$$

$$C_1 = (\lambda^w J_{f_k}^T Q_k^{-1} + \lambda^f P_{f_k}^{-1})(\lambda^w Q_k^{-1} + \lambda^f P_{f_k}^{-1})^{-1}, \tag{37}$$

$$C_2 = \lambda^v J_{h_k}^T R_k^{-1}(\lambda^v R_k^{-1} + \lambda^h P_{h_k}^{-1} + \lambda_1^g I)^{-1}. \tag{38}$$

The optimal values $\lambda_2^g, \lambda^u, \lambda^w, \lambda_1^g, \lambda^v, \lambda^f, \lambda^h$ of the problem variables can be obtained by solving the optimization problem:

$$\min_{\lambda_2^g, \lambda^u, \lambda^w, \lambda_1^g, \lambda^v, \lambda^f, \lambda^h} f(B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} B^T) \tag{39}$$

$$\begin{aligned} \textit{subject to } & \lambda_2^g \in \mathbb{R}, \quad \lambda^u \geq 0, \ \lambda^w \geq 0, \ \lambda_1^g \geq 0, \\ & \lambda^v \geq 0, \quad \lambda^f \geq 0, \ \lambda^h \geq 0, \end{aligned} \tag{40}$$

$$\begin{bmatrix} \Xi_{11} + \Psi_1^T \Xi_{22} \Psi_1 & \Psi_1^T \Xi_{22} \Psi_2 - \Xi_{12} \Psi_2 \\ \Psi_2^T \Xi_{22} \Psi_1 - \Psi_2^T \Xi_{12}^T & \Psi_2^T \Xi_{22} \Psi_2 \end{bmatrix} \succeq 0, \tag{41}$$

where $B = [I, 0, 0]$.

Proof: Refer to Appendix A.

Remark 2: The part-analytical formula (32) and (33) intuitively shows the role of constraints in the state estimation. $P_{k+1|k}$ in (32) indicates that when the variables λ_1^g and λ_2^g take 0, the shape matrix and center of the optimal predicted ellipsoid derived by Proposition 2 degenerate to that of the nonlinear filtering without the constraints [33]. This means that the feasible set of the algorithm without the constraints is included in the feasible set of the quadratic inequality constrained set-membership filter. Thus, the size of the predicted ellipsoid derived by the quadratic inequality constrained set-membership filter is smaller than that of the algorithm without the constraints. If $\hat{G}_k$ is positive definite, then this also shows that the larger λ_1^g and λ_2^g are, the smaller the predicted ellipsoid size is. In addition, it can be proven that the computational complexity of Proposition 2 is significantly lower than that of Proposition 1 [30].

B. MEASUREMENT UPDATE STEP

Proposition 3: The state bounding ellipsoid $\mathcal{E}_{k+1}(\hat{x}_{k+1}, P_{k+1})$ at time $k+1$ can be obtained by solving the following optimization problem:

$$\min f(P_{k+1}) \tag{42}$$

$$\text{subject to } \lambda_2^g \in \mathbb{R}, \quad \lambda^u \geq 0, \ \lambda^v \geq 0, \ \lambda_1^g \geq 0, \ \lambda^h \geq 0, \tag{43}$$

$$-P_{k+1} \prec 0, \tag{44}$$

$$\begin{bmatrix} -P_{k+1} & \Phi_{k+1}(\hat{x}_{k+1})(\Psi_{k+1}(y_{k+1}))^{\perp} \\ (\Phi_{k+1}(\hat{x}_{k+1})(\Psi_{k+1}(y_{k+1}))^{\perp})^T & -((\Psi_{k+1}(y_{k+1}))^{\perp})^T \Xi (\Psi_{k+1}(y_{k+1}))^{\perp} \end{bmatrix} \preceq 0, \tag{45}$$

where

$$\Phi_{k+1}(\hat{x}_{k+1}) = [\hat{x}_{k+1|k} - \hat{x}_{k+1}, E_{k+1|k}, 0, 0, 0], \tag{46}$$

$$\begin{aligned} \Psi_{k+1}(y_{k+1}) &= [h_{k+1}(\hat{x}_{k+1|k}) - y_{k+1} + e_{h_{k+1}}, \\ &\quad J_{h_{k+1}} E_{k+1|k}, I, B_{h_{k+1}}, 0], \end{aligned} \tag{47}$$

$$\Gamma_1 = \begin{bmatrix} \hat{x}_{k+1|k}^T G_{k+1}\hat{x}_{k+1|k} + \beta_{k+1}^T \hat{x}_{k+1|k} + \alpha_{k+1} & \hat{x}_{k+1|k}^T G_{k+1}E_{k+1|k} + \frac{1}{2}\beta_{k+1}^T E_{k+1|k} \\ (\hat{x}_{k+1|k}^T G_{k+1}E_{k+1|k} + \frac{1}{2}\beta_{k+1}^T E_{k+1|k})^T & E_{k+1|k}^T G_{k+1} E_{k+1|k} \end{bmatrix}, \quad (48)$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \quad (49)$$

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^T & 0 \end{bmatrix}, \quad (50)$$

$$P_{k+1|k} = E_{k+1|k}(E_{k+1|k})^T, \quad (51)$$

$$\Xi = diag(1 - \lambda^u - \lambda^v - \lambda_1^g - \lambda^h, \lambda^u I, \lambda^v R_{k+1}^{-1}, \lambda^h I, \lambda_1^g I) + \lambda_2^g \Gamma, \quad (52)$$

where zero matrices and identity matrices have compatible dimensions among the optimization problem, and $(\cdot)^\perp$ denotes the orthogonal complement.

Proof: Refer to Appendix B.

Similar to Proposition 1, in order to further reduce the computation complexity, the part-analytical formula of the optimization problem in Proposition 3 can be derived by the decoupled technique, and an appropriate form of the orthogonal complement of $\Psi_{k+1}(y_{k+1})$ can be chosen as:

$$(\Psi_{k+1}(y_{k+1}))^\perp = \begin{bmatrix} -1 & 0 \\ \Psi_1 & \Psi_2 \end{bmatrix}, \quad (53)$$

where

$$\Psi_1 = [0, (h_{k+1}(\hat{x}_{k+1|k}) - y_{k+1})^T, (B_{h_{k+1}}^{-1} e_{h_{k+1}})^T, 0]^T, \quad (54)$$

$$\Psi_2 = \begin{bmatrix} E_{k+1|k}^{-1} & 0 \\ -J_{h_{k+1}} & I \\ 0 & -B_{h_{k+1}}^{-1} \\ 0 & I \end{bmatrix}. \quad (55)$$

Denote

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix}, \quad (56)$$

where

$$\Xi_{11} = 1 - \lambda^u - \lambda^v - \lambda_1^g - \lambda^h + \lambda_2^g(\hat{x}_{k+1|k}^T G_{k+1}\hat{x}_{k+1|k} + \beta_{k+1}^T \hat{x}_{k+1|k} + \alpha_{k+1}), \quad (57)$$

$$\Xi_{12} = [\lambda_2^g(\hat{x}_{k+1|k}^T G_{k+1}E_{k+1|k} + \frac{1}{2}\beta_{k+1}^T E_{k+1|k}), 0, 0, \frac{1}{2}\lambda_2^g], \quad (58)$$

$$\Xi_{22} = diag(\lambda^u I + \lambda_2^g(E_{k+1|k}^T G_{k+1}E_{k+1|k}), \lambda^v R_{k+1}^{-1}, \lambda^h I, \lambda_1^g I). \quad (59)$$

Proposition 4: Let all symbols be defined as in Proposition 3. If $f(P_{k+1})$ is either trace, or the logdet function of P_{k+1}, then the shape matrix and center of the updated ellipsoid in Proposition 3 can be decoupled. Specifically, the part-analytical formula of the optimization problem is given by:

$$P_{k+1}^{-1} = \lambda^u P_{k+1|k}^{-1} + \lambda_2^g G_{k+1} + J_{h_{k+1}}^T \left(\frac{R_{k+1}}{\lambda^v} + (\lambda^h P_{h_{k+1}}^{-1} + \lambda_1^g I)^{-1}\right)^{-1} J_{h_{k+1}}, \quad (60)$$

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + P_{k+1}X_1 + P_{k+1}CX_2, \quad (61)$$

where

$$X_1 = -\lambda^v J_{h_{k+1}}^T R_{k+1}^{-1}(h_{k+1}(\hat{x}_{k+1|k}) - y_{k+1}) - \lambda_2^g(G_{k+1}\hat{x}_{k+1|k} + \frac{1}{2}\beta_{k+1}), \quad (62)$$

$$X_2 = \lambda^v R_{k+1}^{-1}(h_{k+1}(\hat{x}_{k+1|k}) - y_{k+1}) - \lambda^h P_{h_{k+1}}^{-1} e_{h_{k+1}} - \frac{1}{2}\lambda_2^g I, \quad (63)$$

$$C = \lambda^v J_{h_{k+1}}^T R_{k+1}^{-1}(\lambda^v R_{k+1}^{-1} + \lambda^h P_{h_{k+1}}^{-1} + \lambda_1^g I)^{-1}. \quad (64)$$

The optimal values $\lambda_2^g, \lambda^u, \lambda^v, \lambda_1^g, \lambda^h$ of the problem variables can be obtained by solving the optimization problem:

$$\min_{\lambda_2^g, \lambda^u, \lambda^v, \lambda_1^g, \lambda^h} f(B(\Psi_2^T \Xi_{22}\Psi_2)^{-1}B^T) \quad (65)$$

$$\text{subject to } \lambda_2^g \in \mathbb{R}, \quad \lambda^u \geq 0, \ \lambda^v \geq 0, \ \lambda_1^g \geq 0, \ \lambda^h \geq 0, \quad (66)$$

$$\begin{bmatrix} \Xi_{11} + \Psi_1^T \Xi_{12}\Psi_1 & \Psi_1^T \Xi_{22}\Psi_2 - \Xi_{12}\Psi_2 \\ \Psi_2^T \Xi_{22}\Psi_1 - \Psi_2^T \Xi_{12}^T & \Psi_2^T \Xi_{22}\Psi_2 \end{bmatrix} \succeq 0, \quad (67)$$

where $B = [I, 0]$.

Proof: Refer to Appendix B.

Remark 3: Similar to Proposition 1 and Proposition 2, Proposition 4 has a lower computational complexity than Proposition 3, and the part-analytical formula (60)-(61) in Proposition 4 clarifies how the constraints affect the state estimation.

Corollary 1: If d_k^1 or d_k^2 in the state bilateral constraint (3) takes infinity, it becomes a unilateral constraint. Then, in the prediction step, the state bounding ellipsoid can be obtained by removing the constraint $\lambda_2^g \in \mathbb{R}$ of the optimization problem (13) and let

$$\Phi_{k+1|k}(\hat{x}_{k+1|k}) = [f_k(\hat{x}_k) - \hat{x}_{k+1|k} + e_{f_k}, J_{f_k}E_k, I, 0, B_{f_k}, 0], \quad (68)$$

$$\Psi_{k+1|k}(y_k) = [h_k(\hat{x}_k) + e_{h_k} - y_k, J_{h_k}E_k, 0, I, 0, B_{h_k}], \quad (69)$$

$$\Gamma = diag(\Gamma_1, 0, 0, 0, 0), \quad (70)$$

$$\Xi = diag(1 - \lambda^u - \lambda^w - \lambda_1^g - \lambda^v - \lambda^f - \lambda^h, \lambda^u I, \lambda^w Q_k^{-1}, \lambda^v R_k^{-1}, \lambda^f I, \lambda^h I) + \lambda_1^g \Gamma. \quad (71)$$

Similarly, in the measurement update step, the state bounding ellipsoid can be obtained by removing constraint $\lambda_2^g \in \mathbb{R}$ of the optimization problem (42) and let

$$\Phi_{k+1}(\hat{x}_{k+1}) = [\hat{x}_{k+1|k} - \hat{x}_{k+1}, E_{k+1|k}, 0, 0], \quad (72)$$

$$\Psi_{k+1}(y_{k+1}) = [h_{k+1}(\hat{x}_{k+1|k}) + e_{h_{k+1}} - y_{k+1}, J_{h_{k+1}}E_{k+1|k}, I, B_{h_{k+1}}], \quad (73)$$

$$\Gamma = diag(\Gamma_1, 0, 0), \tag{74}$$
$$\Xi = diag(1 - \lambda^u - \lambda^v - \lambda_1^g - \lambda^h, \lambda^u I, \lambda^v R_{k+1}^{-1}, \lambda^h I) + \lambda_1^g \Gamma. \tag{75}$$

Proof: Refer to Appendix C.

Corollary 2: If $\hat{G}_k$ in (3) takes the zero matrix, it becomes a linear inequality constraint. Then, in the prediction step, the state bounding ellipsoid can be obtained by removing the constraint $\lambda_2^g \in \mathbb{R}$ of the optimization problem (13) and let

$$\Phi_{k+1|k}(\hat{x}_{k+1|k}) = [f_k(\hat{x}_k) - \hat{x}_{k+1|k} + e_{f_k}, J_{f_k}E_k, I, 0, B_{f_k}, 0], \tag{76}$$
$$\Psi_{k+1|k}(y_k) = [h_k(\hat{x}_k) + e_{h_k} - y_k, J_{h_k}E_k, 0, I, 0, B_{h_k}], \tag{77}$$
$$\Gamma = [\beta_k^T \hat{x}_k + \alpha_k, \beta_k^T E_k, 0, 0, 0, 0], \tag{78}$$
$$\Xi = diag(1 - \lambda^u - \lambda^w - \lambda_1^g - \lambda^v - \lambda^f - \lambda^h, \lambda^u I, \lambda^w Q_k^{-1}, \lambda^v R_k^{-1}, \lambda^f I, \lambda^h I) + \lambda_1^g \Gamma^T \Gamma. \tag{79}$$

Similarly, in the measurement update step, the state bounding ellipsoid can be obtained by removing constraint $\lambda_2^g \in \mathbb{R}$ of the optimization problem (42) and let

$$\Phi_{k+1}(\hat{x}_{k+1}) = [\hat{x}_{k+1|k} - \hat{x}_{k+1}, E_{k+1|k}, 0, 0], \tag{80}$$
$$\Psi_{k+1}(y_{k+1}) = [h_{k+1}(\hat{x}_{k+1|k}) + e_{h_{k+1}} - y_{k+1}, J_{h_{k+1}}E_{k+1|k}, I, B_{h_{k+1}}], \tag{81}$$
$$\Gamma = [\beta_{k+1}^T \hat{x}_{k+1|k} + \alpha_{k+1}, \beta_{k+1}^T E_{k+1|k}, 0, 0], \tag{82}$$
$$\Xi = diag(1 - \lambda^u - \lambda^v - \lambda_1^g - \lambda^h, \lambda^u I, \lambda^v R_k^{-1}, \lambda^h I) + \lambda_1^g \Gamma^T \Gamma. \tag{83}$$

Proof: Refer to Appendix C.

C. SET-MEMBERSHIP FILTER WITH QUADRATIC INEQUALITY CONSTRAINTS

Based on Propositions 1-4, the algorithm of the set-membership filtering with quadratic inequality constraints can be summarized, as follows:

Algorithm 1 The Set-Membership Filtering Recursive Algorithm

Step 1: Set $k = 0$. Given the initial value $(\hat{x}_0, P_0)$.
Step 2: Determine the bounding ellipsoids of the remainders Δ_{f_k} and Δ_{h_k} on-line [33], respectively.
Step 3: Compute the predicted state bounding ellipsoid $\mathcal{E}_{k+1|k}(\hat{x}_{k+1|k}, P_{k+1|k})$ by (13)-(17) or (39)-(41).
Step 4: Determine the bounding ellipsoid of the remainder $\Delta_{h_{k+1}}$ on-line [33].
Step 5: Compute the updated state bounding ellipsoid $\mathcal{E}_{k+1}(\hat{x}_{k+1}, P_{k+1})$ by (42)-(45) or (65)-(67).
Step 6: Set $k = k + 1$ and go to Step 2.

IV. NUMERICAL EXAMPLES IN TARGET TRACKING

In this section, the performance between the set-membership filter without constraints and the proposed quadratic inequality constrained set-membership filter is compared. The following simulation results are based on Matlab R2017b with YALMIP.

Considering a two-dimensional dynamic system, a moving target is tracked using the range and bearing measurements [44]:

$$x_{k+1} = \begin{bmatrix} 1 & 0 & \frac{sinwT}{w} & -\frac{1-coswT}{w} \\ 0 & 1 & \frac{1-coswT}{w} & \frac{sinwT}{w} \\ 0 & 0 & coswT & -sinwT \\ 0 & 0 & sinwT & coswT \end{bmatrix} x_k + w_k, \tag{84}$$

$$y_k = \begin{bmatrix} \sqrt{(x_k(1)-a)^2 + (x_k(2)-b)^2} \\ arctan(\frac{x_k(2)-b}{x_k(1)-a}) \end{bmatrix} + v_k, \tag{85}$$

where x is $(x, y, \dot{x}, \dot{y})$, including position and velocity, a and b are the sensor position of location, and T is the sampling time. w_k and v_k are assumed to be restricted in the following ellipsoidal sets:

$$W_k = \{w_k : w_k^T Q_k^{-1} w_k \le 1\}, \tag{86}$$
$$V_k = \{v_k : v_k^T R_k^{-1} v_k \le 1\}, \tag{87}$$

where (88) and (89), as shown at the bottom of this page.

In this example, assume $T = 0.2$, $S_w = 50$, $w = 0.1$, $a = -750$, $b = 500$, and the initial state is (0, 25, 20, 20), which belongs to the ellipsoid $\mathcal{E}(\hat{x}_0, P_0)$, where $\hat{x}_0 = \begin{bmatrix}-10 & 10 & 25 & 25\end{bmatrix}^T$ and

$$P_0 = 40^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q_k = S_w \cdot \begin{bmatrix} \frac{2(wT - sinwT)}{w^3} & 0 & \frac{1-coswT}{w^2} & \frac{wT - sinwT}{w^2} \\ 0 & \frac{2(wT - sinwT)}{w^3} & -\frac{wT - sinwT}{w^2} & \frac{1-coswT}{w^2} \\ \frac{1-coswT}{w^2} & -\frac{wT - sinwT}{w^2} & T & 0 \\ \frac{wT - sinwT}{w^2} & \frac{1-coswT}{w^2} & 0 & T \end{bmatrix}, \tag{88}$$

$$R_k = \begin{bmatrix} 25^2 & 0 \\ 0 & 0.5^2 \end{bmatrix} \tag{89}$$

respectively. Assume that the state and measurement noises are truncated Gaussian with a mean $\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\end{bmatrix}$, and a covariance $Q_k/3^2$ and $R_k/3^2$ on the ellipsoidal sets, respectively.

The true trajectories in this example is generated by the state transform equation with satisfying the constraint. Specifically, given the true state at time k, the state at time $k+1$ is computed by the state transform function $f_k(x_k)$ and the process noise $w_k \in W_k$, which is selected to guarantee that the generated true state satisfies the quadratic inequality constraint at time $k+1$. The quadratic inequality constraint is as follows:

$$0 \le x_k^T \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} -30 \\ 50 \\ 0 \\ 0 \end{bmatrix}^T x_k - 1250 \le 200. \tag{90}$$

From the above description, it can be concluded that the conditions of the set-membership filtering are satisfied. Hence, set-membership filtering without constraints and the new method can be used to determine the bounding ellipsoid of the true state. The simulation results are based on 100 Monte Carlo runs of measurements.

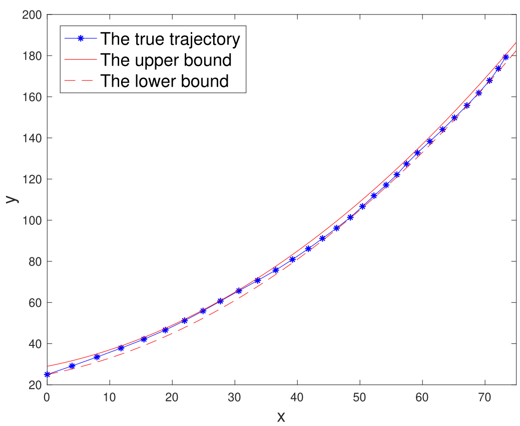

FIGURE 1. The true trajectory and the quadratic constraint.

Fig. 1 presents the true trajectory with the quadratic constraint. Fig. 2 demonstrates a comparison of the size of the state bounding ellipsoids between the new method and set-membership filtering without the constraints, where the size is the sum of the ellipsoid semi-axis lengths defined in [45]. Fig. 2 indicates that the size of the state bounding ellipsoids estimated by the new method is smaller than the size of the state bounding ellipsoids estimated by the set-membership filtering without the constraints, which means that the performance of the new method is better. Fig. 3 presents the root mean square error (RMSE) of the new method and set-membership filtering without the constraints, respectively. Fig. 3 shows that the new method can offer a smaller RMSE than the RMSE of set-membership filtering without the constraints, and the reason is that the constraint information is

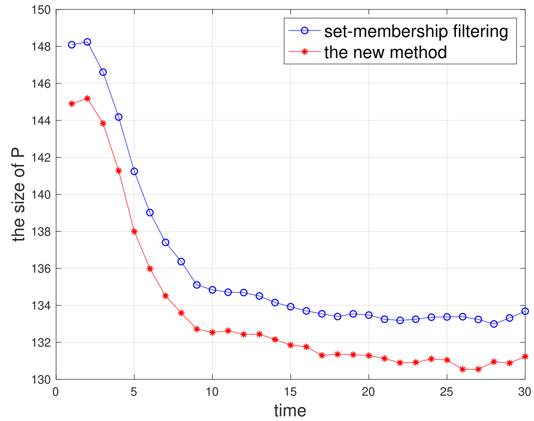

FIGURE 2. Comparison of the size of the state bounding ellipsoids.

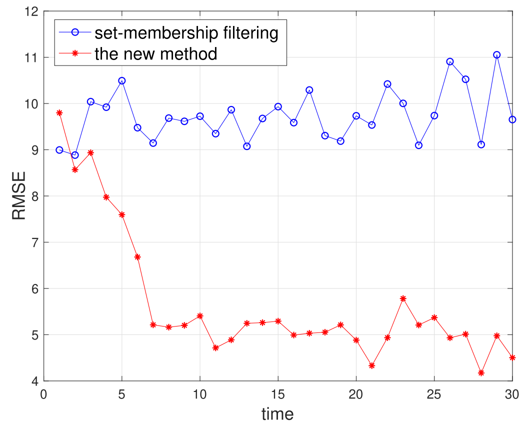

FIGURE 3. Comparison of RMSE of the trajectory estimate.

considered in the new method, which is also consistent with the discussion in Remark 2.

V. CONCLUSION

The present study investigated the problem of set-membership filtering for nonlinear dynamic systems with general nonconvex inhomogeneous quadratic inequality constraints. The bounding ellipsoid for the remainder can be derived *via* the current state bounding ellipsoid on-line. Based on the S-procedure and Schur complements, both the prediction step and measurement update step of the quadratic inequality constrained set-membership filtering problem can be transformed into an SDP problem. In order to further reduce the computational complexity, the part-analytical formula of the state bounding ellipsoid shape matrix and center was derived, and this would be helpful in clarifying how the constraints affect the state estimation. Consistent with the discussion about the role of constraints in the state estimation, the numerical example illustrates that the performance of the new method is better than the performance of set-membership filtering without using constraints. Future work may include more general inequality constraints and multisensor set-membership filter fusion.

APPENDIXES

APPENDIX A

Proof 1.1 (Proof of Proposition 1): Note that $x_k \in \mathcal{E}_k$ is equivalent to $x_k = \hat{x}_k + E_k u_k$, $\|u_k\| \leq 1$, where $P_k = E_k(E_k)^T$ (i.e., E_k is the Cholesky factorization of P_k). By Taylor's Theorem and the remainder bounding techniques [33], we have:

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1|k} &= f_k(x_k) + w_k - \hat{x}_{k+1|k} \\ &= f_k(\hat{x}_k + E_k u_k) + w_k - \hat{x}_{k+1|k} \\ &= f_k(\hat{x}_k) + J_{f_k} E_k u_k \\ &\quad + e_{f_k} + B_{f_k} \Delta_{f_k} + w_k - \hat{x}_{k+1|k}, \end{aligned} \tag{91}$$

$$\begin{aligned} y_k &= h_k(x_k) + v_k \\ &= h_k(\hat{x}_k) + J_{h_k} E_k u_k + e_{h_k} + B_{h_k} \Delta_{h_k} + v_k. \end{aligned} \tag{92}$$

In defining

$$\xi = [1, u_k^T, w_k^T, v_k^T, \Delta_{f_k}^T, \Delta_{h_k}^T, \Delta]^T, \tag{93}$$

(91) and (92) can be rewritten as:

$$x_{k+1} - \hat{x}_{k+1|k} = \Phi_{k+1|k}(\hat{x}_{k+1|k})\xi, \tag{94}$$

$$0 = \Psi_{k+1|k}(y_k)\xi, \tag{95}$$

the constraint of x_k (i.e., (12)) can be rewritten as:

$$\xi^T \Gamma \xi = 0, \tag{96}$$

where $\Phi_{k+1|k}$, $\Psi_{k+1|k}$ and Γ are denoted by (18), (19) and (22), with compatible dimension zero matrices among these, respectively.

In addition, the condition of $x_{k+1} \in \mathcal{E}_{k+1|k}$

$$(x_{k+1} - \hat{x}_{k+1|k})^T (P_{k+1|k})^{-1} (x_{k+1} - \hat{x}_{k+1|k}) \leq 1 \tag{97}$$

is equivalent to

$$\begin{aligned} &\xi^T [\Phi_{k+1|k}(\hat{x}_{k+1|k})^T (P_{k+1|k})^{-1} \Phi_{k+1|k}(\hat{x}_{k+1|k}) \\ &\quad - diag(1, 0, 0, 0, 0, 0, 0)]\xi \leq 0, \end{aligned} \tag{98}$$

with compatible dimension zero matrices among this. Based on the definition of ξ, the conditions of the unknown variables, and the S-procedure [46], [47], a sufficient condition of (98) to hold is that there exist scalars $\lambda^y \in \mathbb{R}$, $\lambda_2^g \in \mathbb{R}$, $\lambda^u \geq 0$, $\lambda^w \geq 0$, $\lambda_1^g \geq 0$, $\lambda^v \geq 0$, $\lambda^f \geq 0$, $\lambda^h \geq 0$, such that:

$$\begin{aligned} &\Phi_{k+1|k}(\hat{x}_{k+1|k})^T (P_{k+1|k})^{-1} \Phi_{k+1|k}(\hat{x}_{k+1|k}) - \Xi \\ &\quad - \lambda^y (\Psi_{k+1|k}(y_k))^T \Psi_{k+1|k}(y_k) \preceq 0, \end{aligned} \tag{99}$$

where Ξ is denoted by (24) with compatible dimension zero matrices among this.

By denoting $(\Psi_{k+1|k}(y_k))^\perp$ as the orthogonal complement of $\Psi_{k+1|k}(y_k)$, the inequality can be obtained:

$$\begin{aligned} &((\Psi_{k+1|k}(y_k))^\perp)^T \Phi_{k+1|k}(\hat{x}_{k+1|k})^T (P_{k+1|k})^{-1} \\ &\quad \Phi_{k+1|k}(\hat{x}_{k+1|k})(\Psi_{k+1|k}(y_k))^\perp \\ &\quad - ((\Psi_{k+1|k}(y_k))^\perp)^T \Xi (\Psi_{k+1|k}(y_k))^\perp \preceq 0. \end{aligned} \tag{100}$$

Based on Schur complements [46], (100) can be rewritten as:

$$\begin{bmatrix} -P_{k+1|k} & \Phi_{k+1|k}(\hat{x}_{k+1|k})(\Psi_{k+1|k}(y_k))^\perp \\ (\Phi_{k+1|k}(\hat{x}_{k+1|k})(\Psi_{k+1|k}(y_k))^\perp)^T & -((\Psi_{k+1|k}(y_k))^\perp)^T \Xi (\Psi_{k+1|k}(y_k))^\perp \end{bmatrix} \preceq 0, \tag{101}$$

$$-P_{k+1|k} \prec 0. \tag{102}$$

The above analysis outlines the method of determining the predicted state bounding ellipsoid. The optimal predicted state bounding ellipsoid can be derived by minimizing $f(P_{k+1|k})$.

Proof 1.2 (Proof of Proposition 2): By using Schur complements [46] and reordering of the blocks, (17) is equivalent to:

$$\begin{bmatrix} P_{k+1|k} & Z & B \\ Z^T & \Xi_{11} + \Psi_1^T \Xi_{22} \Psi_1 & \Psi_1^T \Xi_{22} \Psi_2 - \Xi_{12} \Psi_2 \\ B^T & \Psi_2^T \Xi_{22} \Psi_1 - \Psi_2^T \Xi_{12}^T & \Psi_2^T \Xi_{22} \Psi_2 \end{bmatrix} \succeq 0, \tag{103}$$

$$Z = \hat{x}_{k+1|k} - f_k(\hat{x}_k), \tag{104}$$

$$B = [I, 0, 0], \tag{105}$$

where zero matrices and identity matrices have compatible dimensions. By using the decoupled method [30], the optimization problem in Proposition 1 is equivalent to:

$$\min_{\lambda_2^g, \lambda^u, \lambda^w, \lambda_1^g, \lambda^v, \lambda^f, \lambda^h} f(B(\Psi_2^T \Xi_{22} \Psi_2)^+ B^T) \tag{106}$$

$$\textit{subject to } (40), (41), (I - (\Psi_2^T \Xi_{22} \Psi_2)^+ \Psi_2^T \Xi_{22} \Psi_2) B^T = 0, \tag{107}$$

and the optimal ellipsoid $\mathcal{E}_{k+1|k}$ can be computed as:

$$P_{k+1|k} = B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} B^T, \tag{108}$$

$$Z = B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} (\Psi_2^T \Xi_{22} \Psi_1 - \Psi_2^T \Xi_{12}^T). \tag{109}$$

Through the definition in (25)-(31) and (34)-(38), we have:

$$\begin{aligned} &B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} B^T \\ &= (\lambda^u P_k^{-1} + \lambda_2^g G_k + (J_{f_k} - I)^T (\frac{Q_k}{\lambda^w} + \frac{P_{f_k}}{\lambda^f})^{-1} (J_{f_k} - I) \\ &\quad + J_{h_k}^T (\frac{R_k}{\lambda^v} + (\lambda^h P_{h_k}^{-1} + \lambda_1^g I)^{-1})^{-1} J_{h_k})^{-1}, \end{aligned} \tag{110}$$

$$\begin{aligned} Z &= \hat{x}_{k+1|k} - f_k(\hat{x}_k) \\ &= P_{k+1|k} X_1 + P_{k+1|k} C_1 X_2 + P_{k+1|k} C_2 X_3. \end{aligned} \tag{111}$$

Thus, (32) and (33) can be achieved.

APPENDIX B

Proof 2.1 (Proof of Proposition 3): $x_{k+1} \in \mathcal{E}_{k+1|k}$ is equivalent to that $x_{k+1} = \hat{x}_{k+1|k} + E_{k+1|k}u_{k+1|k}$, $\|u_{k+1|k}\| \leq 1$, where $E_{k+1|k}$ is the Cholesky factorization of $P_{k+1|k}$. Hence we have:

$$x_{k+1} - \hat{x}_{k+1} = \hat{x}_{k+1|k} + E_{k+1|k}u_{k+1|k} - \hat{x}_{k+1}, \quad (112)$$

and

$$\begin{aligned} y_{k+1} &= h_{k+1}(x_{k+1}) + v_{k+1} \\ &= h_{k+1}(\hat{x}_{k+1|k}) + J_{h_{k+1}} E_{k+1|k}u_{k+1|k} \\ &\quad + e_{h_{k+1}} + B_{h_{k+1}}\Delta_{h_{k+1}} + v_{k+1}. \end{aligned} \quad (113)$$

In defining

$$\xi = [1, u_{k+1|k}^T, v_{k+1}^T, \Delta_{h_{k+1}}^T, \Delta]^T, \quad (114)$$

(112) and (113) can be rewritten as:

$$x_{k+1} - \hat{x}_{k+1} = \Phi_{k+1}(\hat{x}_{k+1})\xi, \quad (115)$$

$$0 = \Psi_{k+1}(y_{k+1})\xi, \quad (116)$$

the constraint of x_k (i.e., (12)) can be rewritten as:

$$\xi^T \Gamma \xi = 0, \quad (117)$$

where $\Phi_{k+1}(\hat{x}_{k+1})$, $\Psi_{k+1}(y_{k+1})$ and Γ are denoted by (46), (47) and (50), with compatible dimension zero matrices among these, respectively.

In addition, the condition of $x_{k+1} \in \mathcal{E}_{k+1}$

$$(x_{k+1} - \hat{x}_{k+1})^T (P_{k+1})^{-1} (x_{k+1} - \hat{x}_{k+1}) \leq 1 \quad (118)$$

is equivalent to

$$\xi^T [\Phi_{k+1}(\hat{x}_{k+1})^T (P_{k+1})^{-1} \Phi_{k+1}(\hat{x}_{k+1}) - diag(1, 0, 0, 0, 0)]\xi \leq 0, \quad (119)$$

with compatible dimension zero matrices among this. Based on the definition of ξ, the conditions of the unknown variables, and the S-procedure [46], [47], a sufficient condition of (119) to hold is that there exist scalars $\lambda^y \in \mathbb{R}$, $\lambda_2^g \in \mathbb{R}$, $\lambda^u \geq 0$, $\lambda^v \geq 0$, $\lambda_1^g \geq 0$, $\lambda^h \geq 0$, such that:

$$\Phi_{k+1}(\hat{x}_{k+1})^T (P_{k+1})^{-1} \Phi_{k+1}(\hat{x}_{k+1}) - \Xi - \lambda^y (\Psi_{k+1}(y_{k+1}))^T \Psi_{k+1}(y_{k+1}) \preceq 0, \quad (120)$$

where Ξ is denoted by (52) with compatible dimension zero matrices among this.

By denoting $(\Psi_{k+1}(y_{k+1}))^\perp$ as the orthogonal complement of $\Psi_{k+1}(y_{k+1})$, the following inequality can be obtained:

$$\begin{aligned} &((\Psi_{k+1}(y_{k+1}))^\perp)^T \Phi_{k+1}(\hat{x}_{k+1})^T (P_{k+1})^{-1} \\ &\quad \Phi_{k+1}(\hat{x}_{k+1})(\Psi_{k+1}(y_{k+1}))^\perp \\ &\quad - ((\Psi_{k+1}(y_{k+1}))^\perp)^T \Xi (\Psi_{k+1}(y_{k+1}))^\perp \preceq 0. \end{aligned} \quad (121)$$

Based on Schur complements [46], (121) can be rewritten as:

$$\begin{bmatrix} -P_{k+1} & \Phi_{k+1}(\hat{x}_{k+1})(\Psi_{k+1}(y_{k+1}))^\perp \\ (\Phi_{k+1}(\hat{x}_{k+1})(\Psi_{k+1}(y_{k+1}))^\perp)^T & -((\Psi_{k+1}(y_{k+1}))^\perp)^T \Xi (\Psi_{k+1}(y_{k+1}))^\perp \end{bmatrix} \preceq 0, \quad (122)$$

$$-P_{k+1} \prec 0. \quad (123)$$

The above analysis outlines the principle of determining the state bounding ellipsoid. The optimal state bounding ellipsoid can be derived by minimizing $f(P_{k+1})$.

Proof 2.2 (Proof of Proposition 4): By using Schur complements [46] and reordering of the blocks, (45) is equivalent to:

$$\begin{bmatrix} P_{k+1} & Z & B \\ Z^T & \Xi_{11} + \Psi_1^T \Xi_{22} \Psi_1 & \Psi_1^T \Xi_{22} \Psi_2 - \Xi_{12} \Psi_2 \\ B^T & \Psi_2^T \Xi_{22} \Psi_1 - \Psi_2^T \Xi_{12}^T & \Psi_2^T \Xi_{22} \Psi_2 \end{bmatrix} \succeq 0, \quad (124)$$

$$Z = \hat{x}_{k+1} - \hat{x}_{k+1|k}, \quad (125)$$

$$B = [I, 0], \quad (126)$$

where zero matrices and identity matrices have compatible dimensions. Through the decoupled method [30], the optimization problem in Proposition 3 can be equivalent to:

$$\min_{\lambda_2^g, \lambda^u, \lambda^w, \lambda_1^g, \lambda^v, \lambda^f, \lambda^h} f(B(\Psi_2^T \Xi_{22} \Psi_2)^+ B^T) \quad (127)$$

$$\textit{subject to } (66), (67), (I - (\Psi_2^T \Xi_{22} \Psi_2)^+ \Psi_2^T \Xi_{22} \Psi_2) B^T = 0, \quad (128)$$

and the optimal ellipsoid $\mathcal{E}_{k+1}$ can be computed as:

$$P_{k+1} = B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} B^T, \quad (129)$$

$$Z = B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} (\Psi_2^T \Xi_{22} \Psi_1 - \Psi_2^T \Xi_{12}^T). \quad (130)$$

Through the definition in (53)-(59) and (62)-(64), we have:

$$\begin{aligned} &B(\Psi_2^T \Xi_{22} \Psi_2)^{-1} B^T \\ &= (\lambda^u P_{k+1|k}^{-1} + \lambda_2^g G_{k+1} \\ &\quad + J_{h_{k+1}}^T (\frac{R_{k+1}}{\lambda^v} + (\lambda^h P_{h_{k+1}}^{-1} + \lambda_1^g I)^{-1})^{-1} J_{h_{k+1}})^{-1}, \end{aligned} \quad (131)$$

$$Z = \hat{x}_{k+1} - \hat{x}_{k+1|k} = P_{k+1}X_1 + P_{k+1}CX_2. \quad (132)$$

Thus, (60) and (61) can be achieved.

APPENDIX C

Proof 3.1 (Proof of Corollary 1): Without loss of generality, assume that d_k^1 in (3) takes infinity and $d_k^2 \geq 0$, then the unilateral constraint has the form:

$$x_k^T \hat{G}_k x_k + \hat{\beta}_k^T x_k + \hat{\alpha}_k \leq d_k^2. \quad (133)$$

Hence, this can be rewritten as:

$$x_k^T G_k x_k + \beta_k^T x_k + \alpha_k \leq 1, \quad (134)$$

where $G_k = \frac{1}{d_k^2}\hat{G}_k$, $\beta_k^T = \frac{1}{d_k^2}\hat{\beta}_k^T$, and $\alpha_k = \frac{1}{d_k^2}\hat{\alpha}_k$.

Prediction step: If we define

$$\xi = [1, u_k^T, w_k^T, v_k^T, \Delta_{f_k}^T, \Delta_{h_k}^T]^T, \tag{135}$$

(134) can be rewritten as:

$$\xi^T(\Gamma - diag(1, 0, 0, 0, 0, 0))\xi \leq 0, \tag{136}$$

where Γ is denoted by (70) with compatible dimension zero matrices among this.

In addition, the condition of $x_{k+1} \in \mathcal{E}_{k+1|k}$ is equivalent to:

$$\xi^T[\Phi_{k+1|k}(\hat{x}_{k+1|k})^T(P_{k+1|k})^{-1}\Phi_{k+1|k}(\hat{x}_{k+1|k}) - diag(1, 0, 0, 0, 0, 0)]\xi \leq 0, \tag{137}$$

where $\Phi_{k+1|k}$ is denoted by (68) with compatible dimension zero matrices among this. Similar to Proposition 1, based on the definition of ξ, the conditions of the unknown variables, S-procedure [46], [47], and Schur complements [46], the prediction step can be transformed into an SDP problem with the definition in Corollary 1.

Update step: If we define

$$\xi = [1, u_{k+1|k}^T, v_{k+1}^T, \Delta_{h_{k+1}}^T]^T, \tag{138}$$

the constraint of x_k can be rewritten as:

$$\xi^T(\Gamma - diag(1, 0, 0, 0))\xi \leq 0, \tag{139}$$

where Γ is denoted by (74) with compatible dimension zero matrices among this.

Furthermore, the condition of $x_{k+1} \in \mathcal{E}_{k+1}$ can be rewritten as:

$$\xi^T[\Phi_{k+1}(\hat{x}_{k+1})^T(P_{k+1})^{-1}\Phi_{k+1}(\hat{x}_{k+1}) - diag(1, 0, 0, 0)]\xi \leq 0, \tag{140}$$

where Φ_{k+1} is denoted by (72) with compatible dimension zero matrices among this. Similar to Proposition 3, based on the definition of ξ, conditions of the unknown variables, S-procedure [46], [47], and Schur complements [46], the update step can be transformed into an SDP problem with the definition in Corollary 1.

Proof 3.2 (Proof of Corollary 2): If $\hat{G}_k$ in (3) takes zero matrix, (11) can be written as

$$-1 \leq \beta_k^T x_k + \alpha_k \leq 1. \tag{141}$$

Prediction step: If we define

$$\xi = [1, u_k^T, w_k^T, v_k^T, \Delta_{f_k}^T, \Delta_{h_k}^T]^T, \tag{142}$$

(141) can be rewritten as:

$$\xi^T(\Gamma^T\Gamma - diag(1, 0, 0, 0, 0, 0))\xi \leq 0, \tag{143}$$

where Γ is denoted by (78) with compatible dimension zero matrices among this.

In addition, the condition of $x_{k+1} \in \mathcal{E}_{k+1|k}$ is equivalent to:

$$\xi^T[\Phi_{k+1|k}(\hat{x}_{k+1|k})^T(P_{k+1|k})^{-1}\Phi_{k+1|k}(\hat{x}_{k+1|k}) - diag(1, 0, 0, 0, 0, 0)]\xi \leq 0, \tag{144}$$

where $\Phi_{k+1|k}$ is denoted by (76) with compatible dimension zero matrices among this. Similar to Proposition 1, based on the definition of ξ, the conditions of the unknown variables, S-procedure [46], [47], and Schur complements [46], the prediction step can be transformed into an SDP problem with the definition in Corollary 2.

Update step: if we define

$$\xi = [1, u_{k+1|k}^T, v_{k+1}^T, \Delta_{h_{k+1}}^T]^T, \tag{145}$$

the constraint of x_k can be rewritten as:

$$\xi^T(\Gamma^T\Gamma - diag(1, 0, 0, 0))\xi \leq 0, \tag{146}$$

where Γ is denoted by (82) with compatible dimension zero matrices among this.

Furthermore, the condition of $x_{k+1} \in \mathcal{E}_{k+1}$ is equivalent to:

$$\xi^T[\Phi_{k+1}(\hat{x}_{k+1})^T(P_{k+1})^{-1}\Phi_{k+1}(\hat{x}_{k+1}) - diag(1, 0, 0, 0)]\xi \leq 0, \tag{147}$$

where Φ_{k+1} is denoted by (80) with compatible dimension zero matrices among this. Similar to Proposition 3, based on the definition of ξ, the conditions of the unknown variables, S-procedure [46], [47], and Schur complements [46], the update step can be transformed into an SDP problem with the definition in Corollary 2.

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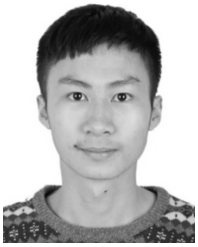

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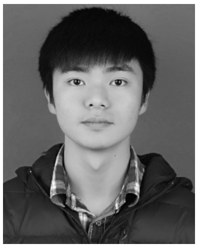

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