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A Theoretical Question in the Optimal Design of Matrix Decomposition Based FIR Filter

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ABSTRACT The matrix decomposition (MD) based finite impulse response (FIR) filter is a low-complexity FIR filter. It has been tested the coefficients of the MD-FIR filter can be effectively optimized by the trust-region-iterative-gradient-searching (TR-IGS) algorithm. This algorithm solves the convex-approximation-problem of the original coefficients optimization problem. In this study, we deal with the relationship between the theoretical termination point of the TR-IGS and the optimal solution of the original coefficients optimization problem.

INDEX TERMS FIR, low-complexity, matrix decomposition, optimal design, trust-region-iterative-gradient-searching.

I. INTRODUCTION

Finite impulse response (FIR) filters [1]–[25] can achieve strict linear-phase (LP) and have guaranteed stability. They are widely used in digital signal processing (e.g., filtering and Hilbert transformer design [9]) and communication systems (e.g., pulse shaping [10] and equalizer). Traditional methods of designing an FIR filter include window method, frequency sampling method and direct optimal design method. The hardware implementation complexity [11] of a traditional FIR filter is high due to the coefficient multiplications. However, it has the advantage that it can be implemented using the well developed direct implementation structure. Particularly, for the window method, the filter coefficients can be analytically obtained.

Various techniques have been developed to decrease the hardware implementation complexity [9], [11]–[24] of a traditional FIR filter. The popular ones of these techniques include sparse traditional FIR filter technique [12], [13], [18], [20], frequency response mask technique [8], and the matrix decomposition based technique [7]. For the sparse traditional FIR filter technique, the designed FIR filter can be implemented using the well developed direct implementation structure. For the other two techniques, the designed FIR filters have to be implemented using different structures.

By utilizing a different FIR filter structure, matrix decomposition (MD) based technique can synthesize any FIR filter (including non-frequency-selective FIR filters), with much lower hardware implementation complexity, affecting the frequency performance metrics very scarcely and with no impact on the group-delay performance metric [6], [7].

The optimal design of a MD-FIR filter is generally a high-dimensional, non-convex and non-differential-able (for mini-max design) optimization problem. Thus, it is not easy to analyze and locate its local optimum. The trust-region iterative-gradient-searching (TR-IGS) is an effective technique to optimize the coefficients of a MD-FIR filter [6].

In [7], a convergent implementation of TR-IGS is proposed for the first time. It is pointed out in [7], the TR-IGS may converge to a non-local-minimum.

A theoretical question regarding TR-IGS is: what is the relationship between the optimal solution of TR-IGS and that of the original filter coefficients optimization problem? In this study, we address this issue for the first time. The theoretical results provide insight into the TR-IGS algorithm. The challenge in addressing this theoretical issue is: the original filter coefficients optimization problem is high-dimensional, non-convex and and non-differential-able (for mini-max design).

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II. THE THEORETICAL QUESTION

The frequency response of a non-linear-phase MD-FIR filter can be given as follows [7]:

$$\begin{aligned} H'(\mathbf{x} \quad \omega) &= \mathbf{x}_{\text{rem}}(0) + \sum_{n=1}^{M}\left[\mathbf{x}_{\text{rem}}(n)\cdot\exp\left(-\sqrt{-1}\cdot\omega\cdot n\right)\right] \\ &+ \sum_{j=1}^{R}\mathbf{s}_1(j) \\ &\cdot \sum_{i=1}^{d}\left[\mathbf{r}_1(i)\cdot\exp\left(-\sqrt{-1}\cdot\omega\cdot(M+(j-1)\cdot d+i)\right)\right] \\ &+ \sum_{j=1}^{R}\mathbf{s}_2(j) \\ &\cdot \sum_{i=1}^{d}\left[\mathbf{r}_2(i)\cdot\exp\left(-\sqrt{-1}\cdot\omega\cdot(M+(j-1)\cdot d+i)\right)\right] \\ &+ \ldots\ldots + \sum_{j=1}^{R}\mathbf{s}_{M_1+M_2+q}(j) \\ &\cdot \sum_{i=1}^{d}\left[\mathbf{r}_{M_1+M_2+q}(i)\cdot\exp\left(-\sqrt{-1}\cdot\omega\cdot(M+(j-1)\right.\right. \\ &\left.\left.\cdot d+i)\right)\right], \end{aligned} \tag{1}$$

where

$$\mathbf{x} = \left[\mathbf{x}_{\text{rem}}^{\text{T}} \quad \mathbf{r}_1^{\text{T}} \quad \mathbf{r}_2^{\text{T}} \quad \ldots \quad \mathbf{r}_p^{\text{T}} \quad \mathbf{s}_1^{\text{T}} \quad \mathbf{s}_2^{\text{T}} \quad \ldots \quad \mathbf{s}_p^{\text{T}}\right]^{\text{T}}. \tag{2}$$

$\mathbf{x}_{\text{rem}}$, $\mathbf{r}_i$ and $\mathbf{s}_i$ are the design parameters of a MD-FIR filter [7]. The frequency response of a linear-phase MD-FIR filter has a similar expression to the above [6]. The optimal design of a MD-FIR filter, in the minimax sense or the least square sense, can be described as follows [6]:

$$\underset{\mathbf{x}_{\text{NZ}}}{\text{minimize}} \left\|\mathbf{w} * (\mathbf{H}(\mathbf{x}_{\text{NZ}}) - \mathbf{H}_d)\right\|_{\infty \text{ or } 2} \qquad \text{(Problem 1)}$$

where $\mathbf{w}$ denotes the frequency weighting vector, $\mathbf{x}_{\text{NZ}}$ denotes the variable coefficients vector in $\mathbf{x}$ and $*$ denotes element-wise multiplication of two vectors.

For some given initial solution $\mathbf{x}_{\text{NZ}}^{\text{Int}} = \mathbf{x}_{\text{NZ}}^{(0)}$ (the superscript 'Int' denotes the initial solution), (Problem 1) can be approximately transformed into a convex optimization problem described as follows:

$$\begin{aligned} &\underset{\nabla\mathbf{x}_{\text{NZ}}}{\text{minimize}} \left\|\mathbf{w} * \left(\mathbf{H}\left(\mathbf{x}_{\text{NZ}}^{\text{Int}}\right) + \mathbf{G}\left(\mathbf{x}_{\text{NZ}}^{\text{Int}}\right)\right.\right. \\ &\left.\left.\cdot\nabla\mathbf{x}_{\text{NZ}} - \mathbf{H}_d\right)\right\|_{\infty \text{ or } 2} && \text{(Problem 2.a)} \\ &\textit{subject to}: \|\nabla x_{\text{NZ}}\|_{\infty} \le \delta && \text{(Problem 2.b)} \end{aligned}$$

where $\mathbf{G}(\mathbf{x}_{\text{NZ}})$ is the Jacobean matrix of the function $\mathbf{H}(\mathbf{x}_{\text{NZ}})$ with respect to $\mathbf{x}_{\text{NZ}}$, $\nabla\mathbf{x}_{\text{NZ}}(\nabla\mathbf{x}_{\text{NZ}} = (\mathbf{x}_{\text{NZ}} - \mathbf{x}_{\text{NZ}}^{\text{Int}}))$ denotes the minor change of variable $\mathbf{x}_{\text{NZ}}$ at some initial point and δ is some prescribed bound. By solving (Problem 2), we could obtain a better solution $\mathbf{x}_{\text{NZ}}^{(1)}$ than $\mathbf{x}_{\text{NZ}}^{(0)}$. Thus, $\mathbf{x}_{\text{NZ}} = \mathbf{x}_{\text{NZ}}^{(0)}$ can be improved iteratively until it cannot be improved (Theoretically speaking, $\mathbf{x}_{\text{NZ}} = \mathbf{x}_{\text{NZ}}^{(0)}$ can be infinitely iteratively improved before reaching the theoretical termination point [7] of TR-IGS (i.e., the optimal solution of TR-IGS). Practically and generally speaking, however, $\mathbf{x}_{\text{NZ}} = \mathbf{x}_{\text{NZ}}^{(0)}$ can only be finitely iteratively improved before reaching the theoretical termination point.).

Note the objective function in Problem (2)/(2.a) is the convex approximation of the objective function in Problem (1). If the optimal solution of Problem (2) is $\nabla\mathbf{x}_{\text{NZ}} = \mathbf{0}$ (i.e., $\mathbf{x}_{\text{NZ}} = \mathbf{x}_{\text{NZ}}^{\text{Int}}$) for some initial solution $\mathbf{x}_{\text{NZ}}^{\text{Int}}$, then the TR-IGS algorithm terminates theoretically at this point $\mathbf{x}_{\text{NZ}}^{\text{Int}}$, and $\nabla\mathbf{x}_{\text{NZ}} = \mathbf{0}$ is the optimally solution of Problem (2)/(2.a). And, this $\nabla\mathbf{x}_{\text{NZ}} = \mathbf{0}$ is the optimal solution of TR-IGS. A theoretical question intuitively arise as follows: what is the relationship between the optimal solution of the TR-IGS and that of the original problem? In this study, a complete relationship between the optimal solution of TR-IGS and that of the original problem is studied.

III. PRELIMINARY WORK

Firstly, we reformulate Problems (1) and (2) by expanding each function in the ℓ_∞ norm or ℓ_2 norm using Taylor series. Let

$$\begin{pmatrix} f\left(\mathbf{x}_{\text{NZ}} \quad \omega_1\right) \\ f\left(\mathbf{x}_{\text{NZ}} \quad \omega_2\right) \\ \cdots \\ f\left(\mathbf{x}_{\text{NZ}} \quad \omega_\Gamma\right) \end{pmatrix} = \mathbf{w} * (\mathbf{H}(\mathbf{x}_{\text{NZ}}) - \mathbf{H}_d), \tag{3}$$

$\nabla f\left(\mathbf{x}_{\text{NZ}}^{\text{Init}} \quad \omega_i\right)$ and $\mathbf{H}\left[f\left(\mathbf{x}_{\text{NZ}}^{\text{Init}} \quad \omega_i\right)\right]$ denote the gradient vector and Hessian matrix of $f\left(\mathbf{x}_{\text{NZ}}\ \omega_i\right)$ for $i = 1, 2, 3, \ldots, \Gamma$, and ω_i are the frequency points of interest. For some initial solution $\mathbf{x}_{\text{NZ}}^{\text{Init}}$, (Problem 1) can be reformulated *equivalently* in (Problem 3-Q), as shown at the bottom of the next page, where $\Delta\mathbf{x}_{\text{NZ}} = \left(\mathbf{x}_{\text{NZ}} - \mathbf{x}_{\text{NZ}}^{\text{Init}}\right)$. And (Problem 2) can be reformulated *equivalently* in (Problem 3-L-a) and (Problem 3-L-b), as shown at the bottom of the next page.

In this paper, "Q" and "L" are used to differentiate the original problem and the convex-approximation problem.

Note the optimal design of the basic frequency response masking (FRM) FIR filter and that of the separable 2-D FIR filter can also be described by (Problem 3-Q).

Then, we consider the reformulated problems only in one direction of $\nabla\mathbf{x}_{\text{NZ}}$. Let $\mathbf{d} = \frac{\nabla\mathbf{x}_{\text{NZ}}}{\|\nabla\mathbf{x}_{\text{NZ}}\|_\infty}$ be the direction of $\nabla\mathbf{x}_{\text{NZ}}$ and $\nabla x_{\text{NZ}} = \|\nabla\mathbf{x}_{\text{NZ}}\|_\infty$ $(\Delta x_{\text{NZ}} \ge 0)$ be the length of $\nabla\mathbf{x}_{\text{NZ}}$. For any given direction $\mathbf{d}$ of $\nabla\mathbf{x}_{\text{NZ}}$, (Problem 3-Q) can be reformulated *equivalently* in (Problem 4-Q), as shown at the bottom of the next page.

Let

$$g_{1i} = \left[\nabla f\left(\mathbf{x}_{\text{NZ}}^{\text{Init}} \quad \omega_i\right)\right]^{\text{T}} \cdot \mathbf{d} \tag{4}$$

And

$$g_{2i} = \mathbf{d}^{\text{T}} \cdot \mathbf{H}\left[f\left(\mathbf{x}_{\text{NZ}}^{\text{Init}} \quad \omega_i\right)\right] \cdot \mathbf{d}. \tag{5}$$

We have (Problem 5-Q), (Problem 5-L-a), and (Problem 5-L-b), as shown at the bottom of this page,

IV. RELATIONSHIP BETWEEN THE OPTIMAL SOLUTION OF THE TR-IGS AND THAT OF THE ORIGINAL PROBLEM, ℓ_∞ NORM

A. THE GENERAL CASE: $f(\mathbf{x}_{NZ}\ \omega_i)$ ARE COMPLEX-COEFFICIENT FUNCTIONS WITH REAL ARGUMENT $\mathbf{x}_{NZ}$ FOR $i = 1, 2, 3, \ldots, \Gamma$

Firstly, some sets of the indexes (i.e., $i = 1, 2, 3, \ldots, \Gamma$) of the functions $f(\mathbf{x}_{\mathrm{NZ}}\ \omega_i)$ are defined, which will be used in simplifying the expressions of the objective functions of Problems (5-Q) and (5-L). (6) and (7), as shown at the bottom of the next page, for $i = 1, 2, 3, \ldots, \Gamma$, where Real [] and Imag [] denote the real and imaginary part of a complex number.

Let $\Phi_1 = \underset{i}{\arg\,\text{maximize}} \left|f(\mathbf{x}_{\mathrm{NZ}}\ \omega_i)\right|^2$, $\Phi_2 = \underset{i\in\Phi_1}{\arg\,\text{maximize}}\, c_{1i}$ and $\Phi_{3-\mathrm{Q}} = \underset{i\in\Phi_2}{\arg\,\text{maximize}}\, c_{2i}$. Let $\Phi_{3-\mathrm{L}} = \underset{i\in\Phi_2}{\arg\,\text{maximize}}\, |g_{1i}|^2$, $\Phi_{4-\mathrm{Q}} = \underset{i\in\Phi_{3-\mathrm{Q}}}{\arg\,\text{maximize}}\, c_{3i}$, and $\Phi_{5-\mathrm{Q}} = \underset{i\in\Phi_{4-\mathrm{Q}}}{\arg\,\text{maximize}}\, c_{4i}$. Let $i-\max-\mathrm{Q}$ denote an element in set $\Phi_{5-\mathrm{Q}}$, and $i-\max-\mathrm{L}$ denote an element in set $\Phi_{3-\mathrm{L}}$. Let $\Phi_{\mathrm{Q-L}} = \Phi_{3-\mathrm{L}} \cap \Phi_{5-\mathrm{Q}}$ and $i-\max-\mathrm{Q}-\mathrm{L}$ denote an element in $\Phi_{\mathrm{Q-L}}$ (i.e., $i-\max-\mathrm{Q}-\mathrm{L} \in \Phi_{\mathrm{Q-L}}$). The relationship between Φ_1, Φ_2, $\Phi_{3-\mathrm{Q}}$, $\Phi_{3-\mathrm{L}}$, $\Phi_{4-\mathrm{Q}}$, $\Phi_{5-\mathrm{Q}}$ and $\Phi_{\mathrm{Q-L}}$ is illustrated in the following Figures 1-3.

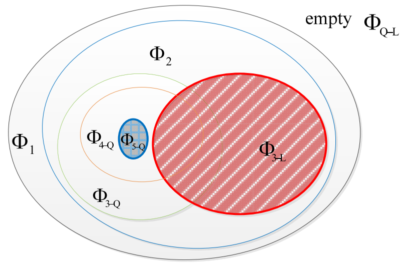

FIGURE 1. Φ_1, Φ_2, $\Phi_{3-\mathrm{Q}}$, $\Phi_{3-\mathrm{L}}$, $\Phi_{4-\mathrm{Q}}$, $\Phi_{5-\mathrm{Q}}$ and $\Phi_{\mathrm{Q-L}}$ (case1).

$$\underset{\Delta\mathbf{x}_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{array}{l} f(\mathbf{x}_{\mathrm{NZ}}\ \omega_1) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}_{\mathrm{NZ}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1)\right]\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_2) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}_{\mathrm{NZ}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2)\right]\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \\ \quad\ldots \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_\Gamma) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}_{\mathrm{NZ}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_N)\right]\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \end{array} \right\|_{2\text{ or }\infty} \qquad \text{(Problem 3-Q)}$$

$$\underset{\Delta\mathbf{x}_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{array}{l} f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_1) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \\ f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_2) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \\ \quad\ldots \\ f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_\Gamma) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma)\right]^{\mathrm{T}} \cdot \Delta\mathbf{x}_{\mathrm{NZ}} \end{array} \right\|_{2\text{ or }\infty} \qquad \text{(Problem 3-L-a)}$$

$$\text{subject to } \|\nabla\mathbf{x}_{\mathrm{NZ}}\|_\infty \le \delta \qquad \text{(Problem 3-L-b)}$$

$$\underset{\Delta x_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{array}{l} f(\mathbf{x}_{\mathrm{NZ}}\ \omega_1) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1)\right]^{\mathrm{T}} \cdot \mathbf{d}\cdot\Delta x_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot\mathbf{d}^{\mathrm{T}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1)\right]\cdot\mathbf{d} \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_2) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2)\right]^{\mathrm{T}} \cdot \mathbf{d}\cdot\Delta x_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot\mathbf{d}^{\mathrm{T}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2)\right]\cdot\mathbf{d} \\ \quad\ldots \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_\Gamma) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma) + \left[\nabla f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma)\right]^{\mathrm{T}} \cdot \mathbf{d}\cdot\Delta x_{\mathrm{NZ}} \\ \quad +0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot\mathbf{d}^{\mathrm{T}}\cdot\mathbf{H}\left[f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma)\right]\cdot\mathbf{d} \end{array} \right\|_{2\text{ or }\infty} \qquad \text{(Problem 4-Q)}$$

$$\underset{\Delta x_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{array}{l} f(\mathbf{x}_{\mathrm{NZ}}\ \omega_1) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} + 0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot g_{2i} \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_2) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} + 0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot g_{2i} \\ \quad\ldots \\ f(\mathbf{x}_{\mathrm{NZ}}\ \omega_\Gamma) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} + 0.5\cdot\Delta x^2_{\mathrm{NZ}}\cdot g_{2i} \end{array} \right\|_{2\text{ or }\infty} \qquad \text{(Problem 5-Q)}$$

$$\underset{\Delta x_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{array}{l} f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_1) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_1) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} \\ f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_2) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_2) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} \\ \quad\ldots \\ f_{\mathrm{L}}(\mathbf{x}_{\mathrm{NZ}}\ \omega_\Gamma) = f(\mathbf{x}^{\mathrm{Init}}_{\mathrm{NZ}}\ \omega_\Gamma) + g_{1i}\cdot\Delta x_{\mathrm{NZ}} \end{array} \right\|_{2\text{ or }\infty} \qquad \text{(Problem 5-L-a)}$$

$$\text{Subject to } \nabla x_{\mathrm{NZ}} \le \delta \qquad \text{(Problem 5-L-b)}$$

Secondly, the objective functions of Problems (5-Q) and (5-L) are simplified. There always exists a positive number $\lambda \succ 0$ such that the following Equations (8) and (9), as shown at the bottom of this page, hold true for $\Delta x_{\mathrm{NZ}} \in \left[0 \quad \lambda\right]$. This can be proved using the definition of infinity norm. In brief words, the left-hand sides of Equations (8) and (9) (i.e., the objective functions in Problems (5-Q) and (5-L)) are only determined by functions $f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{i-\max-\mathrm{Q}}\right)$ ($i-\max-\mathrm{Q} \in \Phi_5$) and $f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{i-\max-\mathrm{L}}\right)$ ($i-\max-\mathrm{L} \in \Phi_{3-\mathrm{L}}$), respectively, as long as Δx_{NZ} is sufficiently small.

A special case: Suppose $\Phi_{\mathrm{Q-L}}$ is not empty for some direction $\mathbf{d}$. Thus, the objective function in (Problem 5-Q) can be determined by $f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{i-\max-\mathrm{Q-L}}\right)$ and that in (Problem 5-L) can be determined by $f_{\mathrm{L}}\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{i-\max-\mathrm{Q-L}}\right)$ for the direction $\mathbf{d}$.

Finally, a series of remarks with respect to the optimums of Problems (3-Q) and (3-L) are obtained:

$$
\begin{aligned}
&\left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\right|^2\\
&\quad=\left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right|^2\\
&\qquad+\Delta x_{\mathrm{NZ}}\cdot\underbrace{\left(2\cdot\mathrm{Real}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Real}\left[g_{1i}\right]+2\cdot\mathrm{Imag}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Imag}\left[g_{1i}\right]\right)}_{c_{1i}}\\
&\qquad+\Delta x_{\mathrm{NZ}}^2\cdot\underbrace{\begin{pmatrix}\mathrm{Real}\left[g_{1i}\right]^2+\mathrm{Imag}\left[g_{1i}\right]^2\\ +\mathrm{Real}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Real}\left[g_{2i}\right]+\mathrm{Imag}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Imag}\left[g_{2i}\right]\end{pmatrix}}_{c_{2i}}\\
&\qquad+\Delta x_{\mathrm{NZ}}^3\cdot\underbrace{\left(\mathrm{Real}\left[g_{1i}\right]\cdot\mathrm{Real}\left[g_{2i}\right]+\mathrm{Imag}\left[g_{1i}\right]\cdot\mathrm{Imag}\left[g_{2i}\right]\right)}_{c_{3i}}\\
&\qquad+\Delta x_{\mathrm{NZ}}^4\cdot\underbrace{\left(0.25\cdot\mathrm{Real}\left[g_{2i}\right]^2+0.25\cdot\mathrm{Imag}\left[g_{2i}\right]^2\right)}_{c_{4i}}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\left.\left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot g_{2i}\cdot\Delta x_{\mathrm{NZ}}^2\right|^2\right|_{g_{2i}=0}\\
&\quad=\left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\right|^2\\
&\quad=\left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right|^2\\
&\qquad+\Delta x_{\mathrm{NZ}}\cdot\underbrace{\left(2\cdot\mathrm{Real}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Real}\left[g_{1i}\right]+2\cdot\mathrm{Imag}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_i\right)\right]\cdot\mathrm{Imag}\left[g_{1i}\right]\right)}_{c_{1i}}\\
&\qquad+\Delta x_{\mathrm{NZ}}^2\cdot\underbrace{\left(\mathrm{Real}\left[g_{1i}\right]^2+\mathrm{Imag}\left[g_{1i}\right]^2\right)}_{c_{2i}=\left|g_{1i}\right|^2}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\left\|\begin{matrix}f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_1\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_1\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_2\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_2\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\\ \cdots\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_\Gamma\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_\Gamma\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\end{matrix}\right\|_\infty\\
&\quad=\sqrt{\begin{matrix}\left|f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{i-\max-\mathrm{Q}}\right)\right|^2+\\ \Delta x_{\mathrm{NZ}}\cdot c_{1i-\max-\mathrm{Q}}+\Delta x_{\mathrm{NZ}}^2\cdot c_{2i-\max-\mathrm{Q}}+\Delta x_{\mathrm{NZ}}^3\cdot c_{3i-\max-\mathrm{Q}}+\Delta x_{\mathrm{NZ}}^4\cdot c_{4i-\max-\mathrm{Q}}\end{matrix}}\\
&\quad=\left\|\begin{matrix}f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{Q}}\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_{i-\max-\mathrm{Q}}\right)+g_{1i-\max-\mathrm{Q}}\cdot\Delta x_{\mathrm{NZ}}\\ +0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i-\max-\mathrm{Q}}\end{matrix}\right\|_\infty
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\left\|\begin{matrix}f_{\mathrm{L}}\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_1\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_1\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\\ f_{\mathrm{L}}\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_2\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_2\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\\ \cdots\\ f_{\mathrm{L}}\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_\Gamma\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_\Gamma\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\end{matrix}\right\|_\infty\\
&\quad=\sqrt{\left|f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{L}}\right)\right|^2+\Delta x_{\mathrm{NZ}}\cdot c_{1i-\max-\mathrm{L}}+\Delta x_{\mathrm{NZ}}^2\cdot\left|g_{1i-\max-\mathrm{L}}\right|^2}\\
&\quad=\left\|f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{L}}\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_{i-\max-\mathrm{L}}\right)+g_{1i-\max-\mathrm{L}}\cdot\Delta x_{\mathrm{NZ}}\right\|_\infty
\end{aligned}
\tag{9}
$$

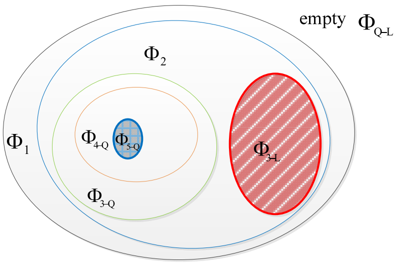

FIGURE 2. Φ_1, Φ_2, Φ_{3-Q}, Φ_{3-L}, Φ_{4-Q}, Φ_{5-Q} and Φ_{Q-L} (case2).

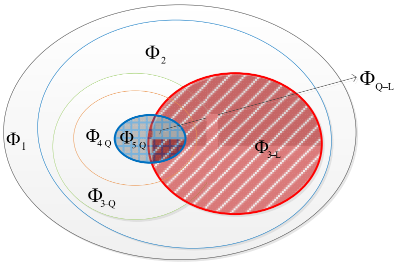

FIGURE 3. Φ_1, Φ_2, Φ_{3-Q}, Φ_{3-L}, Φ_{4-Q}, Φ_{5-Q} and Φ_{Q-L} (case3).

Remark 1: If $\Delta\mathbf{x}_{NZ} = \mathbf{0}$ is the strictly globally/locally optimal solution of (Problem 3-L), it may be/not be the strictly locally optimal solution of (Problem 3-Q).

Proof: Two examples such that the strictly globally/locally optimal solution $\Delta\mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) is not the (strictly) locally optimal solution of (Problem 3-Q) are provided as follows:

The strictly globally/locally optimal solution $\Delta x = 0$ of the TR-IGS-convex-approximation-problem is not the (strictly) locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} (1+2\cdot j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(-7-3\cdot j) \\ (1+2\cdot j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(-8-3\cdot j) \\ (1+2\cdot j)+(-2-j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(-7-3\cdot j) \\ (1+j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(-7-3\cdot j) \end{matrix} \right\|_\infty \right\}$$
(e.g.-L_∞-Complex-1)

The following Figure 4 describes the curves of the objective functions of the above two problems.

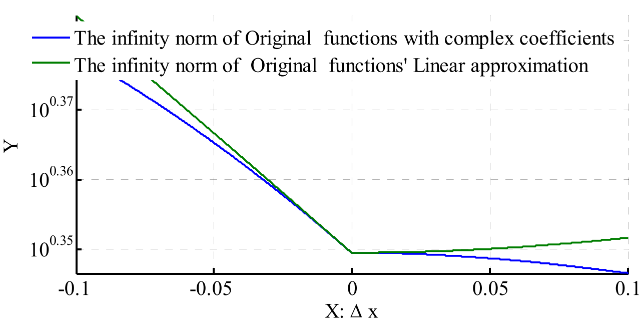

FIGURE 4. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L∞-Complex-1).

The strictly locally optimal solution $\Delta\mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is not the (strictly) locally optimal solution of the original problem

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_\infty$$
(e.g.$-L_\infty$-Complex-3D-1)

where

$$\begin{aligned} \mathbf{grad}_1 &= \mathbf{grad}_2 = \begin{bmatrix} 1+2\cdot j & -1-2\cdot j & 1+2\cdot j \end{bmatrix}^T, \\ \mathbf{grad}_3 &= \begin{bmatrix} 1+2\cdot j & 1+2\cdot j & 0 \end{bmatrix}^T, \\ \mathbf{grad}_4 &= \begin{bmatrix} 0 & 1+2\cdot j & 1+2\cdot j \end{bmatrix}^T, \\ \mathbf{Hess}_1 &= \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 \\ &= \begin{bmatrix} -18-18\cdot j & 0 & 0 \\ 0 & -18-18\cdot j & 0 \\ 0 & 0 & -18-18\cdot j \end{bmatrix}. \end{aligned}$$

The strictly globally/locally optimal solution $\Delta\mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) may also be the strictly locally optimal solution of (Problem 3-Q). Four examples are provided as follows:

The strictly globally/locally optimal solution $\Delta x = 0$ of the TR-IGS-convex-approximation-problem is also the strictly globally/locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} (1+2\cdot j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(3+4\cdot j) \\ (1+2\cdot j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(5+6\cdot j) \\ (1+2\cdot j)+(-2-j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(3+j) \\ (1+j)+(-2+j)\cdot\Delta x+0.5\cdot\Delta x^2\cdot(-7-3\cdot j) \end{matrix} \right\|_\infty \right\}$$
(e.g.$-L_\infty$-Complex-2).

The following Figure 5 describes the curves of the objective functions of the above two problems.

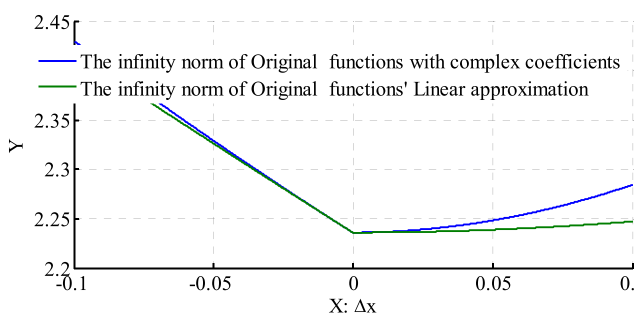

FIGURE 5. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L∞-Complex-3).

The strictly globally/locally optimal solution $\Delta x = 0$ of the TR-IGS-convex-approximation-problem is also the strictly globally/locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} 1 - 2 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot 3 \\ 1 + 0 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot 5 \\ 0.1 - 2 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot 5 \\ 1 + 2 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot 2 \end{matrix} \right\|_\infty \right\} \tag{e.g.-L$_\infty$-Real-1}$$

The following Figure 6 describes the curves of the objective functions of the above two problems.

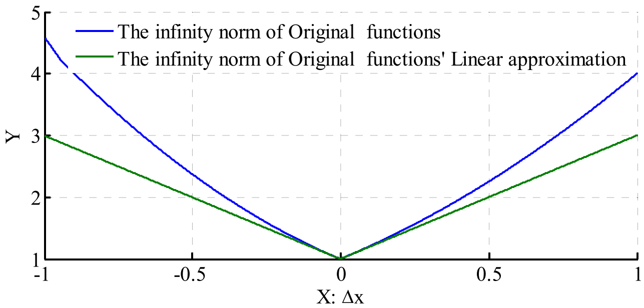

FIGURE 6. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L$_\infty$-Real-1).

The strictly locally optimal solution $\Delta \mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is also the strictly locally optimal solution of the original problem

$$\underset{\Delta \mathbf{x}}{\text{minimize}} \left\| \begin{matrix} 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_1 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_1 \cdot \Delta \mathbf{x} \\ 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_2 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_2 \cdot \Delta \mathbf{x} \\ 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_3 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_3 \cdot \Delta \mathbf{x} \\ 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_4 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_4 \cdot \Delta \mathbf{x} \\ 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_5 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_5 \cdot \Delta \mathbf{x} \\ 2 + \Delta \mathbf{x}^T \cdot \mathbf{grad}_6 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_6 \cdot \Delta \mathbf{x} \end{matrix} \right\|_\infty \tag{e.g.-L$_\infty$-Real-3D-1}$$

where

$$\mathbf{grad}_1 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T, \quad \mathbf{grad}_2 = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \quad \mathbf{grad}_4 = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix}^T,$$
$$\mathbf{grad}_5 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^T, \quad \mathbf{grad}_6 = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}^T$$

and $\mathbf{Hess}_i$ (3×3) can be any matrix of real elements for $i = 1, 2, 3, \ldots, 6$.

The strictly locally optimal solution $\Delta \mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is also the strictly locally optimal solution of the original problem

$$\underset{\Delta \mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_1 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_1 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_2 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_2 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_3 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_3 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_4 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_4 \cdot \Delta \mathbf{x} \end{matrix} \right\|_\infty \tag{e.g.$-L_\infty$-Complex-3D-2}$$

where

$$\mathbf{grad}_1 = \mathbf{grad}_2 = \begin{bmatrix} 1 + 2 \cdot j & -1 - 2 \cdot j & 1 + 2 \cdot j \end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix} 1 + 2 \cdot j & 1 + 2 \cdot j & 0 \end{bmatrix}^T,$$
$$\mathbf{grad}_4 = \begin{bmatrix} 0 & 1 + 2 \cdot j & 1 + 2 \cdot j \end{bmatrix}^T,$$
$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} 18 - 18 \cdot j & 0 & 0 \\ 0 & 18 - 18 \cdot j & 0 \\ 0 & 0 & 18 - 18 \cdot j \end{bmatrix}.$$

The strictly globally/locally optimal solution $\Delta \mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) may also be the non-strictly locally optimal solution of (Problem 3-Q). One example is provided as follows:

The strictly locally optimal solution $\Delta \mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is the non-strictly locally optimal solution of the original problem

$$\underset{\Delta \mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_1 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_1 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_2 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_2 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_3 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_3 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_4 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_4 \cdot \Delta \mathbf{x} \\ (2-j) + \Delta \mathbf{x}^T \cdot \mathbf{grad}_5 + 0.5 \cdot \Delta \mathbf{x}^T \cdot \mathbf{Hess}_5 \cdot \Delta \mathbf{x} \end{matrix} \right\|_\infty \tag{e.g.-L$_\infty$-Complex-3D-3}$$

where

$$\mathbf{grad}_1 = \mathbf{grad}_2 = \begin{bmatrix} 1 + 2 \cdot j & -1 - 2 \cdot j & 1 + 2 \cdot j \end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix} 1 + 2 \cdot j & 1 + 2 \cdot j & 0 \end{bmatrix}^T,$$
$$\mathbf{grad}_4 = \mathbf{grad}_5 = \begin{bmatrix} 0 & 1 + 2 \cdot j & 1 + 2 \cdot j \end{bmatrix}^T, ,$$
$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} -18 - 18 \cdot j & 0 & 0 \\ 0 & -18 - 18 \cdot j & 0 \\ 0 & 0 & -18 - 18 \cdot j \end{bmatrix}$$

and

$$\mathbf{Hess}_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 - 10 \cdot j & 0 \\ 0 & 0 & 10 - 10 \cdot j \end{bmatrix}.$$

Remark 2: The strictly globally/locally optimal solution $\Delta \mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) may be/not be the non-strictly locally optimal solution of (Problem 3-Q).

Proof: Please see the examples of Remark 1.

Remark 3: The strictly globally/locally optimal solution $\Delta \mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) may be/not be the locally optimal solution of (Problem 3-Q).

Proof: Please see the examples of Remark 1:

Remark 4: The non-strictly globally/locally optimal solution $\Delta \mathbf{x}_{NZ} = \mathbf{0}$ of (Problem 3-L) may be/not be the strictly locally optimal solution of (Problem 3-Q).

Proof: The non-strictly globally/locally optimal solution $\Delta x = 0$ of the TR-IGS-convex-approximation-problem is not the (strictly) locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} 1 + 0 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot (-3) \\ 1 + 0 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot (-5) \\ 0.1 + 2 \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot (-5) \\ 1 + (-2) \cdot \Delta x + 0.5 \cdot \Delta x^2 \cdot (-2) \end{matrix} \right\|_\infty \right\} \tag{e.g.-L$_\infty$-Real-1}$$

The following Figure 7 describes the curves of the objective functions of the above two problems.

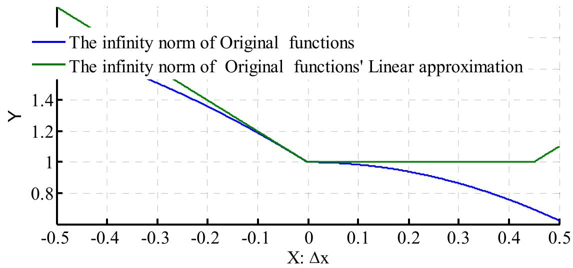

FIGURE 7. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L_∞-Real-1).

The non-strictly globally/locally optimal solution $\Delta\mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is not the (strictly) locally optimal solution of the original problem in the following two examples: (e.g.-L_∞-Real-3D-2) and (e.g.-L_∞-Complex-3D-4)

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_\infty \quad \text{(e.g.-}L_\infty\text{-Real-3D-2)}$$

where

$$\mathbf{grad}_1 = \begin{bmatrix}1 & -1 & 1\end{bmatrix}^T, \quad \mathbf{grad}_2 = \begin{bmatrix}-1 & 1 & -1\end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T, \quad \mathbf{grad}_4 = \begin{bmatrix}-1 & -1 & 0\end{bmatrix}^T,$$

$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix}-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{bmatrix}$$

$$\cdot\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ (1-j)+\Delta\mathbf{x}^T\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_\infty \quad \text{(e.g.-}L_\infty\text{-Complex-3D-4)}$$

where

$$\mathbf{grad}_1 = \mathbf{grad}_2 = \begin{bmatrix}1+2\cdot j & -1-2\cdot j & 1+2\cdot j\end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix}1+2\cdot j & 1+2\cdot j & 0\end{bmatrix}^T,$$
$$\mathbf{grad}_4 = \begin{bmatrix}0 & 1+2\cdot j & 1+2\cdot j\end{bmatrix}^T,$$
$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix}-10+j & 0 & 0 \\ 0 & -10+j & 0 \\ 0 & 0 & -10+j\end{bmatrix}.$$

The non-strictly globally/locally optimal solution of the TR-IGS-convex-approximation-problem is the strictly locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} (1+2\cdot j)+0\cdot\Delta x+0.5\cdot\Delta x^2\cdot(3+3\cdot j) \\ (1+j)+0\cdot\Delta x+0.5\cdot\Delta x^2\cdot(3+3\cdot j) \end{matrix} \right\|_\infty \right\} \quad \text{(e.g.-}L_\infty\text{-Complex-3)}$$

The following Figure 8 describes the curves of the objective functions of the above two problems.

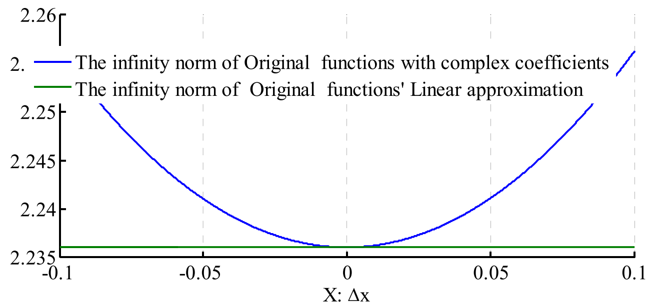

FIGURE 8. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L_∞-Complex-3).

The non-strictly locally optimal solution $\Delta x = 0$ of the TR-IGS-convex-approximation-problem is the strictly globally/locally optimal solution of the original problem

$$\underset{\Delta x}{\text{minimize}} \left\{ \left\| \begin{matrix} 1+0\cdot\Delta x+0.5\cdot\Delta x^2\cdot 3 \\ 1+0\cdot\Delta x+0.5\cdot\Delta x^2\cdot 5 \\ 0.1+2\cdot\Delta x+0.5\cdot\Delta x^2\cdot 5 \\ 1+(-2)\cdot\Delta x+0.5\cdot\Delta x^2\cdot 2 \end{matrix} \right\|_\infty \right\} \quad \text{(e.g.-}L_\infty\text{-Real-2)}$$

The following Figure 9 describes the curves of the objective functions of the above two problems.

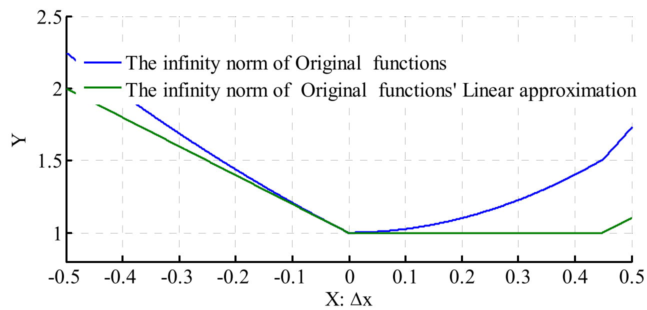

FIGURE 9. The curves of the objective functions of the original problem and its convex approximation. (e.g.-L_∞-Real-2).

The non-strictly globally/locally optimal solution $\Delta\mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is the strictly locally optimal solution of the original problem in the following two examples: ((e.g.-L_∞-Real-3D-3) and (e.g.-L_∞-Complex-3D-5))

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^T\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^T\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_\infty \quad \text{(e.g.-}L_\infty\text{-Real-3D-3)}$$

where

$$\mathbf{grad}_1 = \begin{bmatrix}1 & -1 & 1\end{bmatrix}^T, \quad \mathbf{grad}_2 = \begin{bmatrix}-1 & 1 & -1\end{bmatrix}^T,$$
$$\mathbf{grad}_3 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T, \quad \mathbf{grad}_4 = \begin{bmatrix}-1 & -1 & 0\end{bmatrix}^T,$$

$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ (1-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_{\infty}$$

(e.g.-L_∞-Complex-3D-5)

where

$$\mathbf{grad}_1 = \mathbf{grad}_2 = \begin{bmatrix} 1+2\cdot j & -1-2\cdot j & 1+2\cdot j \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{grad}_3 = \begin{bmatrix} 1+2\cdot j & 1+2\cdot j & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{grad}_4 = \begin{bmatrix} 0 & 1+2\cdot j & 1+2\cdot j \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \begin{bmatrix} 1+j & 0 & 0 \\ 0 & 1+j & 0 \\ 0 & 0 & 1+j \end{bmatrix},$$
$$\mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} 2+j & 0 & 0 \\ 0 & 2+j & 0 \\ 0 & 0 & 2+j \end{bmatrix}.$$

Remark 5: The non-strictly globally/locally optimal solution $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ of (Problem 3-L) may be/not be the non-strictly locally optimal solution of (Problem 3-Q).

Proof: The non-strictly globally/locally optimal solution $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ of (Problem 3-L) may not be the non-strictly locally optimal solution of (Problem 3-Q). Please see (e.g.-L_∞-Real-1), (e.g.-L_∞-Real-3D-2), (e.g.-L_∞-Complex-3D-4), (e.g.-L_∞-Complex-3), (e.g.-L_∞-Real-2), (e.g.-L_∞-Real-3D-3) and (e.g.-L_∞-Complex-3D-5).

The non-strictly globally/locally optimal solution $\Delta\mathbf{x} = \mathbf{0}$ of the TR-IGS-convex-approximation-problem is the non-strictly locally optimal solution of the original problem in the following two examples ((e.g.-L_∞-Real-3D-4) and (e.g.-L_∞-Complex-3D-6))

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} 2+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ 2+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_{\infty}$$

(e.g.-L_∞-Real-3D-4)

where

$$\mathbf{grad}_1 = \mathbf{grad}_3 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{grad}_2 = \mathbf{grad}_4 = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix}^{\mathrm{T}},$$

and

$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\Delta\mathbf{x}}{\text{minimize}} \left\| \begin{matrix} (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_1+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_1\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_2+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_2\cdot\Delta\mathbf{x} \\ (2-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_3+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_3\cdot\Delta\mathbf{x} \\ (1-j)+\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{grad}_4+0.5\cdot\Delta\mathbf{x}^{\mathrm{T}}\cdot\mathbf{Hess}_4\cdot\Delta\mathbf{x} \end{matrix} \right\|_{\infty}$$

(e.g.-L_∞-Complex-3D-6)

where

$$\mathbf{grad}_1 = \mathbf{grad}_2 = \begin{bmatrix} 1+2\cdot j & -1-2\cdot j & 1+2\cdot j \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{grad}_3 = \begin{bmatrix} 1+2\cdot j & 1+2\cdot j & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{grad}_4 = \begin{bmatrix} 0 & 1+2\cdot j & 1+2\cdot j \end{bmatrix}^{\mathrm{T}}$$

and

$$\mathbf{Hess}_1 = \mathbf{Hess}_2 = \mathbf{Hess}_3 = \mathbf{Hess}_4 = \begin{bmatrix} 10-j & 0 & 0 \\ 0 & 10-j & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark 6: The non-strictly globally/locally optimal solution $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ of (Problem 3-L) may be/not be the locally optimal solution of (Problem 3-Q).

Proof: Please see the examples of Remarks (4) and (5).

Remark 7: The globally/locally optimal solution $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ of (Problem 3-L) may be/not be the locally optimal solution of (Problem 3-Q).

Proof: Please see the examples of Remarks (1)-(5).

Remark 8: The locally optimal solution $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ of (Problem 3-Q) must also be the globally/locally optimal solution of (Problem 3-L). *(If (Problem 3-L) is a linear programming problem (please see Part B of this section), a useful necessary condition for the locally optimal solution of the original problem can be obtained.) [7]*

Proof: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the locally optimal solution of (Problem 3-Q), then $c_{1i-\max-\mathrm{Q}} \geq 0$ holds true for any direction $\mathbf{d}$. Because $i-\max-\mathrm{Q} \in \Phi_2$, $i-\max-\mathrm{L} \in \Phi_2$ and $\Phi_2 = \arg\underset{i\in\Phi_1}{\text{maximize}}\, c_{1i}$, $\left(c_{1i-\max-\mathrm{L}} = c_{1i-\max-\mathrm{Q}}\right) \geq 0$ holds true for any direction $\mathbf{d}$. This remark is thus proved.

Remark 9: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the strictly optimal solution of the following linear programming problem

$$\underset{\Delta\mathbf{x}_{\mathrm{NZ}}}{\text{minimize}} \left\| \begin{matrix} \left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_1\right)\right|^2 + \mathbf{c}_1^{\mathrm{T}}\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \\ \left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_2\right)\right|^2 + \mathbf{c}_2^{\mathrm{T}}\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \\ \cdots \\ \left|f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_\Gamma\right)\right|^2 + \mathbf{c}_\Gamma^{\mathrm{T}}\cdot\Delta\mathbf{x}_{\mathrm{NZ}} \end{matrix} \right\|_{\infty} \quad \text{(Problem 6)}$$

where $\underset{i}{\text{maximize}} \left|f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_i\right)\right|^2 \succ 0$ and

$$\mathbf{c}_i^{\mathrm{T}} = \begin{pmatrix} \mathrm{Real}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_i\right)\right]\cdot\mathrm{Real}\left[\left[\nabla f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_i\right)\right]^{\mathrm{T}}\right] \\ +\mathrm{Imag}\left[f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_i\right)\right]\cdot\mathrm{Imag}\left[\left[\nabla f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}} \quad \omega_i\right)\right]^{\mathrm{T}}\right] \end{pmatrix} \tag{10}$$

for $i = 1, 2, 3, \ldots, \Gamma$. Then, $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is also the strictly locally optimal solution of the original problem (Problem 3-Q). *(A sufficient condition for the strictly locally optimal solution of the original problem can be obtained. This condition is of theoretical and practical value in viewing that (Problem 6) is a linear programming problem.) [7]*

Proof: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the strictly optimal solution of Problem (6), then it is strictly optimal in any direction $\mathbf{d}$. So, $\underset{i\in\Phi_1}{\text{maximize}}\, \mathbf{c}_i^{\mathrm{T}}\cdot\mathbf{d} \succ 0$ holds true for any direction $\mathbf{d}$

TABLE 1. The relationship between the optimal solutions of the original problem (Q) and the convex approximation problem (L), L_∞.

	strictly locally optimal solution (Q)		non-strictly locally optimal solution (Q)		locally optimal solution (Q)	
strictly locally optimal solution (L)	may be/not be L→Q (complex) (L∞-Complex-3D-1) (L∞-Complex-3D-2)	must be L→Q (real) (L∞-Real-3D-1)	may be/not be L→Q (complex) (L∞-Complex-3D-2) (L∞-Complex-3D-3)	cannot be L→Q (real) (L∞-Real-3D-1)	may be/not be L→Q (complex) (L∞-Complex-3D-1) (L∞-Complex-3D-2)	must be L→Q (real) (L∞-Real-3D-1)
	may be/not be Q→L (complex) (L∞-Complex-3D-5) (L∞-Complex-3D-2)	may be/not be Q→L (real) (L∞-Real-3D-3) (L∞-Real-3D-1)	may be/not be Q→L (complex) (L∞-Complex-3D-6) (L∞-Complex-3D-3)	cannot be Q→L (real) (L∞-Real-3D-1)	may be/not be Q→L (complex) (L∞-Complex-3D-6) (L∞-Complex-3D-2)	may be/not be Q→L (real) (L∞-Real-3D-4) (L∞-Real-3D-1)
non-strictly locally optimal solution (L)	may be/not be L→Q (complex) (L∞-Complex-3D-6) (L∞-Complex-3D-5)	may be/not be L→Q (real) (L∞-Real-3D-4) (L∞-Real-3D-3)	may be/not be L→Q (complex) (L∞-Complex-3D-5) (L∞-Complex-3D-6)	may be/not be L→Q (real) (L∞-Real-3D-3) (L∞-Real-3D-4)	may be/not be L→Q (complex) (L∞-Complex-3D-4) (L∞-Complex-3D-6)	may be/not be L→Q (real) (L∞-Real-3D-2) (L∞-Real-3D-4)
	may be/not be Q→L (complex) (L∞-Complex-3D-2) (L∞-Complex-3D-5)	may be/not be Q→L (real) (L∞-Real-3D-1) (L∞-Real-3D-3)	may be/not be Q→L (complex) (L∞-Complex-3D-3) (L∞-Complex-3D-6)	must be Q→L (real) (L∞-Real-3D-4)	may be/not be Q→L (complex) (L∞-Complex-3D-2) (L∞-Complex-3D-6)	may be/not be Q→L (real) (L∞-Real-3D-1) (L∞-Real-3D-4)
locally optimal solution (L)	may be/not be L→Q (complex) (L∞-Complex-3D-1) (L∞-Complex-3D-2)	may be/not be L→Q (real) (L∞-Real-3D-4) (L∞-Real-3D-3)	may be/not be L→Q (complex) (L∞-Complex-3D-2) (L∞-Complex-3D-3)	may be/not be L→Q (real) (L∞-Real-3D-3) (L∞-Real-3D-4)	may be/not be L→Q (complex) (L∞-Complex-3D-1) (L∞-Complex-3D-2)	may be/not be L→Q (real) (L∞-Real-3D-2) (L∞-Real-3D-4)
	must be Q→L (complex) (L∞-Complex-3D-2)	must be Q→L (real) (L∞-Real-3D-3)	must be Q→L (complex) (L∞-Complex-3D-3)	must be Q→L(real) (L∞-Real-3D-4)	must be Q→L (complex) (L∞-Complex-3D-2)	must be Q→L (real) (L∞-Real-3D-3)

(Note $\underset{i}{\text{maximize}} \left| f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_i\right)\right|^2 \succ 0$.). Then, $\underset{i\in\Phi_1}{\text{maximize}}\, c_{1i}$ (in the corresponding (Problem 5-Q)) is a positive number for any direction $\mathbf{d}$. Thus, $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is also the strictly optimal solution of the original problem (Problem 3-Q).

Remark 10: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the locally optimal solution of (Problem 3-Q), then it must be the locally optimal solution of (Problem 6). *(A necessary condition for the locally optimal solution of the original problem can be obtained, which is of theoretical and practical value in viewing that (Problem 6) is a linear programming problem.) [7]*

Tip for the Proof: Please see the proof of Remark (8).

B. SPECIAL CASE: $f(\mathbf{x}_{NZ}\ \omega_i)$ ARE REAL-COEFFICIENT FUNCTIONS WITH REAL ARGUMENT $\mathbf{x}_{NZ}$ FOR $i = 1, 2, 3, \ldots, \Gamma$

In this case, (Problem 3-L) is a linear programming problem. Remarks (4)-(10) in Part A still hold true in Part B. However, Remarks (1)-(3) in Part A should be modified in Part B as follows:

Remark 11: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the strictly globally/locally optimal solution of (Problem 3-L), it must also be the strictly locally optimal solution of (Problem 3-Q). *(A useful sufficient condition for the strictly locally optimal solution of the original problem can be obtained.) [7]*

Proof: Firstly, some sets of the indexes (i.e., $i = 1, 2, 3, \ldots, \Gamma$) of the functions $f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_i\right)$ are defined.$sign\,(X)$ is utilized to denote the sign of X (1 for positive number and -1 for negative number.). Let $\Phi_{1-\mathrm{real}} = \arg\underset{i}{\text{maximize}} \left|f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_i\right)\right|$, $\Phi_{2-\mathrm{real}} = \arg\underset{i\in\Phi_{1-\mathrm{real}}}{\text{maximize}} \left[g_{1i}\cdot sign\left(f\left(\mathbf{x}_{\mathrm{NZ}}\ \omega_{\mathrm{i}}\right)\right)\right]$, and $\Phi_{3-\mathrm{Q-real}} = \arg\underset{i\in\Phi_{2-\mathrm{real}}}{\text{maximize}} \left[g_{2i}\cdot sign\left(f\left(\mathbf{x}_{\mathbf{NZ}}\ \omega_i\right)\right)\right]$. Let $i-\max-\mathrm{Q}-\mathrm{real}$ denote an element in set $\Phi_{3-\mathrm{Q-real}}$, and $i-\max-\mathrm{L}-\mathrm{real}$ denote an element in set $\Phi_{2-\mathrm{real}}$. The relationship between $\Phi_{1-\mathrm{real}}$, $\Phi_{2-\mathrm{real}}$ and $\Phi_{3-\mathrm{Q-real}}$ is illustrated in the following Figure 10

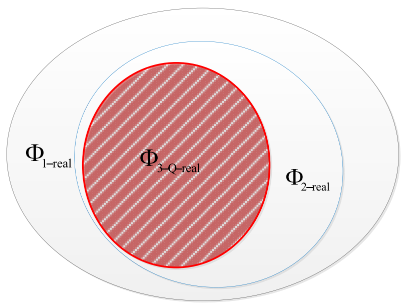

FIGURE 10. Φ_{1-real}, Φ_{2-real} **and** $\Phi_{3-Q-real}$.

Afterwards, the objective functions of Problems (5-Q) and (5-L) are simplified. There always exists a positive number $\lambda \succ 0$ such that the following Equations (3-Q) and (3-L) hold true for $\Delta x_{\mathrm{NZ}} \in \left[0\ \lambda\right]$. This can be proved by the definition of infinity norm (11) and (12), as shown at the bottom of the next page.

Then, according to the assumption that $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the strictly optimal solution of (Problem 3-L) (i.e., it is strictly optimal in any direction $\mathbf{d}$), the following inequality must hold true

$$\left\{\left\{\begin{matrix}\left[g_{1i-\max-\mathrm{L-real}}\cdot sign\left(f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{L-real}}\right)\right)\right] \\ = \underset{i\in\Phi_{1-\mathrm{real}}}{\text{maximize}}\left[g_{1i}\cdot sign\left(f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_i\right)\right)\right]\end{matrix}\right\}\right\} \succ 0 \tag{13}$$

for any direction $\mathbf{d}$. Because $i-\max-\mathrm{Q}-\mathrm{real} \in \Phi_{3-\mathrm{Q-real}}$ and $\Phi_{3-\mathrm{Q-real}} \subseteq \Phi_{2-\mathrm{real}}$, $i-\max-\mathrm{Q}-\mathrm{real} \in \Phi_{2-\mathrm{real}}$; please also see Figure 12. **As** $i-\max-\mathrm{L}-\mathrm{real}$ denotes any element in $\Phi_{2-\mathrm{real}}$ and $i-\max-\mathrm{Q}-\mathrm{real} \in \Phi_{2-\mathrm{real}}$,

TABLE 2. The relationship between the optimal solutions of the original problem (Q) and the convex approximation problem (L), L_2.

	strictly locally optimal solution (Q)		non-strictly locally optimal solution (Q)		locally optimal solution (Q)	
strictly locally optimal solution (L)	may be/not be L→Q (complex) (L_2-Complex-3D-1) (L_2-Complex-3D-2)	may be/not be L→Q (real) (L_2-Real-3D-1) (L_2-Real-3D-2)	cannot be L→Q (complex) (L_2-Complex-3D-2)	cannot be L→Q (real) (L_2-Real-3D-2)	may be/not be L→Q (complex) (L_2-Complex-3D-1) (L_2-Complex-3D-2)	may be/not be L→Q (real) (L_2-Real-3D-1) (L_2-Real-3D-2)
	may be/not be Q→L (complex) (L_2-Complex-3D-4) (L_2-Complex-3D-2)	may be/not be Q→L (real) (L_2-Real-3D-4) (L_2-Real-3D-2)	cannot be Q→L (complex) (L_2-Complex-3D-5)	cannot be Q→L (real) (L_2-Real-3D-5)	may be/not be Q→L (complex) (L_2-Complex-3D-5) (L_2-Complex-3D-2)	may be/not be Q→L (real) (L_2-Real-3D-5) (L_2-Real-3D-2)
non-strictly locally optimal solution (L)	may be/not be L→Q (complex) (L_2-Complex-3D-5) (L_2-Complex-3D-4)	may be/not be L→Q (real) (L_2-Real-3D-5) (L_2-Real-3D-4)	may be/not be L→Q (complex) (L_2-Complex-3D-4) (L_2-Complex-3D-5)	may be/not be L→Q (real) (L_2-Real-3D-4) (L_2-Real-3D-5)	may be/not be L→Q (complex) (L_2-Complex-3D-3) (L_2-Complex-3D-4)	may be/not be L→Q (real) (L_2-Real-3D-3) (L_2-Real-3D-4)
	may be/not be Q→L (complex) (L_2-Complex-3D-2) (L_2-Complex-3D-4)	may be/not be Q→L (real) (L_2-Real-3D-2) (L_2-Real-3D-4)	must be Q→L (complex) (L_2-Complex-3D-5)	must be Q→L (real) (L_2-Real-3D-5)	may be/not be Q→L (complex) (L_2-Complex-3D-2) (L_2-Complex-3D-4)	may be/not be Q→L (real) (L_2-Real-3D-2) (L_2-Real-3D-4)
locally optimal solution (L)	may be/not be L→Q (complex) (L_2-Complex-3D-5) (L_2-Complex-3D-4)	may be/not be L→Q (real) (L_2-Real-3D-5) (L_2-Real-3D-4)	may be/not be L→Q (complex) (L_2-Complex-3D-4) (L_2-Complex-3D-5)	may be/not be L→Q (real) (L_2-Real-3D-4) (L_2-Real-3D-5)	may be/not be L→Q (complex) (L_2-Complex-3D-3) (L_2-Complex-3D-4)	may be/not be L→Q (real) (L_2-Real-3D-3) (L_2-Real-3D-4)
	must be Q→L (complex) (L_2-Complex-3D-4)	must be Q→L (real) (L_2-Real-3D-4)	must be Q→L (complex) (L_2-Complex-3D-5)	must be Q→L(real) (L_2-Real-3D-5)	must be Q→L (complex) (L_2-Complex-3D-4)	must be Q→L (real) (L_2-Real-3D-4)

the following inequality must hold true

$$\left\{\left\{\begin{array}{l}\left[g_{1i-\max-\mathrm{Q}-\mathrm{real}}\cdot sign\left(f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{Q}-\mathrm{real}}\right)\right)\right]\\ =\underset{i\in\Phi_{1-\mathrm{real}}}{\text{maximize}}\left[g_{1i}\cdot sign\left(f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_i\right)\right)\right]\end{array}\right.\right\}\succ 0 \tag{14}$$

for any direction **d**. **So**, $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is also the strictly globally/locally optimal solution of (Problem 3-Q).

According to Remarks (8) and (11), the following Remark (12) can be obtained.

Remark 12: If $\Delta\mathbf{x}_{\mathrm{NZ}} = \mathbf{0}$ is the non-strictly globally/locally optimal solution of (Problem 3-Q), it must also be the non-strictly locally optimal solution of (Problem 3-L).

V. RELATIONSHIP BETWEEN THE OPTIMAL SOLUTION OF THE TR-IGS AND THAT OF THE ORIGINAL PROBLEM, ℓ_2 NORM

The relationship between the optimal solution of the TR-IGS and that of the original problem for the ℓ_2 norm case is provided in the supporting material.

Finally, a complete relationship between the optimal solution of the TR-IGS and that of the original problem is listed in the following Tables 1 (L_∞) and 2 (L_2), which can be obtained based on all the above Remarks (in Sections III and IV). Note for each relationship, the corresponding examples are also provided in these two tables. And, the proofs or tips for the proofs of all the examples in this paper are provided in the supporting material.

VI. CONCLUSION

The MD-FIR filter has been tested to be an effective low-complexity FIR filter [7]. The optimal design of a MD-FIR filter is a high-dimensional non-convex optimization problem. It has been experimentally tested that the coefficients of the MD-FIR filter can be effectively optimized by the TR-IGS algorithm [6], [7]. This algorithm solves a series of the convex-approximation-problems (Problem 2) of the original problem (Problem 1). The relationship between the optimal solution (i.e., theoretical termination point) of the

$$\begin{aligned}&\left\|\begin{array}{l}f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_1\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_1\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_2\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_2\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\\ \cdots\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_\Gamma\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_\Gamma\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i}\end{array}\right\|_\infty\\ &=\left\|\begin{array}{l}f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{Q}-\mathrm{real}}\right)\\ =f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_{i-\max-\mathrm{Q}-\mathrm{real}}\right)+g_{1i-\max-\mathrm{Q}-\mathrm{real}}\cdot\Delta x_{\mathrm{NZ}}+0.5\cdot\Delta x_{\mathrm{NZ}}^2\cdot g_{2i-\max-\mathrm{Q}-\mathrm{real}}\end{array}\right\|_\infty\end{aligned} \tag{11}$$

$$\begin{aligned}&\left\|\begin{array}{l}f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_1\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_1\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_2\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_2\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\\ \cdots\\ f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_\Gamma\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_\Gamma\right)+g_{1i}\cdot\Delta x_{\mathrm{NZ}}\end{array}\right\|_\infty\\ &=\left\|f\left(\mathbf{x}_{\mathrm{NZ}}\quad \omega_{i-\max-\mathrm{L}-\mathrm{real}}\right)=f\left(\mathbf{x}_{\mathrm{NZ}}^{\mathrm{Init}}\quad \omega_{i-\max-\mathrm{L}-\mathrm{real}}\right)+g_{1i-\max-\mathrm{L}-\mathrm{real}}\cdot\Delta x_{\mathrm{NZ}}\right\|_\infty\end{aligned} \tag{12}$$

TR-IGS and that of the original problem is theoretically investigated in this study. A practical issue with respect to TR-IGS is practical TR-IGS generally terminates at a point that is not a theoretical termination point. It will be our future research work to investigate the distance between the practical termination point and a local minimum point.

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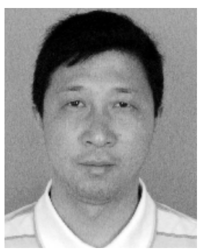

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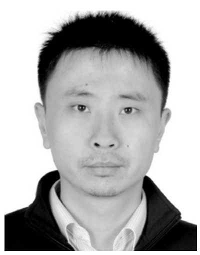

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