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Fuzzy Swing Up Control and Optimal State Feedback Stabilization for Self-Erecting Inverted Pendulum

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ABSTRACT This paper presents the realisation of self-erecting inverted pendulum controls via two switched control approaches, a rule based fuzzy control for swing up inverted pendulum rod to pose upright position from downright position and an optimal state feedback control for stabilization as pendulum on upright position close to its equilibrium vertical line. The aim of this study is to solve two important problems on self-erecting inverted pendulum; swing up and stability in its upright balance position. Simulation and experimental results showed that control methods enabled the inverted pendulum swinging up and reaching its stable attitude in upright position even though small impulse and pulse disturbances were given.

INDEX TERMS Fuzzy swing up, optimal state feedback stabilization, self erecting inverted pendulum.

I. INTRODUCTION

Problems of an inverted pendulum stabilization had been attracting many control system engineers and researchers for years [1]–[3]. Since an inverted pendulum is typically nonlinear, high order, multivariable and unstable, then many efforts to achieve balancing condition were proposed [4], [5]. Also, due to its nonlinearity and instability, balancing capability is the important platform to demonstrate various control applications such as running and biped walking robot, Segway riding and propeller rocket operation [6]–[8]. Moreover, an inverted pendulum also could represent model and control of human balance in walking and running [9]. Developing control schemes for inverted pendulum was often presented in some classic control system literatures as fundamental theory and application examples; (see Nise and Franklin in [10], [11]). To mention some newest research on inverted pendulum, we can observe several implementations of control methods for both swing up and stabilization problems. Horibe and Sakamoto developed nonlinear optimal control for inverted pendulum via stable manifold method to solve Hamilton-Jacobi Equation approximately [12]. In other work, non linear optimal control design with State-Dependent Riccati Equation (SDRE) for two-wheeled inverted pendulum was used to reach the stability [6]. Yang and Li studied finite-time control simulation for inverted pendulum systems [13]. A cyber physical system was used to deploy imitation reinforcement learning for simplifying the deep understanding in nonlinearities of an inverted pendulum. The method was applied for rotary inverted pendulum in open flow network [14].

The biggest problem of inverted pendulum dynamics to be solved is how to swing it up from downright position to upright position against gravitational forces and be kept for a moment to implement linearised control strategy such as a state feedback controller instantly as a pendulum is at stable upright position. The recent work for successful swing up control method can be found in [15], by proposing trajectory planning and inertia effect to make swing up motion and then continued by nonlinear adaptive neural network for stabilization of rotary inverted pendulum. In recent years, neural network, a kind of biological inspired control showed the effectiveness of unknown and nonlinear dynamic control situation [9].

This paper presents the realisation of self-erecting inverted pendulum via two switched control approaches, i) fuzzy logic control (FLC) for swinging up inverted pendulum to reach upright position from downright position and ii) state feedback design control for stabilization as pendulum on upright position. Swinging up and stabilization have done

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by moving cart along horizontal axis. Fuzzy logic control has been widely found in many real applications and used in some inverted pendulum types, for examples were two wheeled type [16], inverted pendulum on a cart [17], fuzzy control of ball-bot [18] and FLC for specialty vehicles that was combined with a phase plane method [19]. Therefore, FLC is believed in extremely movement handling of inverted pendulum from downright to upright position. The critical advantage of FLC use is that we do not need to measure all states accurately and it is robust enough depend on deployed rule based fuzzy. Hence, it also reduced the complexity in deriving mathematic model that was needed for embedded software programming as control algorithm [20]. On the upright position, fuzzy logic control is kept for moments until the inverted pendulum swayed in small angle. In this position, it is necessary to assume that physical model of inverted pendulum is linear so that a state feedback controller is deployed easily. Motivated by the previous study in [15], [20], [21], we implemented two control techniques for self-erecting inverted pendulum; fuzzy logic and optimal state feedback; for swinging up and stabilization.

This study proposed the optimal state feedback was designed by a simple linear matrix inequality (LMI) approach based on Lyapunov's stability theorem. The control method guarantees the closed loop system stability, even though small bounded disturbances occur. A recent work on robust stability for uncertain master and slave chaotic system was investigated and uniformly asymptotical synchronisation was guaranteed by state feedback stabilization using an LMI-based Lyapunov stability approach [22]. Meanwhile in our work, small disturbance of inverted pendulum stabilization can be simply approached by parametric uncertainties [23]. In addition, our study contributed in design and realisation of mechanics, electronics and control instruments were carried out by ourselves from the beginning. Hence, the instruments we built had the potential to be developed freely for testing and validating other models and control strategies. Several potential control designs that can be applied in this inverted pendulum model and instruments are LQR / LQG, adaptive control and others.

The following sections present content of the paper. Section II describes mathematical model of inverted pendulum which was needed in used control approach. Completing with electromechanical forces from dc motor as actuator, state space of inverted pendulum was provided. State feedback gain was obtained by linear matrix inequality approach correspond to stability definition. The model was also used to simulate rule based fuzzy in swinging up before doing real implementation. Control strategies are presented in Section III and realisation that followed by experimental result analyses are provided in Section IV.

II. MATHEMATICAL MODEL

Stability is main problem in self-erecting inverted pendulum control design, both swing up and stabilization controls. To provide mathematical model fairly, it is needed to observe the model of nonlinear dynamic systems and linearise this nonlinear model around its equilibrium point [20]. Electromechanical force is employed to move cart forth and back for swing up and reaching upright stable condition.

TABLE 1. Parameters for inverted pendulum.

Symbol	Variables and constants	Values*
M	mass of cart	0.51 kg
m	mass of pendulum rod	0.05 kg
L	length of pendulum	0.51 m
g	constant of gravitational acceleration	9.8 ms^{-2}
F	force on cart	N
r	radius of dc motor's rotary	0.05 m
R_A	resistance of motor	3 ohm
K_m	motor constant	0.24Nm/A
K_g	gear ratio	1
θ	pendulum angle	rad
V, H	forces on vertical and horizontal axes	N

*kg = kilogram, m = meter, s = seconds, N = Newton, A = Ampere.

A. INVERTED PENDULUM

Physically model and parameters of inverted pendulum moved by cart can be seen in Fig.1. and Table 1.

Based on Fig.1, a point of center of gravity coordinate along horizontal and vertical axes is written

$$x_G = x + l\,sin\,\theta, \quad y_G = y + l\,cos\,\theta. \tag{1}$$

Vertical and horizontal forces are following these equations, (2) and (3)

$$mg - ml\ddot{\theta}\,sin\,\theta - ml\dot{\theta}^2\,cos\,\theta = V \tag{2}$$

$$m\ddot{x} + ml\theta\,cos\,\theta - ml\dot{\theta}^2\,sin\,\theta = H. \tag{3}$$

Balancing force on cart along horizontal direction is following:

$$m\ddot{x} = F - H. \tag{4}$$

Around center of gravity, balancing of rotation motion of pendulum rod is hold in following equation

$$I\ddot{\theta} = Vl\,sin\,\theta - Hl\,cos\,\theta \tag{5}$$

Nonlinear dynamics representing Lagrange's equation based mathematical model of inverted pendulum [21], [24], are obtained by substituting (3) to (4) and (2) to (5):

$$(M+m)\ddot{\theta} + ml\ddot{\theta}\,cos\,\theta - ml\dot{\theta}^2\,sin\,\theta = F \tag{6}$$

$$(I+ml^2)\ddot{\theta} + ml\ddot{\theta}\,cos\,\theta - mgl\,sin\,\theta = 0 \tag{7}$$

where $I = \frac{1}{3}ml^2$

B. LINEARISATION AROUND BALANCE POSITION

In case of the inverted pendulum is in upright position and swinging around vertical line, it is needed to make assumption that position of angle θ is small enough [25]. Therefore, following condition are hold; $\dot{\theta}^2 = 0$, $sin\,\theta = 0$, $cos\,\theta = 1$ and equations (6), (7) are written in following equations:

$$(M+m)\ddot{x} + ml\ddot{\theta} = F \tag{8}$$

$$(I+ml^2)\ddot{\theta} + ml\ddot{x} - mgl\theta = 0 \tag{9}$$

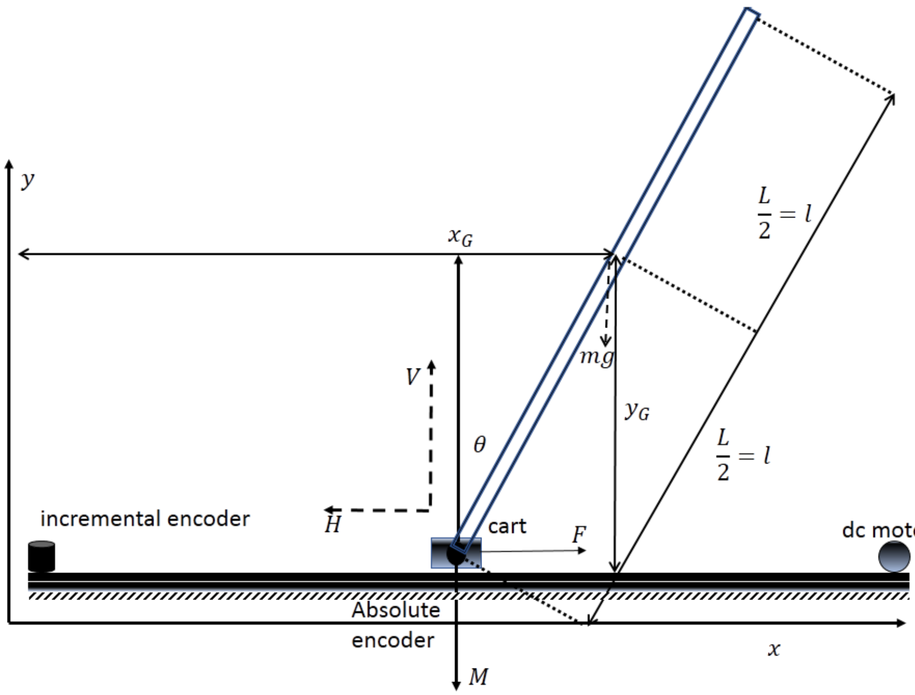


FIGURE 1. Physical model of inverted pendulum moved by cart.

Hence, transfer function of input output relation according to equations (8) and (9) can be written in following

$$G(s) = \frac{ml}{(ml)^2 - (M+m)(I+ml^2)s^2 + (M+m)mlg} \quad (10)$$

C. ELECTROMECHANICAL FORCES

To reach stability condition, dc motor with voltage input V provides electromechanical force F. The equation of dc motor is formulated in:

$$V = R_A I_A + K_m \omega_m \quad (11)$$

where R_A and I_A are resistance and current of dc motor armature whereas K_m and ω_m are torque and rotation constants.

Here, motor torque is

$$\tau_m = K_m K_g I_A \quad (12)$$

where K_g is gear ratio.

Linear force on cart follows dc motor rotation that has radius r is formulated in

$$F = \frac{\tau_m}{r} \quad (13)$$

Then motor angular rotation that corresponds to cart linear motion is

$$\omega_m = \frac{\dot{x}}{r} K_g \quad (14)$$

Hence, motor armature current is

$$I_A = \frac{Fr}{K_m K_g} \quad (15)$$

Substituting (11)-(12) and (14)-(15) to (13) yields electromechanical force:

$$F = \frac{K_m K_g}{r R_A} V - \frac{K_m^2 K_g^2}{r^2 R_A} \dot{x} \quad (16)$$

Finally, an inverted pendulum stabilization system builds a state space form:

$$\dot{\zeta}(t) = A\zeta(t) + BV(t) \quad (17)$$
$$y(t) = C\zeta(t) \quad (18)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \Theta_1 & \Theta_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \Theta_3 & \Theta_4 & 0 \end{bmatrix},$$

$$\Theta_1 = -\frac{(I+ml^2)\frac{(K_m K_g)^2}{r^2 R_A}}{(M+m)(I+ml^2) - (ml)^2},$$

$$\Theta_2 = -\frac{ml(mgl)}{(M+m)(I+ml^2) - (ml)^2},$$

$$\Theta_3 = -\frac{ml\frac{(K_m K_g)^2}{r^2 R_A}}{(M+m)(I+ml^2) - (ml)^2},$$

$$\Theta_4 = \frac{mgl}{I+ml^2}\left(1 + \frac{(ml)^2}{(M+m)(I+ml^2) - (ml)^2}\right),$$

$$B = \begin{bmatrix} 0 \\ -\dfrac{(I+ml^2)\frac{K_m K_g}{rR_A}}{(M+m)(I+ml^2)-(ml)^2} \\ 0 \\ -\dfrac{ml\frac{K_m K_g}{r^2 R_A}}{(M+m)(I+ml^2)-(ml)^2} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 \; 0 \; 1 \; 0 \end{bmatrix}, \quad \text{and } \zeta = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

III. CONTROL METHODS

Explanation for fuzzy theory and applications has been widely discussed [26]. In this paper, the self-erecting inverted pendulum is considered as a multi input single output system with input variable $x_i, (i = 1, 2, 3, \ldots n)$ and output variable y. The universe of x_i and y are $X_i = [-Ux_i, Ux_i]$, $Y = [-Uy, Uy]$ respectively. The fuzzification is a procedure used to map a crisp input x_i into fuzzy set Ax_i in X_i. Ax_i is the fuzzy set label such as ''negative big'', ''negative small'', ''medium'', ''positive small'', ''positive big'', etc; and B is a fuzzy set label of output Y. Next step is deciding IF-THEN rules to reach desired output in fuzzy set Y. An example of IF THEN rules for first rule is,

IF x_1 is Ax_1^k and x_2 is Ax_2^k and x_n is Ax_n^k THEN y is B^k with $k = 1, 2, 3, \ldots N$(N is number of rules).

By defuzzification of Y, crisp output y is obtained as follows

$$y(x) = \frac{\Sigma_{k=1}^{N} y^k \Pi_{i=1}^{n} \mu_{Ax_i}(x_i)}{\Sigma_{k=1}^{N} \Pi_{i=1}^{n} \mu_{Ax_i}(x_i)} \tag{19}$$

where $\mu_{Ax_i}(x_i)$ is membership function degree x_i to Ax_i and y^k is a point as B^k reached maximum value.

A. FUZZY SWING UP CONTROL

Strategy of pendulum's swing up was taken from increasing of controlled energy [21] in following potential energy equation:

$$E_P = \frac{1}{2}(I + ml^2)\dot{\theta}^2 + mgl(cos\theta - 1) \tag{20}$$

In swing up control, controlled inputs are angle θ and velocity angle $\dot{\theta}$ of pendulum that will be mapped for rule based fuzzy. Cart's position $x(t)$ is moved by dc motor as control action for swinging up. Trajectories $\theta, \dot{\theta}$ satisfy E_P = constant and $E_P = 0$ when inverted pendulum is at upright position. In this situation, cart is stopped since cart velocity $\dot{x} = 0$. This study adopted controlled potential energy with mapping of pendulum angle as shown in Fig.2, to develop rule based fuzzy swinging control by considering following algorithm [20]:

1) case 1: for big angle around $\{NS, Z, PS\}$ where $\theta \approx \pm(\frac{3}{4}\pi - \pi)$ rad; two possible strategies are applied
   - For positive/negative/zero angle θ and positive/negative/zero velocity $\dot{\theta}$, the control action is positive/negative/zero.
   - For θ and $\dot{\theta}$ are different sign, the control action was zero.
2) case 2: for medium angle $\{NM, PM\}$ where $\theta \approx \pm(\frac{1}{2}\pi - \frac{3}{4}\pi)$ rad
   - For positive/zero angle θ and positive/zero velocity $\dot{\theta}$, the control action is positive/zero.
   - For θ and $\dot{\theta}$ are different sign, the control action is zero.
3) case 3: for small angle $\{NB, PB\}$ and pendulum rod is around equilibrium vertical line in upright position,
   - If angle velocity $\dot{\theta} \neq 0$, then use case 2.
   - If angle velocity $\dot{\theta} = 0$, then swing up is stopped.

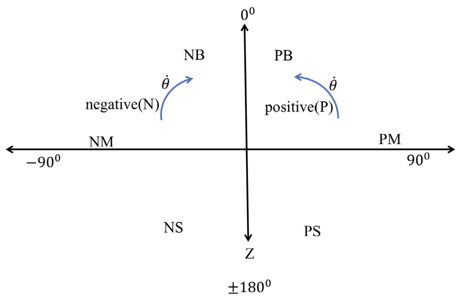


FIGURE 2. Illustration of pendulum angle and velocity used in rule based fuzzy for inverted pendulum swing up control.

TABLE 2. Rule based fuzzy.

		θ NB	NM	NS	Z	PS	PM	PB
	N	N	N	N	N	Z	Z	Z
$\dot{\theta}$	Z	N	N	N	Z	P	P	P
	P	Z	Z	Z	P	P	P	P

The abbreviation of membership functions of pendulum angle are as follows, NS = Negative Small, NM = Negative Medium, NB = Negative Big, Z = Zero, PS = Positive Small, PM = Positive Medium, PB = Positive Big.

For years, fuzzy logic had been proven to have significance advantage in real implementations that it did not need accurate all states measurements [19], [20]. This paper use input output relation according to above swinging algorithm that also can be presented in Table 2.

Fig.2 illustrates mapping of pendulum angle corresponds to the membership function. Fig.3 shows membership functions of inputs $\theta, \dot{\theta}$ and output force F.

By applying rule based fuzzy to the force on cart movement, inverted pendulum does swinging motion shown in Fig. 4.

B. STABILIZING CONTROL

After the inverted pendulum reached a stable position around equilibrium, the linearisation of the model can be done referring to equation (17)-(18). Design of optimal state feedback controller via linear matrix inequality (LMIs) is provided in advanced. Before designing a state feedback controller for stabilization of inverted pendulum, we need to consider the following definition and lemmas:

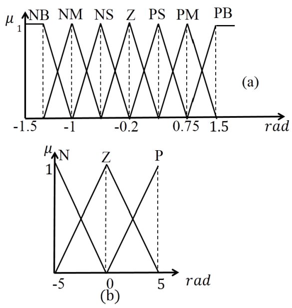


FIGURE 3. Illustration of pendulum angle and velocity used in rule based fuzzy for inverted pendulum swing up control. Figure shows membership functions of (a) angle, (b) angle velocity, and (c) force.

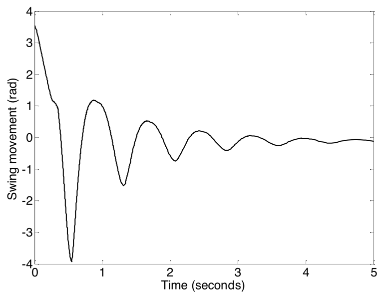


FIGURE 4. Rule based fuzzy swing up movements.

Definition 1 [27]: If state $x_a(t)$ is a solution of a differential equation system (18) with a given initial $x_a(t_0)$, this system is said to be stable if other solution $x_a(t)$ that begin at near of $x_a(t_0)$ will stay close to $x_a(t) \forall t$.

Lemma 2 Barbalat's Lemma [27]: If $x(t)$ has a limit finite for $t \to \infty$ and $\dot{x}(t)$ is uniformly continuous, then $\dot{x}(t)$ towards to zero as $t \to \infty$.

Lemma 3 Schur's Complement [28]: Given constants matrices Ψ_1, Ψ_2, Ψ_3 where $\Psi_1 = \Psi_1^T$ and $\Psi_2 = \Psi_2^T > 0$ then $\Psi_1 + \Psi_3^T \Psi_2^{-1} \Psi_3 < 0$ if and only if

$$\begin{bmatrix} \Psi_1 & \Psi_3^T \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^T & \Psi_1 \end{bmatrix} < 0$$

Use of LMIs in control research area had started when Lyapunov introduced "Lyapunov theory". Consider a continuous time linear time invariant (LTI) system in differential equation

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0 \tag{21}$$

where $x(t) \in \mathbb{R}^n$ is said asymptotically stable i.e. all trajectories go to near zero ($\lim_{\infty} x(t) = 0, \forall x_0$) if and only if there exists a quadratic Lyapunov function $V(x) = x^T Px$ such that along trajectories these inequalities $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ hold. From the inequality $\dot{V}(x(t)) < 0$, then there exists a positive definite matrix $P > 0$ such that

$$A^T P + PA < 0 \tag{22}$$

Note that form (22) is linear matrix inequality (LMI). Stability of the control system can be derived in LMI. For application in this paper, let us solve a state feedback controller for the system (17)-(18) with initial value $\zeta(t) = 0 = \zeta_0$.

The cost function to be minimized is

$$J = \int_0^{\infty} (\zeta^T(t)Q\zeta(t) + V^T(t)RV(t))dt \tag{23}$$

where Q and R are given positive definite matrices.

Consider a candidate of Lyapunov function and its time derivative

$$L = \zeta^T(t)P\zeta(t), \quad \dot{L} = 2\zeta^T(t)P\dot{\zeta}(t) \tag{24}$$

for $\zeta, \dot{\zeta} \neq 0$.

By involving cost function (23) to derivative of Lyapunov in (24), then it is obtained that

$$\dot{L} = \zeta^T(t)\Omega\zeta(t) - \zeta^T(t)Q\zeta(t) + V^T(t)RV(t) \tag{25}$$

where $\Omega = P(A + BK_{sf}) + (A + BK_{sf})^T P + Q + K_{sf}^T RK_{sf}$

Then by condition $\Omega < 0$, time derivative of Lyapunov function follows

$$\dot{L} < \{\zeta^T(t)\Omega\zeta(t) - \zeta^T(t)Q\zeta(t) + V^T(t)RV(t)\} < 0 \tag{26}$$

Note that equation (26) implies that $\dot{L}$ is negative definite and the closed loop system is asymptotically stable. By setting $K_{sf} = -R^{-1}B^T P$, pre- and post-multiplying Ω with $P^{-1} = X$ and using Schur's complement (**Lemma 3**), it is equivalent with LMI

$$\begin{bmatrix} AX + XA^T - B^T R^{-1} B & X \\ X^T & -Q^{-1} \end{bmatrix} < 0 \tag{27}$$

The following matrices are known from the system (17)-(18):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.7349282296650 & -0.7033492822967 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2.161553616662 & 30.89220377146 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 3.0622009569378 \\ 0 \\ -9.00647340275823 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

By solving (27) using LMI toolbox of Matlab, we have matrix $X^{-1} = P$ and state feedback gain $K_{sf} = -R^{-1}B^T P$. It is obvious that obtained K_{sf} is similar to an optimal gain from linear quadratic regulator (LQR) design with $Q = diag\,[0.1; 1; 0.1; 0.1]$, and $R = 0.05$. Then, the state feedback gain is $K_{sf} = [-1.414 \quad -2.456 \quad -19.770 \quad -3.896]$. All poles of closed loop are in the Left Half Plane (LHP), and showing guarantee of closed loop stabilization. This gain is guaranteed for optimal and stability including for nonlinear model around equilibrium because it is similar to obtained gain by LQR design [24]. Fig. 5 shows trajectories of pendulum angle and cart position with chosen initial conditions using LQR and guaranteed cost control designs. Phase plane of pendulum angle, cart position and their velocities are shown in Fig. 6. Here, Fig.5 and Fig.6 are shown to verify the stabilization referring to **Definition 1** and **Lemma 2**.

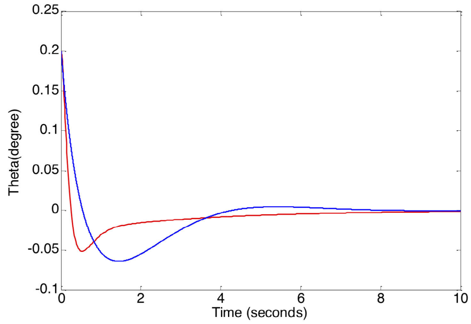
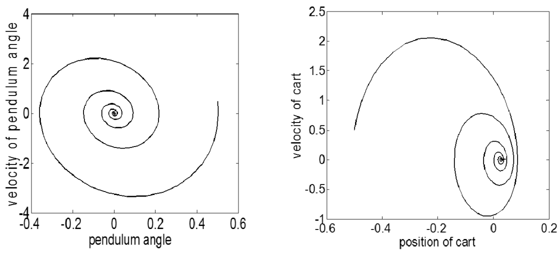


FIGURE 5. State trajectories of pendulum angle θ for LQR (red line) and Guaranteed cost controller (blue line).



(a) (b)

FIGURE 6. Phase plane of (a) pendulum angle vs velocity (b) cart position vs velocity.

C. EXTENSION TO GUARANTEED COST CONTROL

In addition to stability concern, control of inverted pendulum must also consider the uncertainties that may occur. Therefore, one desired controller design that not only guarantees stability, but also guarantees system performance in adequate level [29]. Several methods such as LQR, LQG, H_2 and H_∞ had been used to carry out the problem of uncertainties [30].

Next, assume that uncertainties ΔA, ΔB fulfil the relations: $\Delta A(t) = D_a F_a(t) E_a$ and $\Delta B(t) = D_b F_b(t) E_b$, where D_a, E_a, D_b, E_b are known matrices of real valued constants with appropriate dimensions, while $F_a(t), F_b(t)$ satisfy $F_a^T(t)F_a(t) \leq 0$ and $F_b^T(t)F_b(t) \leq 0$.

The extension of (27) needs following inequality:

$$DF(t)E + E^T F^T(t) D^T \leq \alpha D D^T + \alpha^{-1} E^T E. \quad (28)$$

Now, considering the uncertainties ΔA, ΔB, (17) become

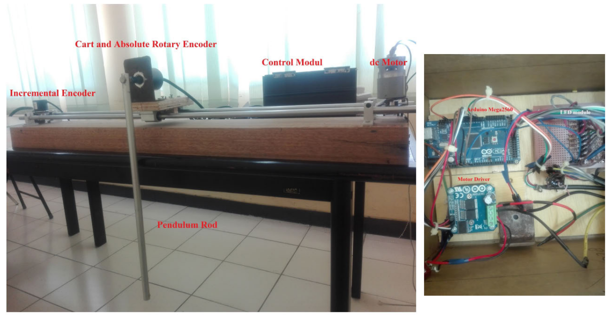
$$\dot{\zeta}(t) = (A + \Delta A)\zeta(t) + (B + \Delta B) + V(t) \quad (29)$$

The closed loop of system (29) is

$$\dot{\zeta}(t) = [(A + \Delta A) + (B + \Delta B)K]\zeta(t) \quad (30)$$

Adopting Lyapunov function from (24) and using (28) to condition $\Omega < 0$ implies that for any $\alpha > 0$ and $\beta > 0$, these inequalities hold:

$$\begin{aligned} &2\zeta^T(t)\Delta A^T P\zeta(t) \\ &= 2\zeta^T(t)E_a^T F_a^T(t) D_a^T P\zeta(t) \\ &= \alpha\zeta^T(t)E_a^T E_a \zeta(t) + \alpha^{-1}\zeta^T(t) P D_a D_a^T P\zeta(t) \quad (31) \end{aligned}$$



(a) (b)

FIGURE 7. Physical systems of (a) inverted pendulum, (b) control module.

$$\begin{aligned} &2\zeta^T(t)K^T \Delta B^T P\zeta(t) \\ &= 2\zeta^T(t)K^T E_b^T F_b^T(t) D_b^T P\zeta(t) \\ &= \beta\zeta^T(t)K^T E_b^T E_b K\zeta(t) + \beta^{-1}\zeta^T(t) P D_b D_b^T P\zeta(t) \quad (32) \end{aligned}$$

By following past design procedure in (24)-(26), considering the cost function (23) and the uncertainties that assumed in previous description, this inequality is an extension of (27)

$$\begin{bmatrix} \Phi & X & XE_a^T & BR^{-1}E_b^T \\ X^T & -Q^{-1} & 0 & 0 \\ E_a X & 0 & -\alpha^{-1}I & 0 \\ E_b R^{-1} B^T & 0 & 0 & -\beta^{-1}I \end{bmatrix} < 0 \quad (33)$$

where $\Phi = AX + XA^T - B^T R^{-1} B + \alpha^{-1} D_a D_a^T + \beta^{-1} D_b D_b^T$ Here, the obtained gain $K = -R^{-1}B^T X$ is a guaranteed cost controller that gives minimum value of guaranteed cost [23]. For nominal systems by suppressing $\Delta A = 0$, $\Delta B = 0$, the derived controller gain is similar to previous K_{sf}.

The signal balance of a noised system output can be written in $y = v - Lv$, where v is measurement noise and L is loop gain. Sensitivity function $S = \frac{1}{1+L}$ corresponds to disturbance attenuation, whereas the smaller $|S(j\omega)|$ means the more attenuation at angular frequency ω. Therefore, a noised system output can be rearranged in terms of sensitivity function:

$$y = \frac{1}{1+L} v = Sv \quad (34)$$

According to (34), it is possible to replace sensitivity function $|S(j\omega)|$ with $|S(j\omega)V(j\omega)|$ at low frequency range for system with uncertainties.

For disturbance attenuation system with uncertainties, it needs bigger control gain to compensate noise so that its loop gain L is also bigger than that of systems without uncertainties or disturbances. As a consequence, sensitivity function of the proposed method (with uncertainties, $V(j\omega) \leq 1$) is smaller than sensitivity function of LQR control method (without uncertainty), see Fig. 10.

IV. EXPERIMENTAL RESULTS

Stability in vertical upright position is crucial problem to be solved in this paper. The experimental testing was done to ensure robustness the designed controllers although a

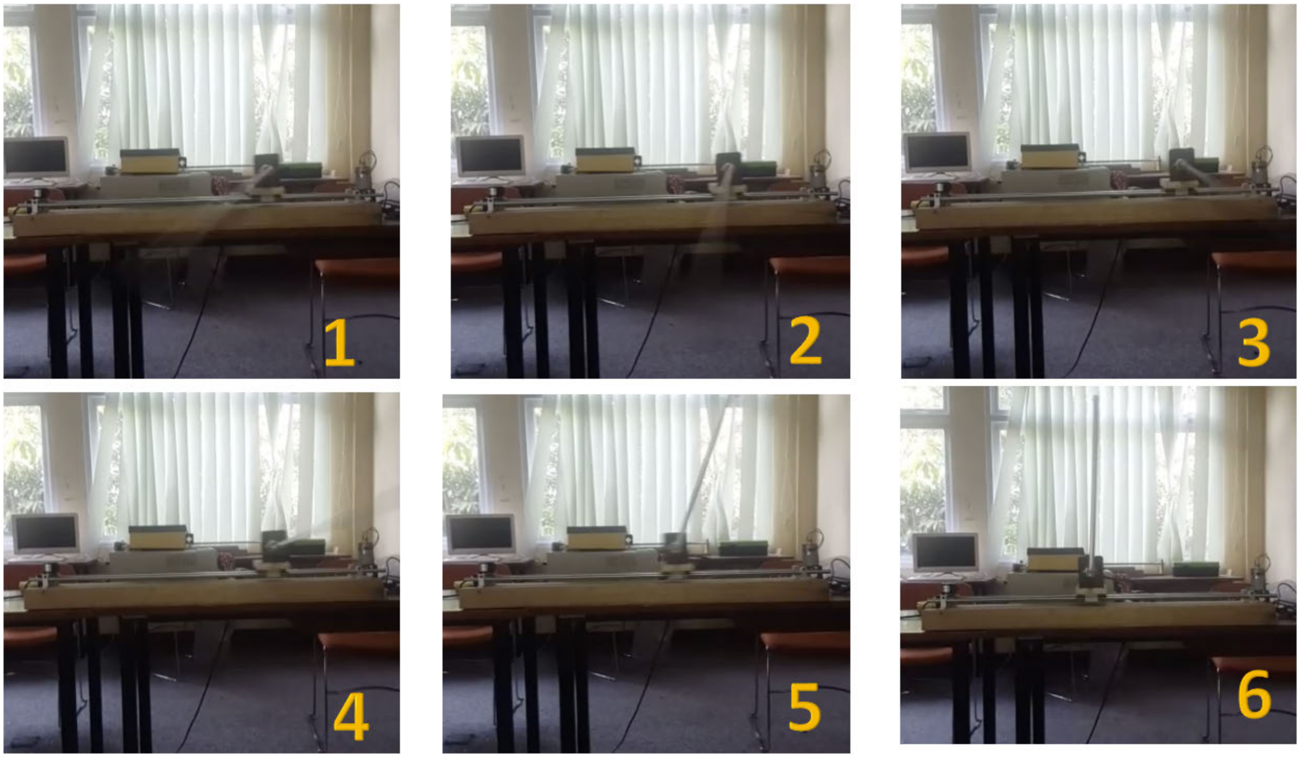
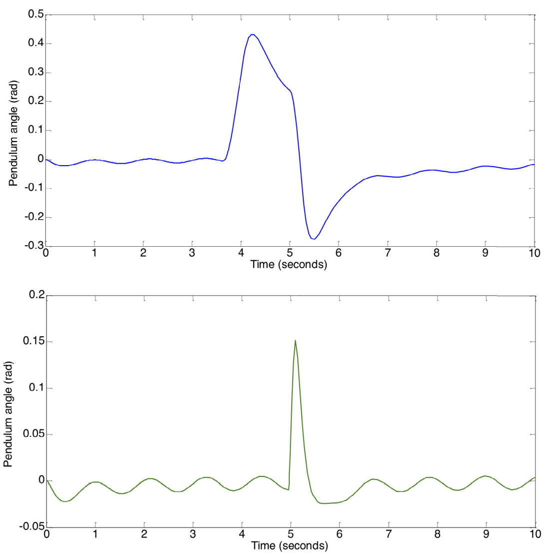


FIGURE 8. Snapshots of self erecting inverted pendulum, from downright to upright.



(b)

FIGURE 9. Responses of inverted pendulum stabilization for (a) pulse disturbance, (b) impulse disturbance.

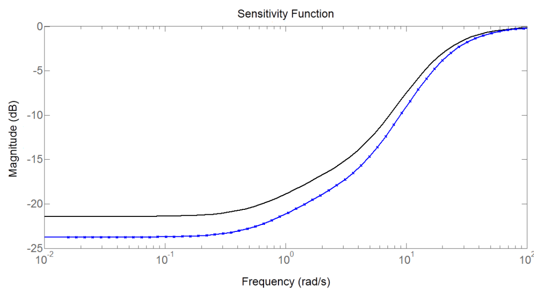


FIGURE 10. Sensitivity Functions of LQR controller(black line) and proposed method (blue line).

small disturbance was given. Instruments that used to build the system consisted of i). DC motor with torque 2 kg.cm, 12-36 Volt input, ii). Motor driver BTS7960 H-bridge with current rating 43 A and input level 3.3-5 Volt, iii). Micro-controller Arduino Mega 2560 Rev3, iv). Incremental Rotary Encoder with 800 Pulse Per Revolution, 5-24 Volt operating voltage, and v). Absolute Rotary Encoder Autonics EP50S with 12-24 Volt, 1024 Pulse Per Revolution and Gray code output.

The absolute rotary encoder detects the angle of the pendulum and activates the motor driver to drive the cart back and forth based on the fuzzy algorithm previously described. The incremental encoder detects the position of the cart to be in the horizontal rail range. When the pendulum is at an upright position, the state feedback control stabilizes the pendulum so that it is kept upright.

On the swing up testings, pendulum rod reached its inverted vertical position in less than 10 seconds. Coding of fuzzy swing up and state feedback stabilization algorithms were embedded in micro controller employing open source Arduino software, Integrated Development Software (IDE). Data of pendulum rod angle and cart position were monitored from serial monitor of Arduino.

Both pulse and impulse disturbances had $\Delta A = \Delta B \approx 6.7\%$ of angle distortion from vertical line. In this situation, all poles of closed loop systems locates in LHP. Time interval of pulse disturbance was set around 2 seconds and 0.5 second for impulse disturbance. System responded pulse and impulse disturbances based on designed controller. Stabilizing time of pulse disturbance was around 4.5 seconds and impulse disturbance was stabilized in 2.2 seconds, see Fig.9. In work of S. Chaterjee and S.K. Das (2018), disturbance was given in the 65.2 second and stabilized in the 68 second [5]. They used optimal tuning of state feedback controller gain with dominant pole structure. In our work, inverted pendulum reacted faster than their work, impulse disturbance occured in the 5 second and stabilized in the 7.2 second.

V. CONCLUSION

Designing controllers for switched condition of self erecting inverted pendulum play crucial role for its extremely dynamic situation. Swing up movements need big enough energy and fast response instantly to draw the pendulum reaching an upright position around its equilibrium line. Fuzzy control methods was implemented to swing pendulum rod from lower straight position 180 deg to upper position around 0 deg. By **Definition 1** and **Lemma 2**, as pendulum was at vertical inverted position, it was shown that pendulum was stabilized. Since linearised was permitted in this situation, a state feedback controller gain was set to stabilize pendulum rod at vertical upright position around equilibrium line by moving cart along horizontal axis. It was shown, by laboratory experiment, swing up and stabilization problem for inverted pendulum can be solved via combination of fuzzy and state feedback controllers. Small disturbances, i.e. impulse and pulse forces on pendulum rod were still overcome by the control methods.

VI. FUTURE WORK

Because the inverted pendulum system is developed by us overall from the beginning, problems of the other control methods and applications are still open to be implemented. For educational purposes in control lecture or practical laboratory, graphical user interface deserves to be developed for monitoring state trajectories such as pendulum angle, angle velocity and cart position.

State feedback guaranteed cost control is effective enough to overcome small disturbances such as when the pendulum is in an upright stabilization position. In the future, robust control methods should be developed for overall inverted pendulum control ranging from swing up condition to upright positions.

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REFERENCES

[1] S. R. Bonifacio, O. O. Patricio, and P. G. Alexander, "Robust stabilizing control for the electromechanical triple–link inverted pendulum system," *IFAC-PapersOnLine*, vol. 51, no. 13, pp. 314–319, 2018.

[2] Z. Sun, N. Wang, and Y. Bi, "Type–1/type-2 fuzzy logic systems optimization with RNA genetic algorithm for double inverted pendulum," *Appl. Math. Model.*, vol. 39, no. 1, pp. 70–85, Jan. 2015.

[3] E. Sivaraman and S. Arulselvi, "Modelling of an inverted pendulum based on fuzzy clustering techniques," *Expert Syst. Appl.*, vol. 38, no. 1, pp. 13942–13949, 2011.

[4] A. Kharola, P. Patil, S. Raiwani, and D. Rajput, "A comparison study for control and stabilisation of inverted pendulum on inclined surface (IPIS) using PID and fuzzy controllers," *Perspect. Sci.*, vol. 8, pp. 187–190, Sep. 2016.

[5] S. Chatterjee and S. K. Das, "An analytical formula for optimal tuning of the state feedback controller gains for the cart-inverted pendulum system," *IFAC-PapersOnLine*, vol. 51, no. 1, pp. 668–672, 2018.

[6] S. Kim and S. Kwon, "Nonlinear optimal control design for underactuated two–wheeled inverted pendulum mobile platform," *IEEE/ASME Trans. Mechatronics*, vol. 22, no. 6, pp. 2803–2808, Dec. 2017.

[7] S. Irfan, A. Mehmood, M. T. Razzaq, and J. Iqbal, "Advanced sliding mode control techniques for inverted pendulum: Modelling and simulation," *Eng. Sci. Technol., Int. J.*, vol. 21, no. 4, pp. 753–759, Aug. 2018.

[8] P. Van Lam and Y. Fujimoto, "A robotic cane for balance maintenance assistance," *IEEE Trans. Ind. Informat.*, vol. 15, no. 7, pp. 3998–4009, Jul. 2019.

[9] W. W. Hilts, N. S. Szczecinski, R. D. Quinn, and A. J. Hunt, "A dynamic neural network designed using analytical methods produces dynamic control properties similar to an analogous classical controller," *IEEE Control Syst. Lett.*, vol. 3, no. 2, pp. 320–325, Apr. 2019.

[10] N.-S. Nise, *Control Systems Engineering*. Hoboken, NJ, USA: Wiley, 2011.

[11] G.-F. Franklin, J.-D. Powell, and A. Emami-Naeni, *Feedback Control Dynamic Systems*. Upper Saddle River, NJ, USA: Pearson, 2010.

[12] T. Horibe and N. Sakamoto, "Optimal swing up and stabilization control for inverted pendulum via stable manifold method," *IEEE Trans. Control Syst. Technol.*, vol. 26, no. 2, pp. 708–715, Mar. 2018.

[13] T. Yang and Y. Li, "Finite–time control for a class of inverted pendulum systems," *IEEE Access*, vol. 7, pp. 129637–129643, Sep. 2019.

[14] J.-B. Kim, H.-K. Lim, C.-M. Kim, M.-S. Kim, Y.-G. Hong, and Y.-H. Han, "Imitation reinforcement learning–based remote rotary inverted pendulum control in OpenFlow network," *IEEE Access*, vol. 7, pp. 36682–36690, Apr. 2019.

[15] X. Yang and X. Zheng, "Swing–up and stabilization control design for an underactuated rotary inverted pendulum system: Theory and experiments," *IEEE Trans. Ind. Electron.*, vol. 65, no. 9, pp. 7229–7238, Sep. 2018.

[16] J. Huang, M. Ri, D. Wu, and S. Ri, "Interval type–2 fuzzy logic modeling and control of a mobile two–wheeled inverted pendulum," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 4, pp. 2030–2038, Aug. 2018.

[17] A.-I. Roose, S. Yahya, and H. Al-Rizzo, "Fuzzy-logic control of an inverted pendulum on cart," *Comput. Electr. Eng.*, vol. 61, pp. 31–47, Jul. 2017.

[18] C. Cai, J. Lu, and Z. Li, "Kinematic analysis and control algorithm for the Ballbot," *IEEE Access*, vol. 7, pp. 38314–38321, Apr. 2019.

[19] L. Qin, J. Hu, H. Li, and W. Chen, "Fuzzy logic controllers for specialty vehicles using a combination of phase plane analysis and variable universe approach," *IEEE Access*, vol. 5, pp. 1579–1588, Jan. 2017.

[20] N. Muskinja and B. Tovornik, "Swinging up and stabilization of a real inverted pendulum," *IEEE Trans. Ind. Electron.*, vol. 53, no. 2, pp. 631–639, Apr. 2006.

[21] J.-H. Yang, S.-Y. Shim, J.-H. Seo, and Y.-S. Lee, "Swing-up control for an inverted pendulum with restricted cart rail length," *Int. J. Control Autom. Syst.*, vol. 7, no. 4, pp. 674–680, Aug. 2009, doi: 10.1007/s12555-009-0419-x.

[22] S. Mobayen and F. Tchier, "Synchronization of a class of uncertain chaotic systems with Lipschitz nonlinearities using state–feedback control design: A matrix inequality approach," *Asian J. Control*, vol. 20, no. 1, pp. 71–85, Jan. 2018.

[23] E. Susanto, M. Ishitobi, S. Kunimatsu, and D. Matsunaga, "A minimal-order observer-based guaranteed cost controller for uncertain time-varying delay systems," *IMA J. Math. Control Inf.*, vol. 29, no. 1, pp. 113–132, Mar. 2012.
[24] R. Liu, J. Wu, and D. Wang, "Sampled-data fuzzy control of two-wheel inverted pendulums based on passivity theory," *Int. J. Control Autom. Syst.*, vol. 16, no. 5, pp. 2538–2548, Oct. 2018.
[25] M. Magdy, A. El Marhomy, and M. A. Attia, "Modeling of inverted pendulum system with gravitational search algorithm optimized controller," *Ain Shams Eng. J.*, vol. 10, no. 1, pp. 129–149, Mar. 2019.
[26] H.-J. Zimmermann, *Fuzzy Set Theory—And Its Applications*. New York, NY, USA: Springer, 2001.
[27] S.-H. Zak, "Tools for analysis of dynamic systems: Lyapunov's method," Purdue Univ., West Lafayette, IN, USA, Tech. Rep., 2013.
[28] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory* Philadelphia, PA, USA: SIAM, 1994.
[29] S. Chang and T. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Trans. Autom. Control*, vol. 17, no. 4, pp. 474–483, Aug. 1972.
[30] M. S. Mahmoud and M. T. Nasir, "Robust control design of wheeled inverted pendulum assistant robot," *IEEE/CAA J. Autom. Sinica*, vol. 4, no. 4, pp. 628–638, Oct. 2017.

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