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On Mixed Metric Dimension of Rotationally Symmetric Graphs

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ABSTRACT A vertex $u \in V(G)$ resolves (distinguish or recognize) two elements (vertices or edges) $v, w \in E(G) \cup V(G)$ if $d_G(u, v) \neq d_G(u, w)$. A subset L_m of vertices in a connected graph G is called a mixed metric generator for G if every two distinct elements (vertices and edges) of G are resolved by some vertex set of L_m. The minimum cardinality of a mixed metric generator for G is called the mixed metric dimension and is denoted by $dim_m(G)$. In this paper, we studied the mixed metric dimension for three families of graphs $\mathcal{D}_n$, $\mathcal{A}_n$, and $\mathcal{R}_n$, known from the literature. We proved that, for $\mathcal{D}_n$ the $dim_m(\mathcal{D}_n) = dim_e(\mathcal{D}_n) = dim(\mathcal{D}_n)$, when n is even, and for $\mathcal{A}_n$ the $dim_m(\mathcal{A}_n) = dim_e(\mathcal{A}_n)$, when n is even and odd. The graph $\mathcal{R}_n$ has mixed metric dimension 5.

INDEX TERMS Mixed metric dimension, metric dimension, edge metric dimension, rotationally-symmetric.

I. INTRODUCTION AND PRELIMINARY RESULTS

The study of standard metric dimension was initiated by Slater [20], where locating sets were called metric generators, under the problem of uniquely identifying the location of an intruder or a thief in the network. When the metric generators were termed as resolving sets of a graph, the notion of metric dimension was introduced by Haray, and Melter [10]. Later, this idea of metric dimension attracted much intention from the researchers, and several articles related to this concept were published; some of them are, for instance, in robot navigation [14], and applications in chemistry [5], [6]. Moreover, some of the recent articles for the reader's convenience are [11], [17], [18], [21], [22].

For a graph $G = (V, E)$, the ordinary distance $d_G(u, v)$ (or $d(u, v)$) between two vertices $u, v \in V(G)$, is the length of shortest path between them. A vertex $a \in V$ resolves two vertices in a graph say b, and c, if $d_G(a, b) \neq d_G(a, c)$ holds. A set $L \subset V$ is the metric generator for a graph G if vertices of L resolves pair of distinct vertices of a graph G.

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The metric generator with least cardinality is the metric basis, and cardinality of its metric basis is called the metric dimension of a graph G. It is denoted by $dim(G)$.

Similar to this concept, edge metric dimension is introduced by [15] which uniquely identifies the edges related to a graph. The distance between the vertex u and edge $e = vw$ is defined as $d(e, u) = min\{d(v, u), d(w, u)\}$. The vertex $u \in V(G)$ resolves or distinguish two edges of a graph $e_1, e_2 \in E(G)$ if $d(e_1, u) \neq d(e_2, u)$. A set $L_e \subset V$ is said to be the edge metric generator for a graph G, if every distinct edges of G are resolved by some vertex of L_e. The minimum cardinality of an edge metric generator of G is known as edge metric dimension of G, and it is represented as $dim_e(G)$. Recently, this variant has been investigated by [19], [24], [25].

Now a new type of dimension is introduced by [16], which is a mixed version of both metric and edge metric dimensions, and authors called it a mixed metric dimension. For a graph G, a set of vertices which can distinguish the elements (vertices and edges). For an ordered subset $L_m = \{s_1, s_2, \ldots, s_w\}$ of the vertex V, and $v \in V(G)$ also $e \in V(G)$, the $m - tuple$ with $r(v|L_m) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_w)\}$, $r(e|L_m) = \{d(e, s_1), d(e, s_2), \ldots, d(e, s_w)\}$ is called mixed metric

TABLE 1. Representation of vertices of C_n.

$r(v\|L_m)$	x_0	x_1	x_m
$x_\ell : 0 \leq \ell \leq 1$	ℓ	$1-\ell$	$m-\ell$
$x_\ell : 2 \leq \ell \leq m$	ℓ	$\ell-1$	$m-\ell$
$x_\ell : m+1 \leq \ell \leq 2m-1$	$2m-\ell$	$2m-\ell+1$	$\ell-m$

TABLE 2. Representation of edges of C_n.

$r(e\|L_m)$	x_0	x_1	x_m
$x_\ell x_{\ell+1} : 0 \leq \ell \leq 1$	ℓ	0	$m-\ell-1$
$x_\ell x_{\ell+1} : 2 \leq \ell \leq m-1$	ℓ	$\ell-1$	$m-\ell-1$
$x_\ell x_{\ell+1} : m \leq \ell \leq m+1$	$2m-\ell-1$	$m-1$	$\ell-m$
$x_\ell x_{\ell+1} : m+2 \leq \ell \leq 2m-1$	$2m-\ell-1$	$2m-\ell$	$\ell-m$

TABLE 3. Representation of vertices of C_n.

$r(v\|L_m)$	x_0	x_1	x_{m+1}
$x_\ell : 0 \leq \ell \leq 1$	ℓ	$1-\ell$	m
$x_\ell : 2 \leq \ell \leq m$	ℓ	$\ell-1$	$m-\ell+1$
$x_\ell : m+1 \leq \ell \leq m+2$	$2m-\ell+1$	m	$\ell-m-1$
$x_\ell : m+3 \leq \ell \leq 2m$	$2m-\ell+1$	$2m-\ell+2$	$\ell-m-1$

TABLE 4. Representation of edges of C_n.

$r(e\|L_m)$	x_0	x_1	x_{m+1}
$x_\ell x_{\ell+1} : 0 \leq \ell \leq 1$	ℓ	0	$m-\ell$
$x_\ell x_{\ell+1} : 2 \leq \ell \leq m$	ℓ	$\ell-1$	$m-\ell$
$x_\ell x_{\ell+1} : m+1 \leq \ell \leq 2m$	$2m-\ell$	$2m-\ell+1$	$\ell-m-1$

representation of a vertex and an edge with respect to L_m. In this sense, L_m is the mixed metric for G, if and only if for every pair of distinct vertices i, j, and edges e_1, e_2 of G, $r(i|L_m) \neq r(j|L_m)$, moreover $r(e_1|L_m) \neq r(e_2|L_m)$. The minimum cardinality of a mixed metric generator is called the mixed metric dimension, and the notation is $dim_m(G)$. A mixed metric basis for a graph G is a mixed metric generator for G with cardinality $dim_m(G)$.

Since metric dimension only resolves vertices of a graph and an edge metric dimension deals with the edges of a graph, the mixed metric dimension deals with both the concepts, so a mixed metric generator is a standard metric generator as well as edge metric generator.

$$dim_m(G) \geq max\{dim(G), dim_e(G)\}$$

It is important to note that a vertex alone in a graph is unable to form a mixed metric generator.

Remark 1 [16]: For any graph G, $2 \leq dim_m \leq n$.

Some of the known results on mixed metric dimension are,

Proposition 1 [16]: For a path graph P_n of order $n \geq 4$, $dim_m(P_n) = 2$.

Proposition 2 [16]: For any two positive integers, g, h,

$$dim_m(K_{g,h}) = \begin{cases} g+h-1, & \text{if } g = 2 \text{ or } h = 2; \\ g+h-2, & \text{otherwise.} \end{cases}$$

Proposition 3 [16]: For grid graphs, $P_m \square P_n$, with $m \geq n \geq 2$. Then we have, $dim_m = 3$.

Proposition 4 [16]: The cycle graph C_n of order $n \geq 4$, has $dim_m(C_n) = 3$.

Proof: We provide the proof and, this technique is implemented for finding the mixed metric dimension for the rest of the graphs generated from cycle graphs. We will divide the proof into two cases.

Case 1: When n is even, then $n = 2m$, where $m \in \mathbb{Z}^+$. Let $L_m = \{x_0, x_1, x_m\}$. We give representation of vertices and edges of C_n with respect to L_m. The tables are shown.

Case 2: In case of odd n, then $n = 2m+1$, where $m \in \mathbb{Z}^+$. Let $L_m = \{x_0, x_1, x_{m+1}\}$. We give representation of vertices and edges of C_n with respect to L_m. The tables are shown.

Note that from the above mentioned tables, vertices and edges do not possess same representation with the resolving set L_m. This shows that L_m distinguishes vertices and edges of C_n, when n is even and odd. So it implies that that $dim_m(C_n) \leq 3$.

Also, it is clear that $dim_m(C_n) \geq 3$. So $dim_m(C_n) = 3$.

Let $\mathcal{F}$ be a family of graphs which are connected, $\mathcal{F}$ has bounded mixed metric dimension if every graph within this family has bounded mixed metric dimension. On the contrary, $\mathcal{F}$ has an unbounded mixed metric dimension. If all graphs within $\mathcal{F}$ have the same unbounded mixed metric dimension, then $\mathcal{F}$ has a constant mixed metric dimension. The path P_n, C_n and grid graphs are the families of graphs with constant mixed metric dimension.

The following proposition shows a comparison between metric, and edge metric dimension among cycle, path, and complete graphs.

Proposition 5 [15]: For any integer $n \geq 2$, $dim(P_n) = dim_e(P_n) = 1$, $dim(C_n) = dim_e(C_n) = 2$, and $dim(K_n) = dim_e(K_n) = n-1$.

In this paper, three families of graphs are considered generated from cycle graphs, denoted as $\mathcal{D}_n$, $\mathcal{A}_n$ and $\mathcal{R}_n$. We proved that for prism graph $\mathcal{D}_n$, $dim_m(\mathcal{P}_n) = dim_e(\mathcal{P}_n) = dim(\mathcal{P}_n)$, for even n, and $dim_m(\mathcal{D}_n) > dim_e(\mathcal{D}_n) > dim(\mathcal{D}_n)$ for odd n. In case of anti-prism graphs $\mathcal{A}_n$, for both even and odd n, we proved that $dim_m(\mathcal{A}_n) = dim_e(\mathcal{A}_n)$. The graph of $\mathcal{R}_n$ has mixed metric dimension of 5.

In the end of this section, we give some known results concerning metric and edge metric dimension of $\mathcal{D}_n$, and $\mathcal{A}_n$.

Lemma 1 [4]: Let $\mathcal{D}_n$ be the prism graph, for $n \geq 4$, then we have,

$$dim(\mathcal{D}_n) = \begin{cases} 2, & \text{n is odd;} \\ 3, & \text{n is even.} \end{cases}$$

Lemma 2 [13]: Let $\mathcal{A}_n$ be the anti-prism graph for $n \geq 3$, then we have $dim(\mathcal{A}_n) = 3$.

Lemma 3 [7]: Let $\mathcal{D}_n$ be the the prism graph which is also called $G(P(n, 1))$, for $n \geq 4$, then we have, $dim_e(\mathcal{P}_n) = 3$.

Lemma 4 [23]: Let $\mathcal{A}_n$ be the anti-prism graph for $n \geq 3$, then we have,

$$dim_e(\mathcal{A}_n) = \begin{cases} 4, & \text{n is even;} \\ 5, & \text{n is odd.} \end{cases}$$

II. THE GRAPH OF PRISM $\mathcal{D}_n$

In this section, mixed metric dimension for $\mathcal{D}_n$, is presented.

The prism graph $\mathcal{D}_n$ is a regular graph with degree 3 studied in [9]. The prism graph is generated by the cartesian product of a cycle graph C_n and a path graph P_2. In this particular case the outer cycle comprises of $y_0, y_1, \ldots, y_{n-1}$, vertices and an inner cycle $x_0, x_1, \ldots, x_{n-1}$. The prism graph is also equivalent to Petersen graph $G(P(n, 1))$. The vertex and edge set for $\mathcal{D}_n$ are as follows .

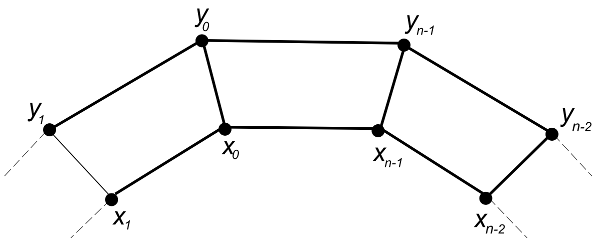
The vertex set of $\mathcal{D}_n$ is,

$$V(\mathcal{D}_n) = \{x_\ell, y_\ell \mid \ell = 0, \ldots, n-1\}$$

and the edge set of $\mathcal{D}_n$ is,

$$E(\mathcal{D}_n) = \{(x_\ell, x_{\ell+1}); (x_\ell, y_\ell); \quad (y_\ell, y_{\ell+1}) \mid \ell{=}0, \ldots, n-1\}.$$

The graph of $\mathcal{D}_n$ is shown below,



Lemma 5: For a mixed metric generator L_m of $\mathcal{D}_n$, L_m must contains vertices of both outer cycle, and inner cycle, respectively.

Proof: Let us assume that, without loss of generality, L_m contains elements of inner cycle, and $\{y_0, y_1, \ldots, y_{n-1}\} \bigcap L_m = \emptyset$. In this scenario, we have $r(x_\ell|L_m) = r(y_\ell x_\ell|L_m)$, for $0 \leq \ell \leq n-1$, which contradicts the definition of mixed metric dimension. So L_m is not the mixed metric generator. It implies that L_m must contain vertices from both outer and inner cycles.

The standard metric dimension and edge metric dimension for prism graphs are studied in see [13], and [7].

Now we give an exact value for mixed metric dimension for $\mathcal{D}_n$.

Theorem 1: Let $(\mathcal{D}_n)$ be the prism graph with $n \geq 5$, then,

$$dim_m(\mathcal{D}_n) = \begin{cases} 3, & \text{n is even;} \\ 4, & \text{n is odd.} \end{cases}$$

Proof: The proof of this theorem is obtained form Lemma 6 to Lemma 8.

Lemma 6: If n is even then $dim_m(\mathcal{D}_n) \leq 3$.

Proof: Now we can write as $n = 2m$, and $m \in \mathbb{Z}^+$. For this particular case, let $L_m = \{y_1, y_{m+1}, x_0\}$. The representation of the vertices and edges of $\mathcal{D}_n$ with respect to L_m are shown in the following tables.

Note that from the above mentioned tables, there are no two vertices and edges having same representation with L_m. This shows that $L_m = \{y_1, y_{m+1}, x_0\}$ resolves vertices and edges of $\mathcal{D}_n$, which indicates that $dim_m(\mathcal{P}_n) \leq 3$. Conversely, we need to show that $dim_m(\mathcal{D}_n) \geq 3$. It is straight forward to see from from Proposition 4, and Lemma 5 so $dim_m(\mathcal{D}_n) \geq 3$, which proves that when n is even $dim_m(\mathcal{D}_n) = 3$.

TABLE 5. Representation of outer vertices of $\mathcal{D}_n$.

$r(v\|L_m)$	y_1	y_{m+1}	x_0
$y_\ell : 0 \leq \ell \leq 1$	$1-\ell$	$m+\ell-1$	$\ell+1$
$y_\ell : 2 \leq \ell \leq m$	$\ell-1$	$m-\ell+1$	$\ell+1$
$y_\ell : m+1 \leq \ell \leq 2m-1$	$2m-\ell+1$	$\ell-m-1$	$2m-\ell+1$

TABLE 6. Representation of inner vertices of $\mathcal{D}_n$.

$r(v\|L_m)$	y_1	y_{m+1}	x_0
$x_\ell : \ell = 0$	2	m	0
$x_\ell : 1 \leq \ell \leq m$	ℓ	$m-\ell+2$	ℓ
$x_\ell : m+1 \leq \ell \leq 2m-1$	$2m-\ell+2$	$\ell-m$	$2m-\ell$

TABLE 7. Representation of outer edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_{m+1}	x_0
$y_\ell y_{\ell+1} : \ell = 0$	0	$m-1$	1
$y_\ell y_{\ell+1} : 1 \leq \ell \leq m-1$	$\ell-1$	$m-\ell$	$\ell+1$
$y_\ell y_{\ell+1} : \ell = m$	$m-1$	0	m
$y_\ell y_{\ell+1} : m+1 \leq \ell \leq 2m-1$	$2m-\ell$	$\ell-m-1$	$2m-\ell$

TABLE 8. Representation of inner edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_{m+1}	x_0
$x_\ell x_{\ell+1} : \ell = 0$	1	m	0
$x_\ell x_{\ell+1} : 1 \leq \ell \leq m-1$	ℓ	$m-\ell+1$	ℓ
$x_\ell x_{\ell+1} : \ell = m$	m	1	$m-1$
$x_\ell x_{\ell+1} : m+1 \leq \ell \leq 2m-1$	$2m-\ell+1$	$\ell-m$	$2m-\ell-1$

TABLE 9. Representation of outer and inner edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_{m+1}	x_0
$y_\ell x_\ell : 0 \leq \ell \leq 1$	$1-\ell$	$m+\ell-1$	ℓ
$y_\ell x_\ell : 2 \leq \ell \leq m$	$\ell-1$	$m-\ell+1$	ℓ
$y_\ell x_\ell : m+1 \leq \ell \leq 2m-1$	$2m-\ell+1$	$\ell-m-1$	$2m-\ell$

TABLE 10. Representation of outer vertices of $\mathcal{D}_n$.

$r(v\|L_m)$	y_1	y_4	y_{m+3}	x_0
$y_\ell : 0 \leq \ell \leq 1$	$1-\ell$	$4-\ell$	$m+\ell-2$	$\ell+1$
$y_\ell : 2 \leq \ell \leq 3$	$\ell-1$	$4-\ell$	m	$\ell+1$
$y_\ell : 4 \leq \ell \leq m$	$\ell-1$	$\ell-4$	$m-\ell+3$	$\ell+1$
$y_\ell : m+1 \leq \ell \leq m+2$	m	$\ell-4$	$m-\ell+3$	$2m-\ell+2$
$y_\ell : m+3 \leq \ell \leq m+4$	$2m-\ell+2$	$\ell-m+3$	$\ell-m-3$	$2m-\ell+2$
$y_\ell : m+5 \leq \ell \leq 2m$	$2m-\ell+2$	$2m-\ell+5$	$\ell-m-3$	$2m-\ell+2$

Remark 2: Let $\mathcal{D}_n$ be a prim graph with $n \geq 3$. If n is even, then $dim_m(\mathcal{D}_n) = dim_e(\mathcal{D}_n) = dim(\mathcal{D}_n)$.

Lemma 7: If n is odd then $dim_m(\mathcal{D}_n) \leq 4$.

Proof: Now we can write as $n = 2m+1$, $m \geq 5$, and $m \in \mathbb{Z}^+$. For this particular case, let $L_m = \{y_1, y_4, y_{m+3}, x_0\}$. The representation of the vertices and edges of $\mathcal{D}_n$ with respect to the L_m are shown in the following tables.

Note that from the above mentioned tables, vertices and edges do not posses same representation with the resolving set L_m. This shows that $dim_m(\mathcal{D}_n) \leq 4$.

Lemma 8: If n is odd then $dim_m(\mathcal{D}_n) \geq 4$.

Proof: Suppose that when n is odd, then $dim_m(\mathcal{D}_n) = 3$. From Lemma 5, there should be at least one vertex in each

TABLE 11. Representation of inner vertices of $\mathcal{D}_n$.

$r(v\|L_m)$	y_1	y_4	y_{m+3}	x_0
$x_\ell : 0 \le \ell \le 1$	$2-\ell$	$5-\ell$	$m+\ell-1$	ℓ
$x_\ell : 2 \le \ell \le 3$	ℓ	$5-\ell$	$m+1$	ℓ
$x_\ell : 4 \le \ell \le m$	ℓ	$\ell-3$	$m-\ell+4$	ℓ
$x_\ell : m+1 \le \ell \le m+2$	$m+1$	$\ell-3$	$m-\ell+4$	$2m-\ell+1$
$x_\ell : m+3 \le \ell \le m+4$	$2m-\ell+3$	$\ell-3$	$\ell-m-2$	$2m-\ell+1$
$x_\ell : m+5 \le \ell \le 2m$	$2m-\ell+3$	$2m-\ell+6$	$\ell-m-2$	$2m-\ell+1$

TABLE 12. (a) Representation of outer edges of $\mathcal{D}_n$. (b) Representation of outer edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_4	y_{m+3}
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	0	$3-\ell$	$m+\ell-2$
$y_\ell y_{\ell+1} : 2 \le \ell \le 3$	$\ell-1$	$3-\ell$	$m-\ell+2$
$y_\ell y_{\ell+1} : 4 \le \ell \le m$	$\ell-1$	$\ell-4$	$m-\ell+2$
$y_\ell y_{\ell+1} : m+1 \le \ell \le m+2$	$2m-\ell+1$	$\ell-4$	$2m-\ell-5$
$y_\ell y_{\ell+1} : m+3 \le \ell \le m+4$	$2m-\ell+1$	$\ell-4$	$\ell-m-3$
$y_\ell y_{\ell+1} : m+5 \le \ell \le 2m$	$2m-\ell+1$	$2m-\ell+4$	$\ell-m-3$

(a)

$r(e\|L_m)$	x_0
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	$\ell+1$
$y_\ell y_{\ell+1} : 2 \le \ell \le 3$	$\ell+1$
$y_\ell y_{\ell+1} : 4 \le \ell \le m$	$\ell+1$
$y_\ell y_{\ell+1} : m+1 \le \ell \le m+2$	$2m-\ell+1$
$y_\ell y_{\ell+1} : m+3 \le \ell \le m+4$	$2m-\ell+1$
$y_\ell y_{\ell+1} : m+5 \le \ell \le 2m$	$2m-\ell+1$

(b)

TABLE 13. Representation of inner edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_4	y_{j+3}	x_0
$x_\ell x_{\ell+1} : 0 \le \ell \le 1$	1	$4-\ell$	$m+\ell-1$	ℓ
$x_\ell x_{\ell+1} : 2 \le \ell \le 3$	ℓ	$4-\ell$	$m-\ell+3$	ℓ
$x_\ell x_{\ell+1} : 4 \le \ell \le m$	ℓ	$\ell-3$	$m-\ell+3$	ℓ
$x_\ell x_{\ell+1} : m+1 \le \ell \le m+2$	$2m-\ell+2$	$\ell-3$	$m-\ell+3$	$2m-\ell$
$x_\ell x_{\ell+1} : m+3 \le \ell \le m+4$	$2m-\ell+2$	$\ell-3$	$\ell-m-2$	$2m-\ell$
$x_\ell x_{\ell+1} : m+5 \le \ell \le 2m$	$2m-\ell+2$	$2m-\ell+5$	$\ell-m-2$	$2m-\ell$

TABLE 14. Representation of outer and inner edges of $\mathcal{D}_n$.

$r(e\|L_m)$	y_1	y_4	y_{m+3}	x_0
$y_\ell x_\ell : 0 \le \ell \le 1$	$1-\ell$	$4-\ell$	$m+\ell-2$	ℓ
$y_\ell x_\ell : 2 \le \ell \le 3$	$\ell-1$	$4-\ell$	m	ℓ
$y_\ell x_\ell : 4 \le \ell \le m$	$\ell-1$	$\ell-4$	$m-\ell+3$	ℓ
$y_\ell x_\ell : m+1 \le \ell \le m+2$	m	$\ell-4$	$2m-\ell-4$	$2m-\ell+1$
$y_\ell x_\ell : m+3 \le \ell \le m+4$	$2m-\ell+2$	$\ell-4$	$\ell-m-3$	$2m-\ell+1$
$y_\ell x_\ell : m+5 \le \ell \le 2m$	$2m-\ell+2$	$2m-\ell+5$	$\ell-m-3$	$2m-\ell+1$

cycle of a graph, so the following possibilities are shown in the table below.

By symmetry of the graph other relations can be considered, they will have same kind of contradictions. Hence, it is proved that, when n is odd, then $dim_e(\mathcal{D}_n) = 4$.

Remark 3: Let $\mathcal{D}_n$ be a prim graph with $n \ge 3$. If n is odd, then $dim_m(\mathcal{D}_n) > dim_e(\mathcal{D}_n) > dim(\mathcal{D}_n)$.

Problem 1: The prism graph is also equivalent to generalized petersen graph $G(P(n, 1))$. It would be interesting to consider mixed metric dimension of other families of Generalized Petersen graphs.

TABLE 15. Contradictions for $\mathcal{D}_n$.

Resolving vertices	Contradictions
$\{y_0, x_t, x_{t+1}\}$ $x_t(0 \le t \le n-2)$	$r(x_0\|\{y_0, x_t, x_{t+1}\}$ $= r(x_{n-1}x_0\|\{y_0, x_t, x_{t+1}\}$ $= (1, t, t+1)$ For $0 \le t \le m-1$. $r(x_{n-1}x_0\|\{y_0, x_t, x_{t+1}\}$ $= r(y_{n-1}x_{n-1}\|\{y_0, x_t, x_{t+1}\}$ $= (1, 2m-t, 2m-t-1)$ For $m \le t \le n-3$. $r(x_{n-1}x_0\|\{y_0, x_t, x_{t+1}\}$ $= r(y_{n-1}x_{n-1}\|\{y_0, x_t, x_{t+1}\}$ $= (1, 1, 0)$ For $t = n-2$.
$\{y_0, y_t, x_0\}$ $y_t(1 \le t \le n-1)$	$r(y_0\|\{y_0, y_t, x_0\}$ $= r(y_{n-1}y_0\|\{y_0, y_t, x_0\}$ $= (0, t, 1)$ For $1 \le t \le m$. $r(y_0\|\{y_0, y_t, x_0\}$ $= r(y_0y_1\|\{y_0, y_t, x_0\}$ $= (0, 2m-t+1, 1)$ For $m+1 \le t \le n-1$.
$\{y_1, y_{m+3}, x_0\}$	$r(y_1\|\{y_1, y_{m+3}, x_0\}$ $= r(y_1y_2\|\{y_1, y_{m+3}, x_0\}$ $= (0, m-1, 2)$
$\{y_1, y_4, x_0\}$	$r(y_0y_1\|\{y_1, y_4, x_0\}$ $= r(y_1x_1\|\{y_1, y_4, x_0\}$ $= (0, 3, 1)$

III. THE GRAPH OF ANTI-PRISM $\mathcal{A}_n$

In this section, we give the mixed metric dimension for anti-prism graph $\mathcal{A}_n$, with $n \ge 3$.

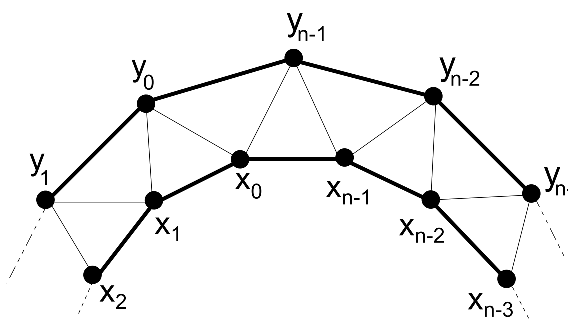
The anti-prism $\mathcal{A}_n$ as defined in [1], [8] is a 4-regular graph. The graph of anti-prism consist of an outer cycle $y_0, y_1, \ldots, y_{n-1}$, and an inner cycle $x_0, x_1, \ldots, x_{n-1}$. Now mathematically the vertex and edge set are represented as, The vertex set of $V(\mathcal{A}_n)$ is,

$$V(\mathcal{A}_n) = \{x_\ell, y_\ell \mid \ell = 0, \ldots, n-1\}$$

and $E(\mathcal{A}_n)$ is,

$$E(\mathcal{A}_n) = \{(x_\ell, x_{\ell+1}); (x_\ell, y_\ell); (x_\ell, y_{\ell+1}); (y_\ell, y_{\ell+1}) \mid \ell = 0, \ldots, n-1\}.$$

The graph of $\mathcal{A}_n$ is shown below,



Lemma 2, and Lemma 4 present the standard metric and edge metric dimension of $\mathcal{A}_n$.

Lemma 9 [23]: If for $\mathcal{A}_n$, the edge metric generator contains two vertices of one cycle than at least it contain two vertices of other cycle respectively.

TABLE 16. Representation of outer vertices of $\mathcal{A}_n$.

$r(v\|L_m)$	y_0	y_m	x_2	x_{m+2}
$y_\ell : 0 \le \ell \le 1$	ℓ	$m-\ell$	$2-\ell$	$m+\ell-1$
$y_\ell : 2 \le \ell \le m$	ℓ	$m-\ell$	$\ell-1$	$m-\ell+2$
$y_\ell : \ell = m+1$	$\ell-2$	1	$\ell-1$	1
$y_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell$	$\ell-m$	$2m-\ell+2$	$\ell-m-1$

TABLE 17. Representation of inner vertices of $\mathcal{A}_n$.

$r(v\|L_m)$	y_0	y_m	x_2	x_{m+2}
$x_\ell : 0 \le \ell \le 1$	1	m	$2-\ell$	$m+\ell-2$
$x_\ell : 2 \le \ell \le m$	ℓ	$m-\ell+1$	$\ell-2$	$m-\ell+1$
$x_\ell : m+1 \le \ell \le m+2$	$2m-\ell+1$	$\ell-m$	$\ell-2$	$m-\ell+2$
$x_\ell : m+3 \le \ell \le 2m-1$	$2m-\ell+1$	$\ell-m$	$2m-\ell+2$	$\ell-m-2$

TABLE 18. Representation of outer edges of $\mathcal{A}_n$.

$r(e\|L_m)$	y_0	y_m	x_2	x_{m+2}
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	ℓ	$m-\ell-1$	1	$m+\ell-1$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell-1$	$\ell-1$	$m-\ell+1$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$2m-\ell-1$	$\ell-m$	$\ell-1$	1
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell-1$	$\ell-m$	$2m-\ell+1$	$\ell-m-1$

TABLE 19. Representation of inner edges of $\mathcal{A}_n$.

$r(e\|L_m)$	y_0	y_m	x_2	x_{m+2}
$x_\ell x_{\ell+1} : 0 \le \ell \le 1$	1	$m-\ell$	$1-\ell$	$m+\ell-2$
$x_\ell x_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell$	$\ell-2$	$m-\ell+1$
$x_\ell x_{\ell+1} : m \le \ell \le m+1$	$2m-\ell$	1	$\ell-2$	$m-\ell+1$
$x_\ell x_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell$	$\ell-m$	$2m-\ell+1$	$\ell-m-2$

TABLE 20. (a) Representation of outer and inner edges of $\mathcal{A}_n$. (b) Representation of outer and inner edges of $\mathcal{A}_n$.

$r(e\|L_m)$	y_0	y_m	x_2	x_{m+2}
$y_\ell x_\ell : 0 \le \ell \le 2$	ℓ	$m-\ell$	$2-\ell$	$m+\ell-2$
$y_\ell x_\ell : 3 \le \ell \le m$	ℓ	$m-\ell$	$\ell-2$	$m-\ell+2$
$y_\ell x_\ell : m+1 \le \ell \le m+2$	$2m-\ell$	$\ell-m$	$\ell-2$	$m-\ell+2$
$y_\ell x_\ell : m+3 \le \ell \le 2m-1$	$2m-\ell$	$\ell-m$	$2m-\ell+2$	$\ell-m-2$

(a)

$r(e\|L_m)$	y_0	y_m	x_2	x_{m+2}
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	ℓ	$m-\ell$	$1-\ell$	$m+\ell-1$
$y_\ell x_{\ell+1} : 2 \le \ell \le m$	ℓ	$m-\ell$	$\ell-1$	$m-\ell+1$
$y_\ell x_{\ell+1} : m+1 \le \ell \le 2m-1$	$2m-\ell$	$\ell-m$	$2m-\ell+1$	$\ell-m-1$

(b)

Proof: Since the mixed metric generator is a standard metric generator as well as edge metric generator, so this will follow for mixed metric generator as well.

In the following theorem, the mixed metric dimension of $\mathcal{A}_n$ is given.

Theorem 2: Let $\mathcal{A}_n$ be an anti-prism graph for $n \ge 3$. Then we have,

$$dim_m(\mathcal{A}_n) = \begin{cases} 4, & \text{n is even ;} \\ 5, & \text{n is odd.} \end{cases}$$

Proof: The proof this theorem will be deduced from Lemma 10 to Lemma 13.

Lemma 10: If n is even then $dim_m(\mathcal{A}_n) \le 4$.

Proof: In case of even n, $n = 2m$, where $m \in \mathbb{Z}^+$. We let $L_m = \{y_0, y_m, x_2, x_{m+2}\}$. In order to show that L_m represents mixed metric generator for $\mathcal{A}_n$. We present representation of any vertices, and edges of $V(\mathcal{A}_n)$ with respect to $L_m$in the following tables.

TABLE 21. Contradictions for $\mathcal{A}_n$.

Resolving vertices	Contradictions
$\{y_0, y_1, x_t\}$ $x_t(0 \le t \le n-1)$	$r(y_0\|\{y_0,y_1,x_t\} = r(y_{n-1}y_0\|\{y_0,y_1,x_t\}, = (0,1,1)$, For $0 \le t \le 1$. $r(y_0\|\{y_0,y_1,x_t\} = r(y_{n-1}y_0\|\{y_0,y_1,x_t\}, = (0,1,t)$, For $2 \le t \le m$. $r(y_0\|\{y_0,y_1,x_t\} = r(y_0x_1\|\{y_0,y_1,x_t\}, = (0,1,2m-t+1)$, For $m+1 \le t \le n-1$.
$\{y_0, y_t, x_0\}$ $y_t(1 \le t \le n-1)$	$r(y_0\|\{y_0,y_t,x_0\} = r(y_{n-1}y_0\|\{y_0,y_t,x_0\}, = (0,t,1)$, For $1 \le t \le m-1$. $r(y_0\|\{y_0,y_t,x_0\} = r(y_0x_1\|\{y_0,y_t,x_0\}, = (0,2m-t,1)$, For $m \le t \le n-1$.
$\{y_0, y_m, x_2\}$	$r(y_0\|\{y_0,y_m,x_2\} = r(y_0x_0\|\{y_0,y_m,x_2\}, = (0,m,2)$.
$\{y_0, y_m, x_{m+2}\}$	$r(y_{n-1}\|\{y_0,y_m,x_2\} = r(y_{n-1}x_0\|\{y_0,y_m,x_2\}, = (1,m-1,m-2)$.

TABLE 22. (a) Representation of outer vertices of $\mathcal{A}_n$. (b) Representation of outer vertices of $\mathcal{A}_n$.

$r(v\|L_m)$	y_0	y_m	x_0
$y_\ell : 0 \le \ell \le 1$	ℓ	$m-\ell$	$\ell+1$
$y_\ell : 2 \le \ell \le m$	ℓ	$m-\ell$	$\ell+1$
$y_\ell : m+1 \le \ell \le m+2$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$
$y_\ell : m+3 \le \ell \le 2m$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$

(a)

$d(.,.)$	x_2	x_{m+1}
$y_\ell : 0 \le \ell \le 1$	$2-\ell$	$m-\ell+1$
$y_\ell : 2 \le \ell \le m$	$\ell-1$	$m-\ell+1$
$y_\ell : m+1 \le \ell \le m+2$	$\ell-1$	$\ell-m$
$y_\ell : m+3 \le \ell \le 2m$	$2m-\ell+3$	$\ell-m$

(b)

TABLE 23. (a) Representation of inner vertices of $\mathcal{A}_n$. (b) Representation of inner vertices of $\mathcal{A}_n$.

$r(v\|L_m)$	y_0	y_m	x_0
$x_\ell : 0 \le \ell \le 1$	1	$m-\ell+1$	ℓ
$x_\ell : 2 \le \ell \le m$	ℓ	$m-\ell+1$	ℓ
$x_\ell : m+1 \le \ell \le m+2$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$
$x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$

(a)

$d(.,.)$	x_2	x_{m+1}
$x_\ell : 0 \le \ell \le 1$	$2-\ell$	m
$x_\ell : 2 \le \ell \le m$	$\ell-2$	$m-\ell+1$
$x_\ell : m+1 \le \ell \le m+2$	$\ell-2$	$\ell-m-1$
$x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+3$	$\ell-m-1$

(b)

Note that from the above mentioned tables, vertices and edges do not show same representation with the L_m. This shows that $L_m = \{y_0, y_m, x_2, x_{m+2}\}$ resolves vertices and edges of $(\mathcal{A}_n)$, which indicates that $dim_m(\mathcal{A}_n) \le 4$.

Lemma 11: If n is even then $dim_m(\mathcal{A}_n) \ge 4$.

TABLE 24. **(a) Representation of outer edges of $\mathcal{A}_n$. (b) Representation of outer edges of $\mathcal{A}_n$.**

$r(e\|L_m)$	y_0	y_m	x_0
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	ℓ	$m-\ell-1$	$\ell+1$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell-1$	$\ell+1$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$2m-\ell$	$\ell-m$	$2m-\ell$
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-k$	$\ell-m$	$2m-\ell$
$y_\ell y_{\ell+1} : \ell = 2m$	0	$\ell-m$	1

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	1	$m-\ell$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	$\ell-1$	$m-\ell$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$\ell-1$	1
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m$
$y_\ell y_{\ell+1} : \ell = 2m$	2	$\ell-m$

(b)

TABLE 25. **Representation of inner edges of $\mathcal{A}_n$.**

$r(e\|L_m)$	y_0	y_m	x_0	x_2	x_{m+1}
$x_\ell x_{\ell+1} : 0 \le \ell \le 1$	1	$m-\ell$	ℓ	$1-\ell$	$m-\ell$
$x_\ell x_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell$	ℓ	$\ell-2$	$m-\ell$
$x_\ell x_{\ell+1} : m \le \ell \le m+1$	m	1	$2m-\ell$	$\ell-2$	0
$x_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$2m-\ell+1$	$\ell-m$	$2m-\ell$	$2m-\ell+2$	$\ell-m-1$

TABLE 26. **(a) Representation of outer and inner edges of $\mathcal{A}_n$. (b) Representation of outer and inner edges of $\mathcal{A}_n$. (c) Representation of outer and inner edges of $\mathcal{A}_n$. (d) Representation of outer and inner edges of $\mathcal{A}_n$.**

$r(e\|L_m)$	y_0	y_m	x_0
$y_\ell x_\ell : 0 \le \ell \le 1$	ℓ	$m-\ell$	ℓ
$y_\ell x_\ell : 2 \le \ell \le m$	ℓ	$m-\ell$	ℓ
$y_\ell x_\ell : m+1 \le \ell \le m+2$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$
$y_\ell x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell x_\ell : 0 \le \ell \le 1$	$2-\ell$	m
$y_\ell x_\ell : 2 \le \ell \le m$	$\ell-2$	$m-\ell+1$
$y_\ell x_\ell : m+1 \le \ell \le m+2$	$\ell-2$	$\ell-m-1$
$y_\ell x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+3$	$\ell-m-1$

(b)

$r(e\|L_m)$	y_0	y_m	x_0
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	ℓ	$m-\ell$	$\ell+1$
$y_\ell x_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell$	$\ell+1$
$y_\ell x_{\ell+1} : m \le \ell \le m+1$	m	$\ell-m$	$2m-\ell$
$y_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$2m-\ell+1$	$\ell-m$	$2m-\ell$

(c)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	$1-\ell$	$m-\ell$
$y_\ell x_{\ell+1} : 2 \le \ell \le m$	$\ell-1$	$m-\ell$
$y_\ell x_{\ell+1} : m+1 \le \ell \le m+2$	$\ell-1$	$\ell-m$
$y_\ell x_{\ell+1} : m+3 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$

(d)

Proof: Suppose, $dim_m(\mathcal{A}_n) = 3$. From Lemma 9, the graph of $\mathcal{A}_n$ have at least two vertices in both the cycles, so the following possibilities are shown in the table below.

By symmetry of the graph other relations can be considered, they will have same kind of contradictions. Hence, it is proved that, when n is even, then $dim_e(\mathcal{A}_n) = 4$.

Remark 4: Let $\mathcal{A}_n$ be a anti-prism graph with $n \ge 5$. If n is even, then $dim_m(\mathcal{A}_n) = dim_e(\mathcal{A}_n)$.

*Lemma 12:*Let $dim_m(\mathcal{A}_n) \le 5$, when n is odd.

TABLE 27. **Contradictions for $\mathcal{A}_n$.**

Resolving vertices	Contradictions
$\{y_0, y_1, x_t, x_{t+1}\}$ $x_t(0 \le t \le n-2)$	$r(y_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= r(y_{n-1}y_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= (0, 1, 1, t+1)$ For $0 \le t \le 1$. $r(y_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= r(y_{n-1}x_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= (0, 1, t, t+1)$, For $2 \le t \le m-1$. $r(x_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= r(y_{n-1}x_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= (1, 2, m, 2m-t)$, For $m \le t \le m+1$. $r(x_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= r(y_{n-1}x_0\|\{y_0, y_1, x_t, x_{t+1}\}$ $= (1, 2, 2m-t+1, 2m-t)$, For $m+2 \le t \le n-1$.
$\{y_0, y_m, x_0, x_2\}$	$r(y_{n-1}y_0\|\{y_0, y_m, x_0, x_2\}$ $= r(y_0\|\{y_0, y_m, x_0, x_2\}$ $= (0, m, 1, 2)$
$\{y_0, y_m, x_0, x_{m+1}\}$	$r(y_{n-1}y_0\|\{y_0, y_m, x_0, x_{m+1}\}$ $= r(y_0x_1\|\{y_0, y_m, x_0, x_{m+1}\}$ $= (0, m, 1, m)$
$\{y_0, y_m, x_2, x_{m+1}\}$	$r(y_{n-1}y_0\|\{y_0, y_m, x_2, x_{m+1}\}$ $= r(y_0x_0\|\{y_0, y_m, x_2, x_{m+1}\}$ $= (0, m, 2, m)$

TABLE 28. **Representation of exterior vertices of $\mathcal{R}_n$.**

$r(v\|L_m)$	z_0	y_1	y_m	x_2	x_{m+2}
$z_\ell : 0 \le \ell \le 1$	ℓ	$2-\ell$	$m-\ell+1$	$3-\ell$	$m+\ell$
$z_\ell : 2 \le \ell \le m$	ℓ	ℓ	$m-\ell+1$	ℓ	$m-\ell+3$
$z_\ell : \ell = m+1$	$2m-\ell$	$2m-\ell+2$	2	$2m-\ell+2$	2
$z_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell$	$2m-\ell+2$	$\ell-m+1$	$2m-\ell+3$	$\ell-m$

TABLE 29. **(a) Representation of central vertices of $\mathcal{R}_n$. Representation of central vertices of $\mathcal{R}_n$.**

$r(v\|L_m)$	z_0	y_1	y_m
$y_\ell : 0 \le \ell \le 1$	$\ell+1$	$1-\ell$	$m-\ell$
$y_\ell : 2 \le \ell \le m$	$\ell+1$	$\ell-1$	$m-\ell$
$y_\ell : \ell = m+1$	$2m-\ell+1$	$2m-\ell+1$	1
$y_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+1$	$2m-\ell+1$	$\ell-m$

(a)

$r(v\|L_m)$	x_2	x_{m+2}
$y_\ell : 0 \le \ell \le 1$	$2-\ell$	$m+\ell-1$
$y_\ell : 2 \le \ell \le m$	$\ell-1$	$m-\ell+2$
$y_\ell : \ell = m+1$	$2m-\ell+1$	1
$y_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m-1$

(b)

Proof: Now $n = 2m+1$, where $m \in \mathbb{Z}^+$. We let $L_m = \{y_0, y_m, x_0, x_2, x_{m+1}\}$. In order to show that L_m represents mixed metric generator for $\mathcal{A}_n$. We present representation of any vertices and edges of $V(\mathcal{A}_n)$ with respect to L_m in the following tables.

Note that from the above mentioned tables, vertices and edges do not show same representation with the resolving set L_m. This shows that $Lm = \{y_0, y_m, x_0, x_2, x_{m+1}\}$ resolves vertices and edges of $(\mathcal{A}_n)$, which indicates that $dim_m(\mathcal{A}_n) \le 4$. Conversely, we will examine that $dim_m(\mathcal{A}_n) \ge 5$.

Lemma 13: If n is odd then $dim_m(\mathcal{A}_n) \ge 5$.

TABLE 30. (a) Representation of interior vertices of $\mathcal{R}_n$. (b) Representation of interior vertices of $\mathcal{R}_n$.

$r(v\|L_m)$	z_0	y_1	y_m
$x_\ell : 0 \le \ell \le 1$	2	$2-\ell$	m
$x_\ell : 2 \le \ell \le m$	$\ell+1$	$\ell-1$	$m-\ell+1$
$x_\ell : \ell = m+1$	$2m-\ell+2$	$2m-\ell+1$	1
$x_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$2m-\ell+2$	$\ell-m$

(a)

$r(v\|L_m)$	x_2	x_{m+2}
$x_\ell : 0 \le \ell \le 1$	$2-\ell$	$m+\ell-2$
$x_\ell : 2 \le \ell \le m$	$\ell-2$	$m-\ell+2$
$x_\ell : \ell = m+1$		$2m-\ell 1$
$x_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m-2$

(b)

TABLE 31. (a) Representation of exterior edges of $\mathcal{R}_n$. (b) Representation of exterior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_1	y_m
$z_\ell z_{\ell+1} : 0 \le \ell \le 1$	ℓ	1	$m-\ell$
$z_\ell z_{\ell+1} : 2 \le \ell \le m-1$	ℓ	ℓ	$m-\ell$
$z_\ell z_{\ell+1} : m \le \ell \le m+1$	$2m-\ell-1$	m	$\ell-m+1$
$z_\ell z_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell-1$	$2m-\ell+1$	$\ell-m+1$

(a)

$r(e\|L_m)$	x_2	x_{m+2}
$z_\ell z_{\ell+1} : 0 \le \ell \le 1$	2	$m+\ell$
$z_\ell z_{\ell+1} : 2 \le \ell \le m-1$	ℓ	$m-\ell+2$
$z_\ell z_{\ell+1} : m \le \ell \le m+1$	ℓ	2
$z_\ell z_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m$

(b)

TABLE 32. (a) Representation of exterior and central edges of $\mathcal{R}_n$. (Representation of exterior and central edges of $\mathcal{R}_n$.)

$r(e\|L_m)$	z_0	y_1	y_m
$z_\ell y_\ell : 0 \le \ell \le 1$	ℓ	$1-\ell$	$m-\ell$
$z_\ell y_\ell : 2 \le \ell \le m$	ℓ	$\ell-1$	$m-\ell$
$z_\ell y_\ell : \ell = m+1$	$2m-\ell$	$2m-\ell+1$	1
$z_\ell y_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell$	$2m-\ell+1$	$\ell-m$

(a)

$r(e\|L_m)$	x_2	x_{m+2}
$z_\ell y_\ell : 0 \le \ell \le 1$	$2-\ell$	$m+\ell-1$
$z_\ell y_\ell : 2 \le \ell \le m$	$\ell-1$	$m-\ell+2$
$z_\ell y_\ell : \ell = m+1$	$\ell-1$	1
$z_\ell y_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m-1$

(b)

Proof: When n is odd, then $dim_m(\mathcal{A}_n) \ge 5$. Then a table is shown.

Hence, it follows from the above table that there are no mixed metric generator with four vertices for $\mathcal{A}_n$, by symmetry of graphs other relations can be obtain which shows the same kind of contradictions, so $dim_m(\mathcal{A}_n) = 5$.

Remark 4: If $\mathcal{A}_n$ is an Anti-prim graph for $n \ge 3$, when n is odd, then $dim_m(\mathcal{A}_n) = dim_e(\mathcal{A}_n)$.

IV. THE GRAPH OF $\mathcal{R}_n$

The graph of $\mathcal{R}_n$ is formed by the combination of anti-prism and prism graph. It is defined in [2].

Mathematically the vertex and edges set are as follows,

$$V(\mathcal{R}_n) = \{x_\ell; y_\ell; z_\ell : \ell = 0, \ldots, n-1\}$$

TABLE 33. (a) Representation of central edges of $\mathcal{R}_n$. (b) Representation of central edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_1	y_m
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	$\ell+1$	0	$m-\ell-1$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	$\ell+1$	$\ell-1$	$m-\ell-1$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$2m-\ell$	$m-1$	$\ell-m$
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell$	$2m-\ell$	$\ell-m$

(a)

$r(e\|L_m)$	x_2	x_{m+2}
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	1	$m+\ell-1$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	$\ell-1$	$m-\ell+1$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$\ell-1$	1
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell+1$	$\ell-m-1$

(b)

TABLE 34. (a) Representation of central and interior edges of $\mathcal{R}_n$. (b) Representation of central and interior edges of $\mathcal{R}_n$. (c) Representation of central and interior edges of $\mathcal{R}_n$. (d) Representation of central and interior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_1	y_m	x_2
$y_\ell x_\ell : 0 \le \ell \le 1$	$\ell+1$	$1-\ell$	$m-\ell$	$2-\ell$
$y_\ell x_\ell : 2 \le \ell \le m$	$\ell+1$	$\ell-1$	$m-\ell$	$\ell-2$
$y_\ell x_\ell : \ell = m+1$	$2m-\ell+1$	$2m-\ell+1$	1	$2m-\ell$
$y_\ell x_\ell : m+2 \le \ell \le 2m-1$	$2m-\ell+1$	$2m-\ell+1$	$\ell-m$	$2m-\ell+2$

(a)

$r(e\|L_m)$	x_{m+2}
$y_\ell x_\ell : 0 \le \ell \le 1$	$m+\ell-2$
$y_\ell x_\ell : 2 \le \ell \le m$	$m-\ell+2$
$y_\ell x_\ell : \ell = m+1$	1
$y_\ell x_\ell : m+2 \le \ell \le 2m-1$	$\ell-m-2$

(b)

$r(e\|L_m)$	z_0	y_1	y_m	x_2
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	$\ell+1$	$1-\ell$	$m-\ell$	$1-\ell$
$y_\ell x_{\ell+1} : 2 \le \ell \le m$	$\ell+1$	$\ell-1$	$m-\ell$	$\ell-1$
$y_\ell x_{\ell+1} : m+1 \le \ell \le 2m-1$	$2m-\ell+1$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$

(c)

$r(e\|L_m)$	x_{m+2}
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	$m+\ell-1$
$y_\ell x_{\ell+1} : 2 \le \ell \le m$	$m-\ell+1$
$y_\ell x_{\ell+1} : m+1 \le \ell \le 2m-1$	$\ell-m-1$

(d)

and edge set

$$E(\mathcal{R}_n) = \{(x_\ell, x_{\ell+1}); (x_\ell, y_\ell); (x_{\ell+1}, y_\ell); (y_\ell, y_{\ell+1}); (y_\ell, z_\ell); (z_\ell, z_{\ell+1}) : \ell = 0, \ldots, n-1\}.$$

In order to avoid any ambiguity, we say cycles induced by x_ℓ, y_ℓ and z_ℓ for $\ell = 0, \ldots, n-1$, the interior cycle, the centre cycle and the exterior cycle respectively. The graph of $\mathcal{R}_n$ is shown below,

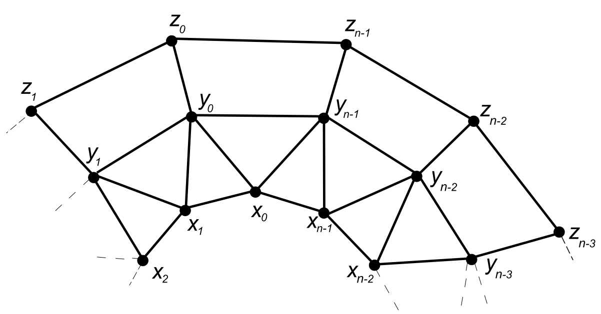


TABLE 35. (a) Representation of interior edges of $\mathcal{R}_n$. (b) Representation of interior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_1	y_m
$x_\ell x_{\ell+1} : 0 \leq \ell \leq 1$	2	1	$m-\ell$
$x_\ell x_{\ell+1} : 2 \leq \ell \leq m-1$	$\ell+1$	$\ell-1$	$m-\ell$
$x_\ell x_{\ell+1} : m \leq \ell \leq m+1$	$2m-\ell+1$	$\ell-1$	1
$x_\ell x_{\ell+1} : m+2 \leq \ell \leq 2m-1$	$2m-\ell+1$	$2m-\ell+1$	$\ell-m$

(a)

$r(e\|L_m)$	x_2	x_{m+2}
$x_\ell x_{\ell+1} : 0 \leq \ell \leq 1$	$1-\ell$	$m+\ell-2$
$x_\ell x_{\ell+1} : 2 \leq \ell \leq m-1$	$\ell-2$	$m-\ell+1$
$x_\ell x_{\ell+1} : m \leq \ell \leq m+1$	$\ell-2$	$m-\ell+1$
$x_\ell x_{\ell+1} : m+2 \leq \ell \leq 2m-1$	$2m-\ell+1$	$\ell-m-2$

(b)

TABLE 36. Contradictions for $\mathcal{R}_n$.

Resolving vertices	Contradictions
$\{z_0, y_t, y_{t+1}, x_0\}$ $y_t(0 \leq t \leq n-2)$	$r(z_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= r(z_{n-1}z_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= (0, t+1, t+2, 2)$ For $0 \leq t \leq 1$. $r(y_{n-1}\|\{z_0, y_t, y_{t+1}, x_0\}$ $= r(y_{n-1}x_{n-1}\|\{z_0, y_t, y_{t+1}, x_0\}$ $= (2, t+1, 2m-t-2, 1)$ For $2 \leq t \leq m-1$. $r(y_{n-1}\|\{z_0, y_t, y_{t+1}, x_0\}$ $= r(y_{n-1}x_{n-1}\|\{z_0, y_t, y_{t+1}, x_0\}$ $= (2, 2m-t-1, 2m-t-2, 1)$ For $m \leq t \leq n-2$.
$\{z_0, y_0, x_t$ $x_{t+1}\}, x_t(0 \leq t \leq n-2)$	$r(y_0\|\{z_0, y_0, x_t, x_{t+1}\}$ $= r(y_{n-1}y_0\|\{z_0, y_0, x_t, x_{t+1}\}$ $= (1, 0, 1, t+1)$ For $0 \leq t \leq 1$. $r(y_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= r(y_0x_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= (1, 0, t, t+1)$ For $2 \leq t \leq m-1$. $r(y_0x_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= r(y_{n-1}y_0\|\{z_0, y_t, y_{t+1}, x_0\}$ $= (1, 0, 2m-t, 2m-t-1)$ For $m \leq t \leq n-2$.

TABLE 37. (a) Representation of exterior vertices of $\mathcal{R}_n$. (b) Representation of exterior vertices of $\mathcal{R}_n$.

$r(v\|L_m)$	z_0	y_m	x_0
$z_\ell : 0 \leq \ell \leq 1$	ℓ	$m-\ell+1$	$\ell+2$
$z_\ell : 2 \leq \ell \leq m$	ℓ	$m-\ell+1$	$\ell+2$
$z_\ell : m+1 \leq \ell \leq m+2$	$2m-\ell+1$	$\ell-m+1$	$2m-\ell+2$
$z_\ell : m+3 \leq \ell \leq 2m$	$2m-\ell+1$	$\ell-m+1$	$2m-\ell+2$

(a)

$r(v\|L_m)$	x_2	x_{m+1}
$z_\ell : 0 \leq \ell \leq 1$	$3-\ell$	$m-\ell+2$
$z_\ell : 2 \leq \ell \leq m$	ℓ	$m-\ell+2$
$z_\ell : m+1 \leq \ell \leq m+2$	ℓ	$\ell-m+1$
$z_\ell : m+3 \leq \ell \leq 2m$	$2m-\ell+4$	$\ell-m+1$

(b)

Lemma 14 [12]: When $n \geq 5$, the standard metric dimension for $\mathcal{R}_n$,is $dim\mathcal{R}_n = 3$.

In the following theorem, we give the exact value of mixed metric dimension of $\mathcal{R}_n$.

TABLE 38. (a) Representation of central vertices of $\mathcal{R}_n$. (b) Representation of central vertices of $\mathcal{R}_n$.

$r(v\|L_m)$	z_0	y_m	x_0	x_2
$y_\ell : 0 \leq \ell \leq 1$	$\ell+1$	$m-\ell$	$\ell+1$	$m-\ell-2$
$y_\ell : 2 \leq \ell \leq m$	$\ell+1$	$m-\ell$	$\ell+1$	$\ell-1$
$y_\ell : m+1 \leq \ell \leq m+2$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$	$\ell-1$
$y_\ell : m+3 \leq \ell \leq 2m$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$	$2m-\ell+3$

(a)

$r(v\|L_m)$	x_{m+1}
$y_\ell : 0 \leq \ell \leq 1$	$m-\ell+1$
$y_\ell : 2 \leq \ell \leq m$	$m-\ell+1$
$y_\ell : m+1 \leq \ell \leq m+2$	$\ell-m$
$y_\ell : m+3 \leq \ell \leq 2m$	$\ell-m$

(b)

TABLE 39. (a) Representation of interior vertices of $\mathcal{R}_n$. (b) Representation of interior vertices of $\mathcal{R}_n$.

$r(v\|L_m)$	z_0	y_m	x_0
$x_\ell : 0 \leq \ell \leq 1$	2	$m-\ell+1$	ℓ
$x_\ell : 2 \leq \ell \leq m$	$\ell+1$	$m-\ell+1$	ℓ
$x_\ell : m+1 \leq \ell \leq m+2$	$2m-\ell+3$	$\ell-m$	$2m-\ell+1$
$x_\ell : m+3 \leq \ell \leq 2m$	$2m-\ell+3$	$\ell-m$	$2m-\ell+1$

(a)

$r(v\|L_m)$	x_2	x_{m+1}
$x_\ell : 0 \leq \ell \leq 1$	$2-\ell$	m
$x_\ell : 2 \leq \ell \leq m$	$\ell-2$	$m-\ell+1$
$x_\ell : m+1 \leq \ell \leq m+2$	$\ell-2$	$\ell-m-1$
$x_\ell : m+3 \leq \ell \leq mj$	$2m-\ell+3$	$\ell-m-1$

(b)

TABLE 40. (a) Representation of exterior edges of $\mathcal{R}_n$. (b) Representation of exterior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_m	x_0
$z_\ell z_{\ell+1} : 0 \leq \ell \leq 1$	ℓ	$m-\ell$	$\ell+2$
$z_\ell z_{\ell+1} : 2 \leq \ell \leq m-1$	ℓ	$m-\ell$	$\ell+2$
$z_\ell z_{\ell+1} : m \leq \ell \leq m+1$	$2m-\ell$	$\ell-m+1$	$2m-\ell+1$
$z_\ell z_{\ell+1} : m+2 \leq \ell \leq 2m-1$	$2m-\ell$	$\ell-m+1$	$2m-\ell+1$
$z_\ell z_{\ell+1} : \ell = 2m$	0	$\ell-m+1$	2

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$z_\ell z_{\ell+1} : 0 \leq \ell \leq 1$	2	$m-\ell+1$
$z_\ell z_{\ell+1} : 2 \leq \ell \leq m-1$	ℓ	$m-\ell+1$
$z_\ell z_{\ell+1} : m \leq \ell \leq m+1$	ℓ	2
$z_\ell z_{\ell+1} : m+2 \leq \ell \leq 2m-1$	$2m-\ell+3$	$\ell-m+1$
$z_\ell z_{\ell+1} : \ell = 2m$	3	$\ell-m+1$

(b)

Theorem 3: Let $\mathcal{R}_n$ be a graph, then for $n \geq 5$, then $dim_m\mathcal{R}_n = 5$.

Proof: The proof of this theorem will follow the proof of Lemma 15 to Lemma 17.

Lemma 15: In case of even n,$dim_m(\mathcal{R}_n) \leq 5$.

Proof: There are two possible cases.

Case 1: When n is even. Now $n = 2m$, where $m \in \mathbb{Z}^+$. We let $L_m = \{z_0, y_1, y_m, x_2, x_{m+2}\}$. In order to show that L_m represents mixed metric generator for $\mathcal{R}_n$. We present representation of any vertices and edges of $V(\mathcal{R}_n)$, and $E(\mathcal{R}_n)$ with respect to L_m in the following tables.

TABLE 41. (a) Representation of exterior and central edges of $\mathcal{R}_n$. (b) Representation of exterior and central edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_m	x_0
$z_\ell y_\ell : 0 \le \ell \le 1$	ℓ	$m-\ell$	$\ell+1$
$z_\ell y_\ell : 2 \le \ell \le m$	ℓ	$m-\ell$	$\ell+1$
$z_\ell y_\ell : m+1 \le \ell \le m+2$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$
$z_\ell y_\ell : m+3 \le \ell \le 2m$	$2m-\ell+1$	$\ell-m$	$2m-\ell+1$

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$z_\ell y_\ell : 0 \le \ell \le 1$	$2-\ell$	$m-\ell+1$
$z_\ell y_\ell : 2 \le \ell \le m$	$\ell-1$	$m-\ell+1$
$z_\ell y_\ell : m+1 \le \ell \le m+2$	$\ell-1$	$\ell-m$
$z_\ell y_\ell : m+3 \le \ell \le 2m$	$2m-\ell+3$	$\ell-m$

(b)

TABLE 42. (a) Representation of central edges of $\mathcal{R}_n$. (b) Representation of central edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_m	x_0
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	$\ell+1$	$m-\ell-1$	$\ell+1$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	$\ell+1$	$m-\ell-1$	$\ell+1$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$2m-\ell+1$	$\ell-m$	$2m-\ell$
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell+1$	$\ell-m$	$2m-\ell$
$y_\ell y_{\ell+1} : \ell = 2m$	1	$\ell-m$	1

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell y_{\ell+1} : 0 \le \ell \le 1$	1	$m-\ell$
$y_\ell y_{\ell+1} : 2 \le \ell \le m-1$	$\ell-1$	$m-\ell$
$y_\ell y_{\ell+1} : m \le \ell \le m+1$	$\ell-1$	1
$y_\ell y_{\ell+1} : m+2 \le \ell \le 2m-1$	$2m-\ell+2$	$\ell-m$
$y_\ell y_{\ell+1} : \ell = 2m$	2	$\ell-m$

(b)

TABLE 43. (a) Representation of central and interior edges of $\mathcal{R}_n$. (b) Representation of central and interior edges of $\mathcal{R}_n$. (c) Representation of central and interior edges of $\mathcal{R}_n$. (d) Representation of central and interior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_m	x_0
$y_\ell x_\ell : 0 \le \ell \le 1$	$\ell+1$	$m-\ell$	ℓ
$y_\ell x_\ell : 2 \le \ell \le m$	$\ell+1$	$m-\ell$	ℓ
$y_\ell x_\ell : m+1 \le \ell \le m+2$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$
$y_\ell x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$	$2m-\ell+1$

(a)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell x_\ell : 0 \le \ell \le 1$	$2-\ell$	m
$y_\ell x_\ell : 2 \le \ell \le m$	$\ell-2$	$m-\ell+1$
$y_\ell x_\ell : m+1 \le \ell \le m+2$	$\ell-2$	$\ell-m-1$
$y_\ell x_\ell : m+3 \le \ell \le 2m$	$2m-\ell+3$	$\ell-m-1$

(b)

$r(e\|L_m)$	z_0	y_m	x_0
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	$\ell+1$	$m-\ell$	$\ell+1$
$y_\ell x_{\ell+1} : 2 \le \ell \le m-1$	$\ell+1$	$m-\ell$	$\ell+1$
$y_\ell x_{\ell+1} : m \le \ell \le m+1$	$m+1$	$\ell-m$	$2m-\ell$
$y_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$	$2m-\ell$

(c)

$r(e\|L_m)$	x_2	x_{m+1}
$y_\ell x_{\ell+1} : 0 \le \ell \le 1$	$1-\ell$	$m-\ell$
$y_\ell x_{\ell+1} : 2 \le \ell \le m-1$	$\ell-1$	$m-\ell$
$y_\ell x_{\ell+1} : m \le \ell \le m+1$	$\ell-1$	$\ell-m$
$y_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$

(d)

As it can be seen from the above tables there are no vertices and edges having the same representation with L_m, so $dim_m\mathcal{R}_n \le 5$.

TABLE 44. (a) Representation of interior edges of $\mathcal{R}_n$. (b) Representation of interior edges of $\mathcal{R}_n$.

$r(e\|L_m)$	z_0	y_m	x_0	x_2
$x_\ell x_{\ell+1} : 0 \le \ell \le 1$	2	$m-\ell$	ℓ	$1-\ell$
$x_\ell x_{\ell+1} : 2 \le \ell \le m-1$	$\ell+1$	$m-\ell$	ℓ	$\ell-2$
$x_\ell x_{\ell+1} : m \le \ell \le m+1$	$m+1$	1	$2m-\ell$	$\ell-2$
$x_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$2m-\ell+2$	$\ell-m$	$2m-\ell$	$2m-\ell+2$

(a)

$r(e\|L_m)$	x_{m+1}
$x_\ell x_{\ell+1} : 0 \le \ell \le 1$	$m-\ell$
$x_\ell x_{\ell+1} : 2 \le \ell \le m-1$	$m-\ell$
$x_\ell x_{\ell+1} : m \le \ell \le m+1$	0
$x_\ell x_{\ell+1} : m+2 \le \ell \le 2m$	$\ell-m-1$

(b)

Lemma 16: If n is even then $dim_m(\mathcal{R}_n) \ge 5$.

Proof: In case of even n, then $dim_m(\mathcal{R}_n) \ge 5$. We let $dim_m(\mathcal{R}_n) = 4$. Then a contradiction table is shown below.

Hence, it can be seen clearly seen from the table that $dim_m\mathcal{R}_n \ge 5$. So from the above lemmas $dim_m(\mathcal{R}_n) = 5$.

Case 2: When n is odd. Now we let $n = 2m + 1$, where $m \in \mathbb{Z}^+$. We let $L_m = \{z_0, y_m, x_0, x_2, x_{m+1}\}$. In order to show that L_m represents mixed metric generator for $\mathcal{R}_n$. We present representation of any vertices and edges of $\mathcal{R}_n$.

From the above mentioned tables, there are no two vertices and edges with same mixed metric dimension for $\mathcal{R}_n$. So $dim_m(\mathcal{R}_n) \le 5$. As shown before the contradiction table for even n, same kind of contradiction occur in case of odd n, so $dim_m(\mathcal{R}_n) \ge 5$. Hence, it is proved that $dim_m(\mathcal{R}_n) = 5$.

Problem 2: Since mixed metric generator is a metric generator as well as edge metric generator. So a natural question arise in case of the graph of $\mathcal{R}_n$. As we know from Lemma 14, the metric dimension is 3. In case of edge metric dimension, is the minimum edge metric dimension is equal to mixed metric dimension for $\mathcal{R}_n$?.

V. CONCLUSION

In this paper, we have found the exact values of the mixed metric dimension of three families of graphs generated from cycle graphs, $\mathcal{P}_n$, $\mathcal{A}_n$, and $\mathcal{R}_n$. We conclude that for $\mathcal{P}_n$ when n is even mixed metric dimension equals the edge and metric dimension. For $\mathcal{A}_n$, n is even and odd; the mixed metric dimension equals the edge metric dimension. Moreover, both $\mathcal{P}_n$, and $\mathcal{A}_n$ has bounded mixed metric dimension while for $\mathcal{R}_n$, it has a constant mixed metric dimension.

The standard and edge metric dimensions have been studied for various well know families of graphs Petersen graphs, Circulant graphs, and many families of Convex polytopes, especially the standard metric dimension. The future research can be thought of as finding the mixed metric dimension for these particular families of graphs. Besides this, it would be interesting to know for which families of graphs $dim_m = dim_e = dim$.

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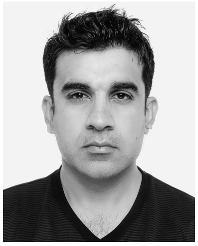

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