Received December 10, 2019, accepted December 26, 2019, date of publication January 6, 2020, date of current version January 14, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2964039

On Pseudo Almost Automorphic Solutions to Quaternion-Valued Cellular Neural Networks With Delays

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This work was supported in part by the National Natural Science Foundation of China under Grant 61673008, in part by the Project of High-Level Innovative Talents of Guizhou Province under Grant [2016]5651, in part by the Major Research Project of the Innovation Group of the Education Department of Guizhou Province under Grant [2017]039, in part by the Innovative Exploration Project of Guizhou University of Finance and Economics under Grant [2017]5736-015, in part by the Project of Key Laboratory of Guizhou Province with Financial and Physical Features under Grant [2017]004, in part by the Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science and Technology, under Grant 2018MMAEZD21, in part by the University Science and Technology Top Talents Project of Guizhou Province under Grant KY[2018]047, in part by the Guizhou University of Finance and Economics under Grant 2018XZD01, and in part by the Foundation of Science and Technology of Guizhou Province under Grant [2019]1051.

ABSTRACT This manuscript considers quaternion-valued cellular neural networks with delays. Applying pseudo almost automorphic solution theory of delayed differential equations and pertinent inequalities, a new sufficient criterion to guarantee the existence and global exponentially stability of pseudo almost automorphic solutions of quaternion-valued cellular neural networks with delays are presented. Different from Xiang and Li, no assumption for kernel functions is necessary. Simulation results is displayed to verify the validity of theoretical analyses. Up to now, few scholars have dealt with this aspect. The obtained results of this manuscript are completely new and complement some previous studies to a certain extent.

INDEX TERMS Quaternion-valued cellular neural networks, pseudo almost automorphic solution, global exponentially stability, time delay.

I. INTRODUCTION

As is known to us, the dynamical property of cellular neural networks has attracted great attention of scholars from all over the world since cellular neural networks can be successfully applied to various areas such as image processing, associative memories, artificial intelligence, disease diagnosis and so on [1]–[16]. In addition, note that time delay usually occur in cellular neural networks and it is an important factor which destroys the stability the networks, thus it is a natural issue for us to investigate the delayed cellular neural networks. Recently, a great deal of work on delayed cellular neural networks has already been done and plenty of research results have been shown [17]–[20]. In particular, remarkable research achievements on periodic solution, almost periodic solution, anti-periodic solution, pseudo almost periodic solution, weighted pseudo almost periodic solution and pseudo almost automorphic solution have been reported. For instance, Wang *et al.* [21] investigated the periodic solution of cellular neural networks; Li and Wang [22] studied the almost periodic solutions to a class of cellular neural networks; Xu and Li [23] discussed the anti-periodic solutions for cellular neural networks; Aouiti *et al.* [24] analyzed the stability of piecewise pseudo almost periodic solution for delayed inertial neural networks; Liu [25] focused on the pseudo almost periodic solutions for cellular neural networks with leakage delays; Abbas *et al.* [26] made a detailed analysis on the almost automorphic solutions for impulsive neural networks with impulsive effect. For more concrete works, one can see [27]–[33].

Generally speaking, almost automorphicity can be regarded as the extension of almost periodicity and pseudo automorphicity can be regarded as the generalization of almost automorphicity [34]. In the objective world, the pseudo automorphicity is more common than periodicity, almost periodicity and almost automorphicity. Therefore the study on pseudo almost automorphic solution has attracted

The associate editor coordinating the review of this manuscript and approving it for publication was Hamid Mohammad-Sedighi.

much attention from many researchers. In recent years, some excellent works on pseudo almost automorphic solutions on neural networks have been displayed. For example, Xiang and Li [34] considered the pesudo almost automorphic solutions of delayed quaternion-valued neural networks; M′hamdiin [35] discussed the pseudo almost automorphic solutions to delayed BAM neural networks; Cieutat and Ezzinbi [36] investigated the pseudo almost automorphic solutions for some dissipative differential equations in Banach spaces; Zhao *et al.* [37] handled the weighted pseudo-almost automorphic solutions for high-order Hopfield neural networks; Aouiti and Dridi [38] made a detailed analysis on piecewise asymptotically almost automorphic solutions of high-order Hopfield neural networks. Zhu *et al.* [39] studied the existence and exponential stability of pseudo almost automorphic solutions to delayed Cohen-Grossberg neural networks.

Complex-valued neural networks are the extension version of real-valued neural networks and quaternion-valued neural networks are the extension form of real-valued neural networks and complex-valued neural networks [40], [41].

The skew of quaternion takes the form: $\mathcal{G} := \{g = g_0 + ig_1 + jg_2 + kg_3\}$, where $g_0, g_1, g_2, g_3 \in R$ and i, j, k satisfy $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = ijk = -1$. $\forall\ g \in \mathcal{G}$, Let $g^* = g_0 - ig_1 - jg_2 - kg_3$ be the conjugate of g. The norm of b is given as follows:

$$||g|| = \sqrt{gg^*} = \sqrt{(g_0)^2 + (g_1^*)^2 + (g_2^*)^2 + (g_3^*)^2}.$$

During the past decades, quaternion-valued neural networks have been found potential application prospect in many areas such as color night vision, spatial rotation, image impression of three dimension geometrical affine transformation [40], [41]. Currently, a great deal of excellent works on dynamics of quaternion-valued neural networks have been available. For example, Abedi Pahnehkolaei *et al.* [42] considered the stability of fractional quaternion-valued delayed neural networks; Li *et al.* [43] focused on the global exponential pseudo almost periodic synchronization for quaternion-valued cellular neural networks; Li *et al.* [44] discussed the quasi-state estimation and quasi-synchronization control for fractional-order quaternion-valued neural networks; Li and Cao [45] analyzed the global dissipativity of quaternion-valued memristive neural networks.

It is a pity that very few authors have investigated the pseudo almost automorphic solutions of quaternion-valued cellular neural networks with delays. In order to make up for the deficiency, we think that it is of importance to discuss pseudo almost automorphic solutions of quaternion-valued cellular neural networks with delays. Stimulated by the above viewpoint, in this manuscript, we consider the dynamics of the following quaternion-valued cellular neural networks with delays:

$$\begin{aligned}\dot{z}_l(t) = -a_l(t)z_l(t) + \sum_{p=1}^{m} b_{lp}(t)h_p(z_p(t))\\ + \sum_{p=1}^{m} c_{lp}(t)h_p(z_p(t-\tau_{lp}(t))) + d_l(t),\end{aligned} \tag{1}$$

where $l = 1, 2, \cdots, m$, $a_l(t) > 0$ denotes the rate at which the lth unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs [46], $z_l(t) \in \mathcal{G}$ denotes the state of the lth neuron, $b_{lp}(t), c_{lp}(t) \in \mathcal{G}$ denote the strength of the pth unit on the lth unit at time t and $t - \tau_{lp}(t)$, respectively [46], $d_l(t) \in \mathcal{B}$ denote the external input on the lth unit at time t, $\tau_{lp}(t) > 0$ denotes the transmission delay, $h_p(z_p(t))$ denotes the output of the pth unit at time t. In model (1), it does involve the kernel functions. Thus no assumption for kernel functions is necessary.

The main task is to establish a new sufficient criteria to guarantee the existence and global exponentially stability of pseudo almost automorphic solutions for system (1).

Let $\mathcal{BC}(R, \mathcal{G}^m)$ denote the set all bounded continuous functions from R to $\mathcal{G}^m$ and $\bar{k} = \sup_{t\in R} k(t), \underline{k} = \inf_{t\in R} k(t)$, where $k(t)$ denotes a bounded continuous function. We given the initial condition of model (1) as follows:

$$z_l(s) = \psi_l(s), s \in [-\tau, 0], \quad l = 1, 2, \cdots, m, \tag{2}$$

where $\psi \in \mathcal{BC}([-\tau, 0], \mathcal{B})$, $\tau = \max_{l,p=1,2,\cdots,m} \bar{\tau}_{lp}$.

We plan the manuscript as follows. In Sect. 2, several basic definitions and lemmas are presented. In Sect. 3, the existence of pseudo almost automorphic solutions for system (1) is discussed and the global exponentially stability of pseudo almost automorphic solutions of system (1) is studied. In Sect. 4, computer simulations are displayed. The conclusion is given in Sect. 5.

II. PRELIMINARIES

In this section, several necessary definitions and lemmas are given.

Definition 1 [34]: A function $g \in \mathcal{BC}(R, \mathcal{G}^m)$ is said to be almost automorphic if for every sequence of real numbers $(r_m^*)_{m\in N}$, $\exists$ a subsequence $(r_m)_{m\in N}$ such that $h(t) := \lim_{m\to\infty} g(t + r_m)$ and $\lim_{m\to\infty} h(t - r_m) = g(t)$ for any $t \in R$.

Let $\mathcal{AA}(R, \mathcal{G}^m)$ be the set of all almost automorphic functions from R to $\mathcal{G}^m$.

Lemma 1 [47]: If $h_1, h_2 \in \mathcal{AA}(R, \mathcal{G})$ and $\xi \in R$, then $h_1 + h_2, h_1h_2, \xi h_1 \in \mathcal{AA}(R, \mathcal{G})$.

Lemma 2 [47]: If $z \in \mathcal{AA}(R, \mathcal{G})$ and $\tau \in R$, then $z(\cdot - \tau) \in \mathcal{AA}(R, \mathcal{G})$.

Lemma 3 [47]: If $h \in C(R, \mathcal{G})$ satisfies the Lipschitz condition and $zz \in \mathcal{AA}(R, \mathcal{G})$, then $h(z(\cdot)) \in \mathcal{AA}(R, \mathcal{G})$.

Set $\mathcal{AA}_0(R,\mathcal{G}) =$

$$\left\{h \in \mathcal{BC}(R,\mathcal{G}) | \lim_{\mathcal{T}\to+\infty} \frac{1}{2\mathcal{T}} \int_{\mathcal{T}}^{\mathcal{T}} ||h(t)||_{\mathcal{G}} dt = 0\right\}.$$

Definition 2: Let $h \in \mathcal{BC}(R,\mathcal{G})$. If $h = h_1 + h_0$, where $h_1 \in \mathcal{AA}(R,\mathcal{G})$ and $h_0 \in \mathcal{AA}_0(R,\mathcal{G})$, then we say that h is pseudo almost automorphic. Let $\mathcal{PAA}(R,\mathcal{G})$ denote the collection of all functions h.

Lemma 4 [47]: Assume that $\psi \in \mathcal{PAA}(R,\mathcal{G})$, then one has $\psi(\cdot - \tau) \in \mathcal{PAA}(R,\mathcal{G})$.

Lemma 5 [47]: Assume that $\psi, \omega \in \mathcal{PAA}(R,\mathcal{G})$, then one has $\psi\omega \in \mathcal{PAA}(R,\mathcal{G})$.

Lemma 6 [47]: Assume that $h \in \mathcal{C}(R,\mathcal{G})$, $\psi \in \mathcal{PAA}(R,\mathcal{G})$ and $\exists$ a constant M which satisfies $||h(u) - h(v)||_{\mathcal{G}} \le M||u - v||_{\mathcal{G}}$ for all $u, v \in \mathcal{G}$), then one has $h(\psi(\cdot)) \in \mathcal{PAA}(R,\mathcal{G})$.

Now we give several notations:

$$b_{lp} = \sup_{t\in R} ||b_{lp}(t)||_{\mathcal{G}}, c_{lp} = \sup_{t\in R} ||c_{lp}(t)||_{\mathcal{G}},$$
$$||z||_\tau = \sum_{p=1}^{m} \sup_{t\in(-\tau,0]} ||z_l(t)||_{\mathcal{G}}.$$

Definition 3: Assume that $z = (z_1, z_2, \cdots, z_m)^T$ is a pseudo almost automorphic solution of model (1.1) with the initial value $\psi = (\psi_1, \psi_2, \cdots, \psi_m)^T \in C((-\tau, 0], \mathcal{G}^m)$ and $u = (u_1, u_2, \cdots, u_m)^T$ is an arbitrary solution of model (1) with the initial value $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m)^T \in C((-\tau, 0], \mathcal{G}^m)$, respectively. If $\exists$ two constants $\alpha > 0$ and $\beta > 0$ which satisfy $|z(t) - u(t)|_{\mathcal{G}^m} \le \beta||\psi - \varphi||_\tau e^{-\alpha t}, t \ge 0$, then one can say that the pseudo almost automorphic solution $z(t)$ of model (1) is globally exponentially stable.

Throughout this manuscript, the following hypotheses are given:

($\mathcal{Z}1$) For $l, p = 1, 2, \cdots, m$, $a_l \in \mathcal{AP}(R, R^+)$, $b_{lp}, c_{lp} \in \mathcal{PAA}(R,\mathcal{G})$, and $a_l^- = \inf_{t\in R} a_l(t)||_{\mathcal{G}} > 0$.

($\mathcal{Z}2$) For $l = 1, 2, \cdots, m$, $h_l \in (\mathcal{G},\mathcal{G})$ and $\exists$ a positive constant M such that

$$||h_l(u) - h_l(v)||_{\mathcal{G}} \le M||u - v||_{\mathcal{G}}, \quad \forall\, u, v \in \mathcal{G}$$

and $h_l(0) = 0$.

($\mathcal{Z}3$) $\kappa = \max_{1\le l\le m}\left\{\frac{1}{a_l^-}\sum_{p=1}^m [b_{lp}^+ + c_{lp}^+]M\right\} < 1$.

III. MAIN RESULTS

Let $\Xi = \mathcal{PAA}(R,\mathcal{G}^m)$. Then $(\Xi, ||\cdot||_0)$ is a Banach space, where $||u||_0 = \sup_{t\in R} ||u(t)||_{\mathcal{G}^m}$ for $u \in \Xi$. Set

$$\psi^0(t) = \left(\int_{-\infty}^{t} e^{\int_s^t -a_1(v)dv} d_1(s)ds, \int_{-\infty}^{t} e^{\int_s^t -a_2(v)dv} d_2(s)ds, \cdots, \int_{-\infty}^{t} e^{\int_s^t -a_m(v)dv} d_m(s)ds\right)^T$$

and choose a constant $\tilde{\psi} > ||\psi^0||_0$.

Lemma 7: Suppose that ($\mathcal{Z}1$) and ($\mathcal{Z}2$) are fulfilled. For $l = 1, 2, \cdots, m$ and each $\psi = (\psi_1, \psi_2, \cdots, \psi_m)^T \in \mathcal{PAA}(R,\mathcal{G}^m)$, then the following function

$$(\Theta_l\psi)(t) = \int_{-\infty}^{t} e^{\int_s^t -a_l(v)dv}\left[\sum_{p=1}^m b_{lp}(s)h_p(\psi_p(s)) + \sum_{p=1}^m c_{lp}(s)h_p(\psi_p(s - \tau_{lp}(s))) + d_l(s)\right] ds$$

is pseudo almost automorphic.

Proof: In view of ($\mathcal{Z}1$), ($\mathcal{Z}2$) and Lemma 2-Lemma 6, one knows that for each $\psi = (\psi_1, \psi_2, \cdots, \psi_m)^T \in \mathcal{PAA}(R,\mathcal{G}^m)$ and $l = 1, 2, \cdots, m$, the following function

$$\Phi_l(t) = \sum_{p=1}^m b_{lp}(t)h_p(z_p(t)) + \sum_{p=1}^m c_{lp}(t)h_p(z_p(t - \tau_{lp}(t))) + d_l(t) \tag{3}$$

is pseudo almost automorphic. Then

$$\Phi_l(t) = \Phi_l^*(t) + \Phi_l^0(t), \tag{4}$$

where $\Phi_l^*(t) \in \mathcal{AA}(R,\mathcal{G})$ and $\Phi_l^0(t) \in \mathcal{AA}_0(R,\mathcal{G})$. Thus

$$\begin{aligned}(\Theta_l\psi)(t) &= \int_{-\infty}^{t} e^{\int_s^t -a_l(v)dv}\Phi_l^*(s)ds + \int_{-\infty}^{t} e^{\int_s^t -a_l(v)dv}\Phi_l^0(s)ds \\ &= (\Theta_l^*\psi)(t) + (\Theta_l^0\psi)(t).\end{aligned} \tag{5}$$

(I) We check that $(\Theta_l^*\psi) \in \mathcal{AA}(R,\mathcal{G})$. Assume that $(t_m^*)_{m\in N}$ is a sequence of real numbers, then there exists a subsequence $(t_m)_{m\in N}$ such that for all $t \in R$ and $l = 1, 2, \cdots, m$,

$$\lim_{m\to\infty} a_l(t + r_m) = \tilde{a}_l(t), \lim_{m\to\infty} \tilde{a}_l(t - r_m) = a_l(t) \tag{6}$$

and

$$\lim_{m\to\infty} \Phi_l^*(t + r_m) = \tilde{\Phi}_l^*(t), \lim_{m\to\infty} \tilde{\Psi}_l^*(t - r_m) = \Phi_l^*(t). \tag{7}$$

Let

$$(\tilde{\Theta}_l^*\psi)(t) = \int_{-\infty}^{t} e^{-\int_s^t \tilde{a}_l(v)dv}\tilde{\Phi}_l^*(s)ds. \tag{8}$$

Then

$$\begin{aligned}&\lim_{m\to+\infty} ||(\Theta_l^*\psi)(t + r_m) - (\tilde{\Theta}_l^*\psi)(t)||_{\mathcal{G}} \\ &= \lim_{m\to+\infty}\left\|\int_{-\infty}^{t+r_m} e^{-\int_s^{t+r_m} a_l(v)dv}\Phi_l^*(s)ds - \int_{-\infty}^{t} e^{-\int_s^t \tilde{a}_l(v)dv}\tilde{\Phi}_l^*(s)ds\right\|_{\mathcal{G}} \\ &= \lim_{m\to+\infty}\left\|\int_{-\infty}^{t} e^{-\int_v^t a_l(u+r_m)du}\Phi_l^*(v + r_m)dv - \int_{-\infty}^{t} e^{-\int_s^t \tilde{a}_l(u)du}\tilde{\Phi}_l^*(s)ds\right\|_{\mathcal{G}} \\ &\le \lim_{m\to+\infty}\int_{-\infty}^{t} e^{-\int_v^t a_l(u+r_m)du}\end{aligned}$$

$$\times \left\|\Phi_l^*(s+r_m)-\tilde{\Phi}_l^*(s)\right\|_{\mathcal{G}} ds + \lim_{m\to+\infty}\left\|\int_{-\infty}^{t}\left(e^{-\int_v^t a_l(u+r_m)du}-e^{-\int_v^t \tilde{a}_l(u)du}\right) \times \tilde{\Phi}_l^*(s)ds\right\|_{\mathcal{G}}. \quad (9)$$

Applying the Lebesgue dominated convergence theorem, one has $\lim_{m\to+\infty}\Theta_l^*\psi)(t+r_m) = (\tilde{\Theta}_l^*\psi)(t)$, where $t\in R$ and $l=1,2,\cdots,m$. In a same way, one can get $\lim_{m\to+\infty}\tilde{\Theta}_l^*\psi)(t-r_m) = (\Theta_l^*\psi)(t)$, where $t\in R$ and $l=1,2,\cdots,m$. Thus $(\Theta_l^*\psi)\in\mathcal{AA}(R,\mathcal{G})$.

(II) We check that $(\Theta_l^0\psi)\in\mathcal{AA}_0(R,\mathcal{G})$. For $l=1,2,\cdots,m$, one has

$$\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\left\|(\Theta_l^0\psi)(s)ds\right\|_{\mathcal{G}}dt \le \Pi_1+\Pi_2, \quad (10)$$

where

$$\Pi_1=\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\mathcal{T}}^{t}\left\|e^{\int_s^t -a_l(v)dv}\Phi_l^0(s)\right\|_{\mathcal{G}}dsdt,$$

$$\Pi_2=\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\infty}^{-\mathcal{T}}\left\|e^{\int_s^t -a_l(v)dv}\Phi_l^0(s)\right\|_{\mathcal{G}}dsdt.$$

Letting $\xi=t-s$ and applying Fubini's theorem, we have

$$\begin{aligned}
&\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\mathcal{T}}^{t}\left\|e^{\int_s^t -a_l(v)dv}\Phi_l^0(s)\right\|_{\mathcal{G}}dsdt\\
&\le \lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}e^{-(t-s)a_l^-}\left\|\Phi_l^0(s)\right\|_{\mathcal{G}}dsdt\\
&= \lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\left(\int_{-\mathcal{T}}^{\mathcal{T}}\int_{0}^{t+\mathcal{T}}e^{-\xi a_l^-}\left\|\Phi_l^0(t-\xi)\right\|_{\mathcal{G}}ds\right)dt\\
&\le \lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\left(\int_{-\mathcal{T}}^{\mathcal{T}}\int_{0}^{+\infty}e^{-\xi a_l^-}\left\|\Phi_l^0(t-\xi)\right\|_{\mathcal{G}}ds\right)dt\\
&= \int_0^{+\infty}e^{-\xi a_l^-}\left(\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\left\|\Phi_l^0(t-\xi)\right\|_{\mathcal{G}}dt\right)d\xi\\
&\le \int_0^{+\infty}e^{-\xi a_l^-}\left(\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}-\xi}^{\mathcal{T}-\xi}\left\|\Phi_l^0(v)\right\|_{\mathcal{G}}dv\right)d\xi. \quad (11)
\end{aligned}$$

Noticing that $\Phi_l^0\in\in\mathcal{AA}_0(R,\mathcal{G})$, one has

$$\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}-\xi}^{\mathcal{T}-\xi}\left\|\Phi_l^0(v)\right\|_{\mathcal{G}}dv=0. \quad (12)$$

In view of the Lebesgue dominated convergence theorem, we get

$$\Pi_1=\lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\mathcal{T}}^{t}\left\|e^{\int_s^t -a_l(v)dv}\Phi_l^0(s)\right\|_{\mathcal{G}}dsdt=0. \quad (13)$$

In addition, notice that Φ_l^0 is bounded, then

$$\begin{aligned}
\Pi_2 &\le \lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{-\infty}^{-\mathcal{T}}e^{-(t-s)a_l^-}\left\|\Phi_l^0(v)\right\|_{\mathcal{G}}dsdt\\
&= \lim_{\mathcal{T}\to+\infty}\frac{1}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{t+\mathcal{T}}^{+\infty}e^{-\xi a_l^-}\left\|\Phi_l^0(t-\xi)\right\|_{\mathcal{G}}d\xi dt\\
&\le \lim_{\mathcal{T}\to+\infty}\frac{\sup_{t\in R}||\Phi_l^0(t)||_{\mathcal{G}}}{2\mathcal{T}}\int_{-\mathcal{T}}^{\mathcal{T}}\int_{\mathcal{T}+t}^{+\infty}e^{-\xi a_l^-}d\xi dt\\
&= \lim_{\mathcal{T}\to+\infty}\frac{\sup_{t\in R}||\Phi_l^0(t)||_{\mathcal{G}}}{2\mathcal{T}}\frac{1}{\xi a_l^-}\int_{-\mathcal{T}}^{\mathcal{T}}e^{-(t+\mathcal{T})\xi a_l^-}dt\\
&= \lim_{\mathcal{T}\to+\infty}\frac{\sup_{t\in R}||\Phi_l^0(t)||_{\mathcal{G}}}{2\mathcal{T}}\frac{1}{\xi a_l^-}\left(1-e^{-2\xi a_l^-\mathcal{T}}\right)=0.
\end{aligned} \quad (14)$$

Thus $(\Theta_l^0\psi)\in\mathcal{AA}_0(R,\mathcal{G})$ for $l=1,2,\cdots,m$. Based on (I) and (II), one can conclude that $(\Theta_l\psi)\in\mathcal{PAA}(R,\mathcal{G})$. This ends the proof of Lemma 7.

Theorem 1: Suppose that ($\mathcal{Z}1$)-($\mathcal{Z}3$) are true. Then model (1) possesses a pseudo almost automorphic solution which lies in $\Xi_0=\{\psi|\psi\in\mathcal{G},||\psi-\psi^0||_0\le\frac{\kappa\tilde{\psi}}{1-\kappa}\}$.

Proof: Assume that $z=(z_1,z_2\cdots,z_m)^T\in\mathcal{C}(R,\mathcal{G}^m)$ such that

$$z_l(t)=\int_{-\infty}^{t}e^{\int_s^t -a_l(v)dv}\left[\sum_{p=1}^{m}b_{lp}(s)h_p(z_p(s)) + \sum_{p=1}^{m}c_{lp}(s)h_p(z_p(s-\tau_{lp}(s)))+d_l(s)\right]ds, \quad (15)$$

where $l=1,2,\cdots,m$. Then

$$\begin{aligned}
\dot{z}_l(t) &= \int_{-\infty}^{t}-a_l(t)e^{\int_s^t -a_l(v)dv}\left[\sum_{p=1}^{m}b_{lp}(s)h_p(z_p(s)) + \sum_{p=1}^{m}c_{lp}(s)h_p(z_p(s-\tau_{lp}(s)))+d_l(s)\right]ds\\
&\quad + e^{\int_t^t -a_l(v)dv}\left[\sum_{p=1}^{m}b_{lp}(s)h_p(z_p(s)) + \sum_{p=1}^{m}c_{lp}(s)h_p(z_p(s-\tau_{lp}(s)))+d_l(s)\right]\\
&= -a_l(t)z_l(t)+\sum_{p=1}^{m}b_{lp}(t)h_p(z_p(t)) + \sum_{p=1}^{m}c_{lp}(t)h_p(z_p(t-\tau_{lp}(t)))+d_l(t),
\end{aligned} \quad (16)$$

which implies that z satisfies (1). Define the following operator: $\Gamma:\Xi_0\to\mathcal{AA}(R,\mathcal{G}^m)$ by $\Gamma=(\Gamma_1,\Gamma_2,\cdots,\Gamma_m)^T$, where

$$(\Gamma_l\psi)(t)=\int_{-\infty}^{t}e^{\int_s^t -a_l(v)dv}\left[\sum_{p=1}^{m}b_{lp}(s)h_p(z_p(s)) + \sum_{p=1}^{m}c_{lp}(s)h_p(z_p(s-\tau_{lp}(s)))+d_l(s)\right]ds, \quad (17)$$

for every $\psi \in \mathcal{AA}(R, \mathcal{G}^m)$ and $l = 1, 2, \cdots, m$. It is not difficult to obtain that

$$||\psi||_0 \le ||\psi - \psi_0||_0 + ||\psi_0||_0 \le \frac{\kappa\tilde{\psi}}{1-\kappa} + \tilde{\psi} = \frac{\tilde{\psi}}{1-\kappa}. \quad (18)$$

Firstly, we check that $\forall\ \psi \in \Xi_0, \Gamma\psi \in \Xi_0$. Notice that

$$\begin{aligned}
&\|\Gamma\psi(t) - \psi^0(t)\|_{\mathcal{G}^m}\\
&\le \max_{1\le l\le m}\left\{\int_{-\infty}^t \left\|e^{\int_s^t -a_l(v)dv}\sum_{p=1}^m b_{lp}(s)h_p(\psi_p(s))\right\|_{\mathcal{G}} ds\right.\\
&\quad\left. + \int_{-\infty}^t \left\|e^{\int_s^t -a_l(v)dv}\sum_{p=1}^m c_{lp}(s)h_p(\psi_p(s-\tau_{lp}(s)))\right\|_{\mathcal{G}} ds\right\}\\
&\le \max_{1\le l\le m}\left\{\sum_{p=1}^m\left[\int_{-\infty}^t e^{-a_l^-(t-s)}b_{lp}^+M||\psi||_0 ds\right.\right.\\
&\quad\left.\left. + \int_{-\infty}^t e^{-a_l^-(t-s)}c_{lp}^+M||\psi||_0 ds\right]\right\}\\
&\le \max_{1\le l\le m}\frac{1}{a_l^-}\left\{\sum_{p=1}^m\left[(b_{lp}^+ + c_{lp}^+)M||\psi||_0\right]\right\}\\
&\le \frac{\tilde{\psi}}{1-\kappa}\max_{1\le l\le m}\left\{\frac{1}{a_l^-}\sum_{p=1}^m\left[(b_{lp}^+ + c_{lp}^+)M\right]\right\}\\
&\le \frac{\kappa\tilde{\psi}}{1-\kappa}. \quad (19)
\end{aligned}$$

Thus $\Gamma\psi \in \Xi_0$.

Secondly, we check that Γ is a contraction mapping of Ξ_0. $\forall\ \theta, \vartheta \in \Xi_0$, one has

$$\begin{aligned}
&\|\Gamma\psi(t) - \psi^0(t)\|_{\mathcal{G}^m}\\
&\le \max_{1\le l\le m}\left\{\int_{-\infty}^t \left\|e^{\int_s^t -a_l(v)dv}\right.\right.\\
&\quad\left.\times\sum_{p=1}^m b_{lp}(s)[h_p(\theta_p(s)) - h_p(\vartheta_p(s))]\right\|_{\mathcal{G}} ds\\
&\quad + \int_{-\infty}^t \left\|e^{\int_s^t -a_l(v)dv}\sum_{p=1}^m c_{lp}(s)[h_p(\theta_p(s-\tau_{lp}(s)))\right.\\
&\quad\left.\left. -h_p(\vartheta_p(s-\tau_{lp}(s)))]\right\|_{\mathcal{G}} ds\right\}\\
&\le \max_{1\le l\le m}\left\{\sum_{p=1}^m\left[\int_{-\infty}^t e^{-a_l^-(t-s)}b_{lp}^+M||\theta-\vartheta||_0 ds\right.\right.\\
&\quad\left.\left. + \int_{-\infty}^t e^{-a_l^-(t-s)}c_{lp}^+M||\theta-\vartheta||_0 ds\right]\right\}\\
&\le \max_{1\le l\le m}\frac{1}{a_l^-}\left\{\sum_{p=1}^m\left[(b_{lp}^+ + c_{lp}^+)M||\theta-\vartheta||_0\right]\right\}\\
&\le \kappa||\theta-\vartheta||_0. \quad (20)
\end{aligned}$$

It follows from (20) that Γ is a contraction mapping of Ξ_0. Then $\exists\ \psi^* \in \Xi_0$ which satisfies $\Gamma\psi^* = \psi^*$. Namely, model (1) possesses a pseudo almost automorphic solution. The proof ends.

Theorem 2: If $(\mathcal{Z}1)$-$(\mathcal{Z}3)$ are satisfied, then model (1) possesses a pseudo almost automorphic solution which is globally exponentially stable.

Proof: In Theorem 1, we have known that model (1) possesses a pseudo almost automorphic solution. Here we assume that $z(t)$ is a pseudo almost automorphic solution (the initial value is $\psi(t)$) and $u(t)$ is an arbitrary solution (the initial value is $\theta(t)$. Let $w_l(t) = u_l(t) - z_l(t)$, $\chi_l(t) = \psi_l(t) - \theta_l(t)$. Then

$$\begin{aligned}
&\dot{w}_l(t) + a_l(t)w(t)\\
&= \sum_{p=1}^m b_{lp}(s)[h_p(w_p(t)+z_p(t)) - h_p(z_p(t))]\\
&\quad + \sum_{p=1}^m c_{lp}(t)[h_p(w_p(t-\tau_{lp}(t)) + z_p(t-\tau_{lp}(t)))\\
&\quad -h_p(z_p(t-\tau_{lp}(t)))], \quad (21)
\end{aligned}$$

where $l = 1, 2, \cdots, m$. Let

$$\Psi_l(\varrho) = a_l^- - \varrho - \sum_{p=1}^m [b_{lp}^+M + c_{lp}^+Me^{\varrho\tau_{lp}}] \quad (22)$$

where $l = 1, 2, \cdots, m$ and $\varrho \ge 0$. It is easy to see that $\lim_{\varrho\to+\infty}\Psi_l(\varrho) = -\infty$. Then $\exists\,\epsilon_l^* > 0$ such that $\Psi_l(\epsilon_l) > 0$, where $\epsilon_l \in (0, \epsilon_l^*)$. Set $\epsilon_0 = \min\{\epsilon_1^*, \epsilon_2^*, \cdots, \epsilon_m^*\}$, then one has $\Psi_l(\epsilon_0) \ge 0, l = 1, 2, \cdots, m$. Now we can choose $0 < \delta < \min\{\epsilon_0, a_1^-, a_2^-, \cdots, a_m^-\}$ such that $\Psi_l(\delta) > 0$. Thus

$$\frac{1}{a_l^- - \delta}\sum_{p=1}^m [b_{lp}^+M + c_{lp}^+Me^{\delta\tau_{lp}}] < 1. \quad (23)$$

By (21), one gets

$$\begin{aligned}
w_l(t) &= \chi_l(0)e^{-\int_0^t a_l(v)dv} + \int_0^t e^{-\int_s^t a_l(v)dv}\\
&\quad\times\left\{\sum_{p=1}^m b_{lp}(t)[h_p(w_p(t)+z_p(t)) - h_p(z_p(t))]\right.\\
&\quad + \sum_{p=1}^m c_{lp}(t)[h_p(w_p(t-\tau_{lp}(t)) + z_p(t-\tau_{lp}(t)))\\
&\quad\left. -h_p(z_p(t-\tau_{lp}(t)))]\right\} ds, \quad (24)
\end{aligned}$$

where $l = 1, 2, \cdots, m$. Set

$$\mathcal{N} = \max_{1\le l\le m}\left[\frac{a_l^-}{\sum_{p=1}^m [b_{lp}^+ + c_{lp}^+]M}\right]. \quad (25)$$

Then one has

$$\frac{1}{\mathcal{N}} - \frac{1}{a_l^- - \delta}\sum_{p=1}^m [b_{lp}^+M + c_{lp}^+Me^{\delta\tau_{lp}}] < 0. \quad (26)$$

Clearly

$$||w(t)||_0 = ||\chi(t)||_\tau \le ||\chi||_\tau \le \mathcal{N}||\chi||_\tau e^{-\delta t}, \quad t \in [-\tau, 0]. \tag{27}$$

Now we prove that

$$||w(t)||_0 \le \mathcal{N}||\chi||_\tau e^{-\delta t}, t > 0. \tag{28}$$

For every $\rho > 1$ one has

$$||w(t)||_0 \le \rho\mathcal{N}||\chi||_\tau e^{-\delta t}, t > 0. \tag{29}$$

Assume that (29) is not true, then $\exists$ some $t^* > 0$ and some $l \in \{1, 2, \cdots, m\}$ such that

$$||w(t^*)||_0 = ||w_l(t^*)||_0 = \rho\mathcal{N}||\chi||_\tau e^{-\delta t}, t > 0. \tag{30}$$

and

$$||w(t)||_0 < \rho\mathcal{N}||\chi||_\tau e^{-\delta t}, t \in (-\tau, t^*). \tag{31}$$

According to (23), (24), (25) and (31), one gets

$$\begin{aligned}
w_l(t^*) &\le ||\chi_l||_\tau e^{-t^* a_l^-} \\
&\quad + \int_0^{t_1} e^{-(t^*-s)a_l^-} \left[\sum_{p=1}^m b_{lp}^+ M ||w_q(s)||_{\mathcal{G}} \right. \\
&\quad \left. + \sum_{p=1}^m c_{lp}^+ M ||w_q(s)||_{\mathcal{G}} \right] ds \\
&\le ||\chi_l||_\tau e^{-t^* a_l^-} + \int_0^{t_1} e^{-(t^*-s)a_l^-} \\
&\quad \times \left[\sum_{p=1}^m b_{lp}^+ M\rho\mathcal{N}||\chi||_\tau e^{-\delta s} \right. \\
&\quad \left. + \sum_{p=1}^m c_{lp}^+ M\rho\mathcal{N}||\chi||_\tau e^{-\delta s} \right] ds \\
&= ||\chi_l||_\tau e^{-t^* a_l^-} + \int_0^{t_1} e^{-(t^*-s)a_l^-} M\rho\mathcal{N}||\chi||_\tau e^{-\delta s} \\
&\quad \times \left[\sum_{p=1}^m (b_{lp}^+ + c_{lp}^+) \right] ds \\
&\le M\rho\mathcal{N}||\chi||_\tau e^{-\delta t^*} \left[\frac{e^{(\delta - a_l^-)t^*}}{\rho\mathcal{N}} + \frac{1}{a_l^- - \delta} \right. \\
&\quad \left. \times \sum_{p=1}^m (b_{lp}^+ + c_{lp}^+ e^{\delta\tau_{lp}}) \left(1 - e^{(\delta - a_l^-)t^*}\right) \right] \\
&\le M\rho\mathcal{N}||\chi||_\tau e^{-\delta t^*} \left[e^{(\delta - a_l^-)t^*} \left(\frac{1}{\mathcal{N}} - \frac{1}{a_l^- - \delta} \right. \right. \\
&\quad \left. \sum_{p=1}^m (b_{lp}^+ + c_{lp}^+ e^{\delta\tau_{lp}}) \right) \\
&\quad \left. + \frac{1}{a_l^- - \delta} (b_{lp}^+ + c_{lp}^+ e^{\delta\tau_{lp}}) \right] \\
&\le M\rho\mathcal{N}||\chi||_\tau e^{-\delta t^*} \left[\frac{1}{a_l^- - \delta} \sum_{p=1}^m (b_{lp}^+ + c_{lp}^+ e^{\delta\tau_{lp}}) \right] \\
&\le M\rho\mathcal{N}||\chi||_\tau e^{-\delta t^*},
\end{aligned} \tag{32}$$

where $l = 1, 2, \cdots, m$. In view of (30), one knows that (29) is true. Letting $\rho \to 1$ one knows that (28) is true. Thus the pseudo almost automorphic solution of model (1) is globally exponentially stable. We end the proof.

Remark 1: The superiority of the quaternion-valued cellular neural networks is that they can provide more convenience than the real-valued cellular neural networks and complex-valued cellular neural networks in handling affine transformation for 3D space, color imagine compression, satellite attitude control, color night vision, etc. [34].

Remark 2: The quaternion-valued cellular neural networks can be regarded as a generalization form of real-valued cellular neural networks. Thus the research on pseudo almost automorphic solution of quaternion-valued cellular neural networks is more complex than that of real-valued cellular neural networks. In fact, a dimensional quaternion-valued cellular neural network can be transformed into four dimensional real-valued cellular neural network.

Remark 3: In [34], the authors discussed the pseudo almost automorphic solution of quaternion-valued cellular neural networks with distributed delays and kernel functions. To obtain the existence and global exponential stability of the pseudo almost automorphic solution, the assumptions for kernel functions are necessary. In this manuscript, we consider the pseudo almost automorphic solution of quaternion-valued cellular neural networks with time-varying delays. We do not need the any assumption for kernel functions and some analytic process is different from that in [34]. In addition, we can better design and optimize the networks to serve the people owing to the lack of kernel functions. Based on this viewpoint, we say that the obtained results of this manuscript are completely new and complement some previous studies to a certain extent.

IV. COMPUTER SIMULATIONS

Give the following system:

$$\begin{cases}
\dot{z}_1(t) = -a_1(t)z_1(t) + \sum_{p=1}^2 b_{1p}(t)h_p(z_p(t)) \\
\qquad + \sum_{p=1}^2 c_{1p}(t)h_p(z_p(t - \tau_{1p}(t))) + d_1(t), \\
\dot{z}_2(t) = -a_2(t)z_2(t) + \sum_{p=1}^2 b_{2p}(t)h_p(z_p(t)) \\
\qquad + \sum_{p=1}^2 c_{2p}(t)h_p(z_p(t - \tau_{2p}(t))) + d_2(t),
\end{cases} \tag{33}$$

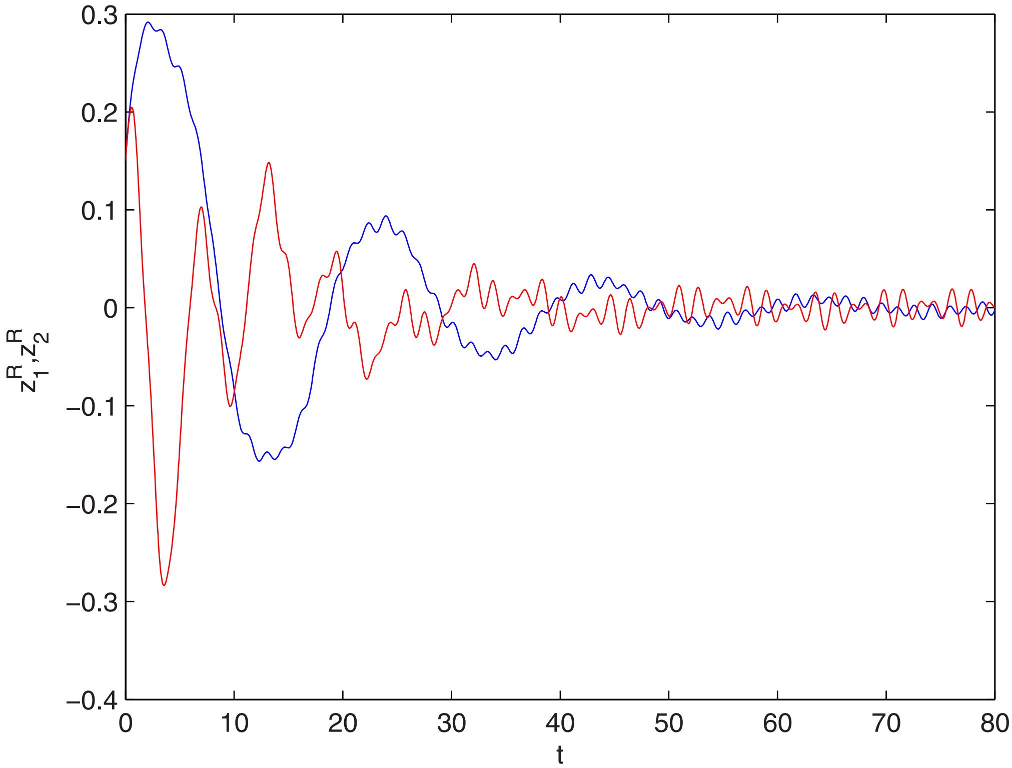

FIGURE 1. Simulation results of model (33): t-z_1^R and t-z_2^R. The blue line and red line denote z_1^R and z_2^R, respectively.

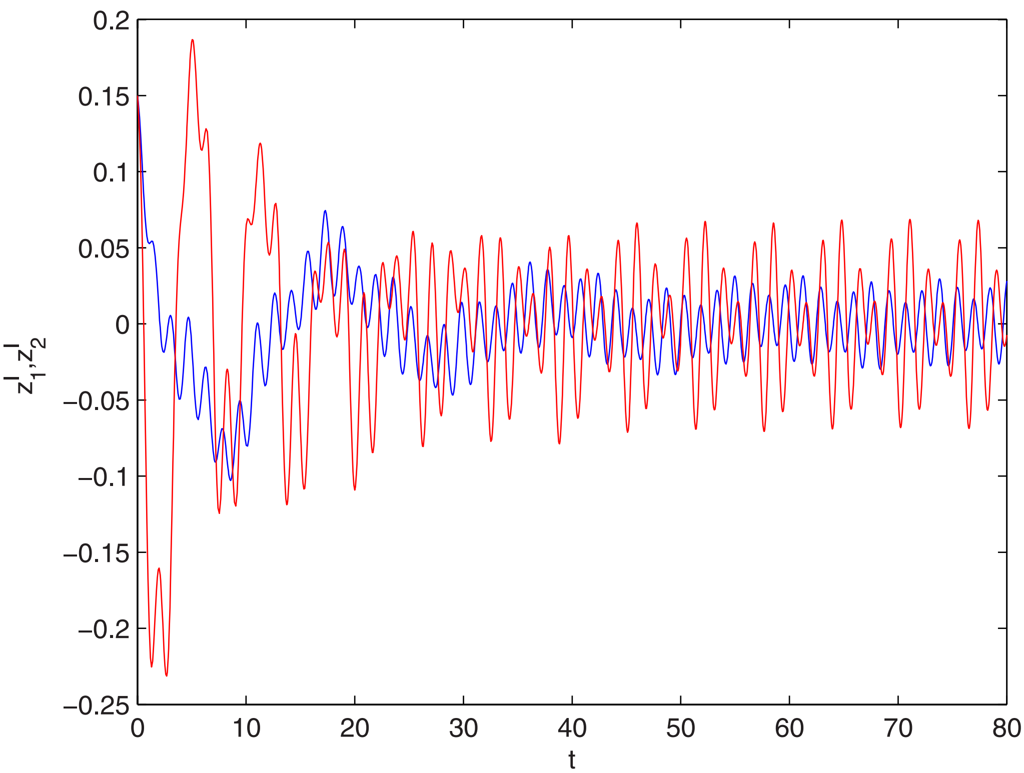

FIGURE 2. Simulation results of model (33): t-z_1^I and t-z_2^I. The blue line and red line denote z_1^I and z_2^I, respectively.

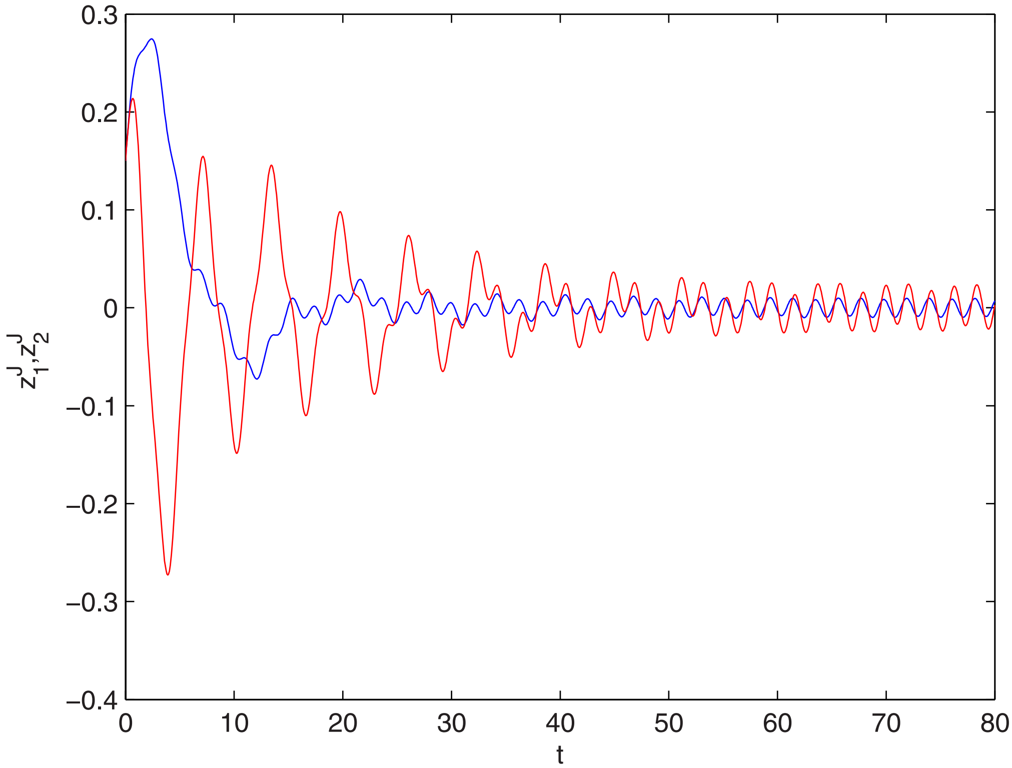

FIGURE 3. Simulation results of model (33): t-z_1^J and t-z_2^J. The blue line and red line denote z_1^J and z_2^J, respectively.

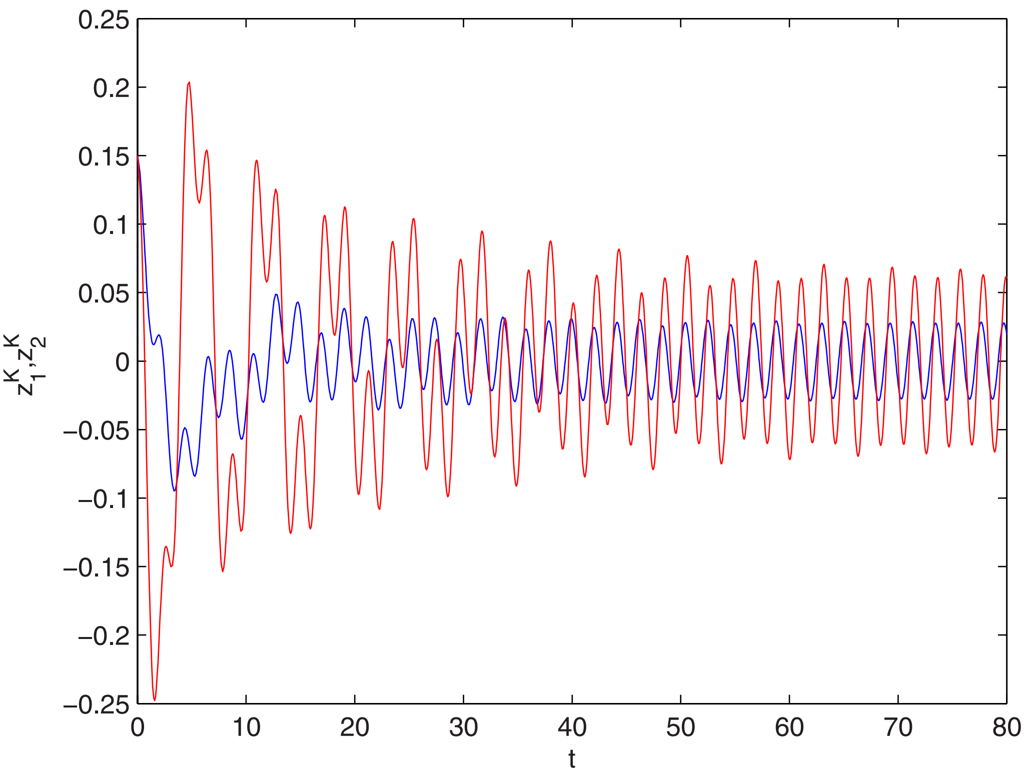

FIGURE 4. Simulation results of model (33): t-z_1^K and t-z_2^K. The blue line and red line denote z_1^K and z_2^K, respectively.

where

$$\begin{cases} h_l(v_l) = \frac{1}{20}\sin(v_l^R + v_l^J) + i\frac{1}{25}\sin(v_l^I - 2v_l^K) \\ \qquad + j\frac{1}{20}\arctan(u_l^R - 2u_l^J) + k\frac{1}{40}\sin(v_l^K + v_l^I), \\ a_1 = 1.5 + 0.2\sin\sqrt{3}t, \\ a_2 = 1.2 + 0.1\cos\sqrt{6}t, \\ d_1 = 0.3\cos\sqrt{6}t + i0.2\sin\sqrt{3}t + j0.2\cos\sqrt{5}t \\ \qquad + k0.4\sin\sqrt{3}t, \\ d_2 = 0.1\sin\sqrt{5}t + i0.1\cos\sqrt{5}t + j0.1\cos\sqrt{7}t \\ \qquad + k0.2\cos\sqrt{5}t, \\ b_{11} = 0.2\cos\sqrt{6}t + i0.3\sin\sqrt{3}t + j0.3\cos\sqrt{3}t \\ \qquad + k0.4\sin\sqrt{6}t, \\ b_{12} = 0.3\sin\sqrt{3}t + i0.2\cos\sqrt{3}t + j0.4\sin\sqrt{2}t \\ \qquad + k0.3\cos\sqrt{11}t, \\ b_{21} = 0.4\sin\sqrt{6}t + i0.1\sin\sqrt{3}t + j0.2\cos\sqrt{6}t \\ \qquad + k0.2\cos\sqrt{6}t, \\ b_{22} = 0.2\cos\sqrt{2}t + i0.2\cos\sqrt{3}t + j0.3\sin\sqrt{5}t \\ \qquad + k0.4\sin\sqrt{3}t, \\ c_{11} = 0.3\sin\sqrt{5}t + i0.2\sin\sqrt{3}t + j0.6\cos\sqrt{7}t \\ \qquad + k0.1\cos\sqrt{7}t, \\ c_{12} = 0.4\cos\sqrt{7}t + i0.5\cos\sqrt{3}t + j0.2\cos\sqrt{5}t \\ \qquad + k0.4\sin\sqrt{3}t, \\ c_{21} = 0.3\sin\sqrt{11}t + i0.2\sin\sqrt{3}t + j0.7\sin\sqrt{11}t \\ \qquad + k0.7\cos\sqrt{11}t, \\ c_{22} = 0.6\cos\sqrt{6}t + i0.7\sin\sqrt{3}t + j0.2\cos\sqrt{3}t \\ \qquad + k0.6\sin\sqrt{2}t, \\ \tau_{11} = 0.5\sin\sqrt{3}t + i0.9\cos\sqrt{3}t + j0.6\cos\sqrt{5}t \\ \qquad + k0.3\cos\sqrt{3}t, \\ \tau_{12} = 0.3\cos\sqrt{6}t + i0.8\sin\sqrt{3}t + j0.2\sin\sqrt{2}t \\ \qquad + k0.2\sin\sqrt{6}t, \\ \tau_{21} = 0.7\cos\sqrt{2}t + i0.7\cos\sqrt{3}t + j0.5\cos\sqrt{5}t \\ \qquad + k0.6\sin\sqrt{3}t, \\ \tau_{22} = 0.4\sin\sqrt{5}t + i0.6\sin\sqrt{3}t + j0.1\sin\sqrt{6}t \\ \qquad + k0.1\cos\sqrt{7}t. \end{cases} \tag{34}$$

Then $M = 0.1, a_1^- = 1.5, a_2^- = 1.2, b_{11}^+ = 0.6164, b_{12}^+ = 0.6164, b_{21}^+ = 0.5, b_{22}^+ = 0.5745, c_{11}^+ = 0.7071, c_{12}^+ = 0.7810, c_{21}^+ = 1.110, c_{22}^+ = 1.1180$ and $\kappa = \max_{1\le l\le 2}\left\{\frac{1}{a_l^-}\sum_{p=1}^m [b_{lp}^+ + c_{lp}^+]M\right\} = 0.2752 < 1$. Hence $\mathcal{Z}_1$-$\mathcal{Z}_3$ of Theorem 1 and Theorem 2 hold true, so one knows that model (33) possesses a globally exponentially stable pseudo almost automorphic solution. The fact can be displayed in figures 1-4.

V. CONCLUSION

In many cases, pseudo almost automorphic solution can give a better description of neural networks. Thus the pseudo almost automorphic solution of neural networks has attracted much attention from many scholars. But numerous authors paid much attention to pseudo almost automorphic solution of real-valued neural networks and complex-valued neural networks. In this manuscript, we concentrate on the pseudo almost automorphic solution of quaternion-valued cellular neural networks with delays. By applying the inequality techniques and pseudo almost automorphic solution theory of delayed differential equations, we have detailedly discussed the existence and global exponentially stability of pseudo almost automorphic solutions to the involved quaternion-valued cellular neural networks. During the process of analysis, we do not decompose the considered quaternion-valued cellular neural networks into four real-valued systems. It can be easily applied in practical design of neural networks. Up to now, only a few authors deal with this aspect. The method of this manuscript is new and the investigation supplements the previous publications.

ACKNOWLEDGMENT

The authors would like to thank the referees and the editor for helpful suggestions incorporated into this paper.

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