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Correlation Coefficients of Interval-Valued Pythagorean Hesitant Fuzzy Sets and Their Applications

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ABSTRACT An interval-valued Pythagorean hesitant fuzzy set (IVPHFS) not only can be regarded as the union of some interval-valued Pythagorean fuzzy sets but also represent the Pythagorean hesitant fuzzy elements in the form of interval values. So IVPHFSs are extensions of Pythagorean hesitant fuzzy sets (PHFSs) and interval-valued Pythagorean fuzzy sets (IVPFSs), which are powerful tools to represent more complicated, uncertain and vague information. This paper focuses on the four kinds of correlation coefficients for PHFSs, and extends them to the correlation coefficients and the weighted correlation coefficients for IVPHFSs. In the processing, we develop the least common multiple expansion (LCME) method to solve the problem that the cardinalities of Pythagorean hesitant fuzzy elements (PHFEs) (or interval-valued Pythagorean hesitant fuzzy elements (IVPHFEs)) are different. In addition, we propose score functions and accuracy functions of Pythagorean fuzzy elements (PFEs) (or interval-valued Pythagorean fuzzy elements (IVPFEs)) to rank all the PFEs (or IVPFEs) in a PHFE (or an IVPHFE). Especially, score functions and accuracy functions of IVPFEs are both presented as interval numbers. Then use the comparison method of interval numbers to compare two revised IVPHFEs in order to keep the original fuzzy information as far as possible. What's more, we define the local correlations and local informational energies which can depict the similarity between two IVPHFEs more meticulously and completely. At last the numerical examples to show the feasibility and applicability of the proposed methods in multiple criteria decision making (MCDM) and clustering analysis.

INDEX TERMS Interval-valued Pythagorean hesitant fuzzy set (IVPHFS), Pythagorean hesitant fuzzy set (PHFS), correlation, informational energy, correlation coefficient.

I. INTRODUCTION

Correlation plays an important role in mathematics, statistics and engineering sciences. The interdependency between two variables can be measured with the aid of correlation analysis. The Karl Pearson coefficient, as a popular correlation coefficient, has been applied widely to various research domains and practical fields, such as data analysis and classification [30], decision-making [33], [37], [45], pattern recognition [20] and so on. Since the information is frequently incomplete, fuzzy and imperfect in many situations, some researchers have developed the correlation coefficients in fuzzy environments. Chiang and Lin [9] defined the concept of correlation of fuzzy sets. Gerstenkorn and Manko [14] discussed the correlation of intuitionistic fuzzy sets. Bustince and Burillo [5] developed the correlation coefficient in interval-valued intuitionistic fuzzy environment. Then Zeng and Wang [47] and Park*et al.* [28] both added indeterminacy degrees to study the correlation coefficient of interval-valued intuitionistic fuzzy sets. Moreover, Mitchell [26] studied the

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correlation coefficient for Type-2 fuzzy sets. It can be seen that the correlation measure is one of the hot spots in the research of fuzzy information. In this paper, we mainly focus on the correlation measures for PHFSs and IVPHFSs.

Torra and Narukawa [35] and Torra [36] introduced the concept of hesitant fuzzy sets (HFSs) which permit the membership degree of an object to a set of several possible values. Because of its outstanding application ability in group decision-making problems, it has been explored in depth. Interval-valued hesitant fuzzy sets [6], [8], [32],dual hesitant fuzzy sets [51], interval-valued dual hesitant fuzzy sets [15], hesitant N-soft sets [3], [4] and interval-valued probabilistic hesitant fuzzy sets [12], [17], [18] have been proposed and applied to decision-making problems successfully. Xu and Xia [42], Chen *et al.* [7] studied the correlation coefficients for classic HFSs. Their methods expand the possible membership degrees through adding some extreme values to the hesitant fuzzy element (HFE) which has the less cardinality of elements. However, if the extreme value is far from the other possible membership values, the extended HFE will be quite different from the original HFE. And if hesitant membership degrees of one HFS is always zero, the correlation coefficients between it and other HFS cannot be calculated. Liao *et al.* [22] defined a novel correlation coefficient formula based on the mean of a HFE and extended the range of the correlation coefficients to the interval $[-1, 1]$. When a HFS is represented by a constant function, the correlation coefficient between it and other HFSs will still not be calculated. Sun *et al.* [34] focused on improving the counter-intuitions of the existing correlation coefficients of HFSs in [7], [22], [42]. Its contribution is mainly in the case of improving the weighted correlation coefficients, but it does not improve the general case. Meng and Chen [24] proposed the correlation coefficients of HFSs based on fuzzy measures and they [25] extended the method to study the case of interval-valued hesitant fuzzy sets. However, the definition is too cumbersome. In addition, Tyagi [37] and Ye [46] studied the correlation coefficients of dual hesitant fuzzy sets. Das *et al.* [10] addressed the correlation coefficients of hesitant fuzzy soft sets. Although there exist several correlation coefficients for HFSs, many unreasonable cases can be made by such concepts. Some unreasonable examples are illustrated in Example 8 and Example 10.

With the development of research, Yager [43], [44] proposed another class of non-standard fuzzy sets, called Pythagorean fuzzy sets (PFSs), which allow the sum of squares of the Pythagorean membership degrees and the corresponding Pythagorean non-membership degree to be less than or equal to 1, rather than the sum of both less than or equal to 1. Obviously a PFS is another extension of an intuitionistic fuzzy set. Due to the broad definition of PFSs, their application has involved various fuzzy problems. For example, Akram *et al.* [1], [2] introduced group decision making methods in Pythagorean fuzzy environment. Yager and Abbasov [44] studied Pythagorean fuzzy aggregation. Li and Zeng [19] developed the distance measure of PFSs and applied them in MCDM. Nguyen [27] proposed the correlation coefficients of PFSs based on the mean of all the Pythagorean fuzzy elements. This method fails when a PFS takes all constant functions. Garg [13] studied other correlation coefficients of PFSs and applied them to decision-making processes, whereas they may encounter situations where the correlation cannot be distinguished. Example 12 shows the drawbacks of his correlation coefficients.

Subsequently, some scholars combined PFSs with HFSs and introduced Pythagorean hesitant fuzzy sets (PHFSs), but their definitions are little different. Liu and He [23], Khan *et al.* [16] and Liang and Xu [21] separated possible Pythagorean membership degrees from possible Pythagorean non-membership degrees and considered possible membership degrees and possible non-membership degrees to form two independent HFEs. Wei *et al.* [38] saw a Pythagorean hesitant fuzzy element (PHFE) as a set of several Pythagorean fuzzy elements (PFEs). Since the evaluations of alternatives based on attributes by some expert are always presented in the form of pairs in real-life decision making, we think Wei's definition is more reasonable than others. However, we have seen few researches on correlation coefficients for PHFSs so far.

In practical problems, we usually encounter possible Pythagorean membership degrees and possible Pythagorean non-membership degrees of an object are represented by several possible interval numbers. On the basis of Wei's definition, Zhang *et al.* [50] put forward the concept of interval-valued Pythagorean fuzzy sets (IVPHFSs), which satisfy each interval-valued Pythagorean hesitant fuzzy element (IVPHFE) is a set of some pairs of possible Pythagorean membership interval values and possible Pythagorean non-membership interval values. The theory depicts the complicated, fuzzy environments more suitably.

The aim of this paper is to study the correlation coefficients for PHFSs and IVPHFSs and their applications. Considering that the existing methods of comparing two HFSs are greatly affected by extreme values, firstly we use the least common multiple expansion (LCME) method to make the cardinalities of two PHFEs (or two IVPHFEs) are consistent. This method does not add any additional extreme information. Then construct score functions and accuracy functions of PFEs in order to sort the PFEs in a PHFE. Especially, while sorting the IVPFEs in an IVPHFE, define score functions and accuracy functions of IVPFEs in the form of interval numbers so as to keep the original fuzzy information as far as possible. What's more, we introduce the concepts of the local correlations and the local informational energies, and then we deduce the four formulas of correlation coefficients for PHFSs and IVPHFSs and the four weighted correlation coefficients for IVPHFSs. Our correlation coefficients can not only degenerate into the correlation coefficients of HFSs or PFSs, but also solve the problem that the existing correlation of HFSs or PFSs cannot handle. Their applications in multiple criteria decision-making (MCDM) and clustering analysis are illustrated.

The structure of this paper is organized as follows: In Section 2, we review some basic concepts and results of HFSs, PHFSs and IVPHFSs. Section 3 gives the definitions of the correlations, informational energies and correlation coefficients of PHFSs and discusses some properties. Furthermore, the correlation coefficients and the weighted correlation coefficients for IVPHFSs are proposed in Section 4. Section 5 solves a MCDM problem and a clustering problem by the proposed weighted correlation coefficients in interval-valued Pythagorean hesitant fuzzy environments. Concluding remarks are made in Section 6.

II. PRELIMINARIES

Let $U = \{x_i | i = 1, 2, \cdots, n\}$ be the universe of discourse in this paper, unless otherwise specified.

A. HESITANT FUZZY SET (HFS)

A hesitant fuzzy set (HFS), introduced by Torra and Narukawa [35], [36], allows the membership degree of an element to a set represented by several possible values. It is very suitable for describing problems that are difficult to be determined by only one membership degree.

Definition 1 [36]: A hesitant fuzzy set (HFS) A on U is described as $A = \{\langle x, h_A(x)\rangle | x \in U\}$, *here* $h_A(x) = \{\mu_A(x) | \mu_A(x) \in [0, 1]\}$ *represents the set of possible membership degrees of A at x.*

For convenience, Xia and Xu [40] call $h = h_A(x)$ a hesitant fuzzy element (HFE). And Torra and Narukawa [35] defined some operators on HFEs, such as:

(1) $h^C = \{1 - \mu | \mu \in h\}$;

(2) $h_1 \cup h_2 = \{\max\{\mu_1, \mu_2\} | \mu_1 \in h_1, \mu_2 \in h_2\}$;

(3) $h_1 \cap h_2 = \{\min\{\mu_1, \mu_2\} | \mu_1 \in h_1, \mu_2 \in h_2\}$.

HFSs have been applied to get the optimal alternatives in decision-making problems with multiple attributes and multiple decision makers.

B. PYTHAGOREAN FUZZY SET (PFS)

A Pythagorean fuzzy set (PFS), introduced by Yager in 2013 [43], is characterized by a membership function and a non-membership function, where the sum of the square of the membership degree and the non-membership degree of x is less than or equal to 1, while an intuitionistic fuzzy set is also characterized by them, where the sum is less than or equal to 1. Obviously PFSs are more general than intuitionistic fuzzy sets. A PFS has emerged as an effective tool to solve multiple criteria decision making problems [31].

Definition 2 [43]: A Pythagorean fuzzy set (PFS) P on U is described as: $P = \{\langle x, \mu_P(x), \nu_P(x)\rangle | x \in U\}$, *here* $\mu_P(x), \nu_P(x) \in [0, 1]$ *are Pythagorean membership degree and Pythagorean non-membership degree of P at x, respectively. They satisfy* $0 \leqslant \mu_P^2(x) + \nu_P^2(x) \leqslant 1$. *The Pythagorean indeterminacy degree is given by* $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)}$.

For convenience, Zhang and Xu [49] called a Pythagorean fuzzy element (PFE), denoted by $P = \langle \mu_P, \nu_P\rangle$.

Yager and Abbasov [44] continued to study the operators on PFSs. Let $A = \langle \mu_A, \nu_A\rangle$ and $B = \langle \mu_B, \nu_B\rangle$ be two PFEs on U. He defined the following operators:

(1) $A^C = \langle \nu_A, \mu_A\rangle$;

(2) $A \bigcup B = \langle \max\{\mu_A, \mu_B\}, \min\{\nu_A, \nu_B\}\rangle$;

(3) $A \bigcap B = \langle \min\{\mu_A, \mu_B\}, \max\{\nu_A, \nu_B\}\rangle$.

Obviously an intuitionistic fuzzy set is a special PFS, or say that a PFS is a generalization of an intuitionistic fuzzy set.

C. PYTHAGOREAN HESITANT FUZZY SET (PHFS)

Based on the concept of PFS and HFS, Wei proposed the concept of Pythagorean hesitant fuzzy sets (PHFSs) [38], whose membership degree is a set of several possible PFEs. That is:

Definition 3 [38]: A Pythagorean hesitant fuzzy set (PHFS) $\mathscr{P}$ *on U is described as:*

$$\mathscr{P} = \{\langle x, h_{\mathscr{P}}(x)\rangle | x \in U\},$$

where $h_{\mathscr{P}}(x) = \{\langle \mu_{\mathscr{P}}(x), \nu_{\mathscr{P}}(x)\rangle | \mu_{\mathscr{P}}^2(x) + \nu_{\mathscr{P}}^2(x) \leqslant 1\}$ *is a set of some PFEs in U, denoting the possible Pythagorean membership degree and possible Pythagorean non-membership degree of* $\mathscr{P}$ *at x. We call* $h_{\mathscr{P}} = h_{\mathscr{P}}(x)$ *a Pythagorean hesitant fuzzy element (PHFE), here* $h_{\mathscr{P}} = \{\langle \mu, \nu\rangle | \mu^2 + \nu^2 \leqslant 1\}$.

Definition 4 [38]: Let $h_{\mathscr{P}} = \{\langle \mu, \nu\rangle\}$, $h_{\mathscr{P}_1} = \{\langle \mu_1, \nu_1\rangle\}$ *and* $h_{\mathscr{P}_2} = \{\langle \mu_2, \nu_2\rangle\}$ *be three PHFEs and* $\lambda > 0$. *The basic operators on PHFEs are defined as:*

(1) $h_{\mathscr{P}}^C = \{\langle \nu, \mu\rangle | \langle \mu, \nu\rangle \in h_{\mathscr{P}}\}$;

(2) $h_{\mathscr{P}}^\lambda = \{\langle \mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda}\rangle | \langle \mu, \nu\rangle \in h_{\mathscr{P}}\}$;

(3) $\lambda h_{\mathscr{P}} = \{\langle \sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda\rangle | \langle \mu, \nu\rangle \in h_{\mathscr{P}}\}$;

(4) $h_{\mathscr{P}_1} \oplus h_{\mathscr{P}_2} = \{\langle \sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1\mu_2)^2}, \nu_1\nu_2\rangle | \langle \mu_i, \nu_i\rangle \in h_{\mathscr{P}_i}, i = 1, 2\}$;

(5) $h_{\mathscr{P}_1} \otimes h_{\mathscr{P}_2} = \{\langle \mu_1\mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1\nu_2)^2}\rangle | \langle \mu_i, \nu_i\rangle \in h_{\mathscr{P}_i}, i = 1, 2\}$;

(6) $h_{\mathscr{P}_1} \bigcup h_{\mathscr{P}_2} = \{\langle \max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}\rangle | \langle \mu_i, \nu_i\rangle \in h_{\mathscr{P}_i}, i = 1, 2\}$;

(7) $h_{\mathscr{P}_1} \bigcap h_{\mathscr{P}_2} = \{\langle \min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\}\rangle | \langle \mu_i, \nu_i\rangle \in h_{\mathscr{P}_i}, i = 1, 2\}$.

D. INTERVAL-VALUED PYTHAGOREAN HESITANT FUZZY SET (IVPHFS)

As mentioned earlier, in many practical problems, it is difficult for decision makers to determine precise membership degrees or non-membership degrees. Interval numbers can avoid the information loss better and enhance the flexibility and applicability of decision-making models in dealing with qualitative information. We extend PHFSs to interval-valued Pythagorean hesitant fuzzy sets (IVPHFSs) in [50].

Definition 5 [50]: An interval-valued Pythagorean hesitant fuzzy set (IVPHFS) $\widetilde{\mathscr{P}}$ *on U is described as*

$$\widetilde{\mathscr{P}} = \{\langle x, h_{\widetilde{\mathscr{P}}}(x)\rangle | x \in U\},$$

where

$$h_{\widetilde{\mathscr{P}}}(x) = \{\langle \widetilde{\mu}_{\widetilde{\mathscr{P}}}(x), \widetilde{\nu}_{\widetilde{\mathscr{P}}}(x)\rangle \; | \widetilde{\mu}_{\widetilde{\mathscr{P}}}(x) = [\mu^-_{\widetilde{\mathscr{P}}}(x), \mu^+_{\widetilde{\mathscr{P}}}(x)] \in D[0, 1],$$

$$\widetilde{\nu}_{\widetilde{\mathscr{P}}}(x) = [\nu^-_{\widetilde{\mathscr{P}}}(x), \nu^+_{\widetilde{\mathscr{P}}}(x)] \in D[0,1], (\mu^+_{\widetilde{\mathscr{P}}}(x))^2 + (\nu^+_{\widetilde{\mathscr{P}}}(x))^2 \leqslant 1\}.$$

$\widetilde{\mu}_{\widetilde{\mathscr{P}}}(x)$ and $\widetilde{\nu}_{\widetilde{\mathscr{P}}}(x)$ are the possible Pythagorean membership intervals and the possible Pythagorean non-membership intervals of $\widetilde{\mathscr{P}}$ at x, respectively. The possible Pythagorean indeterminacy degree of $\widetilde{\mathscr{P}}$ at x is a set of some pairs of intervals defined as

$$\widetilde{\pi}_{\widetilde{\mathscr{P}}}(x) = \{[\pi^-_{\widetilde{\mathscr{P}}}(x), \pi^+_{\widetilde{\mathscr{P}}}(x)]$$
$$|\pi^-_{\widetilde{\mathscr{P}}}(x) = \sqrt{1-(\mu^+_{\widetilde{\mathscr{P}}}(x))^2-(\nu^+_{\widetilde{\mathscr{P}}}(x))^2},$$
$$\pi^+_{\widetilde{\mathscr{P}}}(x) = \sqrt{1-(\mu^-_{\widetilde{\mathscr{P}}}(x))^2-(\nu^-_{\widetilde{\mathscr{P}}}(x))^2},$$
$$\langle[\mu^-_{\widetilde{\mathscr{P}}}(x), \mu^+_{\widetilde{\mathscr{P}}}(x)], [\nu^-_{\widetilde{\mathscr{P}}}(x), \nu^+_{\widetilde{\mathscr{P}}}(x)]\rangle = \langle\widetilde{\mu}_{\widetilde{\mathscr{P}}}(x), \widetilde{\nu}_{\widetilde{\mathscr{P}}}(x)\rangle \in h_{\widetilde{\mathscr{P}}}(x)\}.$$

For convenience, we call each set of pairs $h_{\widetilde{\mathscr{P}}} = h_{\widetilde{\mathscr{P}}}(x)$ as an interval-valued Pythagorean hesitant fuzzy element (IVPHFE), where $h_{\widetilde{\mathscr{P}}} = \{\langle\widetilde{\mu}, \widetilde{\nu}\rangle | \widetilde{\mu} = [\mu^-, \mu^+], \widetilde{\nu} = [\nu^-, \nu^+], (\mu^+)^2 + (\nu^+)^2 \leqslant 1\}$. For each IVPHFE $h_{\widetilde{\mathscr{P}}}$, if $\widetilde{\mu}$ and $\widetilde{\nu}$ both degenerate into one singleton, the IVPHFS is a PHFS; if $h_{\widetilde{\mathscr{P}}}$ includes only one pair of intervals, the IVPHFS degenerates into an IVPFS [29]; if $\widetilde{\nu} \equiv [0,0]$, the IVPHFS is an interval-valued hesitant fuzzy set [6]; if $\mu^+ + \nu^+ \leqslant 1$, the IVPHFS is an interval-valued intuitionistic hesitant fuzzy set, similar to [48].

Now we define some basic operators on IVPHFSs.

Definition 6 [50]: Let $h_{\widetilde{\mathscr{P}}} = \{\langle\widetilde{\mu}, \widetilde{\nu}\rangle | \widetilde{\mu} = [\mu^-, \mu^+], \widetilde{\nu} = [\nu^-, \nu^+]\}$, $h_{\widetilde{\mathscr{P}}_1} = \{\langle\widetilde{\mu}_1, \widetilde{\nu}_1\rangle | \widetilde{\mu}_1 = [\mu^-_1, \mu^+_1], \widetilde{\nu}_1 = [\nu^-_1, \nu^+_1]\}$, $h_{\widetilde{\mathscr{P}}_2} = \{\langle\widetilde{\mu}_2, \widetilde{\nu}_2\rangle | \widetilde{\mu}_2 = [\mu^-_2, \mu^+_2], \widetilde{\nu}_2 = [\nu^-_2, \nu^+_2]\}$ be three IVPHFEs and $\lambda > 0$. The basic operators on IVPHFEs are defined as follows:

(1) $h^C_{\widetilde{\mathscr{P}}} = \{\langle[\nu^-, \nu^+], [\mu^-, \mu^+]\rangle | \langle\widetilde{\mu}, \widetilde{\nu}\rangle \in h_{\widetilde{\mathscr{P}}}\}$;

(2) $h^\lambda_{\widetilde{\mathscr{P}}} = \{\langle[(\mu^-)^\lambda, (\mu^+)^\lambda], [\sqrt{1-(1-(\nu^-)^2)^\lambda}, \sqrt{1-(1-(\nu^+)^2)^\lambda}]\rangle | \langle\widetilde{\mu}, \widetilde{\nu}\rangle \in h_{\widetilde{\mathscr{P}}}\}$;

(3) $\lambda h_{\widetilde{\mathscr{P}}} = \{[\sqrt{1-(1-(\mu^-)^2)^\lambda}, \sqrt{1-(1-(\mu^+)^2)^\lambda}], [(\nu^-)^\lambda, (\nu^+)^\lambda]\rangle | \langle\widetilde{\mu}, \widetilde{\nu}\rangle \in h_{\widetilde{\mathscr{P}}}\}$;

(4) $h_{\widetilde{\mathscr{P}}_1} \oplus h_{\widetilde{\mathscr{P}}_2} = \{\langle[\sqrt{(\mu^-_1)^2 + (\mu^-_2)^2 - (\mu^-_1)^2(\mu^-_2)^2}, \sqrt{(\mu^+_1)^2 + (\mu^+_2)^2 - (\mu^+_1)^2(\mu^+_2)^2}], [\nu^-_1\nu^-_2, \nu^+_1\nu^+_2]\rangle | \langle\widetilde{\mu}_i, \widetilde{\nu}_i\rangle \in h_{\widetilde{\mathscr{P}}_i}, i = 1,2\}$;

(5) $h_{\widetilde{\mathscr{P}}_1} \otimes h_{\widetilde{\mathscr{P}}_2} = \{\langle[\mu^-_1\mu^-_2, \mu^+_1\mu^+_2], [\sqrt{(\nu^-_1)^2 + (\nu^-_2)^2 - (\nu^-_1)^2(\nu^-_2)^2}, \sqrt{(\nu^+_1)^2 + (\nu^+_2)^2 - (\nu^+_1)^2(\nu^+_2)^2}]\rangle, | \langle\widetilde{\mu}_i, \widetilde{\nu}_i\rangle \in h_{\widetilde{\mathscr{P}}_i}, i = 1,2\}$;

(6) $h_{\widetilde{\mathscr{P}}_1} \bigcup h_{\widetilde{\mathscr{P}}_2} = \{\langle[\max\{\mu^-_1, \mu^-_2\}, \max\{\mu^+_1, \mu^+_2\}], [\min\{\nu^-_1, \nu^-_2\}, \min\{\nu^+_1, \nu^+_2\}]\rangle | \langle\widetilde{\mu}_i, \widetilde{\nu}_i\rangle \in h_{\widetilde{\mathscr{P}}_i}, i = 1,2\}$;

(7) $h_{\widetilde{\mathscr{P}}_1} \bigcap h_{\widetilde{\mathscr{P}}_2} = \{\langle[\min\{\mu^-_1, \mu^-_2\}, \min\{\mu^+_1, \mu^+_2\}], [\max\{\nu^-_1, \nu^-_2\}, \max\{\nu^+_1, \nu^+_2\}]\rangle | \langle\widetilde{\mu}_i, \widetilde{\nu}_i\rangle \in h_{\widetilde{\mathscr{P}}_i}, i = 1,2\}$.

Similar to the cases of PHFEs, $h^C_{\widetilde{\mathscr{P}}}$, $h^\lambda_{\widetilde{\mathscr{P}}}$, $\lambda h_{\widetilde{\mathscr{P}}}$, $h_{\widetilde{\mathscr{P}}_1} \oplus h_{\widetilde{\mathscr{P}}_2}$, $h_{\widetilde{\mathscr{P}}_1} \otimes h_{\widetilde{\mathscr{P}}_2}$, $h_{\widetilde{\mathscr{P}}_1} \bigcup h_{\widetilde{\mathscr{P}}_2}$ and $h_{\widetilde{\mathscr{P}}_1} \bigcap h_{\widetilde{\mathscr{P}}_2}$ are all IVPHFEs.

III. CORRELATION COEFFICIENTS FOR PHFSS

A. EXISTING CORRELATION COEFFICIENTS FOR HFSS

Correlation coefficients can reflect the degrees of relationship between two variables. As a probability parameter, it has been successfully applied to many real problems. Many approaches [7], [22], [42] have been introduced to study the correlation coefficients of HFSs. Since the cardinalities of two HFEs may be different, Xu [42] proposed the two methods based on the pessimistic principle and the optimistic principle to make the cardinalities of two HFEs same. Specific methods include: (1) Add multiple minimum elements to the collection with a small cardinality (pessimistic principle); and (2) Add multiple maximum elements to the collection with a small cardinality (optimistic principle). For example, when $|h_A(x_i)| \neq |h_B(x_i)|$ for some $x_i \in U$, assume $|h_A(x_i)| < |h_B(x_i)|$. $h_A(x_i)$ should add the minimum values by pessimistic principle (or the maximum values by optimistic principle) in it until it has the same cardinality with $h_B(x_i)$. Finally the two cardinalities realize $|h_B(x_i)| = \max\{|h_A(x_i)|, |h_B(x_i)|\}$.

At the same time, Chen [7] arranged HFEs in a decreasing order. For any HFE $h = \{\mu_j | j = 1, 2, \cdots, m\}$, let $\sigma:(1,2,\cdots,m) \to (1,2,\cdots,m)$ be a permutation satisfying $\mu_{\sigma(j)} \geq \mu_{\sigma(j+1)}$, $j \in \{1,2,\cdots,m-1\}$, and $\mu_{\sigma(j)}$ be the jth largest value in h. So we have the extended HFE $h' = \{\mu_{\sigma(j)} | j = 1,2,\cdots,m\}$. Then the correlation coefficient for HFSs can be defined as follows.

Definition 7 [7]: Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in U\}$, $B = \{\langle x_i, h_B(x_i)\rangle | x_i \in U\}$ be two HFSs on U. The correlation coefficient between A and B, denoted by $\rho^{(1)}_H(A,B)$, is defined as

$$\rho^{(1)}_H(A,B) = \frac{C_H(A,B)}{\sqrt{E_H(A)}\sqrt{E_H(B)}} = \frac{\sum_{i=1}^{n}(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu_{A\sigma(j)}(x_i)\mu_{B\sigma(j)}(x_i))}{\sqrt{\sum_{i=1}^{n}(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\mu^2_{A\sigma(j)}(x_i))}\sqrt{\sum_{i=1}^{n}(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\mu^2_{B\sigma(j)}(x_i))}},$$

where

$$C_H(A,B) = \sum_{i=1}^{n}(\frac{1}{l_i}\sum_{j=1}^{l_i}\mu_{A\sigma(j)}(x_i)\mu_{B\sigma(j)}(x_i))$$

is called the correlation between A and B.

$$E_H(A) = \sum_{i=1}^{n}(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\mu^2_{A\sigma(j)}(x_i)),$$
$$E_H(B) = \sum_{i=1}^{n}(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\mu^2_{B\sigma(j)}(x_i))$$

are called the informational energy of A and B, respectively.

Here $\mu_{A\sigma(j)}(x_i) \in h'_A(x_i)$, $\mu_{B\sigma(j)}(x_i) \in h'_B(x_i)$, $l_i = \max\{|h_A(x_i)|, |h_B(x_i)|\} = |h'_A(x_i)| = |h'_B(x_i)|$, $l_{Ai} = |h_A(x_i)|$ and $l_{Bi} = |h_B(x_i)|$.

Example 8: Let $A_1 = \{\langle x, \{0\}\rangle\}, B_1 = \{\langle x, \{0.2, 0.3\}\rangle\}$ be two HFSs on $U_1 = \{x\}$.

Since $E_H(A_1) = 0$, we cannot calculate the correlation coefficient between A_1 and B_1 based on Definition 7.

In 2015, Liao [22] presented a novel correlation coefficient for HFSs based on the classical correlation coefficient in statistics.

Definition 9 [22]: Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in U\}, B = \{\langle x_i, h_B(x_i)\rangle | x_i \in U\}$ be two HFSs on U. The correlation coefficient between A and B, denoted by $\rho_H^{(2)}(A, B)$, is defined as

$$\rho_H^{(2)}(A, B) = \frac{\sum_i^n (\overline{h}_A(x_i) - \overline{A})(\overline{h}_B(x_i) - \overline{B})}{\sqrt{\sum_{i=1}^n (\overline{h}_A(x_i) - \overline{A})^2}\sqrt{\sum_{i=1}^n (\overline{h}_B(x_i) - \overline{B})^2}},$$

here $h_A(x_i) = \{\mu_{A1}(x_i), \mu_{A2}(x_2), \cdots, \mu_{Al_{Ai}}(x_i)\}$,

$$h_B(x_i) = \{\mu_{B1}(x_i), \mu_{B2}(x_2), \cdots, \mu_{Bl_{Bi}}(x_i)\},$$

$$\overline{h}_A(x_i) = \frac{1}{l_{Ai}}\sum_{k=1}^{l_{Ai}} \mu_{Ak}(x_i), \quad \overline{h}_B(x_i) = \frac{1}{l_{Bi}}\sum_{k=1}^{l_{Bi}} \mu_{Bk}(x_i),$$

$$\overline{A} = \frac{1}{n}\sum_{i=1}^{n} \overline{h}_A(x_i), \quad \overline{B} = \frac{1}{n}\sum_{i=1}^{n} \overline{h}_B(x_i).$$

Example 10: Let $A_2 = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.2\}\rangle\}$ and $B_2 = \{\langle x_1, \{0.1\}\rangle, \langle x_2, \{0.2, 0.3\}\rangle\}$ be two HFSs on $U_2 = \{x_1, x_2\}$.

Since $\overline{h}_{A_2}(x_1) = \overline{h}_{A_2}(x_2) = \overline{A_2} = 0.2$, $\rho_H(A_2, B_2)$ cannot be calculated by Definition 9.

We can find the above existing two definitions of correlation coefficients for HFSs both have some drawbacks.

B. EXISTING CORRELATION COEFFICIENTS FOR PFSS

Garg [13] discussed the correlation coefficients between two PFSs, which consider not only Pythagorean membership degrees and Pythagorean non-membership degrees, but also Pythagorean indeterminacy degrees.

Definition 11: Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in U\}$ be two PFSs on U. The two correlation coefficients between A and B, denoted by $\rho_P^{(1)}(A, B)$ and $\rho_P^{(2)}(A, B)$, respectively, are defined as

$$\rho_P^{(1)}(A, B) = \frac{C_P(A, B)}{\sqrt{E_P(A)}\sqrt{E_P(B)}};$$

$$\rho_P^{(2)}(A, B) = \frac{C_P(A, B)}{\max\{E_P(A), E_P(B)\}},$$

where

$$C_P(A, B) = \sum_{i=1}^{n}(\mu_A^2(x_i)\mu_B^2(x_i) + \nu_A^2(x_i)\nu_B^2(x_i) + \pi_A^2(x_i)\pi_B^2(x_i)),$$

$$E_P(A) = \sum_{i=1}^{n}(\mu_A^4(x_i) + \nu_A^4(x_i) + \pi_A^4(x_i)),$$

$$E_P(B) = \sum_{i=1}^{n}(\mu_B^4(x_i) + \nu_B^4(x_i) + \pi_B^4(x_i)).$$

Since they add Pythagorean indeterminacy degrees to correlation coefficients, the definitions are more accepted by a large number of researchers. However, they may meet the indistinguishable situation.

Example 12: Let A, B, C be three PFSs on $U = \{x_1, x_2, x_3\}$, here

$$A = \{\langle x_1, 0.6, \sqrt{0.55}\rangle, \langle x_2, 0.5, 0.3\rangle, \langle x_3, 0.4, 0.5\rangle\};$$
$$B = \{\langle x_1, 0.3, \sqrt{0.55}\rangle, \langle x_2, 0.5, 0.4\rangle, \langle x_3, 0.3, 0.5\rangle\};$$
$$C = \{\langle x_1, 0.1, \sqrt{0.98}\rangle, \langle x_2, 1, 0\rangle, \langle x_3, 0, 1\rangle\}.$$

We can compute the correlation coefficients as follows:

$$\rho_P^{(1)}(A, C) = \rho_P^{(1)}(B, C) = 0.5158,$$
$$\rho_P^{(2)}(A, C) = \rho_P^{(2)}(B, C) = 0.3525.$$

Obviously Garg's method cannot distinguish the above situation.

C. CORRELATIONS AND CORRELATION COEFFICIENTS FOR PHFSS

For PHFSs, the first problem is that we should adjust their cardinalities to compare them, since the cardinalities of two PHFEs are often different. Based on the previous analysis, we find the construction of correlation coefficients of HFSs needs to add some minimum values or some maximum values, which may cause the fuzzy information to be inconsistent with the original HFEs. The two methods both have the drawback, which is the methods depend on the decision makers' subjective perception of risk preferences which are affected by the extreme values. In this paper we propose the least common multiple expansion LCME) method similar to [39], which can add information from the original PHFEs evenly and not just add extreme values.

Definition 13: Let $h_{\mathscr{A}_t}(x_i) = \{\langle\mu_{i1}, \nu_{i1}\rangle, \cdots, \langle\mu_{iS_{t_i}}, \nu_{iS_{t_i}}\rangle\}$ $(t = 1, 2, \cdots, T)$ be some PHFEs about x_i on U with $S_{t_i} = |h_{\mathscr{A}_t}(x_i)|$. Take the least common multiple number of all S_{t_i} $(t = 1, 2, \cdots, T)$ and denote it S_i. Then for any t, $h_{\mathscr{A}_t}(x_i)$ can be extended to

$$h'_{\mathscr{A}_t}(x_i) = \{\underbrace{\langle\mu_{i1}, \nu_{i1}\rangle, \cdots, \langle\mu_{i1}, \nu_{i1}\rangle}_{\frac{S_i}{S_{t_i}}\ times}, \cdots, \underbrace{\langle\mu_{iS_{t_i}}, \nu_{iS_{t_i}}\rangle, \cdots, \langle\mu_{iS_{t_i}}, \nu_{iS_{t_i}}\rangle}_{\frac{S_i}{S_{t_i}}\ times}\}.$$

Then $|h'_{\mathscr{A}_1}(x_i)| = |h'_{\mathscr{A}_2}(x_i)| = \cdots = |h'_{\mathscr{A}_T}(x_i)| = S_i$. The method is called the least common multiple expansion (LCME) method.

Then we face the second problem: how to rank the pairs of PFEs in a PHFE? As presented earlier research in

Pythagorean fuzzy environment [49], the score function is used to measure PFEs. The bigger the score values, the larger the PFEs. However, we may face the situation that the score values of two PFEs are equal. We develop a method for Pythagorean fuzzy environments based on the score function and the accuracy function.

Definition 14: For any PFE $P = \langle \mu_P, \nu_P \rangle$ on U, here $\mu_P^2 + \nu_P^2 \leqslant 1$, the accuracy function S is defined to map P to $[-1, 1]$, which satisfies:

$$S(P) = \mu_P^2 - \nu_P^2.$$

The accuracy function H is defined to map P to $[0, 1]$, which satisfies:

$$H(P) = \mu_P^2 + \nu_P^2.$$

Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be two PFEs.
(1) *If $S(A) < S(B)$, then we say $A \prec B$;*
(2) *If $S(A) > S(B)$), then we say $A \succ B$;*
(3) *If $S(A) = S(B)$, furthermore,*
when $H(A) < H(B)$, then we say $A \prec B$;
when $H(A) > H(B)$, then we say $A \succ B$;
when $H(A) = H(B)$, then we say $A = B$.

For two PHFSs $\mathscr{A}$ and $\mathscr{B}$, if for any $x \in U$, extend $h_{\mathscr{A}}(x)$ and $h_{\mathscr{B}}(x)$ based on LCME method and rank all PFEs in $h_{\mathscr{A}}(x)$ and $h_{\mathscr{B}}(x)$ based on the order "$\succ$". Then we have the revised $h'_{\mathscr{A}}(x)$ and $h'_{\mathscr{B}}(x)$.

Example 15: Let $h_{\mathscr{A}}(x) = \{\langle 0.2, 0.4\rangle, \langle 0.3, 0.4\rangle\}$, $h_{\mathscr{B}}(x) = \{\langle 0.4, 0.5\rangle, \langle 0.4, 0.7\rangle, \langle 0.3, 0.6\rangle\}$ and $h_{\mathscr{C}}(x) = \{\langle 0.5, 0.8\rangle, \langle 0.6, 0.8\rangle\}$ be three PHFEs on $U = \{x\}$.

Firstly we use LCME method to extend all the PHFEs:

$$\begin{aligned} h_{\mathscr{A}}(x) &= \{\langle 0.2, 0.4\rangle, \langle 0.2, 0.4\rangle, \langle 0.2, 0.4\rangle, \\ &\quad \langle 0.3, 0.4\rangle, \langle 0.3, 0.4\rangle, \langle 0.3, 0.4\rangle\}; \\ h_{\mathscr{B}}(x) &= \{\langle 0.4, 0.5\rangle, \langle 0.4, 0.5\rangle, \langle 0.4, 0.7\rangle, \\ &\quad \langle 0.4, 0.7\rangle, \langle 0.3, 0.6\rangle, \langle 0.3, 0.6\rangle\}; \\ h_{\mathscr{C}}(x) &= \{\langle 0.5, 0.8\rangle, \langle 0.5, 0.8\rangle, \langle 0.5, 0.8\rangle, \\ &\quad \langle 0.6, 0.8\rangle, \langle 0.6, 0.8\rangle, \langle 0.6, 0.8\rangle\}. \end{aligned}$$

Then we rank all the PHFEs based on "$\succ$":

$$\begin{aligned} h'_{\mathscr{A}}(x) &= \{\langle 0.3, 0.4\rangle, \langle 0.3, 0.4\rangle, \langle 0.3, 0.4\rangle, \\ &\quad \langle 0.2, 0.4\rangle, \langle 0.2, 0.4\rangle, \langle 0.2, 0.4\rangle\}; \\ h'_{\mathscr{B}}(x) &= \{\langle 0.4, 0.5\rangle, \langle 0.4, 0.5\rangle, \langle 0.3, 0.6\rangle, \\ &\quad \langle 0.3, 0.6\rangle, \langle 0.4, 0.7\rangle, \langle 0.4, 0.7\rangle\}; \\ h'_{\mathscr{C}}(x) &= \{\langle 0.6, 0.8\rangle, \langle 0.6, 0.8\rangle, \langle 0.6, 0.8\rangle, \\ &\quad \langle 0.5, 0.8\rangle, \langle 0.5, 0.8\rangle, \langle 0.5, 0.8\rangle\}. \end{aligned}$$

Through the extension and ranking of PHFE, the revised PHFE h' does not add extreme information. Then we propose the correlation coefficients of PHFSs.

Definition 16: Let $\mathscr{A} = \{\langle x_i, h_{\mathscr{A}}(x_i)\rangle | x_i \in U\}$ and $\mathscr{B} = \{\langle x_i, h_{\mathscr{B}}(x_i)\rangle | x_i \in U\}$ be two PHFSs on U, and S_i be the least common multiple number of $|h_{\mathscr{A}}(x_i)|$ and $|h_{\mathscr{B}}(x_i)|$. The total correlation between $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\begin{aligned} &C_{PH}(\mathscr{A}, \mathscr{B}) \\ &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3S_i}\sum_{j=1}^{S_i}(\mu^2_{\mathscr{A}\sigma(j)}(x_i)\mu^2_{\mathscr{B}\sigma(j)}(x_i) \\ &\quad + \nu^2_{\mathscr{A}\sigma(j)}(x_i)\nu^2_{\mathscr{B}\sigma(j)}(x_i) + \pi^2_{\mathscr{A}\sigma(j)}(x_i)\pi^2_{\mathscr{B}\sigma(j)}(x_i))). \end{aligned}$$

And

$$\begin{aligned} C^{\mu}_{PH}(\mathscr{A}, \mathscr{B}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2S_i}\sum_{j=1}^{S_i}(\mu^2_{\mathscr{A}\sigma(j)}(x_i)\mu^2_{\mathscr{B}\sigma(j)}(x_i) \\ &\quad + (1-\mu^2_{\mathscr{A}\sigma(j)}(x_i))(1-\mu^2_{\mathscr{B}\sigma(j)}(x_i))), \\ C^{\nu}_{PH}(\mathscr{A}, \mathscr{B}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2S_i}\sum_{j=1}^{S_i}(\nu^2_{\mathscr{A}\sigma(j)}(x_i)\nu^2_{\mathscr{B}\sigma(j)}(x_i) \\ &\quad + (1-\nu^2_{\mathscr{A}\sigma(j)}(x_i))(1-\nu^2_{\mathscr{B}\sigma(j)}(x_i))), \\ C^{\pi}_{PH}(\mathscr{A}, \mathscr{B}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2S_i}\sum_{j=1}^{S_i}(\pi^2_{\mathscr{A}\sigma(j)}(x_i)\pi^2_{\mathscr{B}\sigma(j)}(x_i) \\ &\quad + (1-\pi^2_{\mathscr{A}\sigma(j)}(x_i))(1-\pi^2_{\mathscr{B}\sigma(j)}(x_i))) \end{aligned}$$

are called the membership, non-membership and indeterminacy correlation between $\mathscr{A}$ and $\mathscr{B}$, respectively. The three kinds of correlations are collectively referred to as the local correlations between $\mathscr{A}$ and $\mathscr{B}$.

Here $\langle \mu_{\mathscr{A}\sigma(j)}(x_i), \nu_{\mathscr{A}\sigma(j)}(x_i)\rangle$ and $\langle \mu_{\mathscr{B}\sigma(j)}(x_i), \nu_{\mathscr{B}\sigma(j)}(x_i)\rangle$ are the jth largest PFEs in $h'_{\mathscr{A}}(x_i)$ and $h'_{\mathscr{B}}(x_i)$, respectively.

Definition 17: Let $\mathscr{A} = \{\langle x_i, h_{\mathscr{A}}(x_i)\rangle | x_i \in U\}$ be an PHFS on U with $T_i = |h_{\mathscr{A}}(x_i)|$. The total informational energy of $\mathscr{A}$ is defined as

$$E_{PH}(\mathscr{A}) = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3T_i}\sum_{j=1}^{T_i}(\mu^4_{\mathscr{A}j}(x_i) + \nu^4_{\mathscr{A}j}(x_i) + \pi^4_{\mathscr{A}j}(x_i))).$$

And

$$\begin{aligned} E^{\mu}_{PH}(\mathscr{A}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2T_i}\sum_{j=1}^{T_i}(\mu^4_{\mathscr{A}j}(x_i) + (1-\mu^2_{\mathscr{A}j}(x_i))^2), \\ E^{\nu}_{PH}(\mathscr{A}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2T_i}\sum_{j=1}^{T_i}(\nu^4_{\mathscr{A}j}(x_i) + (1-\nu^2_{\mathscr{A}j}(x_i))^2), \\ E^{\pi}_{PH}(\mathscr{A}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{2T_i}\sum_{j=1}^{T_i}\pi^4_{\mathscr{A}j}(x_i) + (1-\pi^2_{\mathscr{A}j}(x_i))^2) \end{aligned}$$

are called the membership, non-membership and indeterminacy informational energy of $\mathscr{A}$, respectively. The three kinds of informational energies are collectively referred to as the local informational energies of $\mathscr{A}$.

Here $\langle \mu_{\mathscr{A}j}(x_i), \nu_{\mathscr{A}j}(x_i)\rangle \in h_{\mathscr{A}}(x_i)$.

Remark 18: Let $h'_{\mathscr{A}}(x_i) = \{\langle \mu_{\mathscr{A}\sigma(j)}(x_i), \nu_{\mathscr{A}\sigma(j)}(x_i)\rangle | j = 1, 2, \cdots, S_i\}$ be the revised $h_{\mathscr{A}}(x_i) = \{\langle \mu_{\mathscr{A}j}(x_i), \nu_{\mathscr{A}j}(x_i)\rangle | j = 1, 2, \cdots, T_i\}$, here $S_i = |h'_{\mathscr{A}}(x_i)|$ and $T_i = |h_{\mathscr{A}}(x_i)|$. We can find

$$
\begin{aligned}
&E_{PH}(\mathscr{A})\\
&=\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3T_i}\sum_{j=1}^{T_i}(\mu^4_{\mathscr{A}j}(x_i)+\nu^4_{\mathscr{A}j}(x_i)+\pi^4_{\mathscr{A}j}(x_i)))\\
&=\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3S_i}\sum_{j=1}^{S_i}(\mu^4_{\mathscr{A}\sigma(j)}(x_i)+\nu^4_{\mathscr{A}\sigma(j)}(x_i)+\pi^4_{\mathscr{A}\sigma(j)}(x_i))).
\end{aligned}
$$

So in the following context, we do not emphasize which method is used to calculate the informational energies when we mention them. The case of $E^{\mu}_{PH}, E^{\nu}_{PH}$ and E^{π}_{PH} are similar.

Proposition 19: The correlations and informational energies of PHFSs satisfy:

(1) $0 \leqslant C_{PH}(\mathscr{A},\mathscr{B}) \leqslant 1$,
$0 \leqslant C^{\mu}_{PH}(\mathscr{A},\mathscr{B}), C^{\nu}_{PH}(\mathscr{A},\mathscr{B}), C^{\pi}_{PH}(\mathscr{A},\mathscr{B}) \leqslant 1$;
$0 \leqslant E_{PH}(\mathscr{A}), E^{\mu}_{PH}(\mathscr{A}), E^{\nu}_{PH}(\mathscr{A}), E^{\pi}_{PH}(\mathscr{A}) \leqslant 1$;
(2) $C_{PH}(\mathscr{A},\mathscr{B}) = C_{PH}(\mathscr{B},\mathscr{A})$;
$C^{\mu}_{PH}(\mathscr{A},\mathscr{B}) = C^{\mu}_{PH}(\mathscr{B},\mathscr{A})$;
$C^{\nu}_{PH}(\mathscr{A},\mathscr{B}) = C^{\nu}_{PH}(\mathscr{B},\mathscr{A})$;
$C^{\pi}_{PH}(\mathscr{A},\mathscr{B}) = C^{\pi}_{PH}(\mathscr{B},\mathscr{A})$;
(3) $C_{PH}(\mathscr{A},\mathscr{A}) = E_{PH}(\mathscr{A})$;
$C^{\mu}_{PH}(\mathscr{A},\mathscr{A}) = E^{\mu}_{PH}(\mathscr{A})$;
$C^{\nu}_{PH}(\mathscr{A},\mathscr{A}) = E^{\nu}_{PH}(\mathscr{A})$;
$C^{\pi}_{PH}(\mathscr{A},\mathscr{A}) = E^{\pi}_{PH}(\mathscr{A})$.

Proof: Proposition 19 can be proved by Definition 16 and Definition 17 easily. Here the process is omitted. □

Definition 20: Let $\mathscr{A}$ and $\mathscr{B}$ be two PHFSs on U. Then the correlation coefficients between $\mathscr{A}$ and $\mathscr{B}$, can be defined as the following four forms:

$$
\begin{aligned}
&\rho^{(1)}_{PH}(\mathscr{A},\mathscr{B})=\frac{C_{PH}(\mathscr{A},\mathscr{B})}{\sqrt{E_{PH}(\mathscr{A})}\sqrt{E_{PH}(\mathscr{B})}};\\
&\rho^{(2)}_{PH}(\mathscr{A},\mathscr{B})=\frac{C_{PH}(\mathscr{A},\mathscr{B})}{\max\{E_{PH}(\mathscr{A}),\ E_{PH}(\mathscr{B})\}};\\
&\rho^{(3)}_{PH}(\mathscr{A},\mathscr{B})=\frac{1}{3}(\rho^{\mu}_{PH}(\mathscr{A},\mathscr{B})+\rho^{\nu}_{PH}(\mathscr{A},\mathscr{B})+\rho^{\pi}_{PH}(\mathscr{A},\mathscr{B}));\\
&\rho^{(4)}_{PH}(\mathscr{A},\mathscr{B})=\frac{1}{3}(\rho^{\mu'}_{PH}(\mathscr{A},\mathscr{B})+\rho^{\nu'}_{PH}(\mathscr{A},\mathscr{B})+\rho^{\pi'}_{PH}(\mathscr{A},\mathscr{B})).
\end{aligned}
$$

Here,

$$
\begin{aligned}
\rho^{\mu}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\mu}_{PH}(\mathscr{A},\mathscr{B})}{\sqrt{E^{\mu}_{PH}(\mathscr{A})}\sqrt{E^{\mu}_{PH}(\mathscr{B})}},\\
\rho^{\nu}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\nu}_{PH}(\mathscr{A},\mathscr{B})}{\sqrt{E^{\nu}_{PH}(\mathscr{A})}\sqrt{E^{\nu}_{PH}(\mathscr{B})}},\\
\rho^{\pi}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\pi}_{PH}(\mathscr{A},\mathscr{B})}{\sqrt{E^{\pi}_{PH}(\mathscr{A})}\sqrt{E^{\pi}_{PH}(\mathscr{B})}},\\
\rho^{\mu'}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\mu}_{PH}(\mathscr{A},\mathscr{B})}{\max\{E^{\mu}_{PH}(\mathscr{A}),\ E^{\mu}_{PH}(\mathscr{B})\}},\\
\rho^{\nu'}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\nu}_{PH}(\mathscr{A},\mathscr{B})}{\max\{E^{\nu}_{PH}(\mathscr{A}),\ E^{\nu}_{PH}(\mathscr{B})\}},\\
\rho^{\pi'}_{PH}(\mathscr{A},\mathscr{B}) &= \frac{C^{\pi}_{PH}(\mathscr{A},\mathscr{B})}{\max\{E^{\pi}_{PH}(\mathscr{A}),\ E^{\pi}_{PH}(\mathscr{B})\}}.
\end{aligned}
$$

Proposition 21: The correlation coefficients of PHFSs satisfy the following properties:

(1) $\rho^{(1)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(1)}_{PH}(\mathscr{B},\mathscr{A})$;
$\rho^{(2)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(2)}_{PH}(\mathscr{B},\mathscr{A})$;
$\rho^{(3)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(3)}_{PH}(\mathscr{B},\mathscr{A})$;
$\rho^{(4)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(4)}_{PH}(\mathscr{B},\mathscr{A})$;
(2) $0 \leqslant \rho^{(2)}_{PH}(\mathscr{A},\mathscr{B}) \leqslant \rho^{(1)}_{PH}(\mathscr{A},\mathscr{B}) \leqslant 1$;
$0 \leqslant \rho^{(4)}_{PH}(\mathscr{A},\mathscr{B}) \leqslant \rho^{(3)}_{PH}(\mathscr{A},\mathscr{B}) \leqslant 1$;
(3) $\rho^{(1)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(2)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(3)}_{PH}(\mathscr{A},\mathscr{B}) = \rho^{(4)}_{PH}(\mathscr{A},\mathscr{B}) = 1 \Leftrightarrow \mathscr{A} = \mathscr{B}$.

Proof: Claim (1) and (3) can be proved by Definition 20 easily. Here the processes are omitted. Claim (2) can be proved as following:

Obviously the inequality $\rho^{(1)}_{PH}(\mathscr{A},\mathscr{B}), \rho^{(2)}_{PH}(\mathscr{A},\mathscr{B}) \geqslant 0$.

By Cauchy-Schwarz inequality

$$
\begin{aligned}
&(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\\
&\qquad\leqslant (x_1^2+x_2^2+\cdots+x_n^2)\cdot(y_1^2+y_2^2+\cdots+y_n^2),
\end{aligned}
$$

here $(x_1,x_2,\cdots,x_n),\ (y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$, we have

$$
\begin{aligned}
&C^2_{PH}(\mathscr{A},\mathscr{B})\\
&=(\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{3S_i}\sum_{j=1}^{S_i}(\mu^2_{\mathscr{A}\sigma(j)}(x_i)\mu^2_{\mathscr{B}\sigma(j)}(x_i)\\
&\quad+\nu^2_{\mathscr{A}\sigma(j)}(x_i)\nu^2_{\mathscr{B}\sigma(j)}(x_i)+\pi^2_{\mathscr{A}\sigma(j)}(x_i)\pi^2_{\mathscr{B}\sigma(j)}(x_i)))^2\\
&=(\frac{1}{n}\sum_{j=1}^{S_1}(\frac{\mu^2_{\mathscr{A}\sigma(j)}(x_1)}{\sqrt{3S_1}}\cdot\frac{\mu^2_{\mathscr{B}\sigma(j)}(x_1)}{\sqrt{3S_1}}+\frac{\nu^2_{\mathscr{A}\sigma(j)}(x_1)}{\sqrt{3S_1}}\\
&\quad\cdot\frac{\nu^2_{\mathscr{B}\sigma(j)}(x_1)}{\sqrt{3S_1}}+\frac{\pi^2_{\mathscr{A}\sigma(j)}(x_1)}{\sqrt{3S_1}}\cdot\frac{\pi^2_{\mathscr{B}\sigma(j)}(x_1)}{\sqrt{3S_1}})+\cdots\\
&\quad+\frac{1}{n}\sum_{j=1}^{S_n}(\frac{\mu^2_{\mathscr{A}\sigma(j)}(x_n)}{\sqrt{3S_n}}\cdot\frac{\mu^2_{\mathscr{B}\sigma(j)}(x_n)}{\sqrt{3S_n}}+\frac{\nu^2_{\mathscr{A}\sigma(j)}(x_n)}{\sqrt{3S_n}}\\
&\quad\cdot\frac{\nu^2_{\mathscr{B}\sigma(j)}(x_n)}{\sqrt{3S_n}}+\frac{\pi^2_{\mathscr{A}\sigma(j)}(x_n)}{\sqrt{3S_n}}\cdot\frac{\pi^2_{\mathscr{B}\sigma(j)}(x_n)}{\sqrt{3S_n}}))^2\\
&\leqslant(\frac{1}{n}\sum_{j=1}^{S_1}(\frac{\mu^4_{\mathscr{A}\sigma(j)}(x_1)}{3S_1}+\frac{\nu^4_{\mathscr{A}\sigma(j)}(x_1)}{3S_1}+\frac{\pi^4_{\mathscr{A}\sigma(j)}(x_1)}{3S_1})\\
&\quad+\cdots+\frac{1}{n}\sum_{j=1}^{S_n}(\frac{\mu^4_{\mathscr{A}\sigma(j)}(x_n)}{3S_n}+\frac{\nu^4_{\mathscr{A}\sigma(j)}(x_n)}{3S_n}+\frac{\pi^4_{\mathscr{A}\sigma(j)}(x_n)}{3S_n}))\\
&\quad\cdot(\frac{1}{n}\sum_{j=1}^{S_1}(\frac{\mu^4_{\mathscr{B}\sigma(j)}(x_1)}{3S_1}+\frac{\nu^4_{\mathscr{B}\sigma(j)}(x_1)}{3S_1}+\frac{\pi^4_{\mathscr{B}\sigma(j)}(x_1)}{3S_1})+\cdots\\
&\quad+\frac{1}{n}\sum_{j=1}^{S_n}(\frac{\mu^4_{\mathscr{B}\sigma(j)}(x_n)}{3S_n}+\frac{\nu^4_{\mathscr{B}\sigma(j)}(x_n)}{3S_n}+\frac{\pi^4_{\mathscr{B}\sigma(j)}(x_n)}{3S_n}))\\
&=(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{3S_i}\sum_{j=1}^{S_i}(\mu^4_{\mathscr{A}\sigma(j)}(x_i)+\nu^4_{\mathscr{A}\sigma(j)}(x_i)+\pi^4_{\mathscr{A}\sigma(j)}(x_i)))\\
&\quad\cdot(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{3S_i}\sum_{j=1}^{S_i}(\mu^4_{\mathscr{B}\sigma(j)}(x_i)+\nu^4_{\mathscr{B}\sigma(j)}(x_i)+\pi^4_{\mathscr{B}\sigma(j)}(x_i)))\\
&=E_{PH}(\mathscr{A})\cdot E_{PH}(\mathscr{B}).
\end{aligned}
$$

TABLE 1. Correlation coefficients of PHFSs.

	$(\mathscr{A},\mathscr{B})$	$(\mathscr{B},\mathscr{C})$	$(\mathscr{A},\mathscr{C})$	Ranking
$\rho_{PH}^{(1)}$	0.8730	0.6444	0.2948	$\rho_{PH}^{(1)}(\mathscr{A},\mathscr{B}) > \rho_{PH}^{(1)}(\mathscr{B},\mathscr{C}) > \rho_{PH}^{(1)}(\mathscr{A},\mathscr{C})$
$\rho_{PH}^{(2)}$	0.7131	0.5848	0.2653	$\rho_{PH}^{(2)}(\mathscr{A},\mathscr{B}) > \rho_{PH}^{(2)}(\mathscr{B},\mathscr{C}) > \rho_{PH}^{(2)}(\mathscr{A},\mathscr{C})$
$\rho_{PH}^{(3)}$	0.9227	0.8423	0.6402	$\rho_{PH}^{(3)}(\mathscr{A},\mathscr{B}) > \rho_{PH}^{(3)}(\mathscr{B},\mathscr{C}) > \rho_{PH}^{(3)}(\mathscr{A},\mathscr{C})$
$\rho_{PH}^{(4)}$	0.8310	0.7397	0.5347	$\rho_{PH}^{(4)}(\mathscr{A},\mathscr{B}) > \rho_{PH}^{(4)}(\mathscr{B},\mathscr{C}) > \rho_{PH}^{(4)}(\mathscr{A},\mathscr{C})$

Therefore $C_{PH}(\mathscr{A},\mathscr{B}) \leqslant \sqrt{E_{PH}(\mathscr{A})}\cdot\sqrt{E_{PH}(\mathscr{B})}$, so $0 \leqslant \rho_{PH}^{(1)}(\mathscr{A},\mathscr{B}) \leqslant 1$.

Since $E_{PH}(\mathscr{A}) \leqslant \max\{E_{PH}(\mathscr{A}), E_{PH}(\mathscr{B})\}$, $E_{PH}(\mathscr{B}) \leqslant \max\{E_{PH}(\mathscr{A}), E_{PH}(\mathscr{B})\}$ and $\sqrt{E_{PH}(\mathscr{A})E_{PH}(\mathscr{B})} \leqslant \max\{E_{PH}(\mathscr{A}), E_{PH}(\mathscr{B})\}$. So $\rho_{PH}^{(2)}(\mathscr{A},\mathscr{B}) \leqslant \rho_{PH}^{(1)}(\mathscr{A},\mathscr{B})$. Hence $0 \leqslant \rho_{PH}^{(2)}(\mathscr{A},\mathscr{B}) \leqslant \rho_{PH}^{(1)}(\mathscr{A},\mathscr{B}) \leqslant 1$.

Similarly, we have $0 \leqslant \rho_{PH}^{(4)}(\mathscr{A},\mathscr{B}) \leqslant \rho_{PH}^{(3)}(\mathscr{A},\mathscr{B}) \leqslant 1$. □

Example 22 (Continued From Example 15): Compute the correlation coefficients between $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{B}$ and $\mathscr{C}$, $\mathscr{A}$ and $\mathscr{C}$.

Based on Definition 16 and Definition 17, we have

$$C_{PH}(\mathscr{A},\mathscr{B}) = 0.1502,\quad C_{PH}(\mathscr{B},\mathscr{C}) = 0.0998,$$
$$C_{PH}(\mathscr{A},\mathscr{C}) = 0.0559,$$
$$C_{PH}^{\mu}(\mathscr{A},\mathscr{B}) = 0.4081,\quad C_{PH}^{\mu}(\mathscr{B},\mathscr{C}) = 0.3208,$$
$$C_{PH}^{\mu}(\mathscr{A},\mathscr{C}) = 0.3362,$$
$$C_{PH}^{\nu}(\mathscr{A},\mathscr{B}) = 0.2953,\quad C_{PH}^{\nu}(\mathscr{B},\mathscr{C}) = 0.2313,$$
$$C_{PH}^{\nu}(\mathscr{A},\mathscr{C}) = 0.2024,$$
$$C_{PH}^{\pi}(\mathscr{A},\mathscr{B}) = 0.2471,\quad C_{PH}^{\pi}(\mathscr{B},\mathscr{C}) = 0.2471,$$
$$C_{PH}^{\pi}(\mathscr{A},\mathscr{C}) = 0.1290;$$
$$E_{PH}(\mathscr{A}) = 0.2106,\quad E_{PH}(\mathscr{B}) = 0.1405,$$
$$E_{PH}(\mathscr{C}) = 0.1706,$$
$$E_{PH}^{\mu}(\mathscr{A}) = 0.4398,\quad E_{PH}^{\mu}(\mathscr{B}) = 0.3831,$$
$$E_{PH}^{\mu}(\mathscr{C}) = 0.2911,$$
$$E_{PH}^{\nu}(\mathscr{A}) = 0.3656,\quad E_{PH}^{\nu}(\mathscr{B}) = 0.2774,$$
$$E_{PH}^{\nu}(\mathscr{B}) = 0.2696,$$
$$E_{PH}^{\pi}(\mathscr{A}) = 0.3262,\quad E_{PH}^{\pi}(\mathscr{B}) = 0.2610,$$
$$E_{PH}^{\pi}(\mathscr{C}) = 0.4511.$$

So we have the correlation coefficients in Table 1. It shows the four results of ranking are consistent.

D. COMPARATIVE ANALYSIS WITH THE EXISTING CORRELATION COEFFICIENTS

As we have presented in the introduction, the concept of PHFS is controversial. About the correlation coefficients of Liu's PHFS we have not seen the related reports. Considering PHFSs are the extension of HFSs and PFSs, we will compare the existing correlation coefficients of HFSs and PFSs with our definition.

(1) Let $A = \{\langle x, h_A(x)\rangle | x \in U\}$ be a HFS on U, here $h_A(x) = \{\mu_A(x)\}$. We can generalize $h_A(x)$ to $h_A(x) = \{\langle \mu_A(x), \sqrt{1-\mu_A^2(x)}\rangle\}$. Then A is extended to a PHFS.

Continue to Example 8 as an example. Denote A_1 and B_1 as two PHFSs:

$$h_{A_1}(x) = \{\langle 0, 1\rangle\},\quad h_{B_1}(x) = \{\langle 0.2, \sqrt{0.96}\rangle, \langle 0.3, \sqrt{0.91}\rangle\}.$$

Then compute the correlation coefficients between A_1 and B_1 based on Definition 20 as follows:

$$\rho_{PH}^{(1)}(A_1, B_1) = 0.9969,\quad \rho_{PH}^{(2)}(A_1, B_1) = 0.9350,$$
$$\rho_{PH}^{(3)}(A_1, B_1) = 0.9979,\quad \rho_{PH}^{(4)}(A_1, B_1) = 0.9567.$$

Continue to Example 10 as an example. Denote A_2 and B_2 as two PHFSs:

$$h_{A_2}(x_1) = \{\langle 0.2, \sqrt{0.96}\rangle\},\quad h_{A_2}(x_2) = \{\langle 0.2, \sqrt{0.96}\rangle\},$$
$$h_{B_2}(x_1) = \{\langle 0.1, \sqrt{0.99}\rangle\},$$
$$h_{B_2}(x_2) = \{\langle 0.2, \sqrt{0.96}\rangle, \langle 0.3, \sqrt{0.91}\rangle\}.$$

Then compute the correlation coefficients between A_2 and B_2 based on Definition 20 as follows:

$$\rho_{PH}^{(1)}(A_2, B_2) = 0.9988,\quad \rho_{PH}^{(2)}(A_2, B_2) = 0.9952,$$
$$\rho_{PH}^{(3)}(A_2, B_2) = 0.9992,\quad \rho_{PH}^{(4)}(A_2, B_2) = 0.9968.$$

(2) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in U\}$ be a PFS on U, here $0 \leqslant \mu_A^2(x) + \nu_A^2(x) \leqslant 1$, for any $x \in U$. We also can see A as a PHFS, whose responding PHFEs only include one PFE $\{\langle \mu_A(x), \nu_A(x)\rangle\}$.

Continue to Example 12 as an example and compute the correlation coefficients between A and B based on Definition 20, we have

$$\rho_{PH}^{(1)}(A, C) = \rho_{PH}^{(1)}(B, C) = 0.5158,$$
$$\rho_{PH}^{(2)}(A, C) = \rho_{PH}^{(2)}(B, C) = 0.3525,$$
$$\rho_{PH}^{(3)}(A, C) = 0.7094,\quad \rho_{PH}^{(3)}(B, C) = 0.7066,$$
$$\rho_{PH}^{(4)}(A, C) = 0.5703,\quad \rho_{PH}^{(4)}(B, C) = 0.5702.$$

We can find that our first two definitions of correlation coefficients are consistent with Garg's definition. What's more, our last two definitions of correlation coefficients can distinguish two different situations.

IV. CORRELATION COEFFICIENTS FOR IVPHFSS

A. CORRELATIONS AND CORRELATION COEFFICIENTS OF IVPHFSS

Furthermore, we try to study the correlation coefficients of IVPHFSs. We may use LCME method to solve the inconsistent problem of the cardinalities of IVPHFEs, while we still need to consider the ranking problem in one IVPHFE. Here we recall the operators of interval numbers.

Definition 23 [11]: Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two interval numbers. The interval arithmetic can be defined as:

(1) $a + b = [a^- + b^-, a^+ + b^+]$;
(2) $a - b = [a^- - b^+, a^+ - b^-]$;
(3) $a^n = [(a^-)^n, (a^+)^n]$, *here* $a^- \geqslant 0$, $n \in \mathbb{N}$.

Definition 24 [41]: Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two interval numbers, and $l(a) = a^+ - a^-, l(b) = b^+ - b^-$, then the possibility degree of $a > b$ is defined as follows:

$$P(a \geqslant b) = \max\{1 - \max\{\frac{b^+ - a^-}{l(a) + l(b)}, 0\}, 0\}.$$

The equation of the possibility degree is used to compare two interval numbers. If $P(a \geqslant b) > 0.5$, then a is superior to b, denoted by $a \succ b$; If $P(a \geqslant b) = 0.5$, then a is equivalent to b, denoted by $a = b$.

Similar to the discussion about PHFEs, we develop the score function and the accuracy function to compare IVPHFEs. It is worth noting that an IVPHFE includes some pairs of interval-valued Pythagorean fuzzy elements (IVPFEs). The arithmetic of interval numbers can keep the original fuzzy information more than that of real numbers. So we propose the following concepts of score functions and accuracy functions about IVPFEs.

Definition 25: For any IVPFE $\widetilde{P} = \langle \widetilde{\mu}_{\widetilde{P}}, \widetilde{\nu}_{\widetilde{P}} \rangle = \langle [\mu^-_{\widetilde{P}}, \mu^+_{\widetilde{P}}], [\nu^-_{\widetilde{P}}, \nu^+_{\widetilde{P}}] \rangle$ on U, here $(\mu^+_{\widetilde{P}})^2 + (\nu^+_{\widetilde{P}})^2 \leqslant 1$, the score function S is defined to map $\widetilde{P}$ to $2^{[-1,1]}$, which satisfies

$$S(\widetilde{P}) = (\widetilde{\mu}_{\widetilde{P}})^2 - (\widetilde{\nu}_{\widetilde{P}})^2 = [(\mu^-_{\widetilde{P}})^2 - (\nu^+_{\widetilde{P}})^2, (\mu^+_{\widetilde{P}})^2 - (\nu^-_{\widetilde{P}})^2].$$

The accuracy function H is defined to map $\widetilde{P}$ to $2^{[0,1]}$, which satisfies

$$H(\widetilde{P}) = (\widetilde{\mu}_{\widetilde{P}})^2 + (\widetilde{\nu}_{\widetilde{P}})^2 = [(\mu^-_{\widetilde{P}})^2 + (\mu^+_{\widetilde{P}})^2, (\nu^-_{\widetilde{P}})^2 + (\nu^+_{\widetilde{P}})^2].$$

Let $\widetilde{A} = \langle \widetilde{\mu}_{\widetilde{A}}, \widetilde{\nu}_{\widetilde{A}} \rangle, \widetilde{B} = \langle \widetilde{\mu}_{\widetilde{B}}, \widetilde{\nu}_{\widetilde{B}} \rangle$ be two IVPFEs on U.

(1) If $P(S(\widetilde{A}) \geqslant S(\widetilde{B})) < 0.5$, then $\widetilde{A} \prec \widetilde{B}$;
(2) If $P(S(\widetilde{A}) \geqslant S(\widetilde{B})) > 0.5$, then $\widetilde{A} \succ \widetilde{B}$;
(3) If $P(S(\widetilde{A}) \geqslant S(\widetilde{B})) = 0.5$, furthermore,
when $P(H(\widetilde{A}) \geqslant H(\widetilde{B})) < 0.5$, then $\widetilde{A} \prec \widetilde{B}$;
when $P(H(\widetilde{A}) \geqslant H(\widetilde{B})) > 0.5$, then $\widetilde{A} \succ \widetilde{B}$;
when $P(H(\widetilde{A}) \geqslant H(\widetilde{B})) = 0.5$, then $\widetilde{A} = \widetilde{B}$.

For two IVPHFEs, extend the IVPHFEs by LCME method and rank all the pairs in one IVPHFE based on the order "$\succ$". Then we have the revised IVPHFEs and compare them.

Definition 26: Let $\widetilde{\mathscr{A}} = \{\langle x_i, h_{\widetilde{\mathscr{A}}}(x_i)\rangle | x_i \in U\}$ and $\widetilde{\mathscr{B}} = \{\langle x_i, h_{\widetilde{\mathscr{B}}}(x_i)\rangle | x_i \in U\}$ be two IVPHFSs on U, and S_i be the least common multiple number of $|h_{\widetilde{\mathscr{A}}}(x_i)|$ and $|h_{\widetilde{\mathscr{B}}}(x_i)|$. The total correlation between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ is defined as

$$\begin{aligned} C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{6S_i}\sum_{j=1}^{S_i}((\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2)). \end{aligned}$$

And

$$\begin{aligned} C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4S_i}\sum_{j=1}^{S_i}((\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (1 - (\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) \\ &+ (1 - (\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2)), \end{aligned}$$

$$\begin{aligned} C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4S_i}\sum_{j=1}^{S_i}((\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (1 - (\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) \\ &+ (1 - (\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2)), \end{aligned}$$

$$\begin{aligned} C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4S_i}\sum_{j=1}^{S_i}((\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 \\ &+ (1 - (\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) \\ &+ (1 - (\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1 - (\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2)) \end{aligned}$$

are called the membership, non-membership and indeterminacy correlations between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$, respectively. The three kinds of correlations are collectively referred to as the local correlations between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$.

Here $\langle [\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i), \mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)], [\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i), \nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)]\rangle$ and $\langle [\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i), \mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i)], [\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i), \nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i)]\rangle$ are the jth largest interval-valued PFEs in $h'_{\widetilde{\mathscr{A}}}(x_i)$ and $h'_{\widetilde{\mathscr{B}}}(x_i)$, respectively.

Definition 27: Let $\widetilde{\mathscr{A}} = \{\langle x_i, h_{\widetilde{\mathscr{A}}}(x_i)\rangle | x_i \in U\}$ be an IVPHFS on U and $T_i = |h_{\widetilde{\mathscr{A}}}(x_i)|$. The total informational energy of $\widetilde{\mathscr{A}}$ is defined as

$$\begin{aligned} E_{IVPH}(\widetilde{\mathscr{A}}) &= \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{6T_i}\sum_{j=1}^{T_i}((\mu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\mu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 \\ &+ (\nu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\nu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 \\ &+ (\pi^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\pi^+_{\widetilde{\mathscr{A}}j}(x_i))^4)). \end{aligned}$$

And

$$E^{\mu}_{IVPH}(\widetilde{\mathscr{A}}) = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4T_i}\sum_{j=1}^{T_i}((\mu^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (\mu^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (1-(\mu^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2 + (1-(\mu^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2)),$$

$$E^{\nu}_{IVPH}(\widetilde{\mathscr{A}}) = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4T_i}\sum_{j=1}^{T_i}((\nu^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (\nu^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (1-(\nu^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2 + (1-(\nu^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2)),$$

$$E^{\pi}_{IVPH}(\widetilde{\mathscr{A}}) = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{4T_i}\sum_{j=1}^{T_i}((\pi^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (\pi^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^4 + (1-(\pi^{-}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2 + (1-(\pi^{+}_{\widetilde{\mathscr{A}}_j}(x_i))^2)^2))$$

are called the local membership, non-membership and indeterminacy informational energies of $\widetilde{\mathscr{A}}$*, respectively. The three kinds of informational energies are collectively referred to as the local informational energies of* $\widetilde{\mathscr{A}}$.

Here $\langle[\mu^{-}_{\widetilde{\mathscr{A}}_j}(x_i), \mu^{+}_{\widetilde{\mathscr{A}}_j}(x_i)], [\nu^{-}_{\widetilde{\mathscr{A}}_j}(x_i), \nu^{+}_{\widetilde{\mathscr{A}}_j}(x_i)]\rangle \in h_{\widetilde{\mathscr{A}}}(x_i)$.

Proposition 28: The correlations and informational energies of IVPHFSs satisfy:

(1) $0 \leqslant C_{IVPH}(\mathscr{A}, \mathscr{B}) \leqslant 1$,
$0 \leqslant C^{\mu}_{IVPH}(\mathscr{A}, \mathscr{B}), C^{\nu}_{IVPH}(\mathscr{A}, \mathscr{B}), C^{\pi}_{IVPH}(\mathscr{A}, \mathscr{B}) \leqslant 1$;
$0 \leqslant E_{IVPH}(\mathscr{A}) \leqslant 1$,
$0 \leqslant E^{\mu}_{IVPH}(\mathscr{A}), E^{\nu}_{IVPH}(\mathscr{A}), E^{\pi}_{IVPH}(\mathscr{A}) \leqslant 1$;

(2) $C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$;
$C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$;
$C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$;
$C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$;

(3) $C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{A}}) = E_{IVPH}(\widetilde{\mathscr{A}})$;
$C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{A}}) = E^{\mu}_{IVPH}(\widetilde{\mathscr{A}})$;
$C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{A}}) = E^{\nu}_{IVPH}(\widetilde{\mathscr{A}})$;
$C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{A}}) = E^{\pi}_{IVPH}(\widetilde{\mathscr{A}})$.

Definition 29: Let $\widetilde{\mathscr{A}}$ *and* $\widetilde{\mathscr{B}}$ *be two IVPHFSs on* U*. Then the correlation coefficients between* $\widetilde{\mathscr{A}}$ *and* $\widetilde{\mathscr{B}}$ *can be defined as the following four forms*:

$$\rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E_{IVPH}(\widetilde{\mathscr{A}})}\sqrt{E_{IVPH}(\widetilde{\mathscr{B}})}};$$

$$\rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\max\{E_{IVPH}(\widetilde{\mathscr{A}}),\ E_{IVPH}(\widetilde{\mathscr{B}})\}};$$

$$\rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{1}{3}(\rho^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}));$$

$$\rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{1}{3}(\rho^{\mu'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\nu'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\pi'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})).$$

Here,

$$\rho^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E^{\mu}_{IVPH}(\widetilde{\mathscr{A}})}\sqrt{E^{\mu}_{IVPH}(\widetilde{\mathscr{B}})}},$$

$$\rho^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E^{\nu}_{IVPH}(\widetilde{\mathscr{A}})}\sqrt{E^{\nu}_{IVPH}(\widetilde{\mathscr{B}})}},$$

$$\rho^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E^{\pi}_{IVPH}(\widetilde{\mathscr{A}})}\sqrt{E^{\pi}_{IVPH}(\widetilde{\mathscr{B}})}},$$

$$\rho^{\mu'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\max\{E^{\mu}_{IVPH}(\widetilde{\mathscr{A}}),\ E^{\mu}_{IVPH}(\widetilde{\mathscr{B}})\}},$$

$$\rho^{\nu'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\max\{E^{\nu}_{IVPH}(\widetilde{\mathscr{A}}),\ E^{\nu}_{IVPH}(\widetilde{\mathscr{B}})\}},$$

$$\rho^{\pi'}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\max\{E^{\pi}_{IVPH}(\widetilde{\mathscr{A}}),\ E^{\pi}_{IVPH}(\widetilde{\mathscr{B}})\}}.$$

Proposition 30: The correlation coefficients of IVPHFSs satisfy the following properties:

(1) $\rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(1)}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{A}})$;
$\rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(2)}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{A}})$;
$\rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(3)}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{A}})$;
$\rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(4)}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{A}})$;

(2) $0 \leqslant \rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) \leqslant \rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) \leqslant 1$;
$0 \leqslant \rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) \leqslant \rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) \leqslant 1$;

(3) $\rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = 1 \Leftrightarrow \widetilde{\mathscr{A}} = \widetilde{\mathscr{B}}$.

Example 31: Let $\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}$ be three IVPHFSs on $U = \{x\}$. Here

$$\widetilde{\mathscr{A}} = \{x, \langle[0.2, 0.5], [0.3, 0.4]\rangle\};$$
$$\widetilde{\mathscr{B}} = \{x, \langle[0.2, 0.4], [0.3, 0.7]\rangle, \langle[0.2, 0.4], [0.3, 0.5]\rangle\};$$
$$\widetilde{\mathscr{C}} = \{x, \langle[0.3, 0.5], [0.2, 0.4]\rangle\}.$$

Compute the correlation coefficients between them.

Firstly we give the revised IVPHFEs:

$$h_{\widetilde{\mathscr{A}}}(x) = \{\langle[0.2, 0.5], [0.3, 0.4]\rangle, \langle[0.2, 0.5], [0.3, 0.4]\rangle\};$$
$$h_{\widetilde{\mathscr{B}}}(x) = \{\langle[0.2, 0.4], [0.3, 0.5]\rangle, \langle[0.2, 0.4], [0.3, 0.7]\rangle\};$$
$$h_{\widetilde{\mathscr{C}}}(x) = \{\langle[0.3, 0.5], [0.2, 0.4]\rangle, \langle[0.3, 0.5], [0.2, 0.4]\rangle\}.$$

Based on Definition 26 and 27, we have

$$C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = 0.1905, \quad C_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}) = 0.1901, \quad C_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{C}}) = 0.2000;$$
$$C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = 0.3983, \quad C^{\mu}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}) = 0.3868, \quad C^{\mu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{C}}) = 0.3756;$$
$$C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = 0.3562, \quad C^{\nu}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}) = 0.3664, \quad C^{\nu}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{C}}) = 0.4021;$$
$$C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = 0.3171, \quad C^{\pi}_{IVPH}(\widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}) = 0.3171, \quad C^{\pi}_{IVPH}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{C}}) = 0.3225;$$
$$E_{IVPH}(\widetilde{\mathscr{A}}) = 0.2005, \quad E_{IVPH}(\widetilde{\mathscr{B}}) = 0.1965, \quad E_{IVPH}(\widetilde{\mathscr{C}}) = 0.2005;$$
$$E^{\mu}_{IVPH}(\widetilde{\mathscr{A}}) = 0.3871, \quad E^{\mu}_{IVPH}(\widetilde{\mathscr{B}}) = 0.4136, \quad E^{\mu}_{IVPH}(\widetilde{\mathscr{C}}) = 0.3653;$$
$$E^{\nu}_{IVPH}(\widetilde{\mathscr{A}}) = 0.3919, \quad E^{\nu}_{IVPH}(\widetilde{\mathscr{B}}) = 0.3497,$$

TABLE 2. Correlation coefficients of IVPHFSs in Example 31.

	$(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})$	$(\widetilde{\mathscr{B}},\widetilde{\mathscr{C}})$	$(\widetilde{\mathscr{A}},\widetilde{\mathscr{C}})$	Ranking
$\rho^{(1)}_{IVPH}$	0.9600	0.9579	0.9979	$\rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{C}}) > \rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}) > \rho^{(1)}_{IVPH}(\widetilde{\mathscr{B}},\widetilde{\mathscr{C}})$
$\rho^{(2)}_{IVPH}$	0.9504	0.9483	0.9979	$\rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{C}}) > \rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}) > \rho^{(2)}_{IVPH}(\widetilde{\mathscr{B}},\widetilde{\mathscr{C}})$
$\rho^{(3)}_{IVPH}$	0.9785	0.9788	0.9992	$\rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{C}}) > \rho^{(3)}_{IVPH}(\widetilde{\mathscr{B}},\widetilde{\mathscr{C}}) > \rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})$
$\rho^{(4)}_{IVPH}$	0.9481	0.9312	0.9808	$\rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{C}}) > \rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}) > \rho^{(4)}_{IVPH}(\widetilde{\mathscr{B}},\widetilde{\mathscr{C}})$

$$E^{\nu}_{IVPH}(\widetilde{\mathscr{C}}) = 0.4136,$$
$$E^{\pi}_{IVPH}(\widetilde{\mathscr{A}}) = 0.3225, \quad E^{\pi}_{IVPH}(\widetilde{\mathscr{B}}) = 0.3261,$$
$$E^{\pi}_{IVPH}(\widetilde{\mathscr{C}}) = 0.3225.$$

So we have the correlation coefficients in Table 2. Although the four ranking results are little different, IVPHFS $\widetilde{\mathscr{A}}$ and IVPHFS $\widetilde{\mathscr{C}}$ are always the most similar.

B. WEIGHTED CORRELATION COEFFICIENTS OF IVPHFSS

In the above section, we discuss the case that all objects are equally important. In many practical situations, the different objects may have different weights. This section develops the concept of the weighted correlation coefficients of IVPHFSs.

Definition 32: Let $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ be two IVPHFSs on U. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of U with $\sum_{i=1}^{n} \omega_i = 1$. Then the weighted correlation coefficients between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ can also be defined as the following four forms:

$$\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E_{IVPH_\omega}(\widetilde{\mathscr{A}})}\sqrt{E_{IVPH_\omega}(\widetilde{\mathscr{B}})}};$$
$$\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\max\{E_{IVPH_\omega}(\widetilde{\mathscr{A}}),\ E_{IVPH}(\widetilde{\mathscr{B}})\}};$$
$$\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{1}{3}(\rho^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}));$$
$$\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{1}{3}(\rho^{\mu'}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\nu'}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) + \rho^{\pi'}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})).$$

Here,

$$C_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \sum_{i=1}^{n}(\frac{\omega_i}{6S_i}\sum_{j=1}^{S_i}((\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 +, (\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 +, (\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 +, (\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2));$$

$$C^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4S_i}\sum_{j=1}^{S_i}((\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (1-(\mu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\mu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) + (1-(\mu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\mu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2));$$

$$C^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4S_i}\sum_{j=1}^{S_i}((\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (1-(\nu^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\nu^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) + (1-(\nu^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\nu^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2));$$

$$C^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4S_i}\sum_{j=1}^{S_i}((\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i)\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2 + (1-(\pi^-_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\pi^-_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2) + (1-(\pi^+_{\widetilde{\mathscr{A}}\sigma(j)}(x_i))^2)(1-(\pi^+_{\widetilde{\mathscr{B}}\sigma(j)}(x_i))^2));$$

$$E_{IVPH_\omega}(\widetilde{\mathscr{A}}) = \sum_{i=1}^{n}(\frac{\omega_i}{6\,T_i}\sum_{j=1}^{T_i}((\mu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\mu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\nu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\nu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\pi^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\pi^+_{\widetilde{\mathscr{A}}j}(x_i))^4));$$

$$E^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4\,T_i}\sum_{j=1}^{T_i}((\mu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\mu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 + (1-(\mu^-_{\widetilde{\mathscr{A}}j}(x_i))^2)^2 + (1-(\mu^+_{\widetilde{\mathscr{A}}j}(x_i))^2)^2));$$

$$E^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4\,T_i}\sum_{j=1}^{T_i}((\nu^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\nu^+_{\widetilde{\mathscr{A}}j}(x_i))^4 + (1-(\nu^-_{\widetilde{\mathscr{A}}j}(x_i))^2)^2 + (1-(\nu^+_{\widetilde{\mathscr{A}}j}(x_i))^2)^2));$$

$$E^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}}) = \sum_{i=1}^{n}(\frac{\omega_i}{4\,T_i}\sum_{j=1}^{T_i}((\pi^-_{\widetilde{\mathscr{A}}j}(x_i))^4 + (\pi^+_{\widetilde{\mathscr{A}}j}(x_i))^4 + (1-(\pi^-_{\widetilde{\mathscr{A}}j}(x_i))^2)^2 + (1-(\pi^+_{\widetilde{\mathscr{A}}j}(x_i))^2)^2));$$

$$\rho^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}})}\sqrt{E^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{B}})}};$$

$$\rho^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}) = \frac{C^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})}{\sqrt{E^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}})}\sqrt{E^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{B}})}};$$

$$\rho^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\frac{C^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})}{\sqrt{E^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}})}\sqrt{E^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{B}})}};$$

$$\rho^{\mu'}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\frac{C^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})}{\max\{E^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{A}}),\ E^{\mu}_{IVPH_\omega}(\widetilde{\mathscr{B}})\}};$$

$$\rho^{\nu'}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\frac{C^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})}{\max\{E^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{A}}),\ E^{\nu}_{IVPH_\omega}(\widetilde{\mathscr{B}})\}};$$

$$\rho^{\pi'}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\frac{C^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})}{\max\{E^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{A}}),\ E^{\pi}_{IVPH_\omega}(\widetilde{\mathscr{B}})\}}.$$

Proposition 33: Let $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ be two IVPHFSs on U. Then the weighted correlation coefficients between $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ satisfy:

(1) $\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{B}},\widetilde{\mathscr{A}})$,
$\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{B}},\widetilde{\mathscr{A}})$,
$\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{B}},\widetilde{\mathscr{A}})$,
$\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{B}},\widetilde{\mathscr{A}})$;

(2) $0\leqslant\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})\leqslant\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})\leqslant 1$,
$0\leqslant\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})\leqslant\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})\leqslant 1$;

(3) $\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=1\Leftrightarrow\widetilde{\mathscr{A}}=\widetilde{\mathscr{B}}$.

(4) *If* $\omega=(\frac{1}{n},\cdots\frac{1}{n})^T$,

$$C^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=C^{(1)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$C^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=C^{(2)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$C^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=C^{(3)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$C^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=C^{(4)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$E^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}})=E^{(1)}_{IVPH}(\widetilde{\mathscr{A}}),$$
$$E^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}})=E^{(2)}_{IVPH}(\widetilde{\mathscr{A}}),$$
$$E^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}})=E^{(3)}_{IVPH}(\widetilde{\mathscr{A}}),$$
$$E^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}})=E^{(4)}_{IVPH}(\widetilde{\mathscr{A}}),$$
$$\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(1)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(2)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(3)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}),$$
$$\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}})=\rho^{(4)}_{IVPH}(\widetilde{\mathscr{A}},\widetilde{\mathscr{B}}).$$

V. APPLICATIONS

This section will apply the correlation coefficients for IVPHFSs to MCDM problems and clustering analysis.

A. MCDM PROBLEM

Let $A=\{A_i|i=1,2,\cdots,m\}$ be a finite set of alternatives and $C=\{C_j|j=1,2,\cdots,n\}$ be a set of criteria, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of the criteria, where $\omega_j\in[0,1]$ $(j=1,2,\cdots,n)$ and $\sum_{j=1}^{n}\omega_j=1$. Suppose $M=(h_{\widetilde{\mathscr{A}}_{ij}})_{m\times n}$ is an interval-valued Pythagorean hesitant fuzzy decision matrix (IVPHFDM), where $h_{\widetilde{\mathscr{A}}_{ij}}=\{\langle[\mu^-_{ij},\mu^+_{ij}],[\nu^-_{ij},\nu^+_{ij}]\rangle|(\mu^+_{ij})^2+(\nu^+_{ij})^2\leqslant 1\}$, $(i=1,2,\cdots,m,j=1,2,\cdots,n)$ is an IVPHFE given by decision makers to evaluate the alternative A_i with respect to the criteria C_j.

The concrete algorithm is listed as follows:

Step 1 Input $M=(h_{\widetilde{\mathscr{A}}_{ij}})_{m\times n}$, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ and $\lambda\in[0,1]$.

Step 2 Make M the revised IVPHFDM $M'=(h'_{\widetilde{\mathscr{A}}_{ij}})_{m\times n}$.

The cardinalities of different IVPHFEs may be different even in the same criteria, that is $|h_{\widetilde{\mathscr{A}}_{kj}}|\neq|h_{\widetilde{\mathscr{A}}_{lj}}|$, $(k,l=1,2,\cdots,m,k\neq l,j=1,2,\cdots,n)$. Use the LCME method to make the cardinalities of two IVPHFEs are consistent and order all the interval-valued PFEs in each IVPHFE. Then we have $|h'_{\widetilde{\mathscr{A}}_{kj}}|=|h'_{\widetilde{\mathscr{A}}_{lj}}|$, and the revised IVPHFDM $M'=(h'_{\widetilde{\mathscr{A}}_{ij}})_{m\times n}$ is formed.

Step 3 Compute the weighted correlation coefficients between each $\widetilde{\mathscr{A}}_i$ and $\widetilde{\mathscr{A}}^*$.

Here $\widetilde{\mathscr{A}}_i=(h'_{\widetilde{\mathscr{A}}_{i1}},h'_{\widetilde{\mathscr{A}}_{i2}},\cdots,h'_{\widetilde{\mathscr{A}}_{in}})$ $(i=1,2,\cdots,m)$ can be seen as the IVPHFS which represents the fuzzy degree of the alternative A_i about all the criteria. $\widetilde{\mathscr{A}}^*$ is the IVPHFS which represents the fuzzy degree of the ideal solution A^* about all the criteria. For the benefit criterion C_j, the ideal solution is depicted by the IVPHFE $h_{\widetilde{\mathscr{A}}^*_j}=\{\langle[1,1],[0,0]\rangle\}$; For the cost criterion C_j, the ideal solution is depicted by the IVPHFE $h_{\widetilde{\mathscr{A}}^*_j}=\{\langle[0,0],[1,1]\rangle\}$. Then $\widetilde{\mathscr{A}}^*=(h_{\widetilde{\mathscr{A}}^*_1},h_{\widetilde{\mathscr{A}}^*_2},\cdots,h_{\widetilde{\mathscr{A}}^*_n})$ which satisfies $|h_{\widetilde{\mathscr{A}}^*_j}|=|h'_{\widetilde{\mathscr{A}}_{ij}}|$ for any $j\in\{1,2,\cdots,n\}$.

Step 4 Get the priority of the alternatives A_i by ranking the above correlation coefficients. End.

Example 34: An investment company should evaluate four possible projects A_i $(i=1,2,3,4)$ according to the three criteria C_j $(j=1,2,3)$. Here C_1 and C_2 are both benefit criteria while C_3 is a cost criterion. Suppose that the weight vector of the criteria is $\omega=(0.3,0.45,0.25)^T$. The decision matrix is given by the experts as Table 3.

Considering that $|h_{\widetilde{\mathscr{A}}_{1j}}|\neq|h_{\widetilde{\mathscr{A}}_{2j}}|$ for any $j\in\{1,2,3\}$, we revised the matrix M into M' in Table 4. Table 5 gives the correlations between the four IVPHFSs and the ideal IVPHFS and informational energies of the four IVPHFSs. Furthermore, compute the four correlation coefficients between this alternatives and the ideal alternatives in Table 6. The result shows the project A_3 are always the most optimal, although the four ranking results are little different.

B. CLUSTERING ANALYSIS

Let $\widetilde{\mathscr{A}}_i$ $(i=1,2,\cdots,m)$ be m IVPHFSs on U, and $R=(\rho_{ij})_{m\times m}$ be their interval-valued Pythagorean hesitant fuzzy relation matrix, where $\rho_{ij}=\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_i,\widetilde{\mathscr{A}}_j)$. R obviously satisfies: for any $i,j=1,2,\cdots,m$,

(1)(Boundedness) $0\leqslant\rho_{ij}\leqslant 1$;
(2)(Reflexivity) $\rho_{ii}=1$;
(3)(Symmetry) $\rho_{ij}=\rho_{ji}$.

So R can be regarded as a similarity relation matrix.

TABLE 3. IVPHFDM M in Example 34.

	C_1	C_2	C_3
$\widetilde{\mathscr{A}}_1$	$\{\langle[0.6,0.7],[0.4,0.5]\rangle,$ $\langle[0.5,0.8],[0.1,0.3]\rangle\}$	$\{\langle[0.4,0.6],[0.1,0.3]\rangle,$ $\langle[0.4,0.6],[0.2,0.5]\rangle\}$	$\{\langle[0.5,0.7],[0.2,0.4]\rangle\}$
$\widetilde{\mathscr{A}}_2$	$\{\langle[0.4,0.6],[0.2,0.4]\rangle\}$	$\{\langle[0.5,0.6],[0.3,0.4]\rangle,$ $\langle[0.3,0.5],[0.2,0.3]\rangle,$ $\langle[0.2,0.7],[0.2,0.4]\rangle,$ $\langle[0.3,0.5],[0.2,0.7]\rangle\}$	$\{\langle[0.3,0.6],[0.2,0.3]\rangle,$ $\langle[0.2,0.5],[0.3,0.4]\rangle,$ $\langle[0.2,0.6],[0.1,0.3]\rangle\}$
$\widetilde{\mathscr{A}}_3$	$\{\langle[0.3,0.5],[0.1,0.2]\rangle,$ $\langle[0.5,0.7],[0.3,0.4]\rangle\}$	$\{\langle[0.6,0.8],[0.1,0.2]\rangle,$ $\langle[0.6,0.8],[0.2,0.3]\rangle\}$	$\{\langle[0.3,0.4],[0.4,0.6]\rangle,$ $\langle[0.3,0.4],[0.5,0.7]\rangle,$ $\langle[0.4,0.5],[0.3,0.6]\rangle\}$
$\widetilde{\mathscr{A}}_4$	$\{\langle[0.5,0.6],[0.2,0.4]\rangle\}$	$\{\langle[0.4,0.5],[0.3,0.4]\rangle,$ $\langle[0.3,0.5],[0.3,0.4]\rangle\}$	$\{\langle[0.5,0.8],[0.2,0.3]\rangle\}$

TABLE 4. Revised IVPHFDM M' in Example 34.

	C_1	C_2	C_3
$\widetilde{\mathscr{A}}_1$	$\{\langle[0.5,0.8],[0.1,0.3]\rangle,$ $\langle[0.6,0.7],[0.4,0.5]\rangle\}$	$\{\langle[0.4,0.6],[0.1,0.3]\rangle,$ $\langle[0.4,0.6],[0.1,0.3]\rangle,$ $\langle[0.4,0.6],[0.2,0.5]\rangle,$ $\langle[0.4,0.6],[0.2,0.5]\rangle\}$	$\{\langle[0.5,0.7],[0.2,0.4]\rangle,$ $\langle[0.5,0.7],[0.2,0.4]\rangle,$ $\langle[0.5,0.7],[0.2,0.4]\rangle\}$
$\widetilde{\mathscr{A}}_2$	$\{\langle[0.4,0.6],[0.2,0.4]\rangle,$ $\langle[0.4,0.6],[0.2,0.4]\rangle\}$	$\{\langle[0.5,0.6],[0.3,0.4]\rangle,$ $\langle[0.2,0.7],[0.2,0.4]\rangle,$ $\langle[0.3,0.5],[0.2,0.3]\rangle,$ $\langle[0.3,0.5],[0.2,0.7]\rangle\}$	$\{\langle[0.3,0.6],[0.2,0.3]\rangle,$ $\langle[0.2,0.6],[0.1,0.3]\rangle,$ $\langle[0.2,0.5],[0.3,0.4]\rangle\}$
$\widetilde{\mathscr{A}}_3$	$\{\langle[0.5,0.7],[0.3,0.4]\rangle,$ $\langle[0.3,0.5],[0.1,0.2]\rangle\}$	$\{\langle[0.6,0.8],[0.1,0.2]\rangle,$ $\langle[0.6,0.8],[0.1,0.2]\rangle,$ $\langle[0.6,0.8],[0.2,0.3]\rangle,$ $\langle[0.6,0.8],[0.2,0.3]\rangle\}$	$\{\langle[0.4,0.5],[0.3,0.6]\rangle,$ $\langle[0.3,0.4],[0.4,0.6]\rangle,$ $\langle[0.3,0.4],[0.5,0.7]\rangle\}$
$\widetilde{\mathscr{A}}_4$	$\{\langle[0.5,0.6],[0.2,0.4]\rangle,$ $\langle[0.5,0.6],[0.2,0.4]\rangle\}$	$\{\langle[0.4,0.5],[0.3,0.4]\rangle,$ $\langle[0.4,0.5],[0.3,0.4]\rangle,$ $\langle[0.3,0.5],[0.3,0.4]\rangle,$ $\langle[0.3,0.5],[0.3,0.4]\rangle\}$	$\{\langle[0.5,0.8],[0.2,0.3]\rangle,$ $\langle[0.5,0.8],[0.2,0.3]\rangle,$ $\langle[0.5,0.8],[0.2,0.3]\rangle\}$

TABLE 5. Correlations and informational energies in M' in Example 34.

	C_{IVPH_ω}	$C^{\mu}_{IVPH_\omega}$	$C^{\nu}_{IVPH_\omega}$	$C^{\pi}_{IVPH_\omega}$	E_{IVPH_ω}	$E^{\mu}_{IVPH_\omega}$	$E^{\nu}_{IVPH_\omega}$	$E^{\pi}_{IVPH_\omega}$
$\widetilde{\mathscr{A}}_1$	0.0908	0.2025	0.3464	0.2236	0.1675	0.2959	0.4116	0.2950
$\widetilde{\mathscr{A}}_2$	0.0668	0.1914	0.3388	0.1702	0.1899	0.3416	0.4114	0.3167
$\widetilde{\mathscr{A}}_3$	0.1258	0.2590	0.3893	0.2290	0.1720	0.3119	0.4147	0.2893
$\widetilde{\mathscr{A}}_4$	0.0640	0.1573	0.3400	0.1948	0.1727	0.3186	0.4104	0.2892

TABLE 6. Correlation coefficients of IVPHFSs in Example 34.

	$(\widetilde{\mathscr{A}}_1,\widetilde{\mathscr{A}}^*)$	$(\widetilde{\mathscr{A}}_2,\widetilde{\mathscr{A}}^*)$	$(\widetilde{\mathscr{A}}_3,\widetilde{\mathscr{A}}^*)$	$(\widetilde{\mathscr{A}}_4,\widetilde{\mathscr{A}}^*)$
$\rho^{(1)}_{IVPH_\omega}$	0.3844	0.2655	0.5252	0.2669
$\rho^{(2)}_{IVPH_\omega}$	0.2725	0.2004	0.3773	0.1921
$\rho^{(3)}_{IVPH_\omega}$	0.6241	0.5459	0.7043	0.5524
$\rho^{(4)}_{IVPH_\omega}$	0.5150	0.4669	0.5848	0.4614

	Ranking
$\rho^{(1)}_{IVPH_\omega}$	$\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_3,\widetilde{\mathscr{A}}^*) > \rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_1,\widetilde{\mathscr{A}}^*) > \rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_4,\widetilde{\mathscr{A}}^*) > \rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_2,\widetilde{\mathscr{A}}^*)$
$\rho^{(2)}_{IVPH_\omega}$	$\rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_3,\widetilde{\mathscr{A}}^*) > \rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_1,\widetilde{\mathscr{A}}^*) > \rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_2,\widetilde{\mathscr{A}}^*) > \rho^{(2)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_4,\widetilde{\mathscr{A}}^*)$
$\rho^{(3)}_{IVPH_\omega}$	$\rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_3,\widetilde{\mathscr{A}}^*) > \rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_1,\widetilde{\mathscr{A}}^*) > \rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_4,\widetilde{\mathscr{A}}^*) > \rho^{(3)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_2,\widetilde{\mathscr{A}}^*)$
$\rho^{(4)}_{IVPH_\omega}$	$\rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_3,\widetilde{\mathscr{A}}^*) > \rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_1,\widetilde{\mathscr{A}}^*) > \rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_2,\widetilde{\mathscr{A}}^*) > \rho^{(4)}_{IVPH_\omega}(\widetilde{\mathscr{A}}_4,\widetilde{\mathscr{A}}^*)$

Define $R^2 = R \circ R = (\widetilde{\rho}_{ij})_{m\times m}$, here

$$\widetilde{\rho}_{ij} = \bigvee_{k=1}^{m}\left(\rho_{ik}\bigwedge\rho_{kj}\right), \quad i,j=1,2,\cdots,m.$$

Then we say R^2 a composition matrix of R. Similarly, $R^3 = R^2\circ R, \cdots, R^n = R^{n-1}\circ R, \cdots$ Usually, for any nonnegative integers n_1 and n_2, the composition matrix $R^{n_1+n_2} = R^{n_1}\circ R^{n_2}$ is still a relation matrix.

If $R^2 \subseteq R$, that means $\bigvee_{k=1}^{m}\left(\rho_{ik}\bigwedge\rho_{kj}\right) \leqslant \rho_{ij}$, for any $i,j = 1,2,\cdots,m$. We say R is transitive. When a relation satisfies the reflexivity, symmetry and transitivity, it is an equivalence relation. Then we can say the interval-valued Pythagorean

TABLE 7. IVPHFDM *M* in Example 38.

	C_1	C_2	C_3
$\widetilde{\mathscr{A}_1}$	$\{\langle[0.5,0.6],[0.4,0.5]\rangle\}$	$\{\langle[0.3,0.4],[0.4,0.6]\rangle, \langle[0.2,0.2],[0.7,0.8]\rangle\}$	$\{\langle[0.2,0.4],[0.3,0.6]\rangle, \langle[0.6,0.8],[0.2,0.3]\rangle, \langle[0.5,0.6],[0.4,0.7]\rangle\}$
$\widetilde{\mathscr{A}_2}$	$\{\langle[0.5,0.7],[0.2,0.3]\rangle, \langle[0.3,0.4],[0.6,0.6]\rangle\}$	$\{\langle[0.5,0.7],[0.2,0.3]\rangle\}$	$\{\langle[0.8,0.9],[0.1,0.2]\rangle\}$
$\widetilde{\mathscr{A}_3}$	$\{\langle[0.1,0.3],[0.6,0.8]\rangle\}$	$\{\langle[0.6,0.8],[0.2,0.5]\rangle, \langle[0.3,0.4],[0.6,0.6]\rangle\}$	$\{\langle[0.5,0.6],[0.4,0.5]\rangle, \langle[0.3,0.4],[0.4,0.6]\rangle, \langle[0.3,0.6],[0.2,0.5]\rangle\}$
$\widetilde{\mathscr{A}_4}$	$\{\langle[0.4,0.4],[0.3,0.5]\rangle, \langle[0.1,0.8],[0.2,0.3]\rangle\}$	$\{\langle[0.2,0.5],[0.3,0.6]\rangle\}$	$\{\langle[0.4,0.5],[0.2,0.5]\rangle\}$
$\widetilde{\mathscr{A}_5}$	$\{\langle[0.3,0.6],[0.4,0.7]\rangle\}$	$\{\langle[0.1,0.3],[0.3,0.5]\rangle, \langle[0.3,0.5],[0.1,0.4]\rangle\}$	$\{\langle[0.4,0.4],[0.5,0.6]\rangle, \langle[0.3,0.6],[0.2,0.3]\rangle, \langle[0.2,0.5],[0.3,0.4]\rangle\}$
$\widetilde{\mathscr{A}_6}$	$\{\langle[0.1,0.6],[0.2,0.4]\rangle, \langle[0.1,0.4],[0.3,0.5]\rangle\}$	$\{\langle[0.4,0.7],[0.2,0.3]\rangle\}$	$\{\langle[0.3,0.5],[0.2,0.4]\rangle\}$

TABLE 8. Clustering results of six cars.

Class	Confidence level	Clustering result
1	$0 \leqslant \lambda \leqslant 0.7671$	$\{A_1, A_2, A_3, A_4, A_5, A_6\}$
2	$0.7671 < \lambda \leqslant 0.8965$	$\{A_1, A_3, A_4, A_5, A_6\}, \{A_2\}$
3	$0.8965 < \lambda \leqslant 0.8967$	$\{A_1\}, \{A_2\}, \{A_3, A_4, A_5, A_6\}$
4	$0.8967 < \lambda \leqslant 0.9336$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_5, A_6\}$
5	$0.9336 < \lambda \leqslant 0.9678$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_6\}, \{A_5\}$
6	$0.9678 < \lambda \leqslant 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}$

hesitant fuzzy similarity relation matrix R is an equivalence relation matrix. In fact, a similarity relation matrix can be composited into an equivalence relation matrix.

Proposition 35: Let $R = (\rho_{ij})_{m\times m}$ be an interval-valued Pythagorean hesitant fuzzy relation matrix on U. Then after the finite times of compositions: $R \to R^2 \to R^4 \to \cdots \to R^{2^k} \longrightarrow \cdots$, there must exist a positive integer k such that $R^{2^k} = R^{2^{k+1}}$ and R^{2^k} is also an interval-valued Pythagorean hesitant fuzzy equivalence relation matrix.

Definition 36: Let $R = (\rho_{ij})_{m\times m}$ be an interval-valued Pythagorean hesitant fuzzy relation matrix on U and $\lambda \in [0, 1]$. We say $R_\lambda = (\lambda\rho_{ij})_{m\times m}$ the $\lambda-$cutting matrix of R, where

$$\lambda\rho_{ij} = \begin{cases} 0, & \text{if } \rho_{ij} < \lambda \\ 1, & \text{if } \rho_{ij} \geqslant \lambda \end{cases} \quad i, j = 1, 2, \cdots, m.$$

Proposition 37: Let $R = (\rho_{ij})_{m\times m}$ be an interval-valued Pythagorean hesitant fuzzy relation matrix on U and $\lambda \in [0, 1]$. R is an interval-valued Pythagorean hesitant fuzzy equivalence relation matrix iff R_λ is a classic equivalence relation matrix for any confidence level λ.

Now we propose an algorithm of clustering IVPHFSs as follows:

Step 1 Input $M = (h_{\widetilde{\mathscr{A}_{it}}})_{m\times n}$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$.

Step 2 Compute $\rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}_i}, \widetilde{\mathscr{A}_j})$, for any $i, j = 1, 2, \cdots, m$, and get $R = (\rho_{ij})_{m\times m}$, where $\rho_{ij} = \rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}_i}, \widetilde{\mathscr{A}_j})$.

Step 3 Initiate $D = R$ and $E = \emptyset$.

Step 4 While $E \neq D$, do $E = D$ and $E = E^2$, end.

Step 5 Give any Confidence level λ, and classify all these IVPHFSs $\widetilde{\mathscr{A}_i}$ $(i = 1, 2, \cdots, m)$ based on E_λ. End.

Example 38: With the continuous improvement of people's living standards, the automobile industry is developing faster and faster. To better evaluate six different cars $U = \{A_i | i = 1, 2, \cdots, 6\}$ on the market, we need to cluster them according to the following three attributes: C_1: power performance; C_2: handling stability; and C_3: fuel economy. Because the users have different professions and levels of knowledge, they may give different evaluations of the same car. To clearly reflect the differences of the opinions, we keep all the evaluations by interval-valued Pythagorean hesitant fuzzy information listed in Table 7. Suppose that the weight vector of the criteria is $\omega = (0.3, 0.25, 0.45)^T$.

According to the weighted correlation coefficients $\rho_{ij} = \rho^{(1)}_{IVPH_\omega}(\widetilde{\mathscr{A}_i}, \widetilde{\mathscr{A}_j})$, we derive the relation matrix $R = (\rho_{ij})_{6\times 6}$ as following:

$$R = \begin{pmatrix} 1.0000 & 0.7633 & 0.8703 & 0.8965 & 0.8705 & 0.8490 \\ 0.7633 & 1.0000 & 0.7238 & 0.7671 & 0.7219 & 0.7573 \\ 0.8703 & 0.7238 & 1.0000 & 0.8677 & 0.8967 & 0.8823 \\ 0.8965 & 0.7671 & 0.8677 & 1.0000 & 0.9336 & 0.9678 \\ 0.8705 & 0.7219 & 0.8967 & 0.9336 & 1.0000 & 0.9266 \\ 0.8490 & 0.7573 & 0.8823 & 0.9678 & 0.9266 & 1.0000 \end{pmatrix},$$

Composite R and deduce the equivalence relation matrix:

$$R^8 = \begin{pmatrix} 1.0000 & 0.7671 & 0.8965 & 0.8965 & 0.8965 & 0.8965 \\ 0.7671 & 1.0000 & 0.7671 & 0.7671 & 0.7671 & 0.7671 \\ 0.8965 & 0.7671 & 1.0000 & 0.8967 & 0.8967 & 0.8967 \\ 0.8965 & 0.7671 & 0.8967 & 1.0000 & 0.9336 & 0.9678 \\ 0.8965 & 0.7671 & 0.8967 & 0.9336 & 1.0000 & 0.9336 \\ 0.8965 & 0.7671 & 0.8967 & 0.9678 & 0.9336 & 1.0000 \end{pmatrix} = R^{10},$$

Now classify the six cars based on the $\lambda-$cutting matrix R^8_λ in Table 8:

VI. CONCLUSION

The paper has introduced four correlation coefficients for PHFSs and IVPHFSs. Based on LCME method and the comparison method of interval numbers, the correlations between two IVPHFEs can be derived. In addition, we have added the local correlations and local informational energies for PHFSs and IVPHFSs, so we can more completely depict the similarity between two PHFSs or two IVPHFSs. And the ranking results of the proposed correlation coefficients by local correlations and local informational energies are basically consistent with that by the conventional definitions of correlation coefficients. At the same time, we have applied the correlation coefficients for interval-valued Pythagorean hesitant fuzzy environment in MCDM problems and clustering analysis to demonstrate the effectiveness.

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