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Cumulative Conforming Control Chart Assuming Discrete Weibull Distribution

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ABSTRACT Time Between Events (TBE) charts have advantages over the traditional control charts when monitoring high quality processes with very low defect rates. This article introduces a new discrete TBE control chart following discrete Weibull distribution. The design of the proposed chart is derived analytically and discussed numerically. Moreover, the performance is assessed by using the Average Run Length (ARL) and the Coefficient of Variation of Run Length (CVRL). Besides simulation studies, the proposed scheme is also illustrated using four real data examples.

INDEX TERMS Average run length, discrete Weibull distribution, coefficient of variation, process monitoring.

I. INTRODUCTION

Statistical process control (SPC) is a quality control method, which is used to monitor a process by using statistical methods. There are two-source of variations in statistical process control. The first one is the chance variation also known as the common cause of variation that represents natural or inherent variability of a process. A process with the common cause of variation is known as the in-control process. The variations which are unstable over time are known as the assignable or special cause of variation. For example variation caused due to operator absent, computer crashes, machine malfunction, or poor design, etc.

Control chart is a powerful tool of SPC tool-kit, which is used to monitor whether process is in-control or not. Depending on the nature of data there are two major types of control charts, i.e., variable and attributes control charts. To monitor count data, attribute control charts are used and the most common attribute charts are p (for defective proportion), np (for numbers of defective), c (for unbounded defects), and u (for defects per unit) charts. On the other hand, variable control charts like $\bar{X}, S, S^2$, are commonly used to monitor the continuous data.

The modern development of technology has led to high quality processes with a very low defect rate. Many practical problems, however, may occur while using the traditional control charts for monitoring such processes. These problems include high false alarm rate, absurd control limits, etc. In order to overcome these problems, Time-Between-Events (TBE) charts are used. These charts examined the time interval "T" between successive occurrences of events. These charts are most effective in the case of high yield production with a low defect rate [1]. The discrete version of TBE charts is known as the cumulative conforming control (CCC) charts. Similarly, the continuous version of the TBE charts is known as the cumulative quantity control (CQC) chart proposed by Chan *et al.* [2] based on the exponential distribution and later generalized by Zhang *et al.* [3] using the gamma distribution, while Calvin [4], Goh [5], Xie *et al.* [6], and Chan *et al.* [7] proposed the CCC charts based on the geometric distribution. Zhang *et al.* [8] proposed a generalization of CCC chart by using group inspection denoted by CCC_G, where G denotes the "Group". In order to improve the sensitivity of process deterioration, approximately unbiased ARL design is also introduced. Some recent contribution in this direction can be seen in Ara *et al.* [9], Rahali *et al.* [10], Chen *et al.* [11], Zhang *et al.* [3], Shamsuzzaman *et al.* [12], Ali and Shah [13], Tahir and Xie [14], and references cited therein.

A major problem in the SPC is the estimation of the parameter in the presence of drift at the start-up of process [15]–[18]. Liu *et al.* [19] used TBE charts to present a comparison among the exponential CUSUM, the exponential EWMA, and the CCC-r charts by using ATS as a performance measure. CCC charts have also used for designing economic control chart [20], [21].

In the literature, runs rules are also suggested to improve the sensitivity of the charts [22]. Emura and Lin [23] compared some rules required for the normal approximation to

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binomial and also suggested rules of approximation, including $np > 15$ or $p \geq 0.1$ and $np \geq 10$. Ali *et al.* [24] presented an overview about some recent contributions in high quality process monitoring.

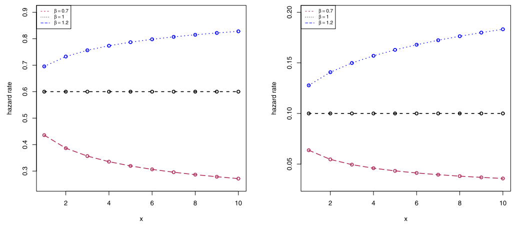
In order to model the lifetime of a component or a device, some well known probability distributions, i.e., exponential, lognormal, Weibull, and normal are commonly used to model continuous random variable. In practice, situations may occur where lifetime is not measured on a continuous scale. For example, the number of runs/cycles to failure when the components are prior to cyclical loading or on/off switching devices, etc. In such situations, the lifetime of a component or a device is modeled as a discrete random variable and geometric and negative binomial are the most common models used in reliability analysis. As a discrete analogue of continuous distribution introduced by Khalique [25], negative binomial and geometric distributions can be used as an alternative for gamma and exponential distributions, respectively. In reliability engineering, however, the Weibull distribution is commonly used because of different values of the shape parameter, i.e., it models failure rate in different dimensions like increasing, decreasing and constant. The discrete analogue of continuous Weibull is the discrete Weibull distribution introduced by Nakagawa and Osaki [26]. This distribution is also known as the Type-I discrete Weibull. Barbiero [27] used Type-III discrete Weibull distribution for modeling failure reliability data. Different methods of parameter estimation including the Maximum Likelihood Estimation (MLE), Method of Moments Estimation (MME), and the Method of Proportion were discussed.

Poisson distribution is used for equidispersed data whereas negative binomial is suitable for overdispersed data. To deal underdispersed data, however, discrete Weibull distribution is more suitable [28]. Peluso *et al.* [29] showed that

- regardless the value of q, the distribution is suitable for overdispersed data for $0 < \beta \leq 1$,
- the distribution is suitable for underdispersed data for $\beta \geq 3$ irrespective the value of q. Furthermore, the distribution approaches to Bernoulli distribution for $\beta \to \infty$.
- depending on the value of q the distribution is suitable for over-and underdispersed data for $1 < \beta < 3$.

Thus, the main aim of this study is to introduce a new CCC chart using discrete Weibull distribution and to study its performance assuming average run length (ARL) and coefficient of variation of run length (CVRL) as the performance measures. It is worth mentioning that other performance measures like median run length (MRL) and standard deviation of run length (SDRL) are also common in practice but these are not scaled free measures [30]. The CVRL is a scale free measure.

The rest of the study is divided as follows. Section 2 presents introduction about the discrete Weibull distribution while Section 3 presents the construction of the discrete Weibull distribution control chart. Section 4 presents applications of discrete Weibull chart using some real data. The performance of the discrete Weibull chart using ARL is discussed in Section 5 whereas Section 6 concludes the study.



(a) $q = 0.4$ (b) $q = 0.9$

FIGURE 1. Hazard rate for different values of β.

II. THE DISCRETE WEIBULL DISTRIBUTION

Let Z be a random variable with probability mass function (PMF) given as

$$P(Z = z) = q^{(z)^\beta} - q^{(z+1)^\beta}, \quad z = 0, 1, 2, \cdots \quad \beta > 0, \quad 0 < q < 1 \tag{1}$$

The PMF given in Eq. 1 is known as the Type-I discrete Weibull distribution [26]. It has two parameters, β and q, representing the shape and scale parameters, respectively. The Cumulative Distribution Function (CDF) of the distribution can be written as

$$F(z) = 1 - q^{(z+1)^\beta} \tag{2}$$

The survival function is

$$S(z) = P(Z \geq z) = 1 - F(z) = q^{(z)^\beta}$$

and the corresponding hazard function is

$$h(z) = \frac{P(z)}{S(z)} = 1 - q^{(z+1)^\beta - (z)^\beta} \tag{3}$$

The distribution has increasing, decreasing, and constant hazard rate for $\beta > 1$, $\beta < 1$ and $\beta = 1$, respectively, as shown in Figure 1. The mean of the distribution can be obtained numerically by specifying β and q, that is,

$$E(Z) = \sum_{i=1}^{\infty} zP(Z = z; q, \beta) = \sum_{i=1}^{\infty} q^{(i)^\beta} \tag{4}$$

By substituting $q = 1 - p$ and $\beta = 1$, the distribution reduces to geometric, i.e.,

$$P(Z = z) = (1-p)^{(z)} - (1-p)^{(z+1)} = pq^z \tag{5}$$

III. CONTROL CHARTS BASED ON DISCRETE WEIBULL DISTRIBUTION

Let Z be the inter arrival time between two nonconformities with PMF given in Eq. 1. In order to construct the CCC chart [2], first fix the probability of Type-I error (false alarm rate) α and then equate $F(z)$ in Eq. 2 to $\frac{\alpha}{2}$, $1 - \frac{\alpha}{2}$ and $\frac{1}{2}$, to get Lower Control Limit (LCL), Upper Control Limit (UCL), and the Central Limit (CL) respectively, as follows.

$$LCL = \left[\frac{\ln(1 - \alpha/2)}{\ln(q)}\right]^{1/\beta} - 1 \tag{6}$$

$$UCL = \left[\frac{\ln(\alpha/2)}{\ln(q)}\right]^{1/\beta} - 1 \tag{7}$$

$$CL = \left[\frac{\ln(1/2)}{\ln(q)}\right]^{1/\beta} - 1 \tag{8}$$

TABLE 1. Numbers of fires in Greece.

Numbers	0	1	2	3	4	5	6	7	8	9	10	11	12	15	16	20	43
Frequency	16	13	14	9	11	13	8	4	9	6	3	4	6	4	1	1	1

TABLE 2. Number of failures of software.

category	0	1	2	3	4	5	6	7	8	9	10	11
frequency	20	10	11	10	2	3	3	0	0	1	1	1

The chart can be constructed by plotting each observed value of Z against the respective sample number. Further, whenever the plotted point is below the LCL, indicates that the defect rate may have increased, i.e., process deterioration. On the other hand, if a plotted point falls above the UCL, it indicates that the defect rate may have been decreased, i.e., process improvement.

In many practical situations, directional changes are of greatest interest, e.g., in manufacturing of air-crafts, it is highly important to detect a manufacturing error increase as a minor fault may lead to a major destruction. On the other hand, decrease in manufacturing error might not be taken seriously and in such cases, the one-sided control chart would be more preferable than two-sided charts. For a one-sided control chart, set $F(z)$ equal to α or $1-\alpha$ to get the lower or upper control limit, i.e.,

$$LCL = \left[\frac{\ln(1-\alpha)}{\ln(q)}\right]^{1/\beta} - 1 \quad (9)$$

$$UCL = \left[\frac{\ln(\alpha)}{\ln(q)}\right]^{1/\beta} - 1 \quad (10)$$

The following are the steps to implement the proposed control chart in practice.

Step-1 Take Phase-I data set and estimate the parameters of the discrete Weibull distribution using the maximum likelihood method. The R code to estimate the parameters is given in the appendix.

Step-2 Establish the control limits using the estimated parameters.

Step-3 Check the Phase-I data set against the established control limits. If there is any out-of-control data point discard it and reconstruct the control limits. Repeat this step until there is no out-of-control data point in the Phase-I data set.

Step-4 Use the control limits constructed in Step-3 for monitoring the future data. Declare the monitoring process out-of-control if any data point falls below the LCL. Similarly, declare the process improved if any data point falls above the UCL.

IV. REAL DATA APPLICATIONS

This section illustrates the real data applications of all the previously defined control charts. In all the following cases the full data are used as the Phase-I sample, and parameters of the discrete Weibull distribution are estimated to construct control limits and monitor data to highlight the superiority of the proposed chart.

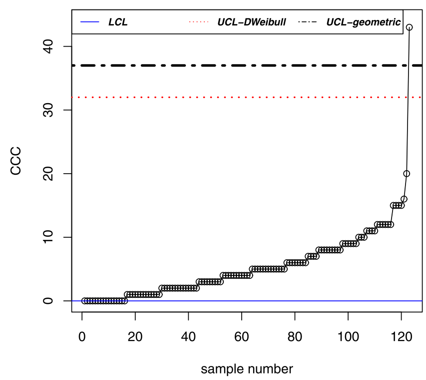


FIGURE 2. Cumulative count control chart for discrete Weibull and geometric distributions.

A. CASE STUDY 1

The first data set is provided by Karlis and Xekalaki [31], which consists of the number of fires in Greece for period from 1^{st} July 1998 to 31^{st} August 1998. The data are reported in Table 1 and to establish a control chart based on the discrete Weibull distribution, it is assumed that the data follow the discrete Weibull distribution. The values of the estimated parameter are $\hat{q} = 0.8798$ and $\hat{\beta} = 1.1306$ with 0.0228 and 0.0823 as the standard errors, respectively. For geometric distribution, $\hat{p} = 0.1563$ and its standard error is 0.0129. For the discrete Weibull distribution, assuming $\alpha = 0.0027$, we have LCL $= 0$, CL $= 3$, and UCL $= 32$, respectively. Similarly for the geometric chart, we have LCL $= 0$, CL $= 3$, and UCL $= 37$, respectively. From Fig. 2, it is clear that only one sample point, i.e., 43rd point, lies outside the UCL of both charts and the chart constructed assuming the discrete Weibull detect it more quickly than the geometric chart.

B. CASE STUDY 2

Nikora [32] provided data about the number of software failures investigated over 62 weeks. The data set is listed in Table 2, and the proposed control scheme is applied to the data in order to monitor the failure rate. The estimated values of parameters, assuming the discrete Weibull distribution are $\hat{q} = 0.6948$ and $\hat{\beta} = 1.0354$ with 0.0544 and 0.1222 as the standard errors, respectively. For the geometric distribution,

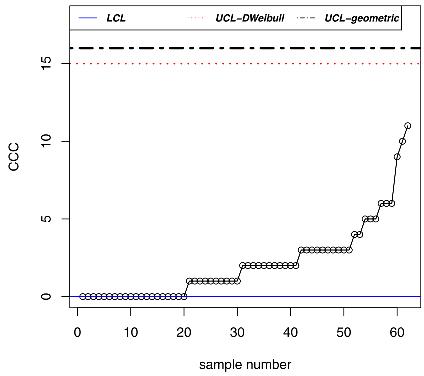


FIGURE 3. Cumulative count control chart for discrete Weibull and geometric distributions.

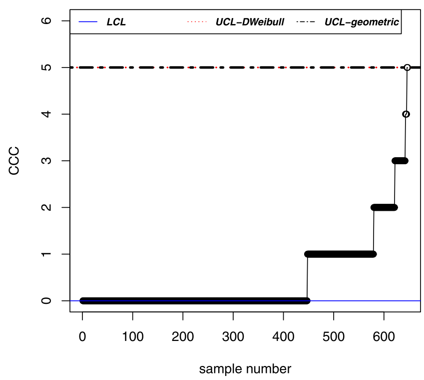


FIGURE 4. Cumulative count control chart for discrete Weibull and geometric distributions.

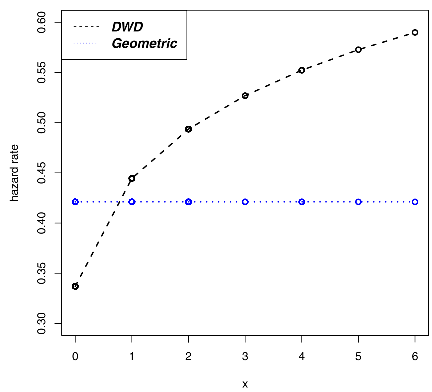


FIGURE 5. Hazard plot for the Dengue Fever Data.

we have $\hat{p} = 0.3179$ with standard error 0.0333. By assuming $\alpha = 0.0027$, the discrete Weibull chart's limits are LCL = 0, CL = 1, UCL = 15 while LCL = 0, CL = 1, UCL = 16 for the geometric chart.

From Figure 3, it is clear that all the points lie within the control limits for both charts (discrete Weibull and geometric), and thus, we conclude that the software failure rate is in-control.

C. CASE STUDY 3

In this example, a data set is taken from Greenwood and Yule [33], reporting the accidents of women working on the H.E. Shells for five weeks. The frequency distribution is given in Table 3. The estimated parameter value using the geometric distribution is $\hat{p} = 0.6825$ with standard error 0.0151. For the discrete Weibull, the estimated parameter values are $\hat{q} = 0.3114$ and $\hat{\beta} = 0.9673$, whereas the standard errors are 0.0181 and 0.0536, respectively. For the discrete Weibull chart, we have LCL = 0, CL = 0, UCL = 5 while LCL = 0,CL = 0, UCL = 5, respectively, for the geometric chart using $\alpha = 0.0027$.

From Figure 4, it is evident that both charts, neither discrete Weibull nor geometric chart signalled any sign regarding the

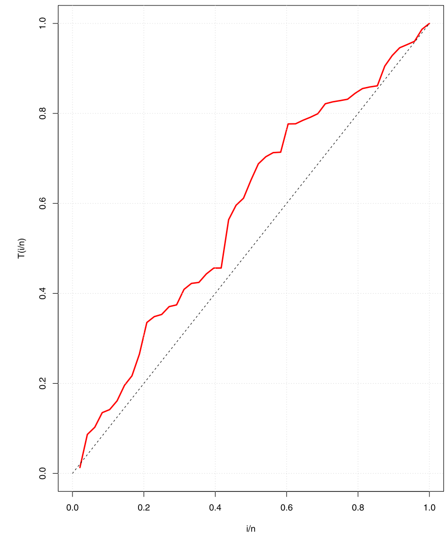


FIGURE 6. TTT plot for the Dengue Fever Data.

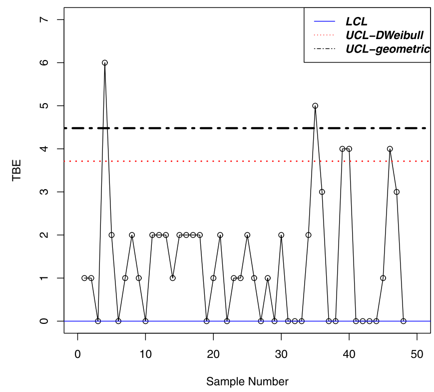


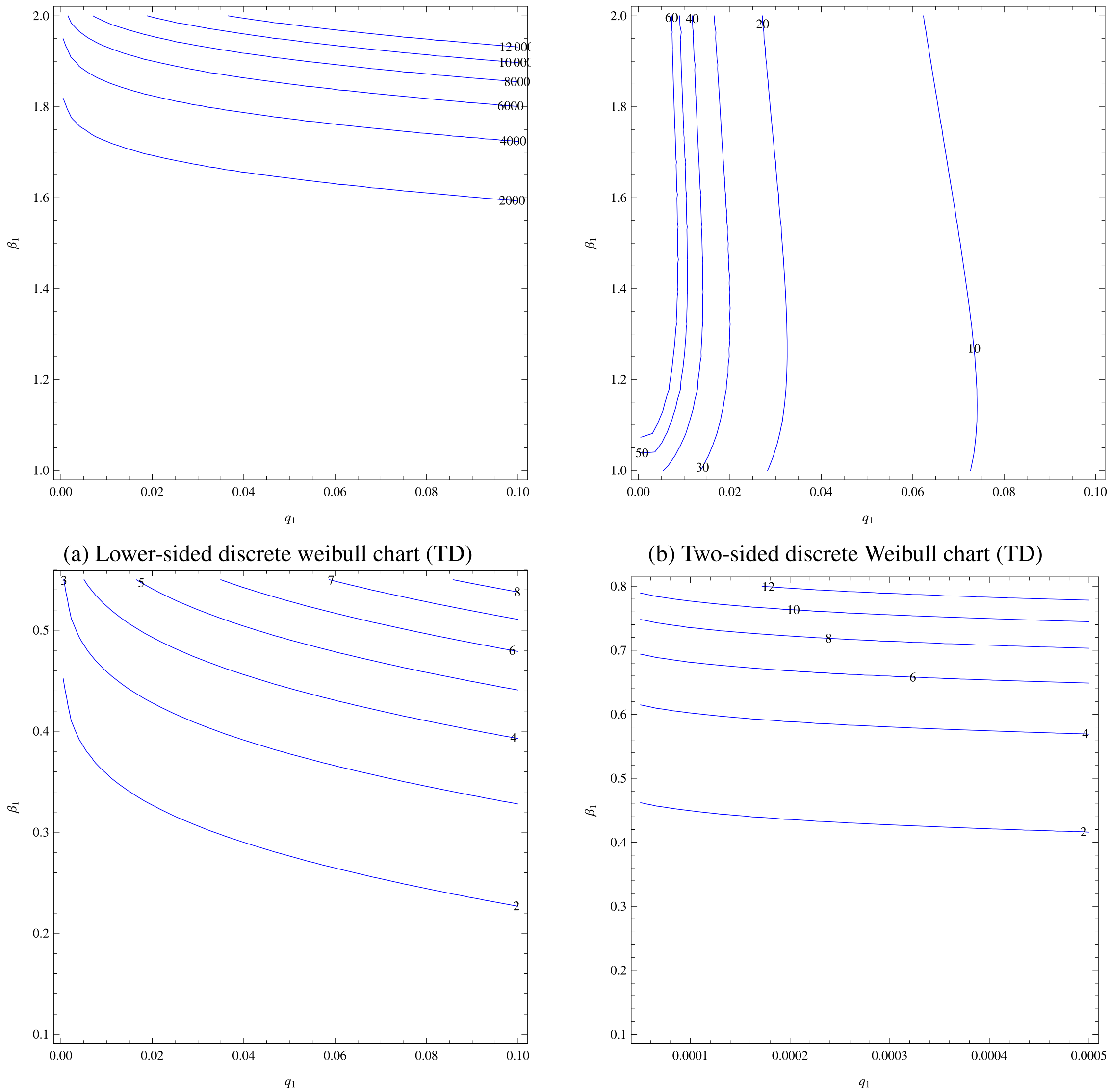
FIGURE 7. Control Charts for the Dengue Fever Data.

TABLE 3. Numbers of accidents on H.E. shells.

category	0	1	2	3	4	5
frequency	447	132	42	21	3	2

TABLE 4. TBE (in hours) for Dengue Fever Patients Registration.

category	0	1	2	3	4	5	6
frequency	17	11	13	2	3	1	1



(a) Lower-sided discrete weibull chart (TD)

(b) Two-sided discrete Weibull chart (TD)

(c) Lower-sided discrete Weibull chart (PD)

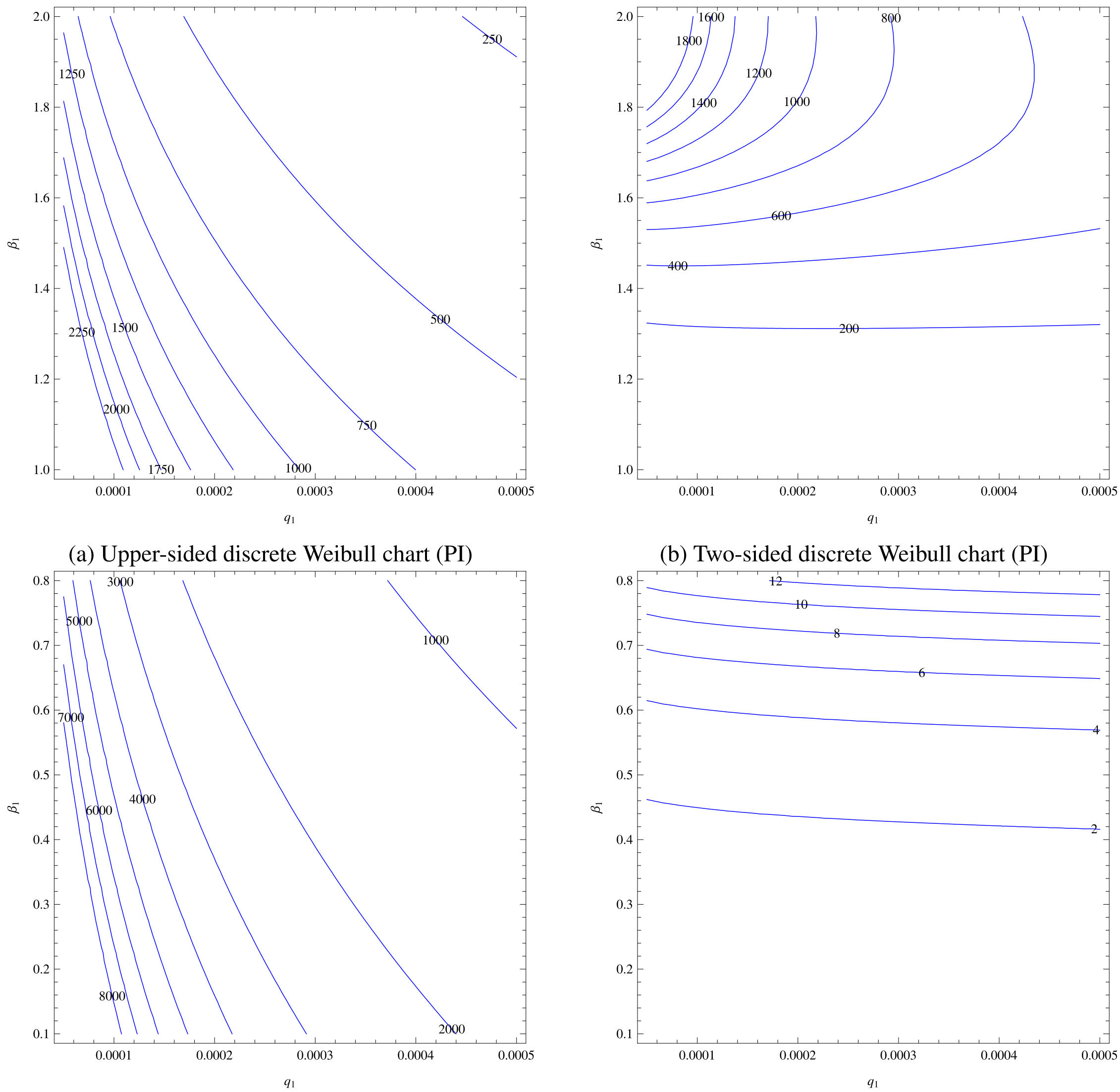
(d) Two-sided discrete Weibull chart (PD)

FIGURE 8. Detecting process deterioration using $\alpha = 0.0027$.

improvement or deterioration of the process. Thus, the process is in-control.

D. DENGUE FEVER OUTBREAK MONITORING

Since 2010, Pakistan has been experiencing an epidemic of dengue fever which is an infectious fever. Despite the efforts of the Government of Pakistan, the high cost of prevention has limited the ability of Pakistan to control epidemics. During the current year, 42 deaths (https://reliefweb.int/report/pakistan/over-25000-dengue-cases-reported-year) have been caused by dengue so far and health practitioners attribute the lower mortality rate to better availability of surveillance and curative measures. To study the outbreak of the dengue fever in Islamabad, a TBE data set of dengue fever patients is collected from a hospital and listed in Table 4. The estimated parameters of the discrete Weibull and geometric distributions with their standard errors are listed in Table 5. Furthermore, model selection criteria, like Akaike Information Criterion (AIC), Bayesian Information criterion (BIC), and the logarithm of the likelihood are also listed in the same table. From the model selection criterion AIC, it is clear that both distributions fit equally well. However, the hazard function plot, Figure-5, indicates increasing hazard and hence, the discrete Weibull distribution should be used rather than the geometric distribution which has a constant hazard function. This conclusion is also supplemented by the smoothed TTT plot [34], [35] depicted in Figure 6, which suggests increasing hazard rate distribution will be suitable for the data. In Figure-7 the proposed and geometric charts are depicted and it is clear that the discrete Weibull chart detects outbreak signal more quickly.



(a) Upper-sided discrete Weibull chart (PI)

(b) Two-sided discrete Weibull chart (PI)

(c) Upper-sided discrete Weibull chart (TI)

(d) Two-sided discrete Weibull chart (TI)

FIGURE 9. Detecting process improvement using $\alpha = 0.0027$.

TABLE 5. Estimated Parameters using Dengue Fever Data with Model Selection Criteria.

Distribution	q	β	AIC	BIC	Log-Likelihood	LCL	UCL
DWD	0.6631	1.2814	156.3931	160.1355	-76.1965	0	3.7137
	(0.0647)	(0.1778)					
Geometric	0.5789	-	157.1835	159.0547	-77.59176	0	4.4804
	(0.0462)						

V. PERFORMANCE EVALUATION USING ARL AND CVRL

There are many criteria which can be used to assess the performance of control charts at a particular shift or for a range of shifts. Among them, Average Run Length (ARL) is the most commonly used criterion to evaluate performance of a chart at a particular shift. The number of observations taken before a signal of point beyond control limits is known as the Run Length (RL). The ARL is defined as the expected number of points plotted within limits of a control chart until an out-of-control signal occurs. The ARL is a discrete random variable because it takes integer values. Let the CCC chart is constructed such that α_L and α_U denotes the false alarm probabilities for LCL and UCL, respectively, and $\alpha = \alpha_L + \alpha_U$ is the overall Type-I error for the two-sided chart.

TABLE 6. Study of ARL for case-1 based on $\beta_0 = 1.5$, $q_0 = 0.0005$, $\beta_1 \in \{1, 1.2, 1.4, 1.5, 2\}$, $q_1 \in \{0.005, 0.01, 0.1\}$ and $\alpha = 0.0027$.

		Two-sided				Lower-sided			
β	q	0.0005	0.005	0.01	0.1	0.0005	0.005	0.01	0.1
1	ARL	40.44409	40.60549	33.86684	7.69029	26.70981	38.09793	43.75636	87.0098
	CV	0.98756	0.99295	0.98513	0.93271	0.98110	0.98679	0.98851	0.99424
1.2	ARL	115.10437	71.18047	47.88287	7.69598	76.05666	108.89186	125.20624	249.91148
	CV	0.99565	0.99295	0.98950	0.93277	0.99340	0.99531	0.99591	0.99791
1.4	ARL	272.60033	88.94533	52.53353	7.50071	218.3324	312.99940	360.03520	719.57000
	CV	0.99816	0.99436	0.99044	0.93096	0.99771	0.99840	0.99861	0.99930
1.5	ARL	370.37040	91.46316	52.43330	7.37923	370.37040	531.11140	610.97640	1221.45250
	CV	0.99865	0.99452	0.99042	0.92978	0.99865	0.99906	0.99918	0.99959
2	ARL	526.11378	80.77022	45.54523	6.75476	5220.703	7489.341	8616.524	17232.548
	CV	0.99905	0.99379	0.98896	0.92301	0.99990	0.99993	0.99994	0.99997

Let ARL_L, ARL_U, and $ARL_{L\cap U}$ denote the ARLs for the lower, upper and two sided control charts, respectively, then,

$$ARL_L = \frac{1}{F(LCL)}$$

Using Eq. 2 and Eq. 9, we get

$$ARL_L = \frac{1}{1 - q_1^{\left(\frac{\ln(1-\alpha_L)}{\ln(q_0)}\right)^{\beta_1/\beta_0}}} \tag{11}$$

Similarly,

$$ARL_U = \frac{1}{1 - F(UCL)}$$

Using Eq. 2 and Eq. 10 we have,

$$ARL_U = \frac{1}{q_1^{\left(\frac{\ln(\alpha_U)}{\ln(q_0)}\right)^{\beta_1/\beta_0}}} \tag{12}$$

Also,

$$ARL_{L\cap U} = \frac{1}{1 - F(UCL) + F(LCL)}$$

After simplification, we get

$$ARL_{L\cap U} = \frac{1}{1 + q_1^{\left(\frac{\ln(\alpha)}{\ln(q_0)}\right)^{\beta_1/\beta_0}} - q_1^{\left(\frac{\ln(1-\alpha)}{\ln(q_0)}\right)^{\beta_1/\beta_0}}} \tag{13}$$

In control chart design, shape parameter has a significant effect and its value must be selected carefully while constructing control charts. As control chart design is based on ARL, a large value of ARL_0 (in control ARL) is always desirable but on the other hand, its variance may increase. Due to large variations in the frequency of false alarm, a signal raised by the control chart may be overlooked. Hence, in cases where performance of control chart is under consideration, the coefficient of variation (CV) of ARL should also be recorded.

To assess the effect of rate and shape parameter for the discrete Weibull distribution chart, the ARL can be studied in four different cases as discussed below.

A. CASE-1 (TD-IHR)

For discrete Weibull distribution, the hazard rate is increasing when $\beta > 1$ and can be denoted as IHR. If q is increasing then process is totally deteriorating and is denoted by TD and hence, the overall situation can be labeled as the TD-IHR.

Let the in-control values of q and β are $q_0 = 0.0005$ and $\beta_0 = 1.5$. To study the performance, let q increases from 0.0005 to $q_1 \in \{0.005, 0.01, 0.1\}$ and $\beta_0 = 1.5$ to $\beta_1 \in \{1, 1.2, 1.4, 1.5, 2\}$. Table-6 lists the computed value of ARL and CV of the run length.

In order to identify deterioration, the lower-sided chart and the two-sided chart should be compared. From Table-6, it is clear that the two-sided chart quickly detects shift in the scale parameter "q" for different values of the shape parameter. In the lower-sided chart, assuming different values of q, the ARL increases with the increase in β, whereas the ARL increases for all values of β at $q = 0.0005$ and it decreases for $q = 0.1$ for the two-sided chart. The results of Table-6 are also depicted in Figure 8.

B. CASE-2 (PI-IHR)

For discrete Weibull distribution, the hazard rate is increasing when $\beta > 1$ and this case is denoted as the IHR. If q is decreasing, then the system shows partial improvement and can be named as the PI. Thus, the overall situation can be labeled as the PI-IHR. Let $q_0 = 0.0005$ and $\beta_0 = 1.5$ and q is decreasing from 0.0005 to $q_1 \in \{0.0003, 0.0001, 0.00005\}$ while $\beta = 1.5$ to $\beta_1 \in \{1, 1.2, 1.4, 1.5, 2\}$ with $\alpha = 0.0027$. The ARL and CV of run length are reported in Table-7.

From Table-7, it is observed that the two-sided chart performs better than the upper-sided chart as it quickly detects

TABLE 7. Study of ARL for case-2 based on $\beta_0 = 1.5$, $q_0 = 0.0005$, $\beta_1 \in \{1, 1.2, 1.4, 1.5, 2\}$, $q_1 \in \{0.0003, 0.0001, 0.00005\}$ and $\alpha = 0.0027$.

		Two-sided				Upper-sided			
β	q	0.0005	0.0003	0.0001	0.00005	0.0005	0.0003	0.0001	0.00005
1	ARL	40.44409	38.55883	34.57513	32.31848	620.3846	955.7502	2420.9592	4351.6896
	CV	0.98756	0.98695	0.98543	0.98441	0.99919	0.99948	0.99979	0.99989
1.2	ARL	115.1044	113.8417	106.0505	100.0687	502.1081	762.6175	1873.5795	3303.4311
	CV	0.99565	0.99551	0.99527	0.99491	0.99900	0.99934	0.99973	0.99985
1.4	ARL	272.6003	296.6433	310.9517	303.6278	409.2185	613.0474	1462.2394	2530.5205
	CV	0.99816	0.99831	0.99839	0.99835	0.99878	0.99918	0.99966	0.99980
1.5	ARL	370.3704	433.5428	507.9193	515.1992	370.3704	551.1424	1295.7672	2222.1224
	CV	0.99865	0.99885	0.99901	0.99902	0.99865	0.99909	0.99961	0.99978
2	ARL	526.1138	784.0382	1747.9873	2709.5659	230.4531	332.1712	729.1935	1197.5488
	CV	0.99905	0.99936	0.99971	0.99982	0.99783	0.99849	0.99931	0.99958

TABLE 8. ARL study of PI-IHR case based on $\beta_0 = 1.5$, $q_0 = 0.0005$, $\beta_1 \in \{0.1, 0.3, 0.45, 0.50, 0.55, 0.8\}$, $q_1 \in \{0.0003, 0.0001, 0.00005\}$ and $\alpha = 0.0027$.

		Two-sided				Upper-sided			
β	q	0.0005	0.0003	0.0001	0.00005	0.0005	0.0003	0.0001	0.00005
0.1	ARL	1.348133	1.308901	1.241160	1.207467	1378.787	2241.248	6371.824	12318.650
	CV	0.50816	0.48571	0.44071	0.41451	0.99964	0.99978	0.99992	0.99996
0.3	ARL	23.39098	22.14676	19.73593	18.42884	691.4458	1073.0172	2760.9405	5012.1307
	CV	0.97839	0.97716	0.97433	0.97249	0.99928	0.99953	0.99982	0.99990
0.45	ARL	225.8125	238.2593	239.6847	231.1045	430.4155	646.9948	1554.5162	2702.6309
	CV	0.99778	0.99781	0.99791	0.99783	0.99884	0.99923	0.99968	0.99982
0.5	ARL	370.3704	433.5428	507.9193	515.1992	370.3704	551.1424	1295.7672	2222.1224
	CV	0.99865	0.99885	0.99901	0.99902	0.99865	0.99909	0.99961	0.99978
0.55	ARL	487.9458	639.8466	941.2072	1055.3811	319.8903	471.3592	1084.9695	1835.9273
	CV	0.99897	0.99922	0.99947	0.99952	0.99844	0.99894	0.99954	0.99973
0.8	ARL	433.4783	650.7570	1551.7289	2667.1626	162.0712	228.1452	475.9877	757.0157
	CV	0.99885	0.99923	0.99968	0.99981	0.99691	0.99781	0.99895	0.99934

shift in the scale parameter for different values of β. Further the values of ARL increases with the increase of β for the two-sided chart while it decreases in the case of upper-sided chart. For $\beta > 1.5$, the ARL values in the last two columns of the table are much larger than for $\beta < 1.5$ in the two-sided chart. Similarly, in the case of the one-sided chart, for $\beta > 1.5$, the ARL values in the last two columns are smaller than for $\beta < 1.5$. Table-7 is graphically shown in Figure 9.

C. CASE-3 (TI-DHR)

For discrete Weibull distribution, the hazard rate decreases when $\beta < 1$ and can be denoted as the DHR. If q is decreasing then process is totally improving and is denoted by TI, thus, the overall situation can be labeled as the TI-DHR. Assuming the in-control values $q_0 = 0.0005$ and $\beta_0 = 0.5$, suppose that q decreases from 0.0005 to $q_1 \in \{0.0003, 0.0001, 0.00005\}$ and $\beta = 1.5$ to $\beta_1 \in \{0.1, 0.3, 0.45, 0.50, 0.55, 0.8\}$ for out-of-control situations. Then, the resulting values of ARL are tabulated in Table-8.

For detecting process improvement, the upper-sided and the two-sided charts are investigated. From Table-8, it can be seen that, again, the two-sided chart detects shift quickly as compared to the one-sided chart. In two-sided chart case, the ARL is less sensitive to variations for different values of q at $\beta = 0.1, 0.3, 0.45$, however, for $\beta > \beta_0$, the ARL is much sensitive to the shape parameter for $q = 0.0001$ and 0.00005. This conclusion is supplemented by the CVRL. That is, the efficient chart has small CVRL values. In the

TABLE 9. Study of ARL for case-4 based on $\beta_0 = 0.5$, $q_0 = 0.0005$, $\beta_1 \in \{0.1, 0.3, 0.45, 0.5, 0.55\}$, $q_1 \in \{0.005, 0.01, 0.1\}$ and $\alpha = 0.0027$.

		Two-sided				Lower-sided			
β	q	0.0005	0.005	0.01	0.1	0.0005	0.05	0.01	0.1
0.1	ARL	1.34813	1.62344	1.75312	2.25973	1.26852	1.51242	1.64027	2.66507
	CV	0.50817	0.61961	0.65543	0.74664	0.46009	0.58207	062478	0.79043
0.3	ARL	23.39098	27.02057	24.96894	7.50465	15.93958	22.64545	25.97761	51.45032
	CV	0.97839	0.98132	0.97977	0.93099	0.96812	0.97767	0.98056	0.99023
0.45	ARL	225.81255	86.24844	52.10818	7.55768	167.6703	240.3201	276.4166	552.3327
	CV	0.99778	0.99419	0.99036	0.93141	0.99701	0.99791	0.99819	0.99909
0.5	ARL	370.37037	91.46316	52.43330	7.37923	370.3704	531.1114	610.9764	1221.4525
	CV	0.99865	0.99452	0.99042	0.92978	0.99865	0.99906	0.99918	0.99959
0.55	ARL	487.94579	90.49701	50.90651	7.18946	818.8532	1174.4996	1351.2041	2701.9081
	CV	0.99897	0.99446	0.99013	0.92785	0.99939	0.99957	0.99963	0.999815

upper-sided chart, for all values of q, the ARL decreases for different values of the shape parameter β. The graphical depiction of Table-8 is provided in Figure 9.

D. CASE-4 (PD-DHR)

For discrete Weibull distribution, the hazard is called decreasing hazard when $\beta < 1$ and this situation can be labelled as the DHR. If q is increasing then the system is partially deteriorating and can be labeled as the PD. Therefore, in this case, the overall situation can be named as the PD-DHR. Assuming the in-control $q_0 = 0.0005$ and $\beta_0 = 0.5$, while for out-of-control situation q is decreasing from 0.0005 to $q_1 \in \{0.005, 0.01, 0.1\}$ for $\beta_1 \in \{0.1, 0.3, 0.45, 0.5, 0.55\}$, the resulting values of the ARL and the CV are listed in Table-9.

In the case of PD-DHR, the performance of the two-sided and the upper-sided charts can be examined from Table-9 and it can be observed that for the lower sided chart, the value of ARL increases for different values of β. For $q_1 \in \{0.0005, 0.005, 0.01\}$, the lower-sided chart outperforms at $\beta = 0.1, 0.3$ the two-sided chart. For $\beta > 0.3$, the one-sided chart performs better than the two-sided chart and this conclusion is also supplemented by the CVRL.

In all the above reported cases, it is assumed that in control value of ARL is 370 by assuming Type-I error $\alpha = 0.0027$, i.e., $ARL_0 = 1/\alpha$. The CV of the run length is also reported in Tables 6-9 which is $\sqrt{1 - 1/ARL}$. It is noticed that the performance of discrete Weibull control chart is not the same over different values of β and q, e.g., the two-sided chart performs better in the TD-IHR, PI-IHR, TI-DHR cases. Similarly, in the case of PD-DHR, the lower-sided chart performs better for detecting deterioration in the process. The variation in performance is due to the characteristic of the distribution, i.e., IHR, DHR or constant hazard rate.

VI. CONCLUDING REMARKS

In time-between-events (TBE) charts, the occurrence of an event follows Poisson process and it is assumed that the time between two non-conformities follows exponential distribution, which is only suitable for constant failure rate. Contrary to this, here we proposed a new control chart using discrete Weibull distribution due to its wide application. The reason of using this distribution is the collection of discrete data more easier than the continuous data. The performance of the control chart is evaluated through ARL and CV. In reliability analysis when failure data measured as discrete variable, control chart based on discrete Weibull distribution can be used for monitoring of data depending on the property IHR, DHR, and constant hazard rate. In future, the effect of estimated parameter [30], [36]–[38] on the ARL can be investigated using different methods of estimation. Also, memory-type control charts [39] for discrete data can be developed and compared. Furthermore, this study assumed that the data follow discrete Weibull distribution. This assumption may violate in real life situations. Thus, for future research, the CCC charts can be extended to nonparametric CCC charts.

APPENDIXES

APPENDIX A
PARAMETER ESTIMATION

To estimate the unknown parameters of the discrete Weibull discrete distribution, we use the maximum likelihood estimation (MLE) method. First, the likelihood function for $z = (z_1, z_2, \cdots, z_n)$ can be written as

$$L(q, \beta; z) = \prod_{i=1}^{n} P(z_i; q, \beta) = \prod_{i=1}^{n} \left(q^{z_i^{\beta}} - q^{(z_i+1)^{\beta}}\right) \quad (14)$$

The logarithm of the likelihood function can be written as

$$\log L(q, \beta; z) = \sum_{i=1}^{n} \log\left(q^{z_i^\beta} - q^{(z_i+1)^\beta}\right) \quad (15)$$

To find the estimators of the unknown parameters q and β, one is required to take the derivatives of Eq. 15 with respect to the unknown parameters and equate the resulting normal equations to zero and simplify.

$$\frac{\partial \log L(q, \beta; z)}{\partial q} = \sum_{i=1}^{n} \frac{z_i^\beta q^{z_i^\beta - 1} - (z_i+1)^\beta q^{(z_i+1)^\beta - 1}}{\left(q^{z_i^\beta} - q^{(z_i+1)^\beta}\right)} \quad (16)$$

$$\frac{\partial \log L(.)}{\partial \beta} = \sum_{i=1}^{n} \frac{q^{z_i^\beta} \ln(z_i)\ln(q) - q^{(z_i+1)^\beta} \ln(z_i+1)\ln(q)}{\left(q^{z_i^\beta} - q^{(z_i+1)^\beta}\right)} \quad (17)$$

Since Eqs.16-17 cannot be solved further, an iterative procedure like Newton Raphson is required to obtain the MLEs. In the next section, R code using the **fitdistrplus** package is given to obtain the parameters of the discrete Weibull distribution numerically.

APPENDIX B
R CODE

The R code to estimate the parameters of the discrete Weibull distribution is given below.

```
###Dengue Fever Data
xx<-c( 1, 1, 0, 6, 2, 0, 1, 2, 1, 0, 2, 2,
2, 1, 2, 2, 2, 2, 0, 1, 2, 0, 1, 1,
2, 1, 0, 1, 0, 2, 0, 0, 0, 2, 5, 3,
0, 0,4, 4, 0, 0, 0, 0, 1, 4, 3, 0)
####install the package
install.packages('fitdistrplus')
library(fitdistrplus)
##PMF of the discrete Weibull distribution
ddweibull<-function(x,a,beta,zero=TRUE) {
a^((x)^beta)-a^((x+1)^beta)}
##CDF of the discrete Weibull distribution
pdweibull<-function(q,a,beta,zero=TRUE) {
1-a^((q+1)^beta)}
##Quantile function of the discrete
##Weibull distribution
qdweibull<-function(p,a,beta,zero=TRUE) {
(log(1-p)/log(a))^(1/beta)-1}
##Obtain the parameter estimates and
##other summaries
summary(fitdist(xx,"dweibull",
start=list(a=0.5,beta=1)))
```

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