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Distributed Event-Triggered Subgradient Method for Convex Optimization With General Step-Size

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ABSTRACT In this paper, the consensus and optimization of a multiagent system in a distributed optimization problem with bounded constraint is discussed under the general step-size, which is square nonsummable. Firstly, a distributed projective subgradient algorithm is designed for time-varying directed communication topologies under the event-triggered mechanism. Secondly, the consensus and optimization of the system state and the ergodic average sequence are discussed. Finally, the effectiveness of the design algorithm and the correctness of the theoretical results is verified by a simulation example.

INDEX TERMS Multiagent system, distributed optimization, event-triggered, general step-size, time-varying switching digraph.

I. INTRODUCTION

With the emergence of complex systems and large-scale networks, distributed optimization problems have received great attention in recent years, and some achievements in theory and application have been obtained, like consensus problem [1]–[20], [38], [39], containment control problems, tracking control problem, optimization problems [16]–[37] and so on. Meantime, the optimization problem is widely applied to all aspects of life. In medical, disease diagnosis is based on a large number of sample data to establish a corresponding optimization model according to human special medical knowledge, and then obtain the types of disease diagnosis. In the natural world, collective behavior is pretty common, such as bird migration, shoal effect, ant foraging, and bee nesting. In the behavior of predation, individual action is often blind and has high risk and low benefit. On the contrary, neighboring individuals can effectively avoid danger by transmitting danger signals in collective behavior. Meanwhile, collective behavior can maximize group benefits. Each individual in the biological cluster can make independent decisions and can be regarded as an agent. The system in which individuals communicate with each other and coordinate to accomplish tasks together is called the multiagent system.

For the consensus-based multiagent system distributed optimization problems, agents by resource sharing, coordination control and distributed executing cooperatively achieve consensus and optimization of states. Hence, the consensus problem is the basic problem of optimization. The consensus problem was earlier raised in [1]. Then the consensus problem was discussed in practical engineering application about self-driven in [2] and theoretical explanation was given in [3]. Then distributed optimization problems were studied systematically by [4], [5]. Later the consensus was discussed in the perspective of hybrid systems [6] and game theory [7], [8]. The bipartite consensus is discussed in [9] which is based on the relationship of cooperation or antagonistic in the networks.

A. GENERAL STEP-SIZE

It should be noted that the study of consensus and optimization is normally discussed on the step-size which is square summable. However, the algorithm with general step-size which is square nonsummable makes the distributed optimization problem more challenging. Along with the distributed subgradient algorithm of multiagent system in [4], [5], the optimization problem with general step-size was further discussed in [10]–[12]. Under time-varying directed graphs, the relationship between the ergodic average sequence and optimization value was provided in [10] with general step-size under unconstrained. The consensus and optimization of the ergodic average sequence was investigated in [11] under digraph topology and unconstrained

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with general step-size. Furthermore, the discrete-time distributed optimization problems with general step-size are discussed under unconstrained and bounded constrained respectively in [12].

B. COMMUNICATION TOPOLOGY GRAPH

The information transfer between agents is established on the communication topology. A bidirectional communication topology is an undirected graph. If the channel is attacked by the network, then the communication topology becomes a one-way topology, i.e. digraph topology. In the meantime, the asymmetry of the adjacency matrix increases the difficulties of the problems. When communication topology is Markov stochastic, then the communication between the agent and the neighbor depends on the agent's current state [13]. In practical communication topology, packet loss and network attack lead to the re-transmission of information and the asynchronous clock between agents, so the communication delay will inevitably occur, which discussed in [14], [15].

C. GENERAL CONSTRAINT

The feasible set in distributed convex optimization problem can be divided into unconstrained problems [16], [17], [35] and constrained problems [18]–[30]. And the constraint set which contains bounded constraint, equality constraint, and inequality constraint is normally called general constraint set. The discrete-time distributed convex optimization with general constraint is studied in [19]. Besides, the continuous-time distributed convex optimization problems with general constraint is considered in [20]–[22]. Meanwhile, this method can directly apply to neurodynamic networks [23], [24], which regard each nerve as an agent. In addition, the constraint of the feasible set can be discussed on the game problems [25]–[30], under the bounded constraint [25]–[28], equality constraint [29], and general constraint [30].

D. EVENT-TRIGGER MECHANISM

The general time-triggered subgradient methods are studied in [4]–[30], however, they consume a great number of resources for the agent needs to communicate with its neighbors at every communication moment. In order to reduces the burden of communication network, event-triggered control [31]–[34], [36]–[39] has be widely studied in recent few years. The key to the event-triggered mechanism is to reduce the unnecessary information transmission among agents which can effectively improve the running speed and reduce the communication burden. Event-based distributed optimization of the multiagent system with general undirected network and strong requirement of step-size be focused on [32]. And the threshold function is assumed to be proportional to the step-size in [32]. Then, the threshold function of the trigger condition is changed to independent with the step-size [33]. The event-based distributed consensus of the multiagent system with time-varying digraph network is explored in [34], however, the constraint case and general step-size are not considered.

Motivated by these researches [12] and [32]–[34], a discrete-time distributed subgradient event-triggered method with general step-size is proposed. It is verified that the condition of general step-size, square summable, is not necessary under the time-varying switching digraph. Meanwhile, the system state and ergodic average sequence can asymptotically converge to the minimizer of the global objective function.

More precisely, the contribution of this paper mainly includes the following three aspects.

1) A discrete-time distributed event-triggered subgradient algorithm with more general step-size is explored, which relaxes the requirement of step-size.

2) Under the event-triggered mechanism, the ergodic average sequence with general step-size is established and its consensus and optimization under time-varying digraph network are shown. Meanwhile, the convergence rate of $O(\frac{\ln(L+1)}{\sqrt{L}})$ is obtained.

3) When the threshold function is large, the general step-sizes of the update algorithm converge quickly, the convergence precision is high, and the trigger times are few.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries, formulate the distributed convex optimization problem, and give the event-triggered subgradient projection algorithm. Then, the main results of consensus and the convergence about the system state and ergodic average sequence are obtained in section 3. Furthermore, the simulation result is given in section 4, and we conclude this paper in section 5.

Notations: Use $\mathbb{N}^+$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ as the set of positive integer numbers, the set of real numbers, the set of n-dimensional Euclidean space, and the set of $n \times n$-dimensional real matrix, respectively. $\|\cdot\|$ represents the Euclidean vector norm. $I_N \in \mathbb{R}^{N\times N}$ is a identity matrix. $< x, y >= x^T y$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. ALGEBRAIC GRAPH THEORY

A Nth order digraph is denoted by $\mathcal{G}(\mathcal{V}, E, A)$, which composes of a vertex set $\mathcal{V} = \{1, 2, \ldots, N\}$, an edge set $E \subseteq \mathcal{V} \times \mathcal{V}$, and a weight adjacency matrix $A = [a_{ij}]_{N\times N}$, respectively. Let a_{ij} denotes the edge from vertex i to vertex j, and $a_{ij} > 0$ means that there exist a path from vertex i to vertex j. Supposing that there didn't exist repeated edges or self-loops, i.e., $a_{ii} = 0$, $\forall i \in \mathcal{V}$. And the vertex $(i, j) \in E$ represents a directed edge from vertex i to vertex j. We denote the in-neighbors and out-neighbors of vertex i by $N_{in}(i) = \{j \in \mathcal{V} | (j, i) \in E\}$ and $N_{out}(i) = \{j \in \mathcal{V} | (i, j) \in E\}$, respectively. The Laplacian matrix $L = [u_{ij}]_{N\times N}$ of digraph $\mathcal{G}(\mathcal{V}, E, A)$ is defined by $u_{ij} = -a_{ij}$, $i \neq j$; $u_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$, $\forall i, j \in \mathcal{V}$, which satisfies $\sum_{j=1}^{N} u_{ij} = 0$. The in-degree and out-degree of agent i can be defined by the Laplacian matrix L as $d_{in}(i) = -\sum_{j=1, j\neq i}^{N} u_{ji}$ and $d_{out}(i) = u_{ii}$.

Definition 1: A digraph $\mathcal{G}$ is balanced, if it satisfies $d_{in}(i) = d_{out}(i), \forall i \in \mathcal{V}$.

B. CONVEX OPTIMIZATION

A set $\mathcal{C} \subset \mathbb{R}^n$ is called convex if for all $x, y \in \mathcal{C}$, it contains all points $\eta x + (1-\eta)y \in \mathcal{C}, \forall \eta \in (0,1)$. A function f is convex if and only if for all x and y in its domain and for all $0 \leq \eta \leq 1$ we have $f(\eta x + (1-\eta)y) \leq \eta f(x) + (1-\eta)f(y)$. If the objective function f is a convex function and the constraint set is a convex set, then the optimization problem is a convex optimization. A vector $g(x)$ is said to be a subgradient of $f: \mathbb{R}^n \to \mathbb{R}$ at $x \in domf = \{x \in \mathbb{R}^n | f(x) < \infty\}$ if it satisfies

$$f(z) \geq f(x) + g^T(x)(z-x), \quad z \in \mathbb{R}^n. \tag{1}$$

Denote $P_{\mathcal{X}}[z] = \arg\min_{x \in \mathcal{X}} \|z - x\|$, which signifies a projection operator of a vector z on a nonempty and closed convex set $\mathcal{X}$. The projection operator has the following property:

$$\|P_{\mathcal{X}}[u] - P_{\mathcal{X}}[v]\| \leq \|u - v\|, \quad \forall u, v \in \mathbb{R}^n. \tag{2}$$

$$\|P_{\mathcal{X}}[u] - v\|^2 \leq \|u - v\|^2 - \|P_{\mathcal{X}}[u] - u\|^2, \quad \forall v \in \mathcal{X}. \tag{3}$$

$$< P_{\mathcal{X}}(u) - u, v - P_{\mathcal{X}}(u) > \geq 0, \quad u \in \mathbb{R}^n, \ v \in \mathcal{X}. \tag{4}$$

C. PROBLEM FORMULATION

For a time-varying switching direct topology graph $\mathcal{G}(\mathcal{V}, E, A)$ that contains N agents, we regard each agent as a node normally. The element of the weight adjacency matrix A satisfied $a_{ij} > 0$ means that the information can be transferred from agent i to agent j. In this subsection, we consider the following distributed minimization problem under the network $\mathcal{G}(\mathcal{V}, E, A)$:

$$\begin{aligned} &minimize\, f(x) = \sum_{i=1}^{N} f_i(x), \\ &subject\ to\ x \in \mathcal{X} = \bigcap_{i=1}^{N} \mathcal{X}_i, \end{aligned} \tag{5}$$

where x is a global decision vector of a multiagent system. The local objection function $f_i: \mathbb{R}^n \to \mathbb{R}, i \in \mathcal{V}$, is only known by the ith agent, and $\mathcal{X}, \mathcal{X}_i \in \mathbb{R}^n, i \in V$, is common objection constraint and local objection constraint, respectively. In general, the step-size is considered to satisfy the condition of $\alpha(l) > 0$, $\lim_{l\to\infty} \alpha(l) = 0$, $\sum_{l=0}^{\infty} \alpha(l) = \infty$, $\sum_{l=0}^{\infty} \alpha^2(l) < \infty$, which can effectively ensure the global decision vector converges to the optimal point in [4], [5]. In this paper, we relax the demand of step-size to general step size which is not square summable, i.e., $\sum_{l=0}^{\infty} \alpha^2(l) = \infty$. And compare the effect of general step-size on convergence rate.

Assumption 1: The step-size $\alpha(l)$ satisfies $\alpha(l) > 0$, $\lim_{l\to\infty} \alpha(l) = 0$, $\sum_{l=0}^{\infty} \alpha(l) = \infty$, and $\sum_{l=0}^{\infty} \alpha^2(l) = \infty$.

Remark 1: In this paper, we consider the general step-size with the form of $\alpha(l) = \frac{1}{\sqrt{l+1}}$, which satisfies positive, vanishing, and not square summable in Assumption 1.

Assumption 2: The constraint set is nonempty, convex and compact, i.e., for all $x \in \mathcal{X}$ there exists a positive constant C_x such that $\|x\| \leq C_x$.

Assumption 3: The problem (5) exists a nonempty bounded optimal set $\mathcal{X}^*$.

Assumption 4: There exist an infinite sequence $\{l_1, l_2, \ldots, l_m, \ldots\}$, such that the union graph $\bigcup_{l=l_m}^{l_{m+1}-1} \mathcal{G}(l)$ is strongly connected if $0 < l_{m+1} - l_m \leq B$, $B \in \mathbb{N}^+$.

D. EVENT-TRIGGERED SUBGRADIENT PROJECTION ALGORITHM

The event-triggered mechanism is studied in this paper, in which each agent executes the update of states in the way of distributed. Consider the event-triggered subgradient projection algorithm with the constraint on time-varying switching digraph, the algorithm we discussed is inspired by [32]–[34]. The square summable step-size in the algorithm of [34] is modified to the general step-size under the bounded constraint. The algorithm is shown as follows:

$$\begin{aligned} z_i(l) &= x_i(l) + h\sum_{j=1}^{N} a_{ij}(l)(\widetilde{x}_{ji}(l) - \widetilde{x}_{ij}(l)) - \alpha(l)g_i(l), \\ x_i(l+1) &= P_{\mathcal{X}_i}[z_i(l)], \end{aligned} \tag{6}$$

where $x_i(l) \in \mathbb{R}^n$ is the state of ith agent at time l; scalar h is a positive control gain which play a role in converting the weight adjacency matrix of digraph balanced to the doubly stochastic matrix; vector $g_i(l)$ is a subgradient of local objective function $f_i(x)$. The nonnegative scalar elements $a_{ij}(l)$ of weight adjacency matrix $A(l)$ describes each agent communication weight in time-varying digraph, and have an upper bound M, i.e., $M = \sup_{l\in\mathbb{N}} a_{ij}(l)$, in which $M > 1$ is permitted, $\forall i, j \in \mathcal{V}$. The vector $z_i(l)$ is used to store an intermediate value in a projection calculation. Triggered state $\widetilde{x}_{ij}(l)$ describes the state that jth agent receive form ith agent at triggered time l, which is denoted as follows:

$$\widetilde{x}_{ij}(l) = \begin{cases} x_i(l), & if\ l \in \kappa_{ij}, \quad (7) \\ \widetilde{x}_{ij}(l-1), & otherwise. \quad (8) \end{cases}$$

where $\kappa_{ij} = \{l_{ij}^0, l_{ij}^1, \ldots, l_{ij}^n, \ldots\}, n \in \mathbb{N}^+$, is the triggered time set, where nth triggered time of ith agent send information to jth agent is denoted as l_{ij}^n.

Remark 2: Based on the definition of $\widetilde{x}_{ij}(l)$, it is worth to notice that $\widetilde{x}_{ij}(l) \in \mathcal{X}$. For $\widetilde{x}_{ij}(l) = x_i(l)$, $if\ l \in \kappa_{ij}$; else $\widetilde{x}_{ij}(l) = \widetilde{x}_{ij}(l-1) = \widetilde{x}_{ij}(l-2) = \cdots = \widetilde{x}_{ij}(l) = x_i(l)$, where $l \in \kappa_{ij}$ is the biggest triggered time of ith agent.

Assumption 5: The objective function $f_i(x)$ is continuous and differentiable. The subgradient set of local objective function $f_i(x)$ is bounded, i.e., for all $x \in \mathcal{X}$, there exist a positive constant C_g such that $\|g_i(x)\| \leq C_g$.

Assumption 6: For all $i \in \mathcal{V}, j \in N_{out}(i)$, the threshold value $0 < E_{ij}(l) \leq E(l)$ satisfies positive, vanishing, and square summable, i.e., $E(l) > 0$, $\lim_{l\to\infty} E(l) = 0$, $\sum_{l=0}^{\infty} E(l) = \infty$ and $\sum_{l=0}^{\infty} E^2(l) < \infty$.

Remark 3: The threshold function with the form of $E(l) = \frac{c}{l+1}$ is considered in this paper, where c is a positive constant and satisfies Assumption 6.

Denote measurement error of triggered time as $e_{ij}(l) \in \mathbb{R}^n$, which satisfies

$$e_{ij}(l) = \widetilde{x}_{ij}(l) - x_i(l). \quad (9)$$

Next, we can denote the triggering function and nth trigger-time as

$$H_{ij}(l) = \|e_{ij}(l)\| - E_{ij}(l), \quad (10)$$
$$l_{ij}^n = \inf\{l | l > l_{ij}^{n-1}, H_{ij}(l) \geq 0\}. \quad (11)$$

Hence, the agent updates their state if the triggering function is nonnegative, and for each time, all agents satisfy the following condition under the event-triggered mechanism at time l.

$$\|e_{ij}(l)\| \leq E_{ij}(l) < E(l). \quad (12)$$

In the event-triggered mechanism, the communication frequency is decided by each agent, i.e., when to communicate with their neighbors under network link lie on the agent threshold function $E_{ij}(l)$.

Let $\Phi_i(l) \in \mathbb{R}^n$ be the perturbation term at time l caused by the projection operator.

$$\Phi_i(l) = P_{\mathcal{X}}[z_i(l)] - z_i(l). \quad (13)$$

By using the projection perturbation term $\Phi_i(l)$, we can remove the projection operator and simplify the calculation. Then, using the projection perturbation term $\Phi_i(l)$ and measure error $e_{ij}(l)$, the event-triggered algorithm (6) can be rewritten as

$$\begin{aligned} x_i(l+1) &= z_i(l) + \Phi_i(l) \\ &= x_i(l) + h\sum_{j=1}^{N} a_{ij}(l)(x_j(l) - x_i(l)) \\ &\quad + h\sum_{j=1}^{N} a_{ij}(l)(e_{ji}(l) - e_{ij}(l)) - \alpha(l)g_i(l) + \Phi_i(l) \\ &= (1 - hu_{ii}(l))x_i(l) - h\sum_{j=1, j\neq i}^{N} u_{ij}(l)x_j(l) \\ &\quad + \widehat{e}_i - \alpha(l)g_i(l) + \Phi_i(l) \\ &= \sum_{j=1}^{N} \overline{a}_{ij}(l)x_j(l) + \widehat{e}_i(l) - \alpha(l)g_i(l) + \Phi_i(l), \end{aligned} \quad (14)$$

where $\widehat{e}_i(l) = h\sum_{j=1}^N a_{ij}(l)(e_{ji}(l) - e_{ij}(l)) \leq \overline{M}E(l), \overline{M} = 2hNM$, $\overline{a}_{ii}(l) = 1 - hu_{ii}(l), \overline{a}_{ij}(l) = -hu_{ij}(l)$. According to the definition of balanced digraph graph, it is easy to see that $\sum_{j=1}^N \overline{a}_{ij}(l) = \sum_{j=1}^N \overline{a}_{ji}(l) = 1$. In order to ensure that the elements of the new doubly stochastic weight adjacency matrix $\overline{A}$ are nonnegative, we require that the control gain h satisfies $\overline{a}_{ii}(l) = 1 - hu_{ii}(l) > 0$, otherwise there may exist negative elements in the weight adjacency matrix $\overline{A}$.

Therefore, we obtain that

$$x_i(l+1) = v_i(l) + \varepsilon_i(l), \quad (15)$$

where the weight average term of ith agent at time l is denoted as $v_i(l) = \sum_{j=1}^N \overline{a}_{ij}(l)x_j(l)$, which playing a role in enabling all agents to reach a consensus. And $\varepsilon_i(l) = \widehat{e}_i - \alpha(l)g_i(l) + \Phi_i(l)$ is the consensus error term.

Denote $y(l) = \frac{1}{N}\sum_{i=1}^N x_i(l)$ as the state average sequence. By equality (14), it is easy to see that the state average sequence satisfies

$$y(l+1) = y(l) - \frac{\alpha(l)}{N}\sum_{i=1}^{N} g_i(l) + \frac{1}{N}\sum_{i=1}^{N}(\widehat{e}_i(l) + \Phi_i(l)). \quad (16)$$

Besides, the ergodic average sequence is utilized to establish the convergence rate of event-triggered mechanism, which defined as

$$\widehat{x}_i(L) = \frac{\sum_{l=0}^{L} \alpha(l)x_i(l)}{\sum_{l=0}^{L} \alpha(l)}, \widehat{y}(L) = \frac{\sum_{l=0}^{L} \alpha(l)y(l)}{\sum_{l=0}^{L} \alpha(l)}, \quad (17)$$

where $\widehat{x}_i(L)$ and $\widehat{y}(L) \in \mathbb{R}^n$.

Assumption 7: There exist a constant μ with $0 < \mu < 1$, $\forall i, j \in \mathcal{V} = \{1, 2, \ldots, N\}$, $\forall l \in \mathbb{N}$, such that the elements of weight matrix $\overline{A}$ satisfies

(a) $\overline{a}_{ii}(l) > \mu$;

(b) $\overline{a}_{ij}(l) > \mu,\ if\ \overline{a}_{ij}(l) > \mu$.

Then the state transition matrix $\Psi(l, s)$ is introduced as follows:

$$\Psi(l, s) = \overline{A}(l)\overline{A}(l-1)\cdots\overline{A}(s), \quad \forall l, s \in \mathbb{N}^+, l \geq s, \quad (18)$$

where the state transition matrix $\Psi(l, s)$ satisfies $\Psi(l, l) = \overline{A}(l)$, $\Psi(l, l+1) = I_N$, $\forall l \in \mathbb{N}^+$. Mark the jth column of the state transition matrix $\Psi(l, s)$ and the element in ith row and jth column of the state transition matrix $\Psi(l, s)$ as vector $[\Psi(l, s)]_j$ and scalar $[\Psi(l, s)]_{ij}$, respectively. The relevant property of the state transition matrix is given as the following.

Lemma 1: (see [5]) Let Assumption 4, 7 hold in the balanced digraph, then the element of the state transition matrix $[\Psi(l, s)]_{ij}$ converge to scalar $\frac{1}{N}$, $\forall i, j \in \mathcal{V}$, and

$$|[\Psi(l, s)]_{ij} - \frac{1}{N}| \leq 2\frac{1 + \mu^{-B_0}}{1 - \mu^{B_0}}(1 - \mu^{B_0})^{(l-s)/B_0}, \quad (19)$$

where μ is a lower bound of Assumption 7, N is the number of agents, $B_0 = (N-1)B$, and B is the intercommunication interval bound of Assumption 4.

Lemma 2: (see [5]) Let $0 < \lambda < 1$ and let $\{\gamma_l\}$ be a positive scalar sequence. If $\lim_{l\to\infty} \gamma_l = 0$, then

$$\lim_{l\to\infty} \sum_{r=0}^{l} \lambda^{l-r}\gamma_r = 0. \quad (20)$$

Besides, if $\sum_{l=0}^{\infty} \gamma_l < \infty$, then

$$\sum_{l=0}^{\infty} \sum_{r=0}^{l} \lambda^{l-r} \gamma_r < \infty. \quad (21)$$

III. MAIN RESULTS

A. THE PROPERTY OF PROJECTION PERTURBATION TERM AND CONSENSUS ERROR TERM

Lemma 3: Let Assumption 1, 5, 6 hold in the balanced digraph, then projection perturbation term $\Phi_i(l)$ have

$$\|\Phi_i(l)\| \le \overline{M}E(l) + C_g\alpha(l), \ \overline{M} = 2hMN. \quad (22)$$

Proof:

Step 1: State that the vector $v_i(l) = \sum_{j=1}^{N} \overline{a}_{ij}(l)x_j(l)$ is belong to the convex set $\mathcal{X}$. According to the definition of convex set $\mathcal{X}$ and $x_j(l) \in \mathcal{X}$, we have linear combinations $v_i(l) = \sum_{j=1}^{N} \overline{a}_{ij}(l)x_j(l)$ in the set $\mathcal{X}$ either.

Step 2: Find the relationship between the projection perturbation term and the system variable. Based on the property (3) of projection vector $P_{\mathcal{X}}[\cdot]$ and formula (14), we have

$$\begin{aligned}\|P_{\mathcal{X}}[z_i(l)]-v_i(l)\|^2 &\le \|z_i(l) - v_i(l)\|^2 - \|\Phi_i(l)\|^2 \\ &= \|\widehat{e}_i - \alpha(l)g_i(l))\|^2 - \|\Phi_i(l)\|^2. \quad (23)\end{aligned}$$

Step 3: Scaling the inequality (23). Due to $\|P_{\mathcal{X}}[z_i(l)] - \zeta_i(l)\|^2 \ge 0$, the boundedness of $a_{ij}(l)$, property (12) of the measurement error $e_{ij}(l)$ and Assumption 5, 6, we have

$$\begin{aligned}\|\Phi_i(l)\|^2 &\le \|\widehat{e}_i - \alpha(l)g_i(l)\|^2 \\ &\le (\overline{M}E(l) + C_g\alpha(l))^2, \quad (24)\end{aligned}$$

where $\overline{M} = 2hMN$ is a positive constant. Thus, Lemma 3 is proved. ■

Lemma 4: Let Assumption 1, 5, 6 hold in the balanced digraph, then the consensus error term satisfies

$$\lim_{l\to\infty} \|\varepsilon_i(l)\| = 0. \quad (25)$$

Proof: Since $\varepsilon_i(l) = \widehat{e}_i - \alpha(l)g_i(l) + \Phi_i(l)$ is a consensus error term, for all $l > 0$ and formula (24), then we have

$$\|\varepsilon_i(l)\| \le \overline{\varepsilon}(l), \quad (26)$$

where $\overline{\varepsilon}(l) = 2(\overline{M}E(l) + C_g\alpha(l))$, $\overline{M} = 2hMN$. By Assumption 1, 5 and 6, for inequality (26) we have $\lim_{l\to\infty} \|\varepsilon_i(l)\| = 0$. Thus, Lemma 4 is proved. ■

B. THE CONSENSUS AND OPTIMIZATION OF MULTIAGENT SYSTEM

Theorem 1 (Consensus): Let Assumption 1-7 hold in the balanced digraph. Consider the sequence $\{x_i(l)\}$ obtained by event-triggered subgradient projection algorithm (6), and for all $i \in \mathcal{V}$ the state average vector $y(l) = \frac{1}{N}\sum_{i=1}^{N} x_i(l)$ satisfies

$$\lim_{l\to\infty} \|x_i(l) - y(l)\| = 0. \quad (27)$$

Proof: **Step 1:** Describe the state $x_i(l)$ by the state transition matrix $\Psi(l, s)$. Based on the relationship in (15) and the state transition matrix $\Psi(l, s)$ in (18), $\forall i \in \mathcal{V}$, l and $s \in \mathbb{N}^+$ with $l > s$, we can write

$$\begin{aligned}&x(l+1) \\ &= (\overline{A}(l) \otimes I_n)x(l) + \epsilon(l) \\ &= [\overline{A}(l) \otimes I_n][(\overline{A}(l-1) \otimes I_n)x(l-1) + \epsilon(l-1)] + \epsilon(l) \\ &= \cdots \\ &= [(\overline{A}(l)\overline{A}(l-1)\cdots\overline{A}(0)) \otimes I_n]x(0) + [(\overline{A}(l)\overline{A}(l-1) \\ &\quad \cdots\overline{A}(1)) \otimes I_n]\epsilon(0) + \cdots + (\overline{A}(l) \otimes I_n)\epsilon(l-1) + \epsilon(l) \\ &= (\Psi(l, 0) \otimes I_n)x(0) + \sum_{r=1}^{l+1}(\Psi(l, r) \otimes I_n)\epsilon(r-1), \quad (28)\end{aligned}$$

where $x(l), \epsilon(l) \in \mathbb{R}^{Nn}$ and $x_i(l), \epsilon_i(l) \in \mathbb{R}^n$. Therefore, it yields that

$$x_i(l+1) = \sum_{j=1}^{N}[\Psi(l, 0)]_{ij}x_j(0) + \sum_{r=1}^{l+1}\sum_{j=1}^{N}[\Psi(l, r)]_{ij}\epsilon_j(r-1). \quad (29)$$

Step 2: Describe the state average vector $y(l) = \frac{1}{N}\sum_{i=1}^{N} x_i(l)$ by the state of $x_i(l)$. According to weight adjacency matrix $\overline{A}(l)$ is doubly stochastic matrix, we have

$$\begin{aligned}y(l) &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}[\Psi(l-1, 0)]_{ij}x_j(0) \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{r=1}^{l}\sum_{j=1}^{N}[\Psi(l-1, r)]_{ij}\epsilon_j(r-1) \\ &= \frac{1}{N}\sum_{j=1}^{N} x_j(0) + \frac{1}{N}\sum_{r=1}^{l}\sum_{j=1}^{N}\epsilon_j(r-1). \quad (30)\end{aligned}$$

Then, combining (29) with (30), yields

$$\begin{aligned}x_i(l) - y(l) &= \sum_{j=1}^{N}([\Psi(l-1, 0)]_{ij} - \frac{1}{N})x_j(0) \\ &\quad + \sum_{r=1}^{l}\sum_{j=1}^{N}([\Psi(l-1, r)]_{ij} - \frac{1}{N})\epsilon_j(r-1). \quad (31)\end{aligned}$$

Using (31), then we obtain

$$\begin{aligned}&\|x_i(l) - y(l)\| \\ &\le \|\sum_{j=1}^{N}([\Psi(l-1, 0)]_{ij} - \frac{1}{N})x_j(0)\| \\ &\quad + \|\sum_{r=1}^{l}\sum_{j=1}^{N}([\Psi(l-1, r)]_{ij} - \frac{1}{N})\epsilon_j(r-1)\| \\ &\le N\max_{j\in V}|[\Psi(l-1, 0)]_{ij} - \frac{1}{N}|\max_{j\in V}\|x_j(0)\| \\ &\quad + \sum_{r=1}^{l} N\max_{j\in V}|[\Psi(l-1, r)]_{ij} - \frac{1}{N}|\max_{j\in V}\|\epsilon_j(r-1)\|. \quad (32)\end{aligned}$$

Step 3: Scaling the inequation (32). According to the property of Lemma 1, we get

$$|[\Psi(l,s)]_{ij}-\frac{1}{N}|\le \mathcal{D}\mathfrak{X}^{l-s}, \tag{33}$$

with $\mathcal{D}=2\frac{1+\mu^{-B_0}}{1-\mu^{B_0}}$, $\mathfrak{X}=(1-\mu^{B_0})^{1/B_0}$, and $0<\mathfrak{X}<1$. Then, by the upper bound of x in Assumption 2, Lemma 4 and (33), yields that

$$N\max_{j\in V}|[\Psi(l-1,0)]_{ij}-\frac{1}{N}|\max_{j\in V}\|x_j(0)\| \le NC_x\mathcal{D}\mathfrak{X}^{l-1}, \tag{34}$$

$$\sum_{r=1}^{l}N\max_{j\in V}|[\Psi(l-1,r)]_{ij}-\frac{1}{N}|\max_{j\in V}\|\epsilon_j(r-1)\| \le N\mathcal{D}\sum_{r=1}^{l}\mathfrak{X}^{l-r-1}\overline{\varepsilon}(r-1). \tag{35}$$

Combine (34) and (35), we have

$$0\le\|x_i(l)-y(l)\|\le NC_x\mathcal{D}\mathfrak{X}^{l-1}+N\mathcal{D}\sum_{r=1}^{l}\mathfrak{X}^{l-r-1}\overline{\varepsilon}(r-1). \tag{36}$$

Finally, according to Lemma 2 and Lemma 4, formula $\lim_{l\to\infty}\|x_i(l)-y(l)\|=0$ hold. Hence, the Theorem 1 is proved. ■

Theorem 2: (*Consensus*) Let Assumption 1-7 hold in the balanced digraph. For all $i\in\mathcal{V}$, considering the sequence $\{\widehat{x}_i(l)\}$ obtained by (17), it satisfies

$$\lim_{L\to\infty}\|\widehat{x}_i(L)-\widehat{x}_j(L)\|=0, \tag{37}$$

and the convergence rate is $O(\frac{\ln(L+1)}{\sqrt{L}})$.

Proof: **Step 1:** Find the relationship between the norm $\|x_i(l)-y(l)\|$ and the norm $\|\widehat{x}_i(L)-\widehat{y}(L)\|$. Since the convexity of norm, we have

$$\|\widehat{x}_i(L)-\widehat{y}(L)\|\le\frac{\sum_{l=0}^{L}\alpha(l)\|(x_i(l)-y(l)\|}{\sum_{l=0}^{L}\alpha(l)}. \tag{38}$$

Step 2: Based on the size of norm $\|x_i(l)-y(l)\|$ to constrain $\|\widehat{x}_i(L)-\widehat{y}(L)\|$. Use the inequality (36) in Theorem 1 expansion and contraction inequality (38), we yield to

$$\|\widehat{x}_i(L)-\widehat{y}(L)\| \le C_x\mathcal{D}N\frac{\sum_{l=0}^{L}\alpha(l)\mathfrak{X}^{l-1}}{\sum_{l=0}^{L}\alpha(l)}+\mathcal{D}N\frac{\sum_{l=0}^{L}\sum_{r=1}^{l}\alpha(l)\mathfrak{X}^{l-r-1}\overline{\varepsilon}(r-1)}{\sum_{l=0}^{L}\alpha(l)}. \tag{39}$$

Step 3: Scaling the inequality (39). With $\alpha(l)=\frac{1}{\sqrt{l+1}}\le 1$ in Assumption 1 and $\overline{\varepsilon}(r-1)$ in Lemma 4, we obtain that

$$\sum_{l=0}^{L}\alpha(l)\ge\int_1^{L+2}\frac{1}{\sqrt{x}}dx=2(\sqrt{L+2}-1)\ge\sqrt{L}, \tag{40}$$

$$\sum_{l=0}^{L}\alpha(l)\mathfrak{X}^{l-1}\le\sum_{l=0}^{L}\mathfrak{X}^{l-1}\le\frac{1}{1-\mathfrak{X}}, \tag{41}$$

$$\sum_{l=0}^{L}\sum_{r=1}^{l}\alpha(l)\mathfrak{X}^{l-r-1}\overline{\varepsilon}(r-1) =\sum_{l=0}^{L}\sum_{r=1}^{l}\alpha(l)\mathfrak{X}^{l-r-1}(2\overline{M}E(r-1)+2C_g\alpha(r-1)). \tag{42}$$

By the define of threshold value of $E(l)=c\alpha^2(l)$ in Assumption 6, we have

$$2\overline{M}\sum_{l=0}^{L}\sum_{r=1}^{l}\alpha(l)\mathfrak{X}^{l-r-1}E(r-1) \le 2c\overline{M}\alpha(0)\sum_{l=0}^{L}\sum_{r=1}^{l}\mathfrak{X}^{l-r-1}\alpha(l)\alpha(r-1). \tag{43}$$

Scaling the second term of inequality (42), we have

$$\begin{aligned}
&2C_g\sum_{l=0}^{L}\sum_{r=1}^{l}\mathfrak{X}^{l-r-1}\alpha(l)\alpha(r-1)\\
&=\frac{2C_g}{\mathfrak{X}}\sum_{l=0}^{L}\sum_{s=0}^{l-1}\mathfrak{X}^{l-s-1}\alpha(l)\alpha(s)\\
&=\frac{2C_g}{\mathfrak{X}}[1+\sum_{i=1}^{L}\mathfrak{X}^{i-1}(\sum_{j=1}^{L-i+1}\frac{1}{\sqrt{j}\sqrt{j+i}})]\\
&\le\frac{2C_g}{\mathfrak{X}}[1+\sum_{i=1}^{L}\mathfrak{X}^{i-1}\int_0^{L-i+1}\frac{1}{\sqrt{x}\sqrt{x+i}}dx]\\
&\le\frac{2C_g}{\mathfrak{X}}[1+\sum_{i=1}^{L}\mathfrak{X}^{i-1}\cdot 2\ln(\sqrt{L-i+1}+\sqrt{L+1})]\\
&\le\frac{2C_g}{\mathfrak{X}}[1+\frac{2}{1-\mathfrak{X}}(\ln 2+\frac{\ln(L+1)}{2})]\\
&\le\frac{2C_g}{\mathfrak{X}}\cdot\frac{4}{1-\mathfrak{X}}\ln(L+1)=\frac{8C_g\ln(L+1)}{\mathfrak{X}(1-\mathfrak{X})},
\end{aligned} \tag{44}$$

which L ≥ 2. Hence, summing (43) and (44) up, for equality (42) we have

$$\sum_{l=0}^{L}\sum_{r=1}^{l}\alpha(l)\mathfrak{X}^{l-r-1}\overline{\varepsilon}(r-1)\le\frac{8(c\overline{M}\alpha(0)+C_g)\ln(L+1)}{\mathfrak{X}(1-\mathfrak{X})}. \tag{45}$$

For inequation (39), we have

$$\|\widehat{x}_i(L)-\widehat{y}(L)\| \le\frac{C_x\mathcal{D}N}{1-\mathfrak{X}}\frac{1}{\sqrt{L}}+\frac{8(c\overline{M}\alpha(0)+C_g)\mathcal{D}N}{\mathfrak{X}(1-\mathfrak{X})}\frac{\ln(L+1)}{\sqrt{L}}. \tag{46}$$

Taking the limit of both sides of (46), it is easy to know that $\lim_{L\to\infty}\frac{1}{\sqrt{L}}=0$ and $\lim_{L\to\infty}\frac{\ln(L+1)}{\sqrt{L}}=0$, i.e., the convergence rate is $\frac{\ln(L+1)}{\sqrt{L}}$. Hence, the Theorem 2 is proved. ■

Theorem 3 (Optimization): Let Assumption 1-7 hold in the balanced digraph. For problem (5), if the agents state sequence $\{x_i(l)\}$ established by event-triggered subgrident projection algorithm (6), then for all $i \in \mathcal{V}$ there exist an optimal solution $x^* \in \mathcal{X}^*$, such that

$$\lim_{l\to\infty} x_i(l) = x^*, \tag{47}$$

and the convergence rate is $O(\frac{\ln(L+1)}{\sqrt{L}})$.

Proof: **Step 1:** Calculate the error between $y(l+1)$ and x^*. Using equality (16), we have that

$$\begin{aligned}
&\|y(l+1)-x^*\|^2\\
&= \|(y(l)-x^*)-\frac{\alpha(l)}{N}\sum_{i=1}^{N} g_i(l)+\frac{1}{N}\sum_{i=1}^{N}(\widehat{e}_i(l)+\Phi_i(l))\|^2\\
&= \|y(l)-x^*\|^2+\frac{\alpha^2(l)}{N^2}\|\sum_{i=1}^{N} g_i(l)\|^2+\frac{1}{N^2}(\sum_{i=1}^{N}\|\widehat{e}_i(l)\\
&\quad +\Phi_i(l)\|)^2-\frac{2\alpha(l)}{N}\sum_{i=1}^{N} g_i^T(l)(y(l)-x^*)\\
&\quad +\frac{2}{N}(y(l)-x^*)^T\sum_{i=1}^{N}(\widehat{e}_i(l)+\Phi_i(l))\\
&\quad -\frac{2\alpha(l)}{N^2}\sum_{i=1}^{N} g_i^T(l)\sum_{i=1}^{N}(\widehat{e}_i(l)+\Phi_i(l)).
\end{aligned} \tag{48}$$

Step 2: Scaling the inequality (48). According to the bounded of $\widehat{e}_i(l)$ and the property of $\Phi_i(l)$ in Lemma 3. We yield that

$$\frac{1}{N}\sum_{i=1}^{N}\|\widehat{e}_i(l)+\Phi_i(l)\| \le 2\overline{M}E(l)+C_g\alpha(l), \tag{49}$$

$$\frac{1}{N^2}[\sum_{i=1}^{N}\|\widehat{e}_i(l)+\Phi_i(l)\|]^2 \le (2\overline{M}E(l)+C_g\alpha(l))^2. \tag{50}$$

And due to the bounded of constraint set, $\|x\| \le C_x, \forall x \in \mathcal{X}$, in Assumption 2, we have

$$\frac{2}{N}\sum_{i=1}^{N}(y(l)-x^*)^T\widehat{e}_i(l) \le 4C_x\overline{M}E(l). \tag{51}$$

According to the lemma3 and inequality (4), respectively. We have $\|\Phi_i(l)\| \le \overline{M}E(l)+C_g\alpha(l)$ and $< P_{\mathcal{X}}[z_i(l)]-x^*, P_{\mathcal{X}}[z_i(l)]-z_i(l) > \le 0$. Besides, in inequation (4),we set $u = z_i(l)$, $v = x^*$, so

$$\begin{aligned}
&\frac{2}{N}\sum_{i=1}^{N}(y(l)-x^*)^T\Phi_i(l)\\
&= \frac{2}{N}\sum_{i=1}^{N}(y(l)-P_{\mathcal{X}}[z_i(l)])^T(P_{\mathcal{X}}[z_i(l)]-z_i(l))\\
&\quad +\frac{2}{N}\sum_{i=1}^{N}(P_{\mathcal{X}}[z_i(l)]-x^*)^T(P_{\mathcal{X}}[z_i(l)]-z_i(l))\\
&\le \frac{2}{N}\sum_{i=1}^{N}(y(l)-P_{\mathcal{X}}[z_i(l)])^T\Phi_i(l)\\
&\le \frac{2}{N}\sum_{i=1}^{N}\|y(l)-x_i(l+1)\|(\overline{M}E(l)+C_g\alpha(l))\\
&\le \frac{2}{N}\sum_{i=1}^{N}\|y(l)-x_i(l)\|(\overline{M}E(l)+C_g\alpha(l))\\
&\quad +\|x_i(l)-x_i(l+1)\|)(\overline{M}E(l)+C_g\alpha(l)).
\end{aligned} \tag{52}$$

According to the consensus of Theorem 1 and algorithm (6), yield that $\lim_{l\to\infty}\tilde{x}_{ij}(l) = \lim_{l\to\infty}\tilde{x}_{ji}(l)$, $\lim_{l\to\infty}\alpha(l) = 0$, so $\lim_{l\to\infty} z_i(l) = x_i(l)$, and $\lim_{l\to\infty} x_i(l+1) = \lim_{l\to\infty} x_i(l) = \bar{x}$. Hence $\forall\epsilon_1 > 0$, $\exists N_1 \in \mathbb{N}^+$, such that if $l > N_1$, then $\|x_i(l+1)-x_i(l)\| < \epsilon_1$. Then, combine the formula (51) and (52), if $l > N_1$, we can get

$$\begin{aligned}
&\frac{2}{N}(y(l)-x^*)^T\sum_{i=1}^{N}(\widehat{e}_i(l)+\Phi_i(l))\\
&\le 4C_x\overline{M}E(l)+\frac{2}{N}\sum_{i=1}^{N}\|y(l)-x_i(l)\|(\overline{M}E(l)+C_g\alpha(l))\\
&\quad +2(\overline{M}E(l)+C_g\alpha(l))\epsilon_1.
\end{aligned} \tag{53}$$

Since the bounded of subgradient $\|g_i(l)\| \le C_g$, it yields that

$$\begin{aligned}
&-\frac{2\alpha(l)}{N^2}\sum_{i=1}^{N} g_i^T(l)\sum_{i=1}^{N}(\widehat{e}_i(l)+\Phi_i(l))\\
&\quad \le 2C_g\alpha(l)(2\overline{M}E(l)+C_g\alpha(l)),
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\frac{\alpha^2(l)}{N^2}\|\sum_{i=1}^{N} g_i(l)\|^2\\
&\quad \le C_g^2\alpha^2(l),
\end{aligned} \tag{55}$$

$$\begin{aligned}
&g_i^T(l)(y(l)-x^*)\\
&\quad = g_i^T(l)(y(l)-x_i(l))+g_i^T(l)(x_i(l)-x^*)\\
&\quad \ge g_i^T(l)(y(l)-x_i(l))+(f_i(x_i(l))-f_i(x^*))\\
&\quad \ge -C_g\|y(l)-x_i(l)\|+(f_i(x_i(l))-f_i(y(l))\\
&\qquad +f_i(y(l))-f_i(x^*))\\
&\quad \ge -2C_g\|y(l)-x_i(l)\|+(f_i(y(l))-f_i(x^*)),\\
&\sum_{i=1}^{N} g_i^T(l)(y(l)-x^*)\\
&\quad \ge -2C_gN\|y(l)-x_i(l)\|+\sum_{i=1}^{N}(f_i(y(l))-f_i(x^*))\\
&\quad = -2C_gN\|y(l)-x_i(l)\|+(f(y(l))-f(x^*)),\\
&-\frac{2\alpha(l)}{N}\sum_{i=1}^{N} g_i^T(l)(y(l)-x^*)\\
&\quad \le 4C_g\alpha(l)\|y(l)-x_i(l)\|-\frac{2\alpha(l)}{N}(f(y(l))-f(x^*)).
\end{aligned} \tag{56}$$

Then, we sum the inequality (50), (53), (54), (55), (56) up, for equality (48), if $l > N_1$, we get

$$\begin{aligned}&\|y(l+1)-x^*\|^2\\ &\le \|y(l)-x^*\|^2+4\overline{M}^2E^2(l)+6C_g^2\alpha^2(l)\\ &\quad+8C_g\overline{M}E(l)\alpha(l)+2(2C_x+\epsilon_1)\overline{M}E(l)\\ &\quad+2C_g\epsilon_1\alpha(l)-\frac{2\alpha(l)}{N}(f(y(l))-f(x^*))\\ &\quad+\frac{2}{N}\sum_{i=1}^{N}\|y(l)-x_i(l)\|(\overline{M}E(l)+C_g\alpha(l))\\ &\quad+4C_g\alpha(l)\|y(l)-x_i(l)\|. \end{aligned}\tag{57}$$

Next, according to the relation $\lim_{l\to\infty}\|x_i(l)-y(l)\|=0$ in Theorem 1, we have $\forall\epsilon_2>0, \exists N_2\in\mathbb{N}^+$, such that if $l>N_2$, then $\|x_i(l)-y(l)\|<\epsilon_2$. If $l\ge\max\{N_1,N_2\}$, yields that

$$\begin{aligned}&\|y(l+1)-x^*\|^2\\ &\le \|y(l)-x^*\|^2+4\overline{M}^2E^2(l)+6C_g^2\alpha^2(l)\\ &\quad+8C_g\overline{M}E(l)\alpha(l)+2(2C_x+\epsilon_1)\overline{M}E(l)\\ &\quad+2C_g\epsilon_1\alpha(l)-\frac{2\alpha(l)}{N}(f(y(l))-f(x^*))\\ &\quad+(2\overline{M}E(l)+6C_g\alpha(l))\varepsilon_2. \end{aligned}\tag{58}$$

Step 3: Prove the formula $\lim_{L\to\infty}f(\widehat{y}(L))=f(x^*)$. Rearranging the nonnegative term $\frac{2\alpha(l)}{N}(f(y(l))-f(x^*))$, yields that

$$\begin{aligned}&\frac{2\alpha(l)}{N}(f(y(l))-f(x^*))\\ &\le(\|y(l)-x^*\|^2-\|y(l+1)-x^*\|^2)\\ &\quad+4\overline{M}^2E^2(l)+6C_g^2\alpha^2(l)+8C_g\overline{M}E(l)\alpha(l)\\ &\quad+2(2C_x+\epsilon_1+\epsilon_2)\overline{M}E(l)+2C_g(\epsilon_1+3\epsilon_2)\alpha(l). \end{aligned}\tag{59}$$

Taking summation of the discrete-time from $l=0$ to L, we obtain

$$\begin{aligned}&\frac{2}{N}\sum_{l=0}^{L}[\alpha(l)(f(y(l))-f(x^*))]\\ &\le\|y(0)-x^*\|^2+4\overline{M}^2\sum_{l=0}^{L}E^2(l)+6C_g^2\sum_{l=0}^{L}\alpha^2(l)\\ &\quad+8C_g\overline{M}\sum_{l=0}^{L}E(l)\alpha(l)+2(2C_x+\epsilon_1+\varepsilon_2)\overline{M}\sum_{l=0}^{L}E(l)\\ &\quad+2C_g(\epsilon_1+3\varepsilon_2)\sum_{l=0}^{L}\alpha(l). \end{aligned}\tag{60}$$

Dividing both sides of inequality (60) by positive term $\frac{2}{N}\sum_{l=0}^{L}\alpha(l)$, we have

$$\begin{aligned}&\frac{1}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}[\alpha(l)(f(y(l))-f(x^*))]\\ &\le\frac{N}{2\sum_{l=0}^{L}\alpha(l)}\|y(0)-x^*\|^2+\frac{2\overline{M}^2N}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}E^2(l)\\ &\quad+\frac{3C_g^2N}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}\alpha^2(l)+\frac{4C_g\overline{M}N}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}E(l)\alpha(l)\\ &\quad+\frac{(2C_x+\epsilon_1+\varepsilon_2)\overline{M}N}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}E(l)+C_gN(\epsilon_1+3\varepsilon_2). \end{aligned}\tag{61}$$

Then, according to the following inequations

$$\sum_{l=0}^{L}\alpha(l)\ge\int_1^{L+2}\frac{1}{\sqrt{x}}dx=2(\sqrt{L+2}-1)\ge\sqrt{L},\tag{62}$$

$$\sum_{l=0}^{L}\alpha^2(l)\le1+\ln(L+1)\le2\ln(L+1), L\ge2,\tag{63}$$

$$\sum_{l=0}^{L}E(l)=\sum_{l=0}^{L}\frac{c}{l+1}\le2c\ln(L+1), L\ge2,\tag{64}$$

$$\sum_{l=0}^{L}E^2(l)=\sum_{l=0}^{L}\frac{c^2}{(l+1)^2}\le c^2(2-\frac{1}{L+1}),\tag{65}$$

$$\sum_{l=0}^{L}\alpha(l)E(l)=\sum_{l=0}^{L}\frac{c}{(l+1)^{\frac{3}{2}}}\le r,\tag{66}$$

where it is easy to know that if $p>1$, then $\sum_{l=0}^{L}\frac{1}{(l+1)^p}$ converge. Let the upper bound of inequality (66) is r. Combining equalities $(62)-(66)$ to inequality (61), if $l\ge\max\{N_1,N_2\}$, yields that

$$\begin{aligned}&\frac{1}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}[\alpha(l)(f(y(l))-f(x^*))]\\ &\le\frac{2C_x^2N}{\sqrt{L}}+\frac{4c^2\overline{M}^2N}{\sqrt{L}}-\frac{2c^2\overline{M}^2N}{\sqrt{L}(L+1)}+6C_g^2N\frac{\ln(L+1)}{\sqrt{L}}\\ &\quad+\frac{4C_g\overline{M}Nr}{\sqrt{L}}+2c(2C_x+\epsilon_1+\epsilon_2)\overline{M}N\frac{\ln(L+1)}{\sqrt{L}}\\ &\quad+C_gN(\epsilon_1+3\varepsilon_2). \end{aligned}\tag{67}$$

According to the definition of convex function $f(x)$, we have $f(\alpha x+(1-\alpha)y)\le\alpha f(x)+(1-\alpha)f(y)$, then we obtain

$$\begin{aligned}f(\widehat{y}(L))-f(x^*)&=f(\frac{1}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}(\alpha(l)y(l)))-f(x^*)\\ &\le\frac{1}{\sum_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}[\alpha(l)(f(y(l))-f(x^*))]. \end{aligned}\tag{68}$$

Basing on inequality (67) and taking the limit of formula (68), we get

$$\begin{aligned}0 &\le \lim_{L\to\infty} f(\widehat{y}(L)) - f(x^*)\\ &\le \lim_{L\to\infty} \frac{1}{\sum\limits_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}[\alpha(l)(f(y(l)) - f(x^*))]\\ &= 0. \end{aligned} \tag{69}$$

Hence, the convergence rate between $f(\widehat{y}(L))$ and $f(x^*)$ is $\frac{\ln(L+1)}{\sqrt{L}}$.

Step 4**:** Prove equality $\lim\limits_{L\to\infty} \|\widehat{y}(L) - y(L)\| = 0$. We can get that

$$\begin{aligned}\|\widehat{y}(L) - y(L)\| &\le \frac{1}{\sum\limits_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}(\alpha(l)\|y(l) - y(L)\|)\\ &\le \frac{\alpha(0)}{\sum\limits_{l=0}^{L}\alpha(l)}\sum_{l=0}^{L}\|y(l) - y(L)\|. \end{aligned}\tag{70}$$

According to the consensus of Theorem 1, we have $\lim\limits_{l\to\infty} y(l) = \lim\limits_{l\to\infty} x_i(l) = \bar{x}$, then $\epsilon > 0$ satisfy that if $l > N = \max\{N_1, N_2\}$, then $\|y(l) - \bar{x}\| \le \frac{\epsilon}{2(m-n)}$. Let $\Theta(s) = \sum\limits_{l=1}^{s} \|y(l) - y(L)\|$. Using *Cauchy Convergence Criterion*, we obtain that $\forall\epsilon > 0$, $N \in \mathbb{N}^+$, if $n > N$, $m > N$, $m > n$ and $L > N$, yield that

$$\begin{aligned}&|\Theta(m) - \Theta(n)|\\ &= |\|y(m) - y(L)\| + \cdots + \|y(n+1) - y(L)\||\\ &\le |(\|y(m) - \bar{x}\| + \|\bar{x} - y(L)\|) + \cdots\\ &\quad + (\|y(n+1) - \bar{x}\| + \|\bar{x} - y(L)\|)|\\ &\le (\frac{\epsilon}{2(m-n)} + \frac{\epsilon}{2(m-n)}) + \cdots + (\frac{\epsilon}{2(m-n)} + \frac{\epsilon}{2(m-n)})\\ &= \epsilon. \end{aligned}\tag{71}$$

Hence the series $\Theta(s)$ converges, i.e. there exists θ, such that $\lim_{s\to\infty} \Theta(s) = \theta$. If $l \ge N$, $N = \max\{N_1, N_2\}$, from formula (70) we have

$$0 \le \lim_{L\to\infty} \|\widehat{y}(L) - y(L)\| \le \lim_{L\to\infty} \frac{\Theta(L)}{\sqrt{L}} = 0. \tag{72}$$

It is easy to know that $\lim_{L\to\infty} \|\widehat{y}(L) - y(L)\| = 0$, then $\lim_{L\to\infty} f(\widehat{y}(L)) = \lim_{L\to\infty} f(y(L)) = f(x^*)$. According to the consensus of Theorem 1, we have $\lim_{l\to\infty} f(x_i(l)) = f(x^*)$. Hence, the Theorem 3 is proved. ■

Theorem 4: (*Optimization*) Let Assumption 1-7 hold in the balanced digraph. For problem (5), if the ergodic average sequence established by (17), then there exist an optimal solution $x^* \in \mathcal{X}^*$, such that

$$\lim_{L\to\infty} f(\widehat{x}_i(L)) = f(x^*), \tag{73}$$

and the convergence rate is $O(\frac{\ln(L+1)}{\sqrt{L}})$.

Proof: **Step** 1**:** According to formula (69) in the proof of Theorem 3, we know that $\lim_{L\to\infty} f(\widehat{y}(L)) - f(x^*) = 0$.

Step 2**:** Base on the consensus of the ergodic average sequence in Theorem 2, we have $\lim\limits_{L\to\infty} \|\widehat{x}_i(L) - \widehat{y}(L)\| = 0$, then

$$\begin{aligned}&\lim_{L\to\infty} |f(\widehat{x}_i(L)) - f(x^*)|\\ &\le \lim_{L\to\infty} |f(\widehat{x}_i(L)) - f(\widehat{y}(L))| + |f(\widehat{y}(L)) - f(x^*)|\\ &= 0. \end{aligned}\tag{74}$$

Hence, the Theorem 4 is proved. ■

Next, the following Algorithm 1 is given to show the distributed event-triggered mechanism implementation.

Algorithm 1

Initialize: x_i^0(*i*th agent initial state, $i \in \mathcal{V}$); $\widetilde{x}_{ij}(0)$(the initial triggered state of *i*th agent to *j*th agent); $a_{ij}(t)$(the element of weighted adjacent matrix); $T = 1$(discrete sampling period); $l = 0$(initial time); L(total number of iterations).
for $l = 1 : T : L$ **do**
  while in the time-varying switching topology graph A_j **do**
    if $\|x_i(l) - \widetilde{x}_{ij}(l)\| \ge E(l)$ **then**
      $\widetilde{x}_{ij}(l) = x_i(l)$ ⇐Update triggered state;
    end if
    $z_i(l) = x_i(l) + h\sum_{j=1}^{N} a_{ij}(l)(\widetilde{x}_{ji}(l) - \widetilde{x}_{ij}(l)) - \alpha(l)g_i(l)$
    if $z_i(l) \in \mathcal{X}_i$ **then**
      $x_i(l+1) = z_i(l)$
    else
      $x_i(l+1) = P_{\mathcal{X}_i}[z_i(l)]$
    end if
    $j = j + 1$ $(1 \le j \le 3)$
  end while
end for

Remark 4: According to Algorithm 1, we can notice that the state update by distributed event-triggered subgradient projection algorithm only use the triggered state rather than the agent state of each moment, which can effectively reduce the transmission of information. Hence this algorithm can filter out the information that has little impact on the update. That is to say, when the error between the last triggered state and the current state is smaller than given threshold function, which standing for that the distortion of state is within the tolerance, then this current state do not be transfer under the time-varying switching digraph topology.

IV. SIMULATION RESULT

In this section, the simulation examples are provided to compare the difference convergence result between the general step-size and the square summable step-size under the event-triggered subgradient projection algorithm.

Example 1: Under the time-varying switching digraph topology showed in Fig.1, consider the optimization problem

TABLE 1. The triggered times of five agents under different step-size with a sample period T = 120s.

Step-size	$E(l) = \frac{1}{l+1}$	$E(l) = \frac{6}{l+1}$	$E(l) = \frac{15}{l+1}$
$\alpha(l) = \frac{1}{\sqrt{l+1}}$	[40,28,48,43,37]	[37,23,40,39,33]	[40,24,43,38,32]
$\alpha(l) = \frac{1}{l+1}$	[47,38,67,58,52]	[39,26,45,40,41]	[39,32,47,45,41]

Remark: The vector $[t_{x_1}, t_{x_2}, t_{x_3}, t_{x_4}, t_{x_5}]$ shows the triggered times of every agent, where t_{x_i} means triggered times of ith agent.

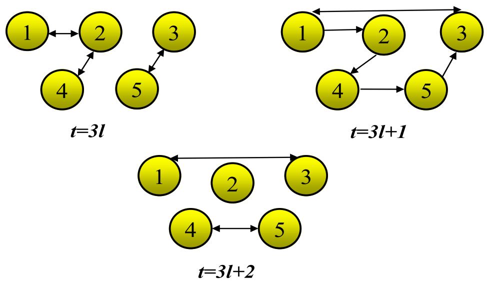
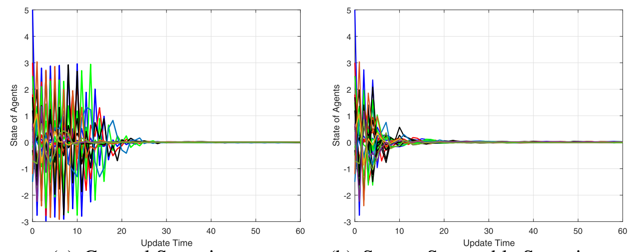


FIGURE 1. Time-varying switching digraph topology.



(a) General Step-size (b) Square Summable Step-size

FIGURE 2. State convergence condition of agent x_i.

$\min_{x\in\mathcal{X}} \sum_{i=1}^{5} f_i(x)$, where $f_i(x) = 0.5log_{10}(1 + x^2) + x^2$, $i \in \mathcal{V} = \{1, 2, 3, 4, 5\}$. The constraint set is $\mathcal{X} = \{x|x \in \mathbb{R}^5, \|x\| \leq 3\}$. Choose the weight adjacency matrix of time-varying switching digraph topology as

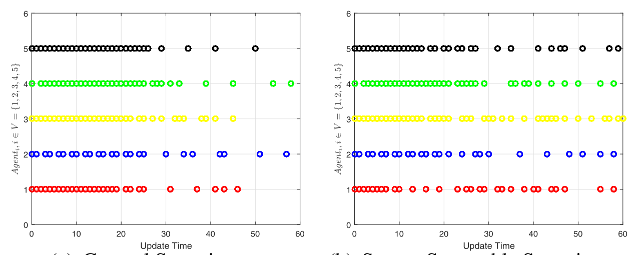
$$A(t) = \begin{cases} A_1, & \text{if } t = 3l, \\ A_2, & \text{if } t = 3l + 1, \\ A_3, & \text{if } t = 3l + 2. \end{cases} \tag{75}$$

where $A_1 = \begin{bmatrix} 0 & 1.2 & 0 & 0 & 0 \\ 1.2 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5 \\ 0 & 0 & 1.5 & 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0 \end{bmatrix}$. Set control gain $h = 0.5$, which satisfies $\overline{a}_{ii}(l) = 1 - hu_{ii}(l) > 0$. This condition ensures that the elements of the new doubly stochastic weight adjacency matrix $\overline{A}$ are nonnegative. The initial values of agents state are $x_1^0 = [5, 0.6, 0.2, 0.5, 1.8]^T$, $x_2^0 = [3, 0.6, 0.2, 0.5, 1.8]^T$, $x_3^0 = [2.5, 1.5, -0.2, -0.5, 0.4]^T$, $x_4^0 = [1.2, 0.5, 1.7, -0.3, 0.5]^T$, $x_5^0 = [-1.5, -1.2, 0.5, 0.7, -0.7]^T$.

Table 1 shows that the general step-size has lesser information transmission compared with the square summable step-size under the event-triggered mechanism when sampling is conducted every second and the overall sampling period is 120 seconds. It can effectively reduce the update times of the actuator, save communication resources and reduce the load of communication flow.

The rest of the simulation graphs show the effect of different step-size for convergence. When the update time is 60 seconds, the convergence situation under the different step-size is revealed in Fig.2. From the simulation graph in Fig.2(a), we can know that the general step-size takes a larger time in the progress of the state update. And when the threshold function is small ($c = 1$), which mean that the error between the state at the triggered time and the current state is small.

Therefore, choosing the larger iterative step makes the algorithm much fluctuation in the process of update. And when a state near to the optimal value, the general step-size may across the optimal value lead to an extension of the convergence time.



(a) General Step-size (b) Square Summable Step-size

FIGURE 3. Trigger time of each agent in different step-sizes.

The trigger time of each agent in different step-size is revealed in Fig.3. Since the algorithm is distributed and parallel, that is the updates of each agent do not interfere with each other and are carried out asynchronously. In the meantime, we know that the information transfer times are lesser under the general step-size from Table 1. The reason why the time-triggered mechanism consumes a great number of resources is that the agent needs to communicate with its neighbors at every moment. That is to say, the triggered times is 120 times in time-triggered mechanism for every agent.

From the Fig.4-Fig.6, it yields that when the threshold function is small($c = 1$), the square summable step-size can be selected to acquire better convergence results (Fig.4).

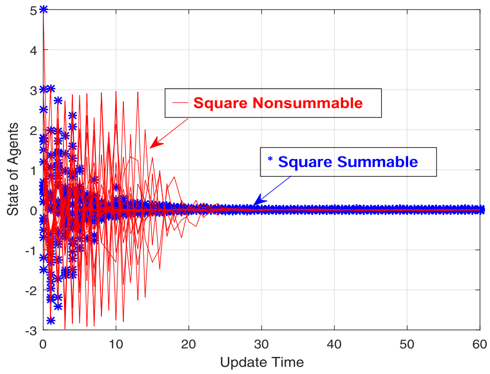


FIGURE 4. Convergence comparison for $E(l) = \frac{1}{l+1}$.

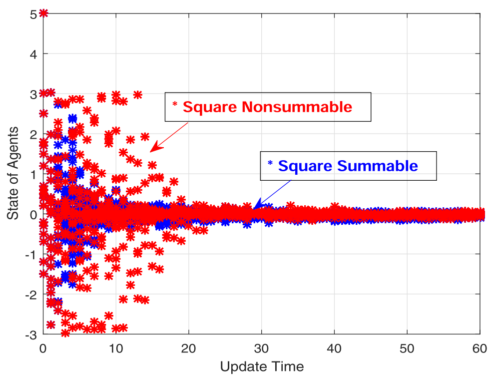


FIGURE 5. Convergence comparison for $E(l) = \frac{6}{l+1}$.

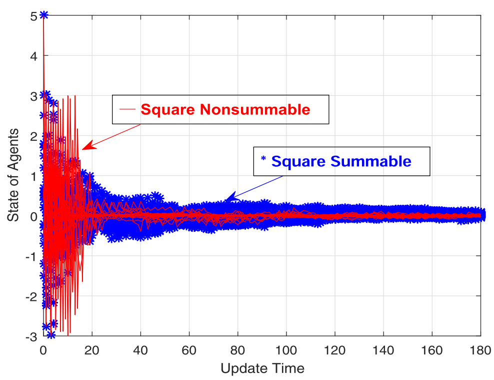


FIGURE 6. Convergence comparison for $E(l) = \frac{15}{l+1}$.

And as threshold function increasing ($c=6$), the convergence case of two types of step-size is the same after 20 seconds (Fig.5). Besides if the threshold function is larger($c = 15$), the general step-size can be selected to acquire a more effective result which demonstrated in Fig.6.

Example 2: Under the time-varying switching digraph topology showed in Fig.7, consider the optimization problem $\min_{x\in\mathcal{X}} \sum_{i=1}^{5} f_i(x)$ with different objective functions, where $f_1(x) = 0.5log_{10}(1 + x^2) + x^2$, $f_2(x) = 2f_1(x)$,$f_3(x) = x^2$,$f_4(x) = x$,$f_5(x) = 3x - 2$. Set the same initial value and constraint set like the Example 1, and select the time-varying

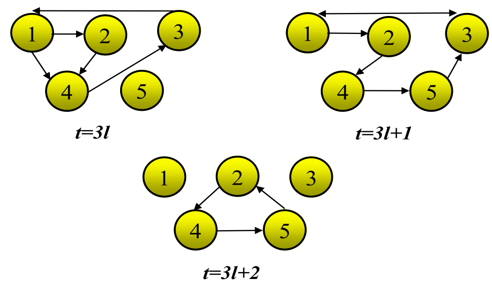


FIGURE 7. Time-varying switching digraph topology.

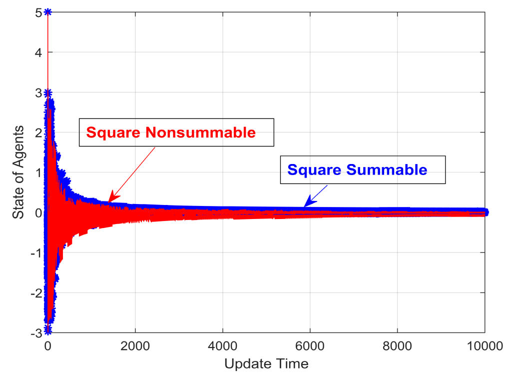


FIGURE 8. Convergence comparison for $E(l) = \frac{300}{l+1}$.

commutative directed graph topology as (75), where $A_1 = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0.8 & 0 & 0 & 0 \end{bmatrix}$, and A_2 is the same as in Example 1. The general step-size can converge to the optimal solution for different objective functions under the time-varying switching digraph showed in Fig.8.

V. CONCLUSION

The discrete-time event-triggered subgradient projection algorithm is considered for constrained convex optimization with general step-size in this paper. It shows that the states of all agents can asymptotically converge to the optimal solution by the proposed algorithm under the time-varying switching digraph network. In the meantime, the convergence rate of the ergodic average sequence is given. Besides, it yields that when the threshold function is large, the general step-size can acquire better convergence results. The subsequent work intents to improve the event-trigger mechanism in this paper into the dynamic event-trigger mechanism.

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