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# Distributed Incremental Cost Consensus-Based Optimization Algorithms for Economic Dispatch in a Microgrid

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**ABSTRACT** The economic dispatch problems (EDPs) in a microgrid (MG) have been extensively investigated by a variety of emerging algorithms. In this paper, we propose two newly distributed dynamic optimization algorithms to respectively study the EDPs under both cases without and with generation constraints under a directed topology network. Two novel dynamic optimization algorithms are based on the distributed incremental cost consensus (ICC), where the mismatch between total demand and power generation is considered. Our algorithms only require the weight matrix of the directed network to be row stochastic. The theoretical analysis on the convergence of the proposed algorithms is presented by using the small gain theorem. It can be found that the algorithms are convergent at the geometric rate. Meanwhile, the power output of the generators are proved to achieve the optimal solution of EDPs based on the proposed algorithms. Finally, the corresponding conditions are also derived, and simulation studies illustrate the correctness of our results.

**INDEX TERMS** Economic dispatch problem, optimization algorithms, smart grid, incremental cost consensus, geometric convergence.

## I. INTRODUCTION

The new emerging micro-grids (MGs) integrate various technologies, concepts, smart infrastructure and advanced management, and they are also equipped with intelligent controllable electrical equipment. Compared with traditional power grids, the micro-grid combines a variety of advantages, such as being more reliable, safe, sustainable, resilient, and so on [1]–[4]. Generators exist in the MG, and each generator produces different amounts of power to supply the entire loads' demand. The economic dispatch problem (EDP) [5], [6] in an MG is derived from this scenario, and its goal is to properly schedule the power output of each generator so as not to waste power resources and at the same time minimize the economic cost [7]–[9]. In this case, it can be viewed as an optimization problem [10]–[12]. For years, the EDP, as one of the most important optimization issues in the micro-grid community, has attracted widespread attention from a number of researchers [13]–[21]. In order to tackle this kind of problems, in the beginning, some centralized optimization algorithms were proposed [22]–[24], the disadvantages of which have: (1) A powerful central controller is needed to gather the global information such that it can deal with huge amounts of data. (2) These algorithms are costly and easy to be subjected to single-point-failure. (3) They lack robustness and need to be re-established when generators and loads need be added or removed in MGs [25]. In order to overcome these deficiencies, some distributed optimization algorithms are currently widely established [26]–[31].

Soon afterwards, some distributed algorithms on the basis of multi-agent consensus are emerged to investigate the EDPs [32]–[34], where the incremental cost (IC) of each generator is selected as the consensus variable. In [32], the authors firstly apply the multi-agent consensus theory into the distributed algorithm, and then, a simplest distributed optimization algorithm is designed to solve the EDP in an MG

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environment. Subsequently, in [33], a similar ICC algorithm is proposed to solve the EDP under the different communication topology network. The proposed algorithm is the simple gradient optimization algorithm that includes two terms: consensus term and optimization term. In order to solve the more complicated EDPs in an MG, an improved power controller framework based on the incremental cost consensus is developed in [34]. In that framework, each generator includes two modules: a distributed economic dispatch module and a cooperative control module. In [35], by developing an ICC algorithm, the EDP for islanded MGs is investigated. In [36], a new ICC algorithm is proposed to deal with the EDP with generation constraints, where the local mismatch between total power generated and total demand is considered as a feedback variable. In above mentioned algorithms, the communication topology network is assumed as undirected and connected, or their topology graph is weight-balanced. Even, their weighted matrix in [35], [36] is doubly stochastic. However, these assumptions are too strict to be implemented in some practical applications and should be relaxed. This motivates researchers to investigate the EDP in an MG under the directed topology network with row or column stochastic matrix. In [37], [38], the authors propose some optimization algorithms with column-stochastic matrix by employing the push-sum technique to solve a class of consensus optimization problem in multi-agent systems (MASs). However, using a column stochastic matrix in optimization algorithms is impractical in many situations. For example, when the network communication uses a broadcast mechanism, each agent does not know itself out-neighbors let alone to adjust its outgoing weights. In this case, a row stochastic matrix is much easier to be implemented in many applications than a column stochastic matrix. Hence, the idea that a row stochastic matrix is used in algorithm is motivated to introduce [39], where each agent can individually determine the weight values from the information that it receives from its neighbors.

Our idea in this paper derives from the referred literatures that have been presented, however, there are some differences. The more advanced algorithms with a row stochastic matrix are motivated to solve the EDPs. Compared with the algorithms [32]–[36], the matrices of our proposed two optimization algorithms are row stochastic. Moreover, a different analysis approach is adopted to prove the geometric convergence of these algorithms. In particular, our algorithms are similar with one in [36], but the main differences have the following: (1) The theoretical analysis is presented by using small gain theorem method such that the geometric convergence rate is obtained; (2) The weighted matrix is row stochastic in our proposed algorithms by introducing an auxiliary variable. From the above discussions, the biggest challenge is to propose the more advanced algorithms with a row stochastic matrix to solve the EDPs. This motivates us to develop improved algorithms to investigate the various kind of EDPs in an MG.

In this paper, two new algorithms are developed for respectively solving EDPs without generation constraints and with generation constraints under the directed topology network. One algorithm is for EDP without generation constraints, and another is for EDP with generation constraints. In those algorithms, the mismatch between total demand and power generation is considered. And, their main advantage is that the weight matrix of the network topology is row stochastic by introducing an auxiliary variable. The main contributions of this paper include:

(1) Two new algorithms with row stochastic matrix are developed to respectively tackle the EDPs in an MG without generation constraints and with generation constraints under the directed topology network, where the IC of each generator is considered as consensus variable.

(2) The theoretical analysis on the geometric convergence of the proposed algorithms is represented by using small gain theorem method. It can be found that the power output of the generators reach the optimal solution of EDPs, and the incremental cost converges to a common constant.

(3) Based on our theoretical analysis, the algebraic conditions related to some parameters including feedback gains, cost coefficients and some constants are derived. Finally, simulation studies illustrate the performance and scalability of our algorithms. We also try to use our algorithm to deal with EDP with varying demand.

## II. PRELIMINARIES

### A. GRAPH THEORY

In an MG, there exist $N$ power generators, and its communication topology is depicted by a graph $g = (V, \varepsilon, A)$, where $V = \{1, 2, \ldots, N\}$ is a set of generators and $\varepsilon \subseteq V \times V$ is a set of edge between generators. $(i, j) \in \varepsilon$ is the directed edge from $i$ to $j$. $N_i = \{j \in \nu | (j, i) \in \varepsilon\}$ is a set of neighbor generators of the $i$-th generator. $A = (a_{ij})_{N\times N}$ is the connection weight matrix of graph $g$. If there are no self-loops, it has $a_{ii} = 0$, and $a_{ij} > 0 \Leftrightarrow j \in N_i$. Otherwise, $a_{ii} > 0$. The directed graph $g$ is strongly connected if and only if for any two distinct generators there is a path from generator $i$ to generator $j$. Here, assume that the network topology graph is directed connected and the weight matrix $A$ is row stochastic, i.e., $\sum_{j=1}^{N} a_{ij} = 1$.

### B. EDP

In this paper, we suppose that the cost objective function $C_i(p_i)$ of each generator is a simple quadratic cost function, that is, $C_i(p_i) = \alpha_i p_i^2 + \beta_i p_i + \gamma_i$. Here, we investigate the ordinary EDP with demand constraint and generator constraints as follows:

$$\begin{aligned} &\min \sum_{i=1}^{N} C_i(p_i), \\ &s.t. \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} p_{li} = P_0, \\ &\quad p_{\min, i} \le p_i \le p_{\max, i}, \end{aligned} \tag{1}$$

where $\alpha_i > 0$, $\beta_i \geq 0$ and $\gamma_i \geq 0$ are the cost coefficients, $p_i$ is the generation power of generator $i$, $p_{li}$ is the load power of generator $i$, $P_0$ is the total power demand, $p_{\min,i}$ and $p_{\max,i}$ are the lower and upper bounds of the generation. Obviously, it has $\sum_{i=1}^{N} p_{\min,i} \leq P_0 \leq \sum_{i=1}^{N} p_{\max,i}$. The first constraint is called demand constraint, and the second constraint $p_{\min,i} \leq p_i \leq p_{\max,i}$ is the generation constraint.

Denote the incremental cost for each generator as $\lambda_i = \frac{d\,C_i(p_i)}{d\,p_i} = 2\alpha_i p_i + \beta_i$, and we select $\lambda_i$ as the consensus variable. If $\lambda_i$ converges to a constant $\lambda^*$, according to the constraints in (1), we have

$$\lambda^* = \frac{P_0 + \sum_{i=1}^{N} \frac{\beta_i}{2\alpha_i}}{\sum_{i=1}^{N} \frac{1}{2\alpha_i}}. \tag{2}$$

It follows from (2) that, without generation constraints, its optimal solution is

$$p_i^* = \frac{\lambda^* - \beta_i}{2\alpha_i}. \tag{3}$$

When we consider the generation constraints in EDP (1), its optimal solution is the following:

$$\lambda^* = \frac{P_0 - \sum_{i\in\Omega} p_i + \sum_{i\notin\Omega} \frac{\beta_i}{2\alpha_i}}{\sum_{i\notin\Omega} \frac{1}{2\alpha_i}},$$
$$p_i^* = \begin{cases} \dfrac{\lambda^* - \beta_i}{2\alpha_i}, & i \notin \Omega_p \\ p_{\min,i} \text{ or } p_{\max,i}, & i \in \Omega_p, \end{cases} \tag{4}$$

where $\Omega_p$ is the subset of the generators with their limits for the optimal assignment. The results (3)-(4) are presented in [34].

## III. DISTRIBUTED ALGORITHMS FOR EDP

Here, we propose respectively two algorithms based on the ICC to solve the EDP without generation constraints and with generation constraints. The convergence analysis ( $\lambda_i(t) \to \lambda^*$ and $p_i(t) \to p_i^*$) of the proposed algorithms are presented, and the theoretical results for solving EDP are obtained.

The following basic lemmas provide us with some known results to be used later in this paper.

*Lemma 1 [40]: If the topology network is strongly connected and the weight matrix A is row stochastic, Then there exists a strictly positive vector $w = [w_1, w_2, \ldots, w_N]^T$ such that $\lim_{t\to\infty} A^t = 1 \cdot w^T$ and $w^T A = w^T$. The vector w is called as the left-Perron eigenvector of A.*

*Lemma 2 [41]: For any $x \in R^N$, define $\hat{x} = A_\infty x$. There exists a constant $0 < \sigma < 1$ such that for all t:*

$$\left\| Ax(t) - \hat{x}(t) \right\| \leq \sigma \left\| x(t) - \hat{x}(t) \right\|.$$

*Lemma 3 [42]: If the objective function $f_i$ satisfies that*

*(a) there exists a positive constant l such that $\|\nabla f_i(x_1) - \nabla f_i(x_2)\| \leq l\,\|x_1 - x_2\|$ holds;*

*(b) there exists a positive constant r such that $r\|x_1 - x_2\|^2 \leq \langle \nabla f_i(x_1) - \nabla f_i(x_2), x_1 - x_2 \rangle$ holds. Then, for any $x \in R$ and $x_+ = x - \alpha \nabla f_i(x)$ with $0 < \alpha < \frac{2}{l}$, we have $\|x_+ - x^*\| \leq \eta\,\|x - x^*\|$ with $\eta = \max(|1-\alpha l|, |1-\alpha r|)$.*

We define $\|s_i\|^{\mu,K} = \max_{k=0,\ldots,K} \frac{1}{\mu^k} \|s_i(k)\|$ and $\|s_i\|^{\mu} = \sup_{k\geq 0} \frac{1}{\mu^k} \|s_i(k)\|$ with $\mu \in (0,1)$ and $K$ is positive integer.

*Lemma 4 (The Small Gain Theorem [43]): Given the infinite sequence $s_i = (s_i(0), s_i(1), s_i(2), \ldots)$. For each $i = 1, 2, \ldots, m$ and positive constant K, we have $s_i \to s_{(i \bmod m)+1}$, i.e.,*

$$\left\| s_{(i \bmod m)+1} \right\|^{\mu,K} \leq r_i \|s_i\|^{\mu,K} + \omega_i$$

*with constants $\omega_i, r_1, r_2, \ldots, r_m \geq 0$ and $r_1 \cdot r_2 \cdots r_m < 1$. Then, it holds*

$$\|s_1\|^{\mu} \leq \frac{\omega_1 r_2 \cdots r_m + \omega_2 r_3 \cdots r_m + \cdots + \omega_{m-1} r_m + \omega_m}{1 - r_1 r_2 \cdots r_m}.$$

*Lemma 5 (Bounded Norm Geometric Rate) [43]: If $\|s_i\|^{\mu}$ is bounded, $\|s_i\|$ converges at a global geometric rate $O(\mu^t)$. That is, if the small gain theorem holds for $\|s_i\|^{\mu}$, $\forall i$, $\|s_i\|$ is convergent at geometric rate $O(\mu^t)$.*

### A. DISTRIBUTED ALGORITHMS FOR EDP WITHOUT GENERATION CONSTRAINTS

In this subsection, we establish a distributed algorithm to tackle the EDP without generation constraints. Let $\lambda_i(t)$ and $p_i(t)$ be the estimations of the optimal incremental cost and the optimal power for iteration $t$th respectively. The distributed iterative algorithm to solve EDP (1) without generation constraints is developed as

$$\begin{aligned}
\lambda_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij}\lambda_j(t) - \epsilon_i s_i(t), \\
s_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij} s_j(t) - \left[\frac{p_i(t+1)}{y_i(t+1)} - \frac{p_i(t)}{y_i(t)}\right], \\
p_i(t+1) &= \frac{\lambda_i(t+1) - \beta_i}{2\alpha_i}, \\
y_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij} y_j(t),
\end{aligned} \tag{5}$$

where $\epsilon_i$ is a nonnegative constant with $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_N$, $s_i(t)$ is the local estimation of the mismatch with $s_i(0) = 0$, $y_i(t)$ is the auxiliary variable with $y_i(0) = 1$. And $y_i(t) > 0$, for $\forall i, \forall t$.

Since the weight matrix $A = (a_{ij})_{N\times N}$ is row stochastic, according to Lemma 1, it follows from the second formula of (5) that

$$\begin{aligned}
&\sum\nolimits_{i=1}^{N} w_i s_i(t+1) + \sum\nolimits_{i=1}^{N} w_i y_i^{-1}(t+1)\, p_i(t+1) \\
&\quad = \sum\nolimits_{i=1}^{N} w_i s_i(t) + \sum\nolimits_{i=1}^{N} w_i y_i^{-1}(t)\, p_i(t) \\
&\quad = \ldots = \sum\nolimits_{i=1}^{N} w_i s_i(0) + \sum\nolimits_{i=1}^{N} w_i p_i(0),
\end{aligned}$$

which implies that

$$\sum\nolimits_{i=1}^{N} w_i s_i(t+1) + \sum\nolimits_{i=1}^{N} w_i y_i^{-1}(t+1)\, p_i(t+1), \quad \forall t$$

is a constant. Here, $w_i, i = 1, 2, \ldots, N$ is the elements of left-Perron eigenvector of the weight matrix $A$, and some introductions on $w_i$ have been given in Lemma 1.

We set $s_i(0) = 0, \forall i$ and take $p_i(0)$ as any values such that $\sum_{i=1}^{N} w_i p_i(0) = P_0$. Hence, the initial values of (5) is:

$$\begin{cases} \lambda_i(0) = 2\alpha_i p_i(0) + \beta_i, \\ \sum_{i=1}^{N} w_i p_i(0) = P_0, \\ s_i(0) = 0, \\ y_i(0) = 1. \end{cases} \tag{6}$$

We can set $w_i p_i(0) = p_{li}$, from which, it satisfies the condition (6).

*Remark 1: Under the initial values (6), $\sum_{i=1}^{N} w_i s_i(t+1) = P_0 - \sum_{i=1}^{N} w_i y_i^{-1}(t+1) p_i(t+1), \forall t$ holds. When $t \to \infty$, $s_i(t+1) = 0$ and $y_i(t+1) = w_i$ (see [39]) hold. It means that $\sum_{i=1}^{N} p_i(t+1) = \sum_{i=1}^{N} p_i^* = P_0, t \to \infty$, which satisfies the equation constraint in (1).*

*From the initial condition (6), $\sum_{i=1}^{N} w_i p_i(0) = P_0$ means that the value $p_i(0)$ for all $i$ must be large enough such that their weighted sum is total demand $P_0$. Besides, $w_i p_i(0) = p_{li}$ means that the values $p_i(0)$ and $p_{li}$ have larger difference.*

In order to solve this initial problem described in Remark 1, the algorithm (5) is slightly modified as follows:

$$\begin{aligned} \lambda_i(t+1) &= \sum_{j\in N_i} a_{ij}\lambda_j(t) - \epsilon_i s_i(t), \\ s_i(t+1) &= \sum_{j\in N_i} a_{ij} s_j(t) \\ &\quad - \left[ N^{\frac{1}{t+2}} \frac{p_i(t+1)}{y_i(t+1)} - N^{\frac{1}{t+1}} \frac{p_i(t)}{y_i(t)} \right], \\ p_i(t+1) &= \frac{\lambda_i(t+1) - \beta_i}{2\alpha_i}, \\ y_i(t+1) &= \sum_{j\in N_i} a_{ij} y_j(t). \end{aligned} \tag{7}$$

It follows from (7) that:

$$\begin{aligned} &\sum_{i=1}^{N} w_i s_i(t+1) + N^{\frac{1}{t+2}} \sum_{i=1}^{N} w_i y_i^{-1}(t+1) p_i(t+1) \\ &= \sum_{i=1}^{N} w_i s_i(t) + N^{\frac{1}{t+1}} \sum_{i=1}^{N} w_i y_i^{-1}(t) p_i(t) \\ &= \ldots = \sum_{i=1}^{N} w_i s_i(0) + N \sum_{i=1}^{N} w_i p_i(0), \end{aligned}$$

then, the corresponding initial values are modified as follows:

$$\begin{cases} \lambda_i(0) = 2\alpha_i p_i(0) + \beta_i, \\ N\sum_{i=1}^{N} w_i p_i(0) = P_0, \\ s_i(0) = 0, \\ y_i(0) = 1. \end{cases} \tag{8}$$

We set $N w_i p_i(0) = p_{li}$, which satisfies the condition (8).

*Remark 2: Under the initial values (8), we can obtain the same results as Remark 1. That is, $\sum_{i=1}^{N} p_i(t+1) = \sum_{i=1}^{N} p_i^* = P_0$, $t \to \infty$ holds since $s_i(t+1) = 0$, $N^{\frac{1}{t+2}} = 1$ and $y_i(t+1) = w_i$ (see [39]) hold for $t \to \infty$.*

The vector form of (7) is

$$\begin{aligned} \lambda(t+1) &= A\lambda(t) - \Lambda s(t), \\ p(t+1) &= \frac{1}{2}\alpha^{-1}\lambda(t+1) - \pi, \\ s(t+1) &= As(t) - [H(t+1) - H(t)], \\ y(t+1) &= Ay(t), \end{aligned} \tag{9}$$

where $\Lambda = diag\{\epsilon_i\}$, $\alpha = diag\{\alpha_i\}$, $\pi = diag\left\{\frac{\beta_i}{2\alpha_i}\right\}$, $H(t) = N^{\frac{1}{t+1}} Y^{-1}(t) p(t)$ and $Y(t) = diag\{y_i(t)\}$.

For the following analysis, we define some notations:

$$\begin{aligned} &A_\infty = \lim_{t\to\infty} A^t, \quad A_\infty = 1 \cdot w^T, \ \hat{\lambda}(t) = A_\infty \lambda(t), \\ &\hat{s}(t) = A_\infty s(t), \quad \bar{s}(t) = \sum_{i=1}^{N} w_i s_i(t), \\ &\lambda_i^* = \lambda^*, \quad \forall i,\ d = \max_t \left\{ \left\| Y^{-1}(t) \right\| \right\},\ \tau = \|A - I_N\|. \end{aligned}$$

where $w = [w_1, w_2, \ldots, w_N]^T$ and $\sum_{i=1}^{N} w_i = 1$.

Our idea to prove the convergence of algorithm is that: Firstly, we construct this circle: $\lambda(t) - \hat{\lambda}(t) \to s(t) - \hat{s}(t) \to \hat{\lambda}(t) - \lambda^* \to \lambda(t) - \hat{\lambda}(t)$ such that each arrow with the norm $\|\cdot\|^{\mu,K}$ is implemented and their norm $\|\cdot\|^{\mu,K}$ is proved to be bounded. Then, according to Lemma 4 and 5, we can achieve that every term in the cycle is convergent at geometric rate $O(\mu^t)$.

*Theorem 1: If the algorithm parameters satisfy $\eta = \max(|1 - \epsilon_i l|, |1 - \epsilon_i r|)$, $0 < \epsilon_i < \frac{2}{l}$ and*

$$\begin{cases} \gamma_1 \cdot \gamma_2 \cdot \gamma_3 < 1, \\ \mu - \left(\eta + \|\Lambda\| N \left(1 + \frac{d\|A_\infty\| \|\alpha^{-1}\|}{2}\right)\right) > 0, \\ 1 - \left(2\sigma(\mu - \sigma) + \tau d N \|\alpha^{-1}\| \|\Lambda\| (1+\mu)\right) > 0, \\ 0 < \sigma < \mu < 1, \end{cases} \tag{10}$$

*where $r, l$ are positive constants, and*

$$\begin{aligned} \gamma_1 &= \frac{\|\Lambda\|}{\mu - \sigma}, \\ \gamma_2 &= \frac{\tau d N (1+\mu)(\mu - \sigma) \|\alpha^{-1}\|}{1 - \left(2\sigma(\mu - \sigma) + \tau d N \|\alpha^{-1}\| \|\Lambda\| (1+\mu)\right)}, \\ \gamma_3 &= \frac{\|\Lambda\| N \left(1 + \frac{d\|A_\infty\|\|\alpha^{-1}\|}{2}\right)}{\mu - \left(\eta + \|\Lambda\| N \left(1 + \frac{d\|A_\infty\|\|\alpha^{-1}\|}{2}\right)\right)}. \end{aligned}$$

*Then, distributed algorithm (7) with initial condition (8) addresses EDP (1) without generation constraints. More specifically, $\lambda_i(t)$ and $p_i(t)$ tend to the optimal values $\lambda^*$ and $p_i^*$ at the geometric rate $O(\mu^t)$, respectively.*

*Proof:* We implement this circle $\lambda(t) - \hat{\lambda}(t) \to s(t) - \hat{s}(t) \to \hat{\lambda}(t) - \lambda^* \to \lambda(t) - \hat{\lambda}(t)$ in its norm $\|\cdot\|^{\mu,K}$.

(1). The implementation of the first arrow $\left\|\lambda(t) - \hat{\lambda}(t)\right\| \to \left\|s(t) - \hat{s}(t)\right\|$ as follows.

According to (9), we have $\lambda(t+1) = A\lambda(t) - \Lambda s(t)$ and $\hat{\lambda}(t+1) = \hat{\lambda}(t) - \Lambda\hat{s}(t)$, then,

$$\begin{aligned} &\left\|\lambda(t+1) - \hat{\lambda}(t+1)\right\| \\ &= \left\|A\lambda(t) - \Lambda s(t) - A\hat{\lambda}(t) + \Lambda\hat{s}(t)\right\| \\ &\le \left\|A\lambda(t) - \hat{\lambda}(t)\right\| + \left\|\Lambda s(t) - \Lambda\hat{s}(t)\right\|. \end{aligned} \tag{11}$$

It follows from lemma 2 that

$$\left\|\lambda(t+1)-\hat{\lambda}(t+1)\right\| \leq \sigma\left\|\lambda(t)-\hat{\lambda}(t)\right\|+\|\Lambda\|\|s(t)-\hat{s}(t)\|. \quad (12)$$

Then, multiplying both sides of (12) by $\mu^{-(t+1)}$ and taking $\max_{t=0,1,\ldots,K-1}\{\cdot\}$ on the both sides of (12) yields:

$$\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-\sigma}\|s-\hat{s}\|^{\mu,K}, \quad (13)$$

where $0<\sigma<\mu<1$.

(2). The implementation of the second arrow $\|s(t)-\hat{s}(t)\| \rightarrow \left\|\hat{\lambda}(t)-\lambda^*\right\|$ is as follows.

It follows from (9) that

$$\begin{aligned}\|s(t+1)-\hat{s}(t+1)\| &= \|As(t)-\hat{s}(t)+[(I_N-A_\infty)(H(t+1)-H(t))]\| \\ &\leq \sigma\|s(t)-\hat{s}(t)\|+\tau\|H(t+1)-H(t)\| \\ &\leq \sigma\|s(t)-\hat{s}(t)\|+\tau dN\|p(t+1)\| \\ &\quad+\tau dN\|p(t)\|. \end{aligned} \quad (14)$$

where $H(t)=N^{\frac{1}{t+1}}Y^{-1}(t)\,p(t)$.

Due to $p(t)=\frac{1}{2}\alpha^{-1}\lambda(t)-\pi$, we have

$$\begin{aligned}\|p(t)\| &= \left\|\frac{1}{2}\alpha^{-1}\lambda(t)-\pi\right\| \\ &= \frac{1}{2}\left\|\alpha^{-1}\lambda(t)-\alpha^{-1}\hat{\lambda}(t)+\alpha^{-1}\hat{\lambda}(t) -\alpha^{-1}\lambda^*+\alpha^{-1}\lambda^*-2\pi\right\| \\ &\leq \frac{1}{2}\left\|\alpha^{-1}\right\|\left\|\lambda(t)-\hat{\lambda}(t)\right\|+\frac{1}{2}\left\|\alpha^{-1}\right\|\left\|\hat{\lambda}(t)-\lambda^*\right\| \\ &\quad+\left\|\frac{1}{2}\alpha^{-1}\lambda^*-\pi\right\|. \end{aligned} \quad (15)$$

Substituting (15) into (14) yields:

$$\begin{aligned}&\|s(t+1)-\hat{s}(t+1)\| \\ &\leq \sigma\|s(t)-\hat{s}(t)\|+\frac{\tau dN\left\|\alpha^{-1}\right\|}{2}\left\|\lambda(t+1)-\hat{\lambda}(t+1)\right\| \\ &\quad+\frac{\tau dN\left\|\alpha^{-1}\right\|}{2}\left\|\hat{\lambda}(t+1)-\lambda^*\right\| \\ &\quad+\frac{\tau dN\left\|\alpha^{-1}\right\|}{2}\left\|\lambda(t)-\hat{\lambda}(t)\right\| \\ &\quad+\frac{\tau dN\left\|\alpha^{-1}\right\|}{2}\left\|\hat{\lambda}(t)-\lambda^*\right\|+\tau dN\left\|\alpha^{-1}\lambda^*-2\pi\right\|. \end{aligned} \quad (16)$$

Multiplying both sides of (16) by $\mu^{-(t+1)}$ and taking $\max_{t=0,1,\ldots,K-1}\{\cdot\}$ on the both sides of (16), we have

$$\begin{aligned}&\|s-\hat{s}\|^{\mu,K} \\ &\leq \frac{\sigma}{\mu}\|s-\hat{s}\|^{\mu,K}+\left(\frac{\tau dN\left\|\alpha^{-1}\right\|(1+\mu)}{2\mu}\right)\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \\ &\quad+\left(\frac{\tau dN\left\|\alpha^{-1}\right\|(1+\mu)}{2\mu}\right)\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \\ &\quad+\tau dN\left\|\alpha^{-1}\lambda^*-2\pi\right\|. \end{aligned} \quad (17)$$

Because of $\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-\sigma}\|s-\hat{s}\|^{\mu,K}$, we have

$$\begin{aligned}&\|s-\hat{s}\|^{\mu,K} \\ &\leq \frac{\tau dN(1+\mu)(\mu-\sigma)\left\|\alpha^{-1}\right\|}{1-\left(2\sigma(\mu-\sigma)+\tau dN\left\|\alpha^{-1}\right\|\|\Lambda\|(1+\mu)\right)}\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \\ &\quad+\frac{\tau dN\mu(\mu-\sigma)\left\|\alpha^{-1}\lambda^*-2\pi\right\|}{1-\left(2\sigma(\mu-\sigma)+\tau dN\left\|\alpha^{-1}\right\|\|\Lambda\|(1+\mu)\right)}. \end{aligned} \quad (18)$$

(3). The implementation of the third arrow $\hat{\lambda}(t)-\lambda^* \rightarrow \lambda(t)-\hat{\lambda}(t)$ is as follows.

According to (9) and Lemma 3, we have

$$\begin{aligned}&\left\|\hat{\lambda}(t+1)-\lambda^*\right\| \\ &= \left\|\hat{\lambda}(t)-\Lambda\hat{s}(t)-\lambda^*\right\| \\ &= \left\|\hat{\lambda}(t)-\Lambda\lambda(t)+\Lambda\lambda(t)-\Lambda\hat{s}(t)-\lambda^*\right\| \\ &\leq \left\|\hat{\lambda}(t)-\Lambda\lambda(t)-\lambda^*\right\|+\left\|\Lambda\lambda(t)-\Lambda\hat{s}(t)\right\| \\ &\leq \eta\left\|\hat{\lambda}(t)-\lambda^*\right\|+\|\Lambda\|\left\|\lambda(t)-\hat{s}(t)\right\| \\ &\leq \eta\left\|\hat{\lambda}(t)-\lambda^*\right\|+\|\Lambda\|\|\lambda(t)\|+\|\Lambda\|\left\|\hat{s}(t)\right\|. \end{aligned} \quad (19)$$

Notice that

$$\begin{aligned}\|\Lambda\|\|\lambda(t)\| &= \|\Lambda\|\left\|\lambda(t)-\hat{\lambda}(t)+\hat{\lambda}(t)-\lambda^*+\lambda^*\right\| \\ &\leq \|\Lambda\|\left\|\lambda(t)-\hat{\lambda}(t)\right\|+\|\Lambda\|\left\|\hat{\lambda}(t)-\lambda^*\right\| \\ &\quad+\|\Lambda\|\left\|\lambda^*\right\|. \end{aligned} \quad (20)$$

Besides,

$$\hat{s}(t+1)=\hat{s}(t)-A_\infty[(H(t+1)-H(t))],$$

then, we have

$$\hat{s}(t)=\hat{s}(0)-A_\infty N^{\frac{1}{t+1}}Y^{-1}(t)\,p(t)+A_\infty NY^{-1}(0)\,p(0).$$

Hence $\hat{s}(t)=A_\infty Np(0)-A_\infty N^{\frac{1}{t+1}}Y^{-1}(t)\,p(t)$ holds due to $\hat{s}(0)=A_\infty s(0)=0$, we have

$$\begin{aligned}&\|\Lambda\|\left\|\hat{s}(t)\right\| \\ &\leq dN\|\Lambda\|\|A_\infty\|\|p(t)\|+N\|\Lambda\|\|A_\infty\|\|p(0)\| \\ &\leq \frac{dN\|\Lambda\|\|A_\infty\|\left\|\alpha^{-1}\right\|}{2}\left\|\lambda(t)-\hat{\lambda}(t)\right\| \\ &\quad+\frac{dN\|\Lambda\|\|A_\infty\|\left\|\alpha^{-1}\right\|}{2}\left\|\hat{\lambda}(t)-\lambda^*\right\| \\ &\quad+dN\|\Lambda\|\|A_\infty\|\left\|\frac{1}{2}\alpha^{-1}\lambda^*-\pi\right\| \\ &\quad+N\|\Lambda\|\|A_\infty\|\|p(0)\|. \end{aligned} \quad (21)$$

Substituting (20) and (21) into (19), we have

$$\begin{aligned}
&\left\|\hat{\lambda}(t+1)-\lambda^*\right\| \\
&\leq \left(\eta+\|\Lambda\| N\left(1+\frac{d\left\|A_\infty\right\|\left\|\alpha^{-1}\right\|}{2}\right)\right)\left\|\hat{\lambda}(t)-\lambda^*\right\| \\
&\quad+\|\Lambda\| N\left(1+\frac{d\left\|A_\infty\right\|\left\|\alpha^{-1}\right\|}{2}\right)\left\|\lambda(t)-\hat{\lambda}(t)\right\| \\
&\quad+\|\Lambda\| N\left\|\lambda^*\right\|+dN\|\Lambda\|\left\|A_\infty\right\|\left\|\frac{1}{2}\alpha^{-1}\lambda^*-\pi\right\| \\
&\quad+\|\Lambda\|\left\|A_\infty\right\| N\|p(0)\|. \qquad (22)
\end{aligned}$$

Multiplying both sides of (22) by $\mu^{-(t+1)}$ and taking $\max_{t=0,1,\ldots,K-1}\{\cdot\}$ on the both sides of (22), we have

$$\begin{aligned}
&\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \\
&\leq \frac{\|\Lambda\| N\left(1+\frac{d\|A_\infty\|\|\alpha^{-1}\|}{2}\right)}{\mu-\left(\eta+\|\Lambda\| N\left(1+\frac{d\|A_\infty\|\|\alpha^{-1}\|}{2}\right)\right)}\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \\
&\quad+\frac{\mu N\left(d\|\Lambda\|\left\|A_\infty\right\|\left\|\frac{1}{2}\alpha^{-1}\lambda^*-\pi\right\|+\Sigma\right)}{\mu-\left(\eta+\|\Lambda\| N\left(1+\frac{d\|A_\infty\|\|\alpha^{-1}\|}{2}\right)\right)}, \qquad (23)
\end{aligned}$$

where $\Sigma=\|\Lambda\|\|\lambda^*\|+\|\Lambda\|\|A_\infty\|\|p(0)\|$.

According to (13), (18) and (23), it has:

(i): $\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \leq \gamma_1\left\|s-\hat{s}\right\|^{\mu,K}+\omega_1$,

(ii): $\left\|s-\hat{s}\right\|^{\mu,K} \leq \gamma_2\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K}+\omega_2$,

(iii): $\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \leq \gamma_3\left\|\lambda-\hat{\lambda}\right\|^{\mu,K}+\omega_3$,

Applying the small gain theorem and Lemma 5 into the circle $\lambda(t)-\hat{\lambda}(t) \rightarrow s(t)-\hat{s}(t) \rightarrow \hat{\lambda}(t)-\lambda^* \rightarrow \lambda(t)-\hat{\lambda}(t)$, we have the convergence condition as follows:

$$\begin{cases}
\gamma_1\cdot\gamma_2\cdot\gamma_3<1, \\
\mu-\left(\eta+\|\Lambda\| N\left(1+\frac{d\left\|A_\infty\right\|\left\|\alpha^{-1}\right\|}{2}\right)\right)>0, \\
1-\left(2\sigma(\mu-\sigma)+\tau dN\left\|\alpha^{-1}\right\|\|\Lambda\|(1+\mu)\right)>0, \\
0<\sigma<\mu<1,
\end{cases} \qquad (24)$$

which implies that the algorithm (7) is geometric convergent at the rate $O\left(\mu^t\right)$ for all $t$. □

## B. DISTRIBUTED ALGORITHM WITH GENERATION CONSTRAINTS

Regarding the EDP (1) with generation constraints, we design the distributed algorithm as follows:

$$\begin{aligned}
\lambda_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij}\lambda_j(t)-\epsilon_i s_i(t) \\
p_i(t+1) &= \begin{cases}
p_{\min,i}, & \lambda_i(t+1)<\lambda_{\min,i} \\
\dfrac{\lambda_i(t+1)-\beta_i}{2\alpha_i}, & \lambda_{\min,i}\leq\lambda_i(t+1)\leq\lambda_{max,i} \\
p_{\max,i}, & \lambda_i(t+1)>\lambda_{\max,i}
\end{cases} \\
s_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij}s_j(t) \\
&\quad-\left[N^{\frac{1}{t+2}}\frac{p_i(t+1)}{y_i(t+1)}-N^{\frac{1}{t+1}}\frac{p_i(t)}{y_i(t)}\right], \\
y_i(t+1) &= \sum\nolimits_{j\in N_i} a_{ij}y_j(t), \qquad (25)
\end{aligned}$$

where $\lambda_{\min,i}=2\alpha_i p_{\min,i}+\beta_i$ and $\lambda_{max,i}=2\alpha_i p_{\max,i}+\beta_i$. The initial conditions are also (8).

For analyzing the convergence of the algorithm (25), the same approach is employed. The same circle $\lambda(t)-\hat{\lambda}(t) \rightarrow s(t)-\hat{s}(t) \rightarrow \hat{\lambda}(t)-\lambda^* \rightarrow \lambda(t)-\hat{\lambda}(t)$ is constructed such that each arrow with the norm $\|\cdot\|^{\mu,K}$ is implemented.

In the following, three cases are considered according to the difference of expression $p_i(t+1)$. The second case is the same as the algorithm (7). In the following, we discuss the two cases: $p_i(t+1)=p_{\min,i}$ and $p_i(t+1)=p_{\max,i}$.

Case 1: If $p_i(t+1)=p_{\min,i}$, the implementation of each arrow in our circle is updated as follows:

(i): $\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-\sigma}\left\|s-\hat{s}\right\|^{\mu,K}$;

(ii): $\left\|s-\hat{s}\right\|^{\mu,K} \leq \frac{\tau}{\mu-\sigma}\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K}+\frac{\tau d\mu}{\mu-\sigma}\|p_{\min}\|$;

(iii): $\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-(\eta+\|\Lambda\|)}\left\|\lambda-\hat{\lambda}\right\|^{\mu,K}$ $+\frac{\|\Lambda\|\|\lambda^*\|+(1+d)\|A_\infty\|\|\Lambda\|N\|p_{\min}\|}{\mu-(\eta+\|\Lambda\|)}$;

where $p_{\min}=\left[p_{\min,1},p_{\min,2},\ldots,p_{\min,N}\right]^T$.

Case 2: If $p_i(t+1)=p_{\max,i}$, the implementation of each arrow in our circle is updated as follows:

(i): $\left\|\lambda-\hat{\lambda}\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-\sigma}\left\|s-\hat{s}\right\|^{\mu,K}$;

(ii): $\left\|s-\hat{s}\right\|^{\mu,K} \leq \frac{\tau}{\mu-\sigma}\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K}+\frac{\tau d\mu}{\mu-\sigma}\|p_{\max}\|$ with $p_{\max}=\left[p_{\max,1},p_{\max,2},\ldots,p_{\max,N}\right]^T$;

(iii): $\left\|\hat{\lambda}-\lambda^*\right\|^{\mu,K} \leq \frac{\|\Lambda\|}{\mu-(\eta+\|\Lambda\|)}\left\|\lambda-\hat{\lambda}\right\|^{\mu,K}$ $+\frac{\|\Lambda\|\|\lambda^*\|+(1+d)\|A_\infty\|\|\Lambda\|N\|p_{\max}\|}{\mu-(\eta+\|\Lambda\|)}$;

where $p_{\max}=\left[p_{\max,1},p_{\max,2},\ldots,p_{\max,N}\right]^T$.

On the basis of discussions above, we apply the small gain theorem to our circle, and we have the following theorem directly.

*Theorem 2: Distributed algorithm (25) with initial condition (8) deals with EDP (1) with generation constraints. Specifically, $\lambda_i(t)$ and $p_i(t)$ converge to the optimal values $\lambda^*$ and $p_i^*$ at the geometric rate $O\left(\mu^t\right)$, respectively, if the algorithm parameters satisfy $\eta=\max\left(|1-\epsilon_i l|,|1-\epsilon_i r|\right)$, $0<\epsilon_i<\frac{2}{l}$ and*

$$\begin{cases}
\gamma_1\cdot\gamma_2\cdot\gamma_3<1, \\
\mu-\left(\eta+\|\Lambda\|\left(1+\frac{d\left\|A_\infty\right\|\left\|\alpha^{-1}\right\|}{2}\right)\right)>0, \\
1-\left(2\sigma(\mu-\sigma)+\tau d\left\|\alpha^{-1}\right\|\|\Lambda\|(1+\mu)\right)>0, \\
\dfrac{\|\Lambda\|}{\mu-\sigma}\cdot\dfrac{\tau}{\mu-\sigma}\cdot\dfrac{\|\Lambda\|}{\mu-(\eta+\|\Lambda\|)}<1, \\
0<\sigma<\mu<1,
\end{cases} \qquad (26)$$

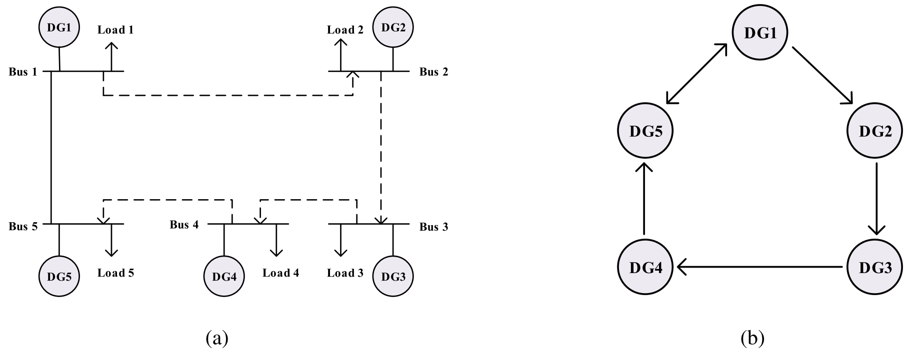


**FIGURE 1.** **Test system. (a) The structure diagram of the micro-grid test system with five buses; (b) The communication topology of the five generators.**

*where $r$, $l$ are positive constants, and*

$$\gamma_1 = \frac{\|\Lambda\|}{\mu - \sigma},$$

$$\gamma_2 = \frac{\tau d N (1+\mu)(\mu-\sigma)\left\|\alpha^{-1}\right\|}{1 - \left(2\sigma(\mu-\sigma) + \tau d N \left\|\alpha^{-1}\right\| \|\Lambda\| (1+\mu)\right)},$$

$$\gamma_3 = \frac{\|\Lambda\| N \left(1 + \frac{d\|A_\infty\|\left\|\alpha^{-1}\right\|}{2}\right)}{\mu - \left(\eta + \|\Lambda\| N \left(1 + \frac{d\|A_\infty\|\left\|\alpha^{-1}\right\|}{2}\right)\right)}.$$

*Proof:* The implementation of each arrow in the circle $\lambda(t) - \hat{\lambda}(t) \to s(t) - \hat{s}(t) \to \hat{\lambda}(t) - \lambda^* \to \lambda(t) - \hat{\lambda}(t)$ are similar with ones in Theorem 1. Here, we omit its proof and directly give the results. □

## IV. SIMULATION EXAMPLES

Two studies are used to verify the effectiveness of algorithms (7) and (25). Based on the results, the convergence of the proposed algorithms are shown. Then, we try to use the algorithm (25) to deal with EDP with time-varying demand.

Fig.1(a) shows the structure diagram of the MG test system with five buses. Each bus has a generator and a load. The solid line and dashed lines between two buses represent the communication links, where the solid line means that the link between two buses is bi-directional. The communication topology is given in Fig. 1(b). And, the connection weight values are: $a_{11} = 1/2, a_{51} = 1/2, a_{12} = 1/2, a_{22} = 1/2, a_{23} = 1/2, a_{33} = 1/2, a_{34} = 1/2, a_{44} = 1/2, a_{14} = 1/4, a_{45} = 1/4, a_{55} = 1/2$.

From the communication topology in Fig. 1(b), we can see that the topology graph is directed and connected, and the weight matrix $A$ is row stochastic and is not column stochastic. By the simple calculation, the left-Perron eigenvector of the weighted matrix $A$ is $w = [0.2857, 0.1429, 0.1429, 0.1429, 0.2857]^T$. The cost parameters, generation capacities, and load demand of the five buses in this MG are given in Table 1. We set the initial values of the proposed algorithms as $p(0) = [50.0, 14.3, 17.9, 21.4, 14.3]^T$, $\lambda(0) = 2 * p(0) * \alpha + \beta = [4.65, 6.53, 9.88, 8.94, 4.65]^T$ and $s(0) = [0, 0, 0, 0, 0]^T$.

**TABLE 1.** **Parameters of the five-bus in this MG.**

| $Bus_i$ | $\alpha_i$ | $\beta_i$ | $\gamma_i$ | $p_{min,i}(kW)$ | $p_{max,i}(kW)$ | $p_{il}(kW)$ |
|---|---|---|---|---|---|---|
| 1 | 0.049 | 1.22 | 51 | 10 | 80 | 35 |
| 2 | 0.078 | 3.41 | 31 | 8 | 60 | 20 |
| 3 | 0.105 | 2.53 | 78 | 3.8 | 40 | 25 |
| 4 | 0.082 | 4.02 | 42 | 5.4 | 45 | 30 |
| 5 | 0.074 | 3.17 | 62 | 4.2 | 18 | 10 |

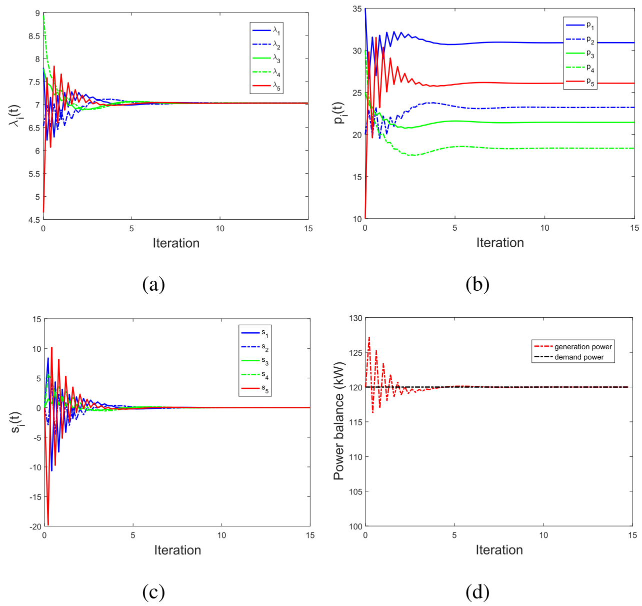


**FIGURE 2.** **Results of the algorithm (5) without generation constraints. (a) The incremental cost of the five generators; (b) The power outputs of the five generators; (c) The estimated mismatch; (d) The balance between the total power generated and total demand.**

Case A: Optimal analysis without Generation Constraints

In this subsection, algorithm (7) is considered to solve the EDP without generation constraints. The feedback gain $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.03$. Based on our algorithm (7), the simulation results are shown in Fig.2. Fig.2(a) shows that the incremental cost converges to the optimal value $\lambda^* = 7.03\ \$/kW$, and the optimal power outputs for the five generators are $p_1^* = 30.8962\ kW$, $p_2^* = 23.2332\ kW$, $p_3^* = 21.4396\ kW$, $p_4^* = 18.3609\ kW$ and $p_5^* = 26.0697\ kW$,

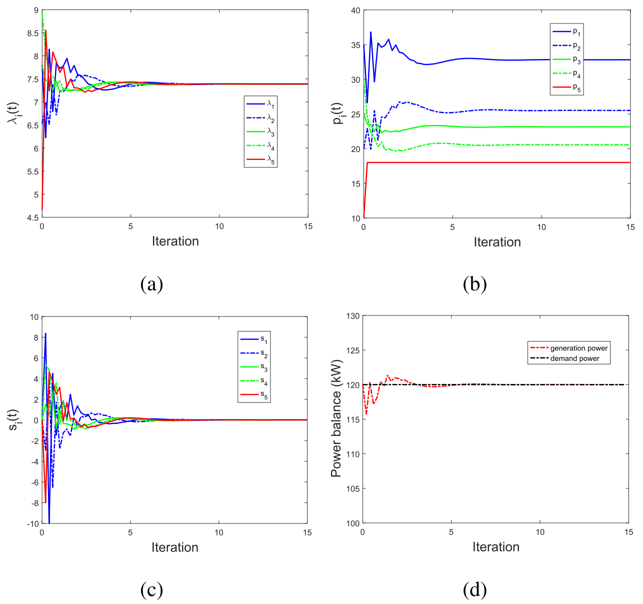


**FIGURE 3. Results of the algorithm (23) with generation constraints. (a) The incremental cost of the five generators; (b) The power outputs of the five generators; (c) The estimated mismatch; (d) The balance between the total power generated and total demand.**

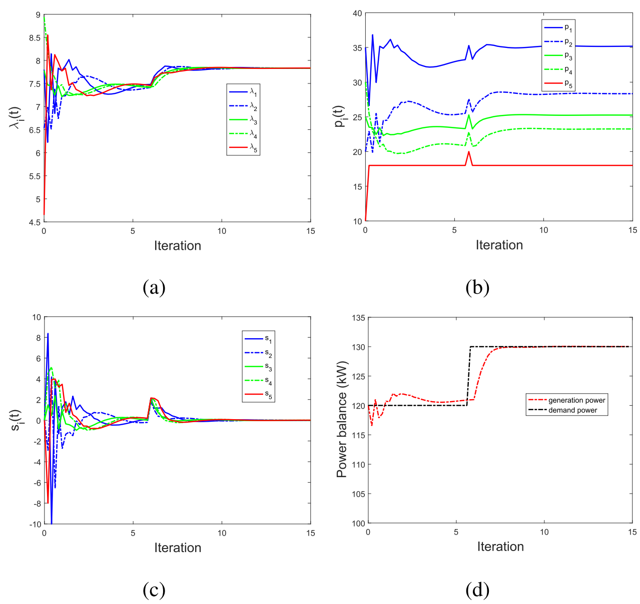


**FIGURE 4. Results of the algorithm (23) with generation constraints. (a) The incremental cost of the five generators; (b) The power outputs of the five generators; (c) The estimated mismatch; (d) The balance between the total power generated and total demand.**

which are seen in Fig.2(b). Fig.2(c) shows that the mismatch $s_i(t)$ between demand and power generated converges to zero, and the balance between the total power generated and total demand is achieved, which is shown in Fig.2(d). Since the generation constraints are not considered, $p_5^* = 26.0697\ kW$ exceeds its generation constraint 18 $kW$. These results verify our algorithm (7).

Case B: Optimal analysis with Generation Constraints

In this subsection, algorithm (25) is applied to solve the EDP with generation constraints. The feedback gain $\epsilon_1 = 0.01, \epsilon_2 = 0.02, \epsilon_3 = 0.07, \epsilon_4 = 0.05, \epsilon_5 = 0.03$. Based on our algorithm (25), the simulation results are shown in Fig.3. Fig.3(a) shows that the incremental cost converges to the optimal value $\lambda^* = 7.3894\ \$/kW$, and the optimal power outputs for the five generators are $p_1^* = 32.8158\ kW$, $p_2^* = 25.4824\ kW$, $p_3^* = 23.1423\ kW$, $p_4^* = 20.5540\ kW$ and $p_5^* = 18.00\ kW$, which are seen in Fig.3(b). Fig.3(c) shows that the mismatch $s_i(t), \forall i$ between demand and generation converges to zero, and the balance between the total power generated and total demand is shown in Fig.3(d). Since the generation constraints are considered, $p_5^*$ is restricted to 18 $kW$, which verify our algorithm (25).

Case C: Optimal analysis with varying demand

In this subsection, we apply the proposed algorithm (25) to address the EDP with varying demand, where the demand of each generator is increased by 2 $kW$. In that case, the total demand changes from 120 $kW$ to 130 $kW$. We set the feedback gain as $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.03$. Based on our algorithm (25), the simulation results are shown in Fig.4. Fig.4(a) shows that the incremental cost converges to the new optimal value $\lambda^* = 7.8285\ \$/kW$, and the optimal power outputs for the five generators are $p_1^* = 35.1513\ kW$, $p_2^* = 28.2290\ kW$, $p_3^* = 25.2764\ kW$, $p_4^* = 23.3107\ kW$ and $p_5^* = 18.00\ kW$, which are seen in Fig.4(b). Fig.4(c) shows that the mismatch $s_i(t)$ between demand and generation converges to zero, and the balance between the total power generated and total demand is achieved, which can be seen in Fig.4(d). $p_5^*$ is still restricted to the generation constraint of 18 $kW$. From the results, the power outputs for each generator increases as the demand increases such that the balance between generation and demand is achieved.

## V. CONCLUSION

In this paper, two newly distributed ICC-based optimization algorithms are proposed to respectively solve the EDPs without generation constraints and with generation constraints. The theoretical analysis on the convergence of the proposed algorithms is presented by using the small gain theorem. It can be found that the algorithms are convergent at the geometric rate. At the same time, the optimal EDPs are achieved under the proposed algorithms. The corresponding conditions are also obtained. Finally, simulation studies illustrate the correctness of our results.

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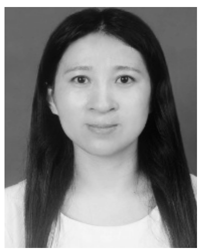

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