IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received November 28, 2019, accepted December 15, 2019, date of publication December 23, 2019, date of current version January 16, 2020.

Digital Object Identifier 10.1109/ACCESS.2019.2961745

Distributed Robust Adaptive Learning Coordination Control for High-Order Nonlinear Multi-Agent Systems With Input Saturation

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This work was supported in part by the Hongliu First-Class Disciplines Development Program of Lanzhou University of Technology, in part by the National Natural Science Foundations of China under Grant 61573013 and Grant 61603286, and in part by the Fundamental Research Funds for the Central Universities under Grant JB190703.

ABSTRACT The paper involves the distributed robust adaptive learning coordination control for high-order nonlinear multi-agent systems, where the leader has nonzero input and followers are subject to input saturation. To solve the problem, two initial assumptions concerning initial state learning and alignment initial condition are introduced, and the distributed learning protocols as well as parameter adaptive laws are designed. It should be noted that the protocols proposed under initial state learning containing the global information are not fully distributed, while the fully distributed protocols can be obtained by the alignment initial condition. Through the rigorous analysis, it is proved that each follower can perfectly track the leader on a finite time interval under both two assumptions. Then, the consensus results under the alignment initial condition are generalized to formation control and two simulation examples verify the correctness and feasibility of the proposed algorithms.

INDEX TERMS Adaptive coordination control, iterative learning control, high-order nonlinear multi-agent systems, input saturation.

I. INTRODUCTION

Over the past several tens of years, great progress has been made in multi-agent systems (MASs) coordination control, and scholars investigated the coordination control from different perspectives [1], [2]. As a basic and significant issue in coordination control, the consensus has gained much attention from the control community [3], [4], especially the leader-follower consensus [5], [6] or consensus tracking [7] which refers to that followers can track a leader in a distributed fashion. Whereas in earlier researches [1], [3] and [4], the coupling gains and spectra of graph matrix must meet some extra requirements, which contained the global information and did not fully utilize the distributed characteristics of MASs. In order to harness the drawback, some researchers applied the adaptive control scheme to design the fully distributed control protocols [8]–[10], where input saturation (IS) was not considered. Actually, most physical systems may subject to IS as the capability of actuators is limited, and this saturated nonlinearity can cause the instability or damage the control systems performance. Hence, it is more practical to take into account IS in the analysis of systems. Fortunately, many efforts have been done regarding MASs with IS [11]–[13].

However, above literatures were implemented when time goes to infinity. If a high precision is needed for the distributed coordination of the MASs with IS from a limited time horizon, those documents are invalid. Iterative learning control (ILC) is an effective approach to realize accurate tracking performance under the repetitive environment [14], [15]. ILC for an individual system, [16] designed an ILC algorithm for the uncertain nonlinear systems having IS, and [17]–[19] proposed the adaptive ILC method for non-linearly parameterised systems having IS. Nevertheless, the results of [16]–[19] having IS were performed under the identical initial conditions (i.i.c) [20] which was a rigorous condition

The associate editor coordinating the review of this manuscript and approving it for publication was Weiguo Xia.

and may not be realized in practice. In addition, until now, ILC has been extensively used to study MASs without IS [21]–[23] for first- and second-order MASs and [24]–[26] for high-order nonlinear MASs (HON-MASs), where [21]–[25] were performed under the alignment initial condition (a.i.c) (a more practical initial condition), whereas [26] assumed the initial state learning containing the global information, and [27] investigated the first-order MASs with IS using a.i.c.

Consequently, based on the aforementioned discussions, it is highly desirable for us to dispose for the distributed robust adaptive learning control (DRALC) problem of HON-MASs with IS by initial state learning and a.i.c. The difficulty lies in how to construct an appropriate Lyapunov-Krasvoskii functional and deal with input saturation in order to obtain the fully distributed protocols with filter errors in the framework of ILC. In this paper, we put forward the DRALC algorithms for HON-MASs with IS under initial state learning as well as a.i.c. The nonzero input of the leader is considered. Since the assumption of initial state learning depends on the global information, the protocols on this occasion are not fully distributed. While the fully distributed protocols can be gained under a.i.c. Through analyzing rigorously, the followers can exactly follow the leader within a limited time interval. Then, the consensus results under a.i.c are extended to formation control. The major highlights fall into three folds.

(1) Different from [11]–[13] for MASs with IS, the perfect consensus tracking can be achieved on a limited time horizon.

(2) Although there exist many existing literatures on IS in ILC frame, most of them including [16]–[19] were for an individual system and under the i.i.c. This paper is first research on the DRALC protocols for the HON-MASs having IS under the initial state learning and a.i.c.

(3) We present a new approach to deal with IS in HON-MASs, which is unlike the work on ILC-based MASs without IS [21]–[26]. Moreover, in the most relevant literature [26], it should be noted that the fuzzy approximation technique was applied to approximate the unknown functions in the dynamics of followers and the parameter γ in initial state learning must satisfy Lemma 2 containing the global information related to the topology structure. Therefore, the protocols designed in [26] were not fully distributed. Nevertheless, to obtain the fully DRALC consensus protocols, the assumption on initial state learning of this paper is changed into the a.i.c and the fully distributed formation control is also investigated.

The rest is displayed as the following. Section 2 states preliminaries and problem formulation. Section 3 demonstrates the DRALC consensus protocols design and Section 4 is the extension to formation control under a.i.c. Two simulation examples are offered in Section 5 and conclusion is given in Section 6.

II. PRELIMINARIES AND PROBLEM DESCRIPTION

A. PRELIMINARIES

Let $\mathcal{G} = (V, E, A)$ denote an undirected graph, where $V = \{v_1, \cdots, v_S\}$ is the set of vertices and $E \subseteq V \times V$ is the set of edges. $A = [a_{ij}] \in R^{S\times S}$ is the weighted adjacency matrix of the graph $\mathcal{G}$. If there is an edge between agents i and j, that is, $(v_j, v_i) \in E$, then $a_{ij} = a_{ji} > 0$, otherwise $a_{ij} = a_{ji} = 0$. Moreover, we assume that $a_{ii} = 0$. The set of neighbors of v_i is $S_i = \{v_j : (v_j, v_i) \in E\}$. The Laplacian matrix of $\mathcal{G}$ is $L = D - A$, where $D = diag\{d_1, \cdots, d_S\}$ with $d_i = \sum_{j=1}^{S} a_{ij}$. A path is a sequence of connected edges in a graph. For the undirected graph $\mathcal{G}$, the adjacency matrix A is symmetric and the graph $\mathcal{G}$ is connected if there is a path between any two vertices.

In what follows, we mainly consider $\bar{\mathcal{G}}$ associated with the system consisting of S agents whose topology graph is denoted by $\mathcal{G}$ and one leader (labeled as 0). Let $\bar{b}_i$ denote the connection weight between the agent i and the leader. If the ith agent can obtain information from the leader, then $\bar{b}_i > 0$, otherwise $\bar{b}_i = 0$. It is obvious that $H = L + B$ is a symmetric matrix associated with $\bar{\mathcal{G}}$, where $B = diag\{\bar{b}_1, \cdots, \bar{b}_n\}$.

Lemma 1 [27]: If the graph $\bar{\mathcal{G}}$ is connected, then the symmetric matrix H associated with $\bar{\mathcal{G}}$ is positive definite.

B. PROBLEM DESCRIPTION

During the kth loop, the dynamics of ith follower are governed by

$$\begin{cases} \dot{x}_{i,s}^k = x_{i,s+1}^k, & s = 1, 2, \ldots, n-1, \\ \dot{x}_{i,s}^k = \eta(\bar{x}_i^k, t) + sat(u_i^k, u^*), \end{cases} \tag{1}$$

where $s = n$, $i = 1, 2, \ldots S$; $\bar{x}_i^k = [x_{i,1}^k, x_{i,2}^k, \ldots, x_{i,n}^k] \in R^n$ and $u_i^k \in R$ are the states and control input of ith follower, respectively; $\eta(\bar{x}_i^k, t) : R^n \times R^+ \to R$ is an unknown time-varying Lipschitz continuously differentiable function in $\bar{x}_i^k(t)$, and piecewise continuous in t; $sat(u_i^k, u^*)$ is a saturation function defined in [16].

Let $x_s^k = [x_{1,s}^k, x_{2,s}^k, \ldots, x_{S,s}^k]^T$, then the MASs (1) can be written as

$$\begin{cases} \dot{x}_s^k = x_{s+1}^k, & s = 1, 2, \ldots, n-1, \\ \dot{x}_n^k = \eta(x^k, t) + \tilde{u}^k, & s = n, \end{cases} \tag{2}$$

where $\eta(x^k, t) = [\eta(\bar{x}_1^k, t), \eta(\bar{x}_2^k, t), \ldots, \eta(\bar{x}_S^k, t)]^T \in R^S$, and $\tilde{u}^k(t) = [\tilde{u}_1^k, \tilde{u}_2^k, \ldots, \tilde{u}_S^k]^T \in R^S$ with $\tilde{u}_i^k = sat(u_i^k, u^*)$.

Assumption 1: There exist constants $l_s > 0$ $(s = 1, 2, \ldots, n)$ which makes

$$|\eta(x_s, t) - \eta(y_s, t)| \leq \sum_{s=1}^{n} l_s |x_s - y_s|,$$

for all $x_s \in R^n$ and $y_s \in R^n$.

The dynamics of the leader are

$$\begin{cases} \dot{x}_{0,s} = x_{0,s+1}, & s = 1, 2, \ldots, n-1, \\ \dot{x}_{0,n} = \eta(x_0, t) + u_0, & s = n, \end{cases} \tag{3}$$

where $x_0 = [x_{0,1}, x_{0,2}, \ldots, x_{0,n}]^T \in R^n$, $u_0 \in R$ are the states and control input of the leader with $|u_0| \leq u^*$.

Denote the consensus errors as

$$\delta_{i,s}^k(t) = x_{i,s}^k(t) - x_{0,s}(t), \tag{4}$$

and the vector form is

$$\delta_s^k(t) = x_s^k(t) - 1_S x_{0,s}(t), \tag{5}$$

where $\delta_s^k(t) = [\delta_{1,s}^k(t), \delta_{2,s}^k(t), \ldots, \delta_{S,s}^k(t)]^T \in R^S$, $x_s^k(t) = [x_{1,s}^k(t), x_{2,s}^k(t), \ldots, x_{S,s}^k(t)]^T \in R^S$ and $1_S = [1, 1, \ldots, 1]^T \in R^S$.

The ultimate aim of this paper is to find the proper protocols u_i^k for $k \in Z^+$, $\forall t \in [0, T]$ and $i = 1, 2, \ldots, S$, which can make each follower track the leader exactly on $[0, T]$ along the iteration axis, that is, the perfect consensus tracking is realized, $\lim_{k\to\infty} \delta_s^k = 0$, $\forall t \in [0, T]$ and $s = 1, 2, \ldots, n$.

Moreover, the distributed errors of ith follower are defined as

$$e_{i,s}^k = \sum_{j=1}^{S} a_{ij}(x_{j,s}^k - x_{i,s}^k) + b_i(x_{0,s} - x_{i,s}^k) \tag{6}$$

and the compact form of the errors is

$$e_s^k = -(L + B)(x_s^k - 1_S x_{0,s}) = -H\delta_s^k, \tag{7}$$

where $e_s^k = [e_{1,s}^k, e_{2,s}^k, \ldots, e_{S,s}^k]^T \in R^S$.

Assumption 2: Assuming the initial learning protocols is

$$x_{i,s}^{k+1}(0) = x_{i,s}^k(0) + \beta e_{i,s}^k(0), \tag{8}$$

where β is a designed parameter.

Lemma 2 [26]: The initial learning protocols can ensure that $\lim_{k\to\infty} \left\|\delta_s^k(0)\right\|_2 = 0$, if β satisfies $\|I - \beta H\|_2 < 1$.

Remark 1: For any iteration, we have $\left\|\delta_s^k(0)\right\|_2 < m$ with m being a positive constant.

To design the distributed adaptive ILC protocols, the filter errors of ith follower are denoted as

$$\sigma_{e_i}^k(t) = \lambda_1 e_{i,1}^k + \lambda_2 e_{i,2}^k + \cdots + \lambda_{n-1} e_{i,n-1}^k + e_{i,n}^k, \tag{9}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the coefficients of Hurwitz polynomial $p^{n-1} + \lambda_{n-1}p^{n-2} + \cdots + \lambda_1$. Then,

$$\begin{aligned}\sigma_e^k(t) &= \lambda_1 e_1^k + \lambda_2 e_2^k + \cdots + \lambda_{n-1} e_{n-1}^k + e_n^k \\ &= -H(\lambda_1\delta_1^k + \lambda_2\delta_2^k + \cdots + \lambda_{n-1}\delta_{n-1}^k + \delta_n^k) \\ &= -H\sigma_\delta^k,\end{aligned} \tag{10}$$

where $\sigma_e^k(t) = [\sigma_{e_1}^k, \sigma_{e_2}^k, \cdots, \sigma_{e_S}^k]^T \in R^S$ and $\sigma_\delta^k = \lambda_1\delta_1^k + \lambda_2\delta_2^k + \cdots + \lambda_{n-1}\delta_{n-1}^k + \delta_n^k \in R^S$.

Furthermore, to promote the subsequent analysis, define $F_1^k = [(\delta_1^k)^T, (\delta_2^k)^T, \cdots, (\delta_{n-1}^k)^T]^T \in R^{S(n-1)\times 1}$, $F_2^k = [(\delta_2)^T, (\delta_3^k)^T, \cdots, (\delta_n^k)^T]^T \in R^{S(n-1)\times 1}$, $h = [0, \cdots, 0, 1]^T \in R^{(n-1)\times 1}$ and $\Omega = \begin{bmatrix} 0 & I_{n-2} \\ -\lambda_1 & -\lambda_2 \ \cdots - \lambda_{n-1}\end{bmatrix} \in R^{(n-1)\times(n-1)}$, then

$$\dot{F}_1^k = F_2^k = (\Omega \otimes I_S)F_1^k + (h \otimes I_S)\sigma_\delta^k. \tag{11}$$

Since Ω is Hurwitz, for any constant $\mu > 0$, there exists $\mathcal{Q} \in R^{(n-1)\times(n-1)}$ and $\mathcal{Q}^T = \mathcal{Q}$, such that

$$\Omega^T\mathcal{Q} + \mathcal{Q}\Omega = -\mu I_{n-1}. \tag{12}$$

Remark 2: From Assumption 3, it is easy to know that $e_s^{k+1}(0) = (I - \beta H)^{k+1}e_s^0(0)$ and $\delta_s^{k+1}(0) = (I - \beta H)^{k+1}\delta_s^0(0)$, $s = 1, 2, \ldots, n$. Then, $\sigma_\delta^{k+1}(0) = (I - \beta H)^{k+1}\sigma_\delta^0(0)$ and $F_1^{k+1}(0) = (I - \beta H)^{k+1}F_1^0(0)$, which means that $\sigma_\delta^k(0)$ and $F_1^k(0)$ are all bounded.

Taking the derivative of $\sigma_\delta^k(t)$ yields

$$\begin{aligned}\dot{\sigma}_\delta^k(t) &= \lambda_1\dot{\delta}_1^k + \lambda_2\dot{\delta}_2^k + \cdots + \lambda_{n-1}\dot{\delta}_{n-1}^k + \dot{\delta}_n^k \\ &= \lambda_1\delta_2^k + \lambda_2\delta_3^k + \cdots + \lambda_{n-1}\delta_n^k + (\dot{x}_n^k - 1_S\dot{x}_{0,n}) \\ &= \lambda_1\delta_2^k + \lambda_2\delta_3^k + \cdots + \lambda_{n-1}\delta_n^k \\ &\quad + (\eta(x^k, t) - 1_S\eta(x_0, t) + \tilde{u}^k - 1_S u_0) \\ &= \rho^k + (\eta(x^k, t) - 1_S\eta(x_0, t) + \tilde{u}^k - 1_S u_0),\end{aligned} \tag{13}$$

where $\rho^k(t) = \lambda^T F_2^k \in R^S$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{n-1}]^T \in R^{n-1}$.

III. DISTRIBUTED ROBUST ADAPTIVE LEARNING CONSENSUS PROTOCOLS DESIGN FOR HON-MASS WITH IS

To tackle the consensus problem of the MASs (1) and (3), the distributed robust learning consensus protocols (DRLCP) for ith follower are devised as

$$u_i^k = sat(u_i^{k-1}, u^*) + \hat{\phi}_i^k(t)\sigma_{e_i}^k, \quad u_i^{-1}(t) = 0, \ \forall t \in [0, T] \tag{14}$$

and

$$\hat{\phi}_i^k(t) = sat(\hat{\phi}_i^{k-1}(t) + q_i(\sigma_{e_i}^k)^2, \phi^*), \tag{15}$$

where $\hat{\phi}_i^{-1}(t) > 0$, $\hat{\phi}_i^0(0) > 0$, $q_i > 0$ and ϕ^* indicates the saturated bound of $\hat{\phi}_i^k$.

Remark 3: It should be noted that the DRLCP (14) consists of two terms. The first term $sat(u_i^{k-1}, u^*)$ is used to deal with the saturation nonlinearity and guarantee the perfect consensus tracking on $[0, T]$. The second term $\hat{\phi}_i^k\sigma_{e_i}^k$ is the adaptive learning term with time varying coupling gains $\hat{\phi}_i^k(t)$ having the fully saturation difference learning laws (15), which can make the DRLCP fully distributed. Meanwhile, from (15), $\hat{\phi}_i^k$ is obviously bounded, and $\hat{\phi}_i^0(0) > 0$ can ensure $\hat{\phi}_i^k > 0$.

The DRLCP (14) can be collectively written as

$$u^k = \tilde{u}^{k-1} + \hat{\Phi}^k\sigma_e^k = \tilde{u}^{k-1} - \hat{\Phi}^k H\sigma_\delta^k \tag{16}$$

where $u^k = [u_1^k, u_2^k, \ldots, u_S^k]^T \in R^S$, $\tilde{u}^{k-1} = [\tilde{u}_1^{k-1}, \tilde{u}_2^{k-1}, \ldots, \tilde{u}_S^{k-1}]^T \in R^S$ with $\tilde{u}_i^{k-1} = sat(u_i^{k-1}, u^*)$, and $\hat{\Phi}^k = diag\{\hat{\phi}_1^k(t), \hat{\phi}_2^k(t), \ldots, \hat{\phi}_S^k(t)\} \in R^{S\times S}$.

Theorem 1: For the MASs (1) and (3) with the connected topology graph $\bar{\mathcal{G}}$, if Assumptions 1 and 2 hold, then the leader can be perfectly tracked by S followers under the protocols (14) and adaptive updating laws (15) along the iteration axis, i.e., $\lim_{k\to\infty} \delta_s^k = 0$ for $\forall t \in [0, T]$, $s = 1, 2, \ldots, n$. Besides, the variables involved in the closed loop systems are all finite.

Proof: At the kth iteration, let us define a Lyapunov function as

$$\bar{V}^k = \frac{1}{2}(\sigma_\delta^k)^T H\sigma_\delta^k + \frac{1}{2}(F_1^k)^T(\mathcal{Q} \otimes I_S)F_1^k. \tag{17}$$

Then, the derivative of $\bar{V}^k$ is calculated as

$$\begin{aligned}\dot{\bar{V}}^k &= (\sigma_\delta^k)^T H \dot{\sigma}_\delta^k + (F_1^k)^T (\mathcal{Q} \otimes I_S) \dot{F}_1^k \\ &= (\sigma_\delta^k)^T H[\lambda^T F_2^k + \eta(x^k, t) - 1_S \eta(x_0, t) + \tilde{u}^k - 1_S u_0] \\ &\quad + (F_1^k)^T (\mathcal{Q} \otimes I_S)[(\Omega \otimes I_S) F_1^k + (h \otimes I_S)\sigma_\delta^k] \\ &= (\sigma_\delta^k)^T H[\lambda^T (\Omega \otimes I_S) F_1^k + \lambda^T (h \otimes I_S)\sigma_\delta^k] \\ &\quad + (\sigma_\delta^k)^T H[\eta(x^k, t) - 1_S \eta(x_0, t)] \\ &\quad + (\sigma_\delta^k)^T H(\tilde{u}^k - u^k + u^k - 1_S u_0) \\ &\quad + \frac{1}{2}(F_1^k)^T [(\mathcal{Q}\Omega + \Omega^T \mathcal{Q}) \otimes I_S] F_1^k + (F_1^k)^T (\mathcal{Q} h \otimes I_S)\sigma_\delta^k \\ &= (\sigma_\delta^k)^T H \lambda^T (\Omega \otimes I_S) F_1^k + (\sigma_\delta^k)^T H \lambda^T (h \otimes I_S)\sigma_\delta^k \\ &\quad + (\sigma_\delta^k)^T H[\eta(x^k, t) - 1_S \eta(x_0, t)] - (\sigma_\delta^k)^T H \hat{\Phi}^k H \sigma_\delta^k \\ &\quad + (\sigma_\delta^k)^T H(\tilde{u}^k - u^k + \tilde{u}^{k-1} - 1_S u_0) \\ &\quad - \frac{1}{2}\mu (F_1^k)^T F_1^k + (F_1^k)^T (\mathcal{Q} h \otimes I_S)\sigma_\delta^k, \end{aligned} \tag{18}$$

where

$$\begin{aligned}&(\sigma_\delta^k)^T H[\eta(x^k, t) - 1_S \eta(x_0, t)] \\ &= \sum_{i=1}^{S} \sigma_{e_i}^k [\eta(x_0, t) - \eta(\bar{x}_i^k, t)] \\ &\le \sum_{i=1}^{S} \left|\sigma_{e_i}^k\right| (l_1 \left|x_{i,1}^k - x_{0,1}\right| + l_2 \left|x_{i,2}^k - x_{0,2}\right| + \cdots + l_n \left|x_{i,n}^k - x_{0,n}\right|) \\ &= \sum_{i=1}^{S} \left|\sigma_{e_i}^k\right| (l_1 \left|\delta_{i,1}^k\right| + l_2 \left|\delta_{i,2}^k\right| + \cdots + l_n \left|\delta_{i,n}^k\right|) \\ &= \left|\sigma_e^k\right|^T (l^T \left|F_1^k\right|) + l_n \left|\sigma_e^k\right|^T \left|\delta_n^k\right| \\ &\le \left\|\sigma_e^k\right\| \|l\| \left\|F_1^k\right\| + l_n \left\|\sigma_e^k\right\| \left\|\sigma_\delta^k - \lambda^T F_1^k\right\| \\ &= \left\|-H\sigma_\delta^k\right\| \|l\| \left\|F_1^k\right\| + l_n \left\|-H\sigma_\delta^k\right\| (\left\|\sigma_\delta^k\right\| + \|\lambda\| \left\|F_1^k\right\|) \\ &\le \lambda_{\max}(H) \|l\| \left\|\sigma_\delta^k\right\| \left\|F_1^k\right\| \\ &\quad + l_n \lambda_{\max}(H) \left\|\sigma_\delta^k\right\| (\left\|\sigma_\delta^k\right\| + \|\lambda\| \left\|F_1^k\right\|) \\ &= l_n \lambda_{\max}(H)(\sigma_\delta^k)^T \sigma_\delta^k + \lambda_{\max}(H)(\|l\| + l_n \|\lambda\|) \left\|\sigma_\delta^k\right\| \left\|F_1^k\right\| \\ &\le l_n \lambda_{\max}(H)(\sigma_\delta^k)^T \sigma_\delta^k \\ &\quad + \frac{\lambda_{\max}(H)(\|l\| + l_n \|\lambda\|)}{2} (\left\|\sigma_\delta^k\right\|^2 + \left\|F_1^k\right\|^2) \\ &= \frac{\lambda_{\max}(H)(2l_n + \|l\| + l_n \|\lambda\|)}{2} (\sigma_\delta^k)^T \sigma_\delta^k \\ &\quad + \frac{\lambda_{\max}(H)(\|l\| + l_n \|\lambda\|)}{2} (F_1^k)^T F_1^k, \end{aligned} \tag{19}$$

and $l = [l_1, l_2, \ldots, l_{n-1}]^T$.

Substituting (19) into (18) results,

$$\begin{aligned}&\dot{\bar{V}}^k(t) \\ &\le \frac{\lambda_{\max}(H)(\|l\| + l_n \|\lambda\|) - \mu}{2} (F_1^k)^T F_1^k - (\sigma_\delta^k)^T H \hat{\Phi}^k H \sigma_\delta^k \\ &\quad + (\sigma_\delta^k)^T H \lambda^T (\Omega \otimes I_S) F_1^k + (F_1^k)^T (\mathcal{Q} h \otimes I_S)\sigma_\delta^k \\ &\quad + (\sigma_\delta^k)^T [H \lambda^T (h \otimes I_S) \\ &\quad + \frac{\lambda_{\max}(H)(2l_n + \|l\| + l_n \|\lambda\|)}{2} I_S] \sigma_\delta^k \\ &\quad + (\sigma_\delta^k)^T H(\tilde{u}^k - u^k + \tilde{u}^{k-1} - 1_S u_0). \end{aligned} \tag{20}$$

At this time, select a Lyapunov-Krasovskii functional as

$$\bar{E}^k(t) = \bar{V}^k + \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t (\tilde{\phi}_i^k(\rho))^2 d\rho + \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k(\rho)} (\delta u_i^k)^2 d\rho, \tag{21}$$

where $\tilde{\phi}_i^k(t) = \phi - \hat{\phi}_i^k(t)$ with $\phi \in [-\phi^*, \phi^*]$, and $\delta u_i^k = u_i^k - u_0$.

The rest of the proof consists of three parts, i.e., calculating the difference between $\bar{E}^k$ and $\bar{E}^{k-1}$, obtaining the boundednesses of all variables and proving the consensus results.

Part I. Difference between $\bar{E}^k$ and $\bar{E}^{k-1}$.

Denote $\Delta\bar{E}^k(t)$ as the difference between $\bar{E}^k$ and $\bar{E}^{k-1}$, we have

$$\begin{aligned}&\Delta\bar{E}^k(t) \\ &= \bar{E}^k - \bar{E}^{k-1} \\ &= \bar{V}^k + \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t [(\tilde{\phi}_i^k)^2 - (\tilde{\phi}_i^{k-1})^2] d\rho - \bar{V}^{k-1} \\ &\quad + \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} (\delta u_i^k)^2 d\rho - \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^{k-1}} (\delta u_i^{k-1})^2 d\rho \\ &= \int_0^t \dot{\bar{V}}^k d\rho + \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t [(\tilde{\phi}_i^k)^2 - (\tilde{\phi}_i^{k-1})^2] d\rho \\ &\quad + \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} (\delta u_i^k)^2 d\rho - \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^{k-1}} (\delta u_i^{k-1})^2 d\rho \\ &\quad + \bar{V}^k(0) - \bar{V}^{k-1}. \end{aligned} \tag{22}$$

According to (15), it should be noted that $\hat{\phi}_i^k \ge \hat{\phi}_i^{k-1}$, (22) becomes as

$$\begin{aligned}\Delta\bar{E}^k &\le \int_0^t \dot{\bar{V}}^k d\rho + \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t [(\tilde{\phi}_i^k)^2 - (\tilde{\phi}_i^{k-1})^2] d\rho \\ &\quad + \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} [(\delta u_i^k)^2 - (\delta u_i^{k-1})^2] d\rho \\ &\quad + \bar{V}^k(0) - \bar{V}^{k-1}. \end{aligned} \tag{23}$$

Utilizing the algebraic relation $(z - k_1)^2 - (z - k_2)^2 = (k_2 - k_1)[2(z - k_1) + (k_1 - k_2)]$ and (15), the second term on the right hand (RHS) of (23) changes into

$$\begin{aligned}&\sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t [(\tilde{\phi}_i^k)^2 - (\tilde{\phi}_i^{k-1})^2] d\rho \\ &= \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t (\hat{\phi}_i^{k-1} - \hat{\phi}_i^k)[2(\phi - \hat{\phi}_i^k) + (\hat{\phi}_i^k - \hat{\phi}_i^{k-1})] d\rho \\ &= \sum_{i=1}^{S} \frac{1}{q_i} \int_0^t \tilde{\phi}_i^k (\hat{\phi}_i^{k-1} - \hat{\phi}_i^k) d\rho - \sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t (\hat{\phi}_i^k - \hat{\phi}_i^{k-1})^2 d\rho \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{S} \frac{1}{q_i} \int_0^t \tilde{\phi}_i^k (\hat{\phi}_i^{k-1} - \hat{\phi}_i^k) d\rho \\
&= \sum_{i=1}^{S} \frac{1}{q_i} \int_0^t \tilde{\phi}_i^k (\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2 - q_i (\sigma_{e_i}^k)^2 - \hat{\phi}_i^k) d\rho \\
&= \sum_{i=1}^{S} \frac{1}{q_i} \int_0^t \tilde{\phi}_i^k [\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2 - \hat{\phi}_i^k] d\rho - \int_0^t \tilde{\phi}_i^k (\sigma_{e_i}^k)^2 d\rho.
\end{aligned} \tag{24}
$$

In accordance with the saturation function property (PSF) 3 in [27], it can be obtained that

$$
\begin{aligned}
&\tilde{\phi}_i^k [\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2 - \hat{\phi}_i^k] \\
&\quad = (\phi - \hat{\phi}_i^k)[\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2 - \hat{\phi}_i^k] \\
&\quad = [\phi - sat(\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2, \phi^*)] \\
&\qquad [\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2 - sat(\hat{\phi}_i^{k-1} + q_i (\sigma_{e_i}^k)^2, \phi^*)] \\
&\quad \leq 0.
\end{aligned} \tag{25}
$$

Hence, according to (24) and (25), we can know that

$$
\begin{aligned}
&\sum_{i=1}^{S} \frac{1}{2q_i} \int_0^t [(\tilde{\phi}_i^k)^2 - (\tilde{\phi}_i^{k-1})^2] d\rho \\
&\quad \leq -\sum_{i=1}^{S} \int_0^t \tilde{\phi}_i^k(\rho) (\sigma_{e_i}^k)^2 d\rho \\
&\quad = -\phi \sum_{i=1}^{S} \int_0^t (\sigma_{e_i}^k)^2 d\rho + \sum_{i=1}^{S} \int_0^t \hat{\phi}_i^k (\sigma_{e_i}^k)^2 d\rho \\
&\quad = -\phi \int_0^t (\sigma_\delta^k)^T H^2 \sigma_\delta^k d\rho + \int_0^t (\sigma_\delta^k)^T H \hat{\Phi}^k(\rho) H \sigma_\delta^k d\rho.
\end{aligned} \tag{26}
$$

From the PSF 1 in [18] and DRLCP (14), the third term on the RHS of (23) is written as

$$
\begin{aligned}
&\sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} [(\delta u_i^k)^2 - (\delta u_i^{k-1})^2] d\rho \\
&\quad \leq \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} [(u_i^k - u_0)^2 - (\tilde{u}_i^{k-1} - u_0)^2] d\rho \\
&\quad = \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} (\tilde{u}_i^{k-1} - u_i^k)[2(u_0 - u_i^k) + (u_i^k - \tilde{u}_i^{k-1})] d\rho \\
&\quad = \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} (-\hat{\phi}_i^k(\rho) \sigma_{e_i}^k [2(u_0 - u_i^k)] d\rho \\
&\qquad - \sum_{i=1}^{S} \int_0^t \frac{1}{2\hat{\phi}_i^k} (u_i^k - \tilde{u}_i^{k-1})^2 d\rho \\
&\quad = \sum_{i=1}^{S} \int_0^t \sigma_{e_i}^k (u_i^k - u_0) d\rho - \sum_{i=1}^{S} \int_0^t \frac{\hat{\phi}_i^k}{2} (\sigma_{e_i}^k)^2 d\rho \\
&\quad = \int_0^t (\sigma_\delta^k)^T H (1_S u_0 - u^k) d\rho - \frac{1}{2} \int_0^t (\sigma_\delta^k)^T H \hat{\Phi}^k H \sigma_\delta^k d\rho.
\end{aligned} \tag{27}
$$

Substituting (20), (26) and (27) into (23) leads to

$$
\begin{aligned}
&\Delta \bar{E}^k(t) \\
&\leq \frac{\lambda_{\max}(H)(\|l\| + l_n \|\lambda\|) - \mu}{2} \int_0^t (F_1^k)^T F_1^k d\rho \\
&\quad + \int_0^t (\sigma_\delta^k)^T H \lambda^T (\Omega \otimes I_S) F_1^k d\rho \\
&\quad + \int_0^t (F_1^k)^T (\mathcal{Q}h \otimes I_S) \sigma_\delta^k d\rho \\
&\quad + \int_0^t (\sigma_\delta^k)^T [H \lambda^T (h \otimes I_S) \\
&\quad + \frac{\lambda_{\max}(H)(2l_n + \|l\| + l_n \|\lambda\|)}{2} I_S] \sigma_\delta^k d\rho \\
&\quad + \int_0^t (\sigma_\delta^k)^T H (\tilde{u}^k - 2u^k + \tilde{u}^{k-1}) d\rho \\
&\quad - \frac{1}{2} \int_0^t (\sigma_\delta^k)^T H \hat{\Phi}^k(\rho) H \sigma_\delta^k d\rho - \phi \int_0^t (\sigma_\delta^k)^T H^2 \sigma_\delta^k d\rho \\
&\quad + \bar{V}^k(0) - \bar{V}^{k-1},
\end{aligned} \tag{28}
$$

and based on PSF 2 in [16] as well as the DRLCP (14), we have $|u_i^k - \tilde{u}_i^k| \leq \hat{\phi}_i^k(t) |\sigma_{e_i}^k|$, it can be attained that

$$
\begin{aligned}
&\int_0^t (\sigma_\delta^k)^T H (\tilde{u}^k - 2u^k + \tilde{u}^{k-1}) d\rho \\
&\quad = -\int_0^t (\sigma_e^k)^T (\tilde{u}^k - 2u^k + \tilde{u}^{k-1}) d\rho \\
&\quad = \sum_{i=1}^{S} \int_0^t \sigma_{e_i}^k (u_i^k - \tilde{u}_i^k + u_i^k - \tilde{u}_i^{k-1}) d\rho \\
&\quad \leq \sum_{i=1}^{S} \int_0^t |\sigma_{e_i}^k| (|\tilde{u}_i^k - u_i^k| + |\tilde{u}_i^{k-1} - u_i^k|) d\rho \\
&\quad \leq \sum_{i=1}^{S} \int_0^t |\sigma_{e_i}^k| (\hat{\phi}_i^k |\sigma_{e_i}^k| + \hat{\phi}_i^k |\sigma_{e_i}^k|) d\rho \\
&\quad = 2 \sum_{i=1}^{S} \int_0^t \hat{\phi}_i^k (\sigma_{e_i}^k)^2 d\rho \\
&\quad = 2 \int_0^t (\sigma_\delta^k)^T H \hat{\Phi}^k H \sigma_\delta^k d\rho.
\end{aligned} \tag{29}
$$

Taking (29) into (28) results

$$
\begin{aligned}
\Delta \bar{E}^k(t) &\leq \frac{\lambda_{\max}(H)(\|l\| + l_n \|\lambda\|) - \mu}{2} \int_0^t (F_1^k)^T F_1^k d\rho \\
&\quad + \int_0^t (\sigma_\delta^k)^T H \lambda^T (\Omega \otimes I_S) F_1^k d\rho \\
&\quad + \int_0^t (F_1^k)^T (\mathcal{Q}h \otimes I_S) \sigma_\delta^k d\rho \\
&\quad + \int_0^t (\sigma_\delta^k)^T [H \lambda^T (h \otimes I_S) \\
&\quad + \frac{\lambda_{\max}(H)(2l_n + \|l\| + l_n \|\lambda\|)}{2} I_S] \sigma_\delta^k d\rho \\
&\quad + \bar{V}^k(0) - \bar{V}^{k-1} + (\frac{3}{2}\phi^* - \phi) \int_0^t (\sigma_\delta^k)^T H^2 \sigma_\delta^k d\rho.
\end{aligned} \tag{30}
$$

Denoting $Z^k = [(F_1^k)^T, (\sigma_\delta^k)^T]^T$ brings about

$$\Delta \bar{E}^k \leq -\int_0^t (Z^k)^T \Omega Z^k d\rho + \bar{V}^k(0) - \bar{V}^{k-1}, \tag{31}$$

where, Ω, as shown at the bottom of this page.

Then, we can select $l_1, l_2, \ldots, l_n$ and ϕ such that $\Omega > 0$. In consequence,

$$\Delta \bar{E}^k(t) \leq -\lambda_{\min}(\Omega) \int_0^t (Z^k)^T Z^k d\rho + \bar{V}^k(0) - \bar{V}^{k-1}, \tag{32}$$

where $\lambda_{\min}(\Omega)$ is the minimum eigenvalue of Ω.

Part II. Boundednesses of all variables.

Let $t = T$, it can be gained from (32) that

$$\begin{aligned} \Delta \bar{E}^k(T) &\leq -\lambda_{\min}(\Omega) \int_0^T (Z^k)^T Z^k d\rho + \bar{V}^k(0) \\ &\leq \bar{V}^k(0). \end{aligned} \tag{33}$$

That is,

$$\bar{E}^k(T) \leq \bar{E}^{k-1}(T) + \bar{V}^k(0), \tag{34}$$

and

$$\begin{aligned} \bar{E}^k(t) &= \Delta \bar{E}^k(t) + \bar{E}^{k-1}(t) \\ &\leq -\lambda_{\min}(\Omega) \int_0^t (Z^k)^T Z^k d\rho + \bar{V}^k(0) - \bar{V}^{k-1} \\ &\quad + \bar{V}^{k-1} + \sum_{i=1}^S \frac{1}{2q_i} \int_0^t (\tilde{\phi}_i^{k-1})^2 d\rho \\ &\quad + \sum_{i=1}^S \int_0^t \frac{1}{2\hat{\phi}_i^{k-1}} (\delta u_i^{k-1})^2 d\rho \\ &\leq \sum_{i=1}^S \frac{1}{2q_i} \int_0^T (\tilde{\phi}_i^{k-1})^2 d\rho + \bar{V}^k(0) + \bar{V}^{k-1}(T) \\ &\quad + \sum_{i=1}^S \int_0^T \frac{1}{2\hat{\phi}_i^{k-1}} (\delta u_i^{k-1})^2 d\rho \\ &= \bar{E}^{k-1}(T) + \bar{V}^k(0) \\ &\leq \bar{E}^{k-2}(T) + \bar{V}^{k-1}(0) + \bar{V}^k(0) \\ &\leq \bar{E}^{k-3}(T) + (\bar{V}^{k-2}(0) + \bar{V}^{k-1}(0) + \bar{V}^k(0)) \\ &\ \ \vdots \\ &\leq \bar{E}^0(T) + \sum_{j=1}^k \bar{V}^j(0) \\ &\leq \bar{E}^0(T) + \frac{\lambda_{\max}(H)}{2} \sum_{j=1}^k (\sigma_\delta^j(0))^T \sigma_\delta^j(0) \\ &\quad + \frac{\lambda_{\max}(\mathcal{Q})}{2} \sum_{j=1}^k (F_1^j(0))^T F_1^j(0), \end{aligned} \tag{35}$$

where

$$\begin{aligned} &\sum_{j=1}^k (\sigma_\delta^j(0))^T \sigma_\delta^j(0) \\ &\quad = \sum_{j=1}^k (\sigma_\delta^0(0))^T (I - \beta H)^{2j} \sigma_\delta^0(0) \\ &\quad \leq (\sigma_\delta^0(0))^T \sigma_\delta^0(0) \sum_{j=1}^k [\rho(I - \beta H)]^{2j} \\ &\quad = (\sigma_\delta^0(0))^T \sigma_\delta^0(0) \frac{[\rho(I - \beta H)]^2}{1 - [\rho(I - \beta H)]^2} \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\sum_{j=1}^k (F_1^j(0))^T F_1^j(0) \\ &\quad = \sum_{j=1}^k (F_1^0(0))^T (I - \beta H)^{2j} F_1^0(0) \\ &\quad \leq (F_1^0(0))^T F_1^0(0) \sum_{j=1}^k [\rho(I - \beta H)]^{2j} \\ &\quad = (F_1^0(0))^T F_1^0(0) \frac{[\rho(I - \beta H)]^2}{1 - [\rho(I - \beta H)]^2}. \end{aligned} \tag{37}$$

As a result,

$$\bar{E}^k(t) \leq \bar{E}^0(T) + \bar{S}, \tag{38}$$

where

$$\begin{aligned} \bar{S} \triangleq &\frac{[\rho(I - \beta H)]^2}{2[1 - (\rho(I - \beta H))^2]} \\ &\cdot [\lambda_{\max}(H)(\sigma_\delta^0(0))^T \sigma_\delta^0(0) + \lambda_{\max}(\mathcal{Q})(F_1^0(0))^T F_1^0(0)]. \end{aligned}$$

From (38), it can be intuitively seen that the finiteness of $\bar{E}^k(t)$ can be guaranteed if $\bar{E}^0(T)$ is bounded.

Next, we will prove the finiteness of $\bar{E}^0(t)$, and then the finiteness of $\bar{E}^0(T)$ is obvious. In the light of the definition of $\bar{E}^k$,

$$\begin{aligned} \bar{E}^0 = \bar{V}^0 + \sum_{i=1}^S \frac{1}{q_i} \int_0^t (\tilde{\phi}_i^0(\rho))^2 d\rho \\ + \sum_{i=1}^S \int_0^t \frac{1}{2\hat{\phi}_i^0(\rho)} (\delta u_i^0)^2 d\rho. \end{aligned} \tag{39}$$

Taking the derivative both sides of (39) leads to

$$\dot{\bar{E}}^0 = \dot{\bar{V}}^0 + \sum_{i=1}^S \frac{1}{q_i} (\tilde{\phi}_i^0)^2 + \sum_{i=1}^S \frac{1}{2\hat{\phi}_i^0} (\delta u_i^0)^2$$

$$\Omega = \begin{bmatrix} \frac{-\lambda_{\max}(H)(\|l\| + l_n\|\lambda\| + \mu}{2}(I_{n-1} \otimes I_S) & -\frac{1}{2}[(\mathcal{Q}h \otimes I_S) + (\mathcal{Q}^T \otimes I_S)\lambda H] \\ -\frac{1}{2}[(\mathcal{Q}h \otimes I_S) + (\mathcal{Q}^T \otimes I_S)\lambda H]^T & \frac{2\phi - 3\phi^*}{2} H^2 - \frac{(2l_n + \|l\| + l_n\|\lambda\|)I_S + H\lambda^T(h \otimes I_S)}{2} \end{bmatrix}$$

$$
\begin{aligned}
&\le \frac{\lambda_{\max}(H)(\|l\|+l_n\|\lambda\|)-\mu}{2}(F_1^0)^T F_1^0 \\
&\quad +(\sigma_\delta^0)^T H\lambda^T(\Omega\otimes I_S)F_1^0 \\
&\quad +(F_1^0)^T(\mathcal{Q}h\otimes I_S)\sigma_\delta^0+(\sigma_\delta^0)^T[H\lambda^T(h\otimes I_S) \\
&\quad +\frac{\lambda_{\max}(H)(2l_n+\|l\|+l_n\|\lambda\|)}{2}I_S]\sigma_\delta^0 \\
&\quad -(\sigma_\delta^0)^T H\hat{\Phi}^0 H\sigma_\delta^0+(\sigma_\delta^0)^T H(\tilde{u}^0-u^0+\tilde{u}^{-1}-1_S u_0) \\
&\quad +\sum_{i=1}^{S}\frac{1}{2q_i}[(\tilde{\phi}_i^0)^2-(\tilde{\phi}_i^{-1})^2]+\sum_{i=1}^{S}\frac{1}{2q_i}(\tilde{\phi}_i^{-1})^2 \\
&\quad +\sum_{i=1}^{S}\frac{1}{2\hat{\phi}_i^0}[(\delta u_i^0)^2-(\delta u_i^{-1})^2]+\sum_{i=1}^{S}\frac{1}{2\hat{\phi}_i^0}(\delta u_i^{-1})^2 \\
&\le -(Z^0)^T\Omega Z^0+\sum_{i=1}^{S}\frac{1}{2q_i}(\tilde{\phi}_i^{-1})^2+\sum_{i=1}^{S}\frac{1}{2\hat{\phi}_i^0}{u_0}^2. \qquad (40)
\end{aligned}
$$

Since $(\tilde{\phi}_i^{-1})^2=(\phi-\hat{\phi}_i^{-1})^2$ is a positive constant and u_0 is continuous on $[0,T]$. Therefore, there exists

$$
P=\max_{\substack{t\in[0,T]\\ 1\le i\le S}}[\sum_{i=1}^{S}\frac{1}{2q_i}(\tilde{\phi}_i^{-1})^2+\sum_{i=1}^{S}\frac{1}{2\hat{\phi}_i^0(t)}u_0^2]<\infty. \qquad (41)
$$

It can be easily got that

$$
\begin{aligned}
\bar{E}^0(t) &\le \left|\bar{E}^0(0)\right|+\int_0^t\left|\dot{\bar{E}}^0\right|d\rho \\
&\le \frac{1}{2}(\sigma_\delta^0(0))^T H\sigma_\delta^0(0)+\frac{1}{2}(F_1^0(0))^T(\mathcal{Q}\otimes I_S)F_1^0(0) \\
&\quad +TP\ <\infty. \qquad (42)
\end{aligned}
$$

Therefore, the finiteness of $\bar{E}^0(T)$ is obtained. In turn, the finiteness of $\bar{E}^k$ is followed. From (17) and (21), we can acquire that σ_δ^k and F_1^k are uniformly bounded, then the finiteness of δ_s^k is gained, $s=1,2,\ldots,n$. (15) renders the finiteness of $\hat{\phi}_i^k$. According to (14), u_i^k is uniformly bounded. Hence, the boundednesses of all signals in closed loop systems are realized.

Part III. Consensus analysis

Lastly, the learning consensus property will be verified. From (33), we can obtain that

$$
\begin{aligned}
\bar{E}^k(T) &= \bar{E}^0(T)+\sum_{j=1}^{k}\Delta\bar{E}^j(T) \\
&\le \bar{E}^0(T)-\lambda_{\min}(\Omega)\sum_{j=1}^{k}\int_0^T(Z^j)^T Z^j d\rho+\sum_{j=1}^{k}\bar{V}^j(0). \qquad (43)
\end{aligned}
$$

On account of the positiveness of $\bar{E}^k(T)$ and finiteness of $\sum_{j=1}^{k}\bar{V}^j(0)$, the series $\sum_{j=1}^{k}\int_0^T(Z^j)^T Z^j d\rho$ is convergent, $\lim_{k\to\infty}\int_0^T(\sigma_\delta^k)^T\sigma_\delta^k dt=0$ is followed. Also we can check (13) $\dot{\sigma}_\delta^k$ is uniformly bounded on $[0,T]$. By virtue of Barbalat-like Lemma [28], $\lim_{k\to\infty}\sigma_\delta^k=0$. Hence, $\lim_{k\to\infty}\delta_s^k(t)=0$, $s=1,2,\ldots,n$. That is to say, the leader is perfectly tracked by each follower along the iteration axis, $\forall t\in[0,T]$. The proof is completed. □

Remark 4: Noting that the selection of parameter β in Assumption 2 needs to satisfy the condition $(\|I-\beta H\|_2<1)$ in Lemma 2, which contains the global information related to the topology graph. Therefore, the designed protocols under Assumption 2 in Theorem 1 are not fully distributed. In order to obtain the fully distributed protocols, another initial condition is proposed as follows.

Assumption 3: The alignment initial condition is satisfied, that is, $x_{i,s}^k(0)=x_{i,s}^{k-1}(T)$, $i=1,2,\ldots,S$,$s=1,2,\ldots,n$; and $x_{0,s}(0)=x_{0,s}(T)$.

Remark 5: It follows from Assumption 3 that $x_s^k(0)=x_s^{k-1}(T)$. Then, $\delta_s^k(0)=\delta_s^{k-1}(T)$, $\sigma_\delta^k(0)=\sigma_\delta^{k-1}(T)$ and $F_1^k(0)=F_1^{k-1}(T)$.

Theorem 2: For the MASs (1) and (3) with the connected topology graph $\bar{\mathcal{G}}$, if Assumptions 1 and 3 hold, then the leader can be perfectly tracked by S followers under the protocols (14) and adaptive updating laws (15) along the iteration axis, i.e., $\lim_{k\to\infty}\delta_s^k=0$, $\forall t\in[0,T]$, $s=1,2,\ldots,n$. Besides, the signals in the closed loop systems are all finite.

Proof: Choose the same Lyapunov function and Lyapunov-Krasovskii functional as in Theorem 1, we have

$$
\Delta\bar{E}^k\le-\lambda_{\min}(\Omega)\int_0^t(Z^k)^T Z^k d\rho+\bar{V}^k(0)-\bar{V}^{k-1} \qquad (44)
$$

Let $t=T$, it is obtained from Remark 5 and (44) that

$$
\Delta\bar{E}^k(T)\le-\lambda_{\min}(\Omega)\int_0^T(Z^k)^T Z^k d\rho. \qquad (45)
$$

That is,

$$
\bar{E}^k(T)\le\bar{E}^{k-1}(T). \qquad (46)
$$

and

$$
\begin{aligned}
\bar{E}^k &= \Delta\bar{E}^k+\bar{E}^{k-1} \\
&\le -\lambda_{\min}(\Omega)\int_0^t(Z^k)^T Z^k d\rho+\bar{V}^k(0)-\bar{V}^{k-1}+\bar{V}^{k-1} \\
&\quad +\sum_{i=1}^{S}\frac{1}{2q_i}\int_0^t(\tilde{\phi}_i^{k-1})^2 d\rho+\sum_{i=1}^{S}\int_0^t\frac{1}{2\hat{\phi}_i^{k-1}}(\delta u_i^{k-1})^2 d\rho \\
&\le \bar{V}^{k-1}(T)+\sum_{i=1}^{S}\frac{1}{2q_i}\int_0^T(\tilde{\phi}_i^{k-1})^2 d\rho \\
&\quad +\sum_{i=1}^{S}\int_0^T\frac{1}{2\hat{\phi}_i^{k-1}}(\delta u_i^{k-1})^2 d\rho \\
&= \bar{E}^{k-1}(T) \\
&\le \bar{E}^{k-2}(T) \\
&\quad \vdots \\
&\le \bar{E}^0(T). \qquad (47)
\end{aligned}
$$

According to (47), if $\bar{E}^0(T)$ is finite, then the finiteness of $\bar{E}^k$ is ensured. In this case, it is only needed to prove the finiteness of $\bar{E}^0(t)$, then the finiteness of $\bar{E}^0(T)$ is followed.

On account of the proof of the finiteness of $\bar{E}^0(t)$ similar to Theorem 1, it is omitted here. Last but not the least, we testify the property of learning consensus. From (45), we know that

$$\bar{E}^k(T) = \bar{E}^0(T) + \sum_{j=1}^{k} \Delta \bar{E}^j(T) \leq \bar{E}^0(T) - \lambda_{\min}(\Omega) \sum_{j=1}^{k} \int_0^T (Z^j)^T Z^j d\rho. \quad (48)$$

Owing to the finiteness of $\bar{E}^0(T)$ and the positiveness of $\bar{E}^k(T)$, the series $\sum_{j=1}^{k} \int_0^T (Z^j)^T Z^j d\rho$ is convergent, and the rest is the same as Theorem 1, the perfect consensus tracking can be achieved. The proof is completed. □

Remark 6: On the one hand, in the proof of Theorem 2, the condition used in (32) are only for the consensus analysis and independent of the protocols design. On the other hand, unlike Assumption 2, Assumption 3 does not utilize any global information. Thus, the establishment of Theorem 2 does not contain any global information, and the designed protocols are fully distributed.

IV. FULLY DISTRIBUTED FORMATION CONTROL ALGORITHM FOR HON-MASS WITH IS

In this part, let us focus on the the formation problem of HON-MASs with IS. If each follower forms an expected distance from the leader for $t \in [0, T]$, then it means that the MASs can realize the leader-follower formation control.

Define

$$\bar{x}_{i,1}^k = x_{i,1}^k - \Delta_i, \quad (49)$$

where Δ_i is the expected formation space for the leader from ith follower.

The formation errors of ith follower are

$$\delta_{i,1}^k(t) = \bar{x}_{i,1}^k(t) - x_{0,1}(t), \quad (50)$$

and $\delta_{i,s}^k(t)$ are the same as $\delta_{i,s}^k(t)$ defined in (4), $s = 2, 3, \ldots, n$. As such, we can reformulate the formation problem as the problem of consensus, that is, $\lim_{k\to\infty} \delta_s^k = 0$.

Besides, the distributed formation errors for ith follower are

$$e_{i,1}^k = \sum_{j=1}^{S} a_{ij}(\bar{x}_{j,1}^k - \bar{x}_{i,1}^k) + b_i(x_{0,1} - \bar{x}_{i,1}^k),$$
$$e_{i,s}^k = \sum_{j=1}^{S} a_{ij}(\bar{x}_{j,s}^k - \bar{x}_{i,s}^k) + b_i(x_{0,s} - \bar{x}_{i,s}^k), \quad (51)$$

where $s = 2, 3, \ldots, n$.

Assumption 4: The alignment initial condition is satisfied, i.e., $\bar{x}_{i,1}^k(0) = \bar{x}_{i,1}^{k-1}(T), x_{i,s}^k(0) = x_{i,s}^{k-1}(T), i = 1, 2, \ldots, S$ and $s = 2, 3, \ldots, n$; and the leader satisfies $x_{0,s}(0) = x_{0,s}(T)$. Then, it follows that $x_s^k(0) = x_s^{k-1}(T)$. Then, $\delta_s^k(0) = \delta_s^{k-1}(T), \sigma_\delta^k(0) = \sigma_\delta^{k-1}(T)$ and $F_1^k(0) = F_1^{k-1}(T)$.

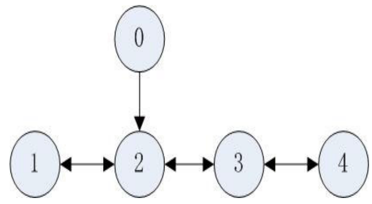

FIGURE 1. Communication topology graph.

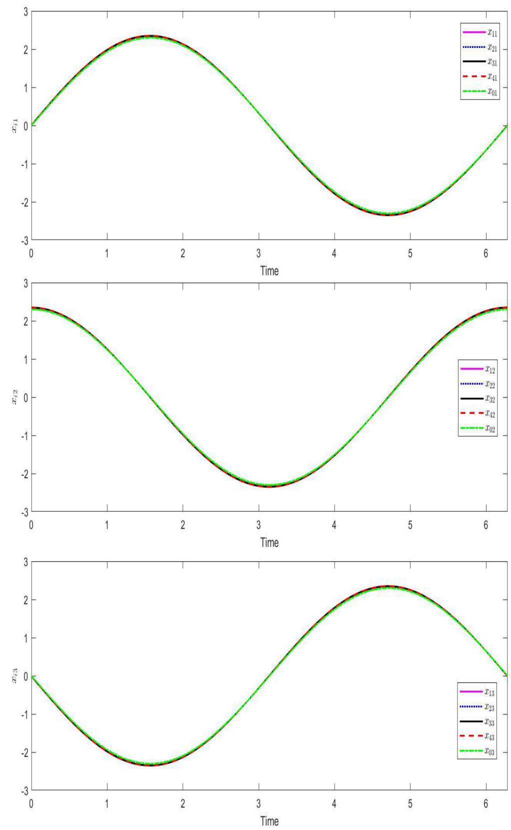

FIGURE 2. Evolutions of agents states in Case 1 of Example 1.

Theorem 3: For the MASs (1) and (3) with the connected topology graph $\bar{\mathcal{G}}$, if Assumptions 1 and 4 hold, then S followers can form the desired formation from the leader under the DRLCP (14) and adaptive updating laws (15) with the distributed formation errors (51) along the iteration axis, $\forall t \in [0, T]$, and the variables in the closed-loop systems are all finite.

V. SIMULATIONS

In the segment, a numerical example and a networked pendulum systems are provided to validate theoretical results under a.i.c. The networked pendulum systems can be considered as the MASs. Besides, suppose that there are four followers and one leader in both two examples. Fig.1 is the communication topology graph.

From Fig.1, Laplace matrix L and B can be obtained that

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad B = diag\{0, 1, 0, 0\}.$$

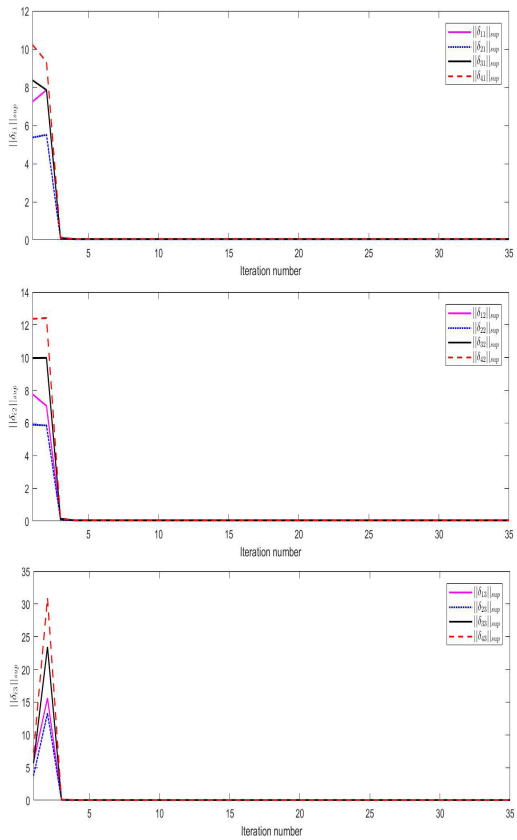

FIGURE 3. **Consensus errors in Case 1 of Example 1.**

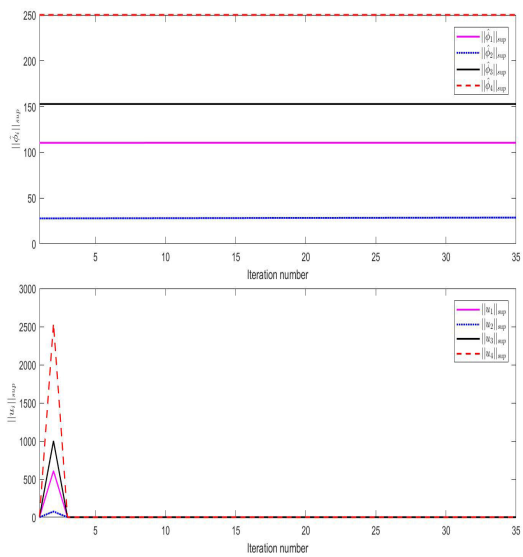

FIGURE 4. **Responses of signals in Case 1 of Example 1.**

Example 1: The dynamics of four followers are $\dot{x}_{i,1}^k = x_{i,2}^k$, $\dot{x}_{i,2}^k = x_{i,3}^k$, $\dot{x}_{i,3}^k = 0.01\cos(x_{i,1}^k - x_{i,2}^k - x_{i,3}^k)\cos(2\ t) + sat(u_i^k(t), u^*)$, where $t \in [0, T]$, $T = 2\pi$, $k = 35$, $u^* = 25$, and $x_{0,1} = 2.3\sin t$, $x_{0,2} = 2.3\cos t$, $x_{0,3} = -2.3\sin t$.

Case 1: Consensus for the HON-MASs of Example 1.

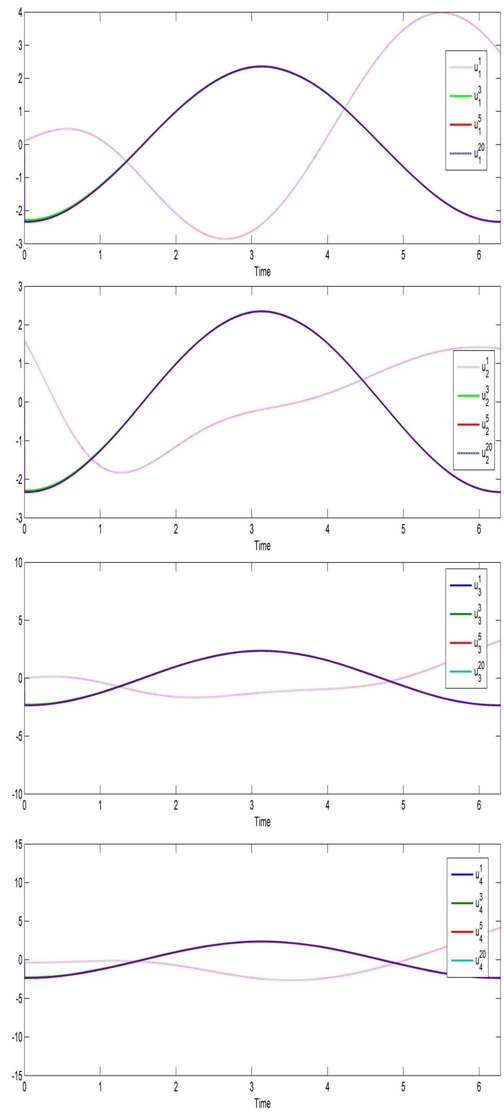

FIGURE 5. **Input curves in Case 1 of Example 1.**

The initial values are $x_1^0(0) = [0.7, 0.6, 0.8, 1]^T$, $x_2^0(0) = [0.7, 0.8, 0.9, 1]^T$, $x_3^0(0) = [0.7, 0.8, 0.9, 1]^T$ and $\hat{\phi}_1^0(0) = 0.5$, $\hat{\phi}_2^0(0) = \hat{\phi}_3^0(0) = \hat{\phi}_4^0(0) = 0.5$. Select $q_1 = q_2 = q_3 = q_4 = 3.6$, $\lambda_1 = 1.8$, $\lambda_2 = 3$ and $\phi^* = 250$. Figs.2-4 show the simulation results.

Figs.2-3 demonstrate the involutions of agents states at the 35*th* loop and consensus errors of all followers, respectively. It is seen that the leader can be perfectly tracked by each follower even if there is IS in the dynamics of each follower. The responses of four followers are displayed in Fig.4, which indicates that all the variables are bounded. In order to illustrate the influence of input saturation, Fig.5 shows the input curves of each follower at the 1st, 3rd, 5th and 20th iterations on $[0, 2\pi]$, which means that the control input of each follower is within the saturation bound $u^* = 25$. Therefore, the results fit into Theorem 2 in the paper.

Case 2: Formation control for the HON-MASs of Example 1.

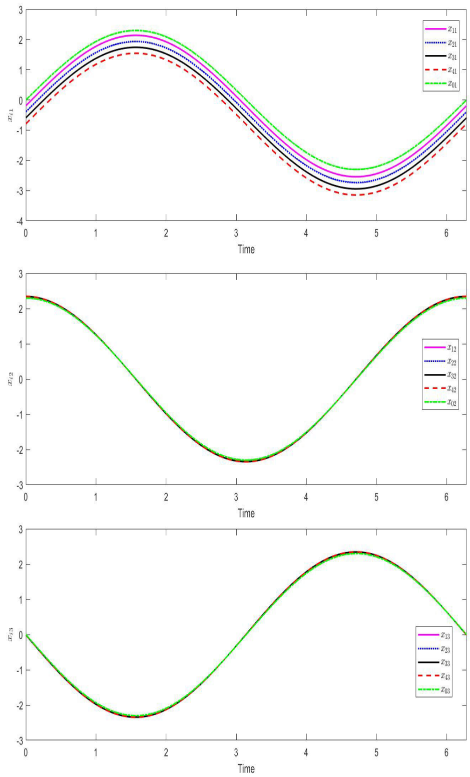

FIGURE 6. Evolutions of agents states in Case 2 of Example 1.

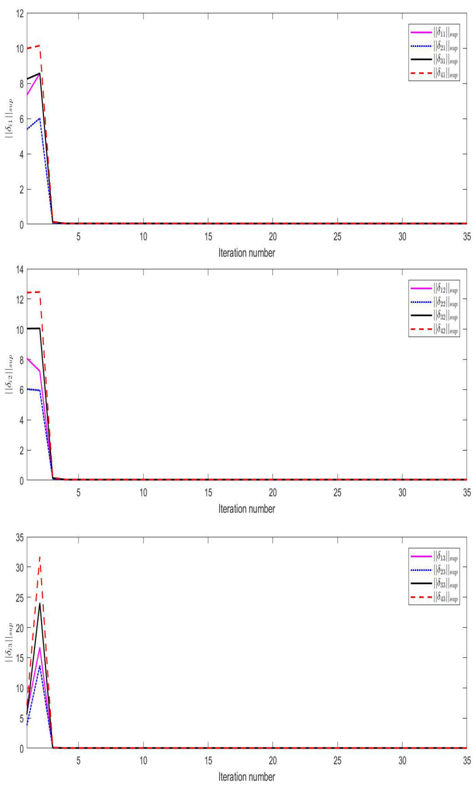

FIGURE 7. Formation errors in Case 2 of Example 1.

Consider the formation control of the HON-MASs and choose the desired relative distances are $\Delta_1 = -0.2$, $\Delta_2 = -0.4$, $\Delta_3 = -0.6$, and $\Delta_4 = -0.8$. Other arguments involved are the same as Case 1.

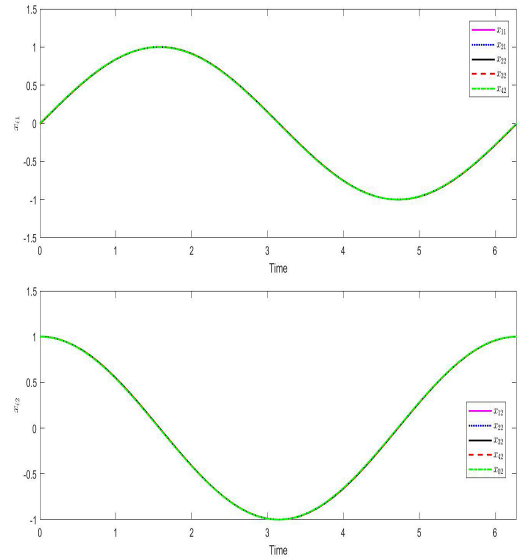

FIGURE 8. States of all pendulums in Case 1 of Example 2.

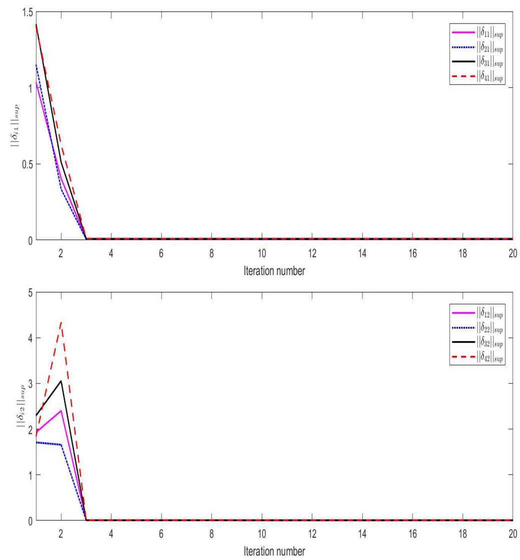

FIGURE 9. Consensus errors in Case 1 of Example 2.

We learn that four followers can form the expected formation in Figs.6-7. The validity of this part is obvious.

Example 2: At present, under the repetitive environment, let us consider a networked pendulum systems [29] comprised of four follower pendulums with one leader pendulum having the communication topology graph Fig.1. Meanwhile, it is assumed that the dynamics of each follower pendulum suffer from IS. At the kth loop, the state equations can be written as

$$\begin{cases} \dot{x}_{i1}^k = x_{i2}^k \\ \dot{x}_{i2}^k = -\frac{g}{r_i}\sin x_{i1}^k - \frac{h_i}{m_i}x_{i2}^k + \frac{1}{r_i}sat(u_i^k, u^*), \end{cases}$$

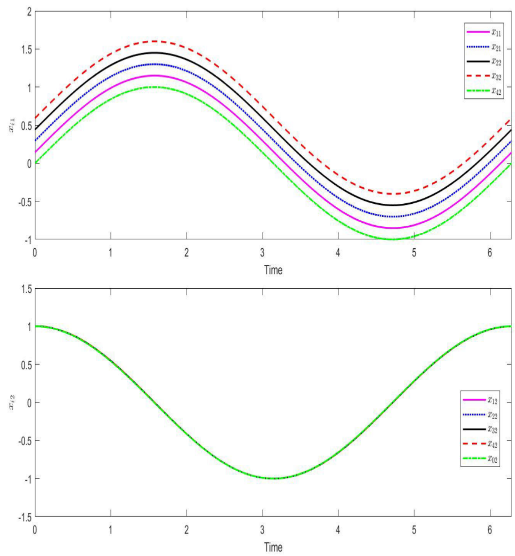

FIGURE 10. States of all pendulums in Case 2 of Example 2.

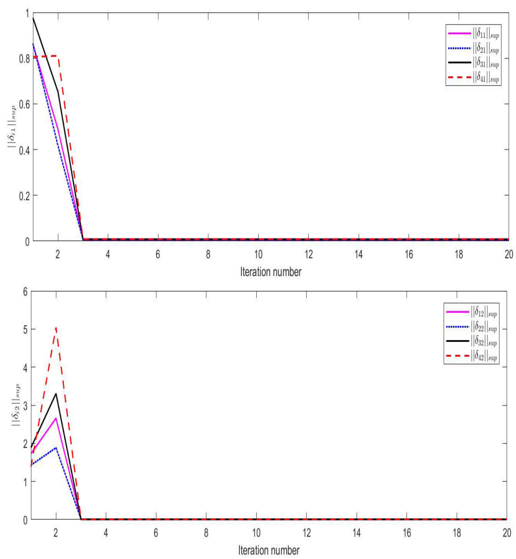

FIGURE 11. Formation errors in Case 2 of Example 2.

where x_{i1}^k and x_{i2}^k are the position and velocity of ith follower pendulum, r_i means the length of the rod, m_i means the mass of the bob, h_i is the friction resistance coefficients and g is the acceleration of gravity, $i = 1, 2, 3, 4$, $u^* = 25$. In addition, $x_{0,1} = sin(t)$ and $x_{0,2} = \cos(t)$.

Case 1: Consensus for the MASs of Example 2 on $[0, 2\pi]$.

Take $\frac{g}{r_i} = 1$, $\frac{h_i}{m_i} = 1$ and $\frac{1}{m_i r_i^2} = 1$, the initial states are $x_1^0(0) = [1, 1.1, 1.2, 1.4]^T$, $x_2^0(0) = [1.2, 2.2, 2.5, 1]^T$, and $\hat{\phi}_1^0(0) = 0.4$, $\hat{\phi}_2^0(0) = 0.3$, $\hat{\phi}_3^0(0) = 0.4$, and $\hat{\phi}_4^0(0) = 0.3$. Select $q_1 = 3.2$, $q_2 = 3.5$, $q_3 = 3.1$, $q_4 = 3.4$, $\lambda_1 = 33$, and $\phi^* = 80$. The results for 20 loops are displayed in Figs.8-9, and the proposed algorithms can perform the networked-pendulum consensus assignment as well.

Case 2: Formation control of the MASs of Example 2 on [0, 2].

Choosing the desired relative distances are $\Delta_1 = 0.15$, $\Delta_2 = 0.3$, $\Delta_3 = 0.45$ and $\Delta_4 = 0.6$. Other variables involved are the same as Case 1.

From Figs.10 and 11, we can see that all the pendulums form the expected formation, and achieve perfect consensus of velocity on $[0, 2\pi]$.

VI. CONCLUSION

In the paper, the DRLCP for the HON-MASs with IS are firstly studied under the initial state learning containing the global information. Secondly, in order to obtain the fully distributed protocols, the a.i.c is proposed. And the leader can be perfectly tracked by each follower on a finite time interval under both two assumptions. Finally, the fully distributed formation control is researched, and two illustrative examples display the efficacy and practicability of the algorithms.

REFERENCES

[1] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 427–438, Feb. 2013.

[2] L. He, J. Zhang, Y. Hou, X. Liang, and P. Bai, "Time-varying formation control for second-order discrete-time multi-agent systems with switching topologies and nonuniform communication delays," *IEEE Access*, vol. 7, pp. 65379–65389, 2019.

[3] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 1, pp. 213–224, Jan. 2010.

[4] H. Zhang and F. L. Lewis, "Adaptive cooperative tracking control of higher-order nonlinear systems with unknown dynamics," *Automatica*, vol. 48, no. 7, pp. 1432–1439, Jul. 2012.

[5] X. Wang, S. Li, X. Yu, and J. Yang, "Distributed active anti-disturbance consensus for leader-follower higher-order multi-agent systems with mismatched disturbances," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5795–5801, Nov. 2017.

[6] J. Xi, J. Yang, H. Liu, and T. Zheng, "Adaptive guaranteed-performance consensus design for high-order multiagent systems," *Inf. Sci.*, vol. 467, pp. 1–14, Jun. 2018.

[7] T. Yu, L. Ma, and N. Qin, "Adaptive cooperative tracking control of multi-agent systems with unknown actuators hysteresis," *IEEE Access*, vol. 6, pp. 33015–33028, 2018.

[8] Z. Li, W. Ren, X. Liu, and M. Fu, "Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1786–1791, Jul. 2013.

[9] M.-C. Fan and W. Yue, "Global leader-following consensus of nonlinear multi-agent systems with unknown control directions and unknown external disturbances," *Appl. Math. Comput.*, vol. 331, pp. 274–286, Aug. 2018.

[10] D. Ye, M. Chen, and H. Yang, "Distributed adaptive event-triggered fault-tolerant consensus of multiagent systems with general linear dynamics," *IEEE Trans. Cybern.*, vol. 49, no. 3, pp. 757–767, Mar. 2019.

[11] Q. Song, F. Liu, H. Su, and A. V. Vasilakos, "Semi-global and global containment control of multi-agent systems with second-order dynamics and input saturation," *Int. J. Robust Nonlinear Control*, vol. 26, no. 16, pp. 3460–3480, Nov. 2016.

[12] X. Wang, X. Wang, H. Su, and G. Chen, "Fully distributed event-triggered semiglobal consensus of multi-agent systems with input saturation," *IEEE Trans. Ind. Electron.*, vol. 64, no. 6, pp. 5055–5064, Jun. 2017.

[13] Z. Fan, H. Su, S. Chen, M. Yu, and J. Ma, "Semi-global leader-following coordination of multi-agent systems with input saturation and aperiodic intermittent communications," *J. Franklin Inst.*, vol. 356, no. 2, pp. 1051–1066, Jan. 2019.

[14] J.-X. Xu, "A survey on iterative learning control for nonlinear systems," *Int. J. Control*, vol. 84, no. 7, pp. 1275–1294, Jul. 2011.

[15] D. Shen, "Iterative learning control with incomplete information: A survey," *IEEE/CAA J. Autom. Sinica*, vol. 5, no. 5, pp. 885–901, Jul. 2018.

[16] J.-X. Xu, Y. Tan, and T.-H. Lee, ''Iterative learning control design based on composite energy function with input saturation,'' *Automatica*, vol. 40, no. 8, pp. 1371–1377, Aug. 2004.
[17] R. Zhang, Z. Hou, H. Ji, and C. Yin, ''Adaptive iterative learning control for a class of non-linearly parameterised systems with input saturations,'' *Int. J. Syst. Sci.*, vol. 47, no. 5, pp. 1084–1094, Nov. 2016.
[18] R. Zhang, Z. Hou, R. Chi, and H. Ji, ''Adaptive iterative learning control for nonlinearly parameterised systems with unknown time-varying delays and input saturations,'' *Int. J. Control*, vol. 88, no. 6, pp. 1133–1141, Jun. 2015.
[19] H. Ji, Z. Hou, L. Fan, and F. L. Lewis, ''Adaptive iterative learning reliable control for a class of non-linearly parameterised systems with unknown state delays and input saturation,'' *IET Control Theory Appl.*, vol. 10, no. 17, pp. 2160–2174, Nov. 2016.
[20] J.-X. Xu and R. Yan, ''On initial conditions in iterative learning control,'' *IEEE Trans. Autom. Control*, vol. 50, no. 9, pp. 1349–1354, Sep. 2005.
[21] J. Li and J. Li, ''Adaptive iterative learning control for consensus of multi-agent systems,'' *IET Control Theory Appl.*, vol. 7, no. 1, pp. 136–142, Jan. 2013.
[22] J. Li and J. Li, ''Coordination control of multi-agent systems with second-order nonlinear dynamics using fully distributed adaptive iterative learning,'' *J. Franklin Inst.*, vol. 352, no. 6, pp. 2441–2463, Jun. 2015.
[23] S. Yang and J.-X. Xu, ''Leader-follower synchronisation for networked Lagrangian systems with uncertainties: A learning approach,'' *Int. J. Syst. Sci.*, vol. 47, no. 4, pp. 956–965, Mar. 2016.
[24] J. Xu, ''Adaptive iterative learning control for high-order nonlinear multi-agent systems consensus tracking,'' *Syst. Control Lett.*, vol. 89, pp. 16–23, Mar. 2016.
[25] D. Shen and J.-X. Xu, ''Distributed learning consensus for heterogenous high-order nonlinear multi-agent systems with output constraints,'' *Automatica*, vol. 97, pp. 64–72, Nov. 2018.
[26] J. Li and J. Li, ''Distributed adaptive fuzzy iterative learning control of coordination problems for higher order multi-agent systems,'' *Int. J. Syst. Sci.*, vol. 47, no. 10, pp. 2318–2329, Jul. 2016.
[27] N. Yang and J. Li, ''Distributed iterative learning coordination control for leader-follower uncertain nonlinear multi-agent systems with input saturation,'' *IET Control Theory Appl.*, vol. 13, pp. 2252–2260, Sep. 2019.
[28] M. Sun, ''A Barbalat-like lemma with its application to learning control,'' *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2222–2225, Sep. 2009.
[29] H. K. Khalil and J. W. Grizzle, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

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