IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received October 6, 2019, accepted December 13, 2019, date of publication December 17, 2019, date of current version January 8, 2020.

Digital Object Identifier 10.1109/ACCESS.2019.2960296

Three-Way Group Decision in a Covering Decision Information System

XIAOLIANG XIE[1], JIALI HE[2], AND DAN HUANG[3]

[1]Key Laboratory of Hunan Province for New Retail Virtual Reality Technology, School of Mathematics and Statistics, Institute of Big Data and Internet Innovation, Hunan University of Technology and Business, Changsha 410205, China
[2]Key Laboratory of Complex System Optimization and Big Data Processing, Department of Guangxi Education, Yulin Normal University, Yulin 537000, China
[3]School of Mathematics and Statistics, Baise University, Baise 533000, China

Corresponding authors: Jiali He (jialihe100@126.com) and Dan Huang (danhuang2255@163.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 11971420, in part by the Special Talent Project in Guangxi under Grant 2019AC20052, in part by the High Level Innovation Team Program from Guangxi Higher Education Institutions of China under Grant [2019] 52, in part by the Natural Science Foundation of Guangxi under Grant 2018GXNSFDA294003 and Grant 2018GXNSFAA294134, and in part by the Research Project of Data Research Institute in Yulin under Grant 2019YJKY03.

ABSTRACT Three-way decision (3WD) provides a new perspective for solving practical decision-making problems, which is in line with human's cognitive pattern. A covering information system (CIS) is an information system (IS) that consists of multiple coverings in the universe. A CIS with decision attributes which is seen as a covering decision information system (CDIS). This paper proposes three-way group decisions in a CDIS, as well as gives its application on the problem of position competition. First of all, the neighbourhood of every point in a CDIS is defined, and corresponding similarity class of this point is also obtained. Then, because of the uncertainty of risks, loss functions are acquired through group decision-making by means of interval numbers. Next, a method of three-way group decisions in a CDIS is presented. Eventually, the position competition is presented as an example to support our proposed decision-making method.

INDEX TERMS Three-way, covering decision information system, group decision-making, interval number, loss function, position competition.

I. INTRODUCTION

Three-way decision (3WD), proposed by Yao [44], is an extended of decision rough sets. Based on loss functions, DTRS model uses Bayesian decision theory to systematically calculate thresholds. Intuitively, this model, divides the universe into many equivalence classes by using equivalence relation, then describes the uncertainty of the system by introducing upper and lower approximations. On the basis of the idea of DTRSs, the thresholds α and β are obtained by means of loss functions and these equivalence classes. They can separate the universe into three domains-disjoint namely 3WD, it endues a good semantic interpretation of rough sets: the rule generated by the positive region indicates the acceptance of something; the rule generated by the negative region indicates the rejection of something; the rule generated by the boundary region indicates deferment decision which means that something cannot be accepted or rejected from judgment. Recently, 3WD has attracted the attention of many scholars. The research results of 3WD can be reflected from the aspects of conditional probability, loss function and so on.

In terms of the research of conditional probability, DTRSs takes conditional probability as an evaluation function and associates the evaluation function with various measurement functions. Conditional probability has been studied by many scholars. For example, Yao and Zhou [50] presented a DTRS model with naive Bayesian decision theory, which can be used to assess conditional probability in 3WD; Mandal and Ranadive [29] discussed 3WDs with multi-granulation interval-valued fuzzy probabilistic rough sets; Grecoa *et al.* [5] investigated the cost of misclassification and put forward three-way probability models; Liu *et al.* [18] used logistic regression to estimate the conditional probability of DTRS and combined logistic regression to propose a new discriminant analysis method.

On the study of loss function, Liu *et al.* [16] taken into account of uncertain decision environments and raised three-way thresholds when the loss function adopted

The associate editor coordinating the review of this manuscript and approving it for publication was Xiang Zhao.

uncertainty measures; Yang and Yao [49] gave a multi-agent decision model by using 3WD's idea; Yu *et al.* [48] considered various loss functionson based on DTRS model, and proposed a cost evaluation method of clustering pattern and a clustering validity index; Herbert and Yao [6] combined the loss function of DTRS with the game theory of classification measurement, so as to optimize the size of each decision domain; Agbodah [1] studied the loss function evaluated based on multiple experts of 3WDs with DTRSs; Liang and Liu [11]–[13], [20] took into account uncertainty of loss functions, then they drew randomness, interval, fuzziness, and triangular fuzzy number into DTRSs, and developed uncertainty 3WD models, so then widened range of loss value; Liu *et al.* [14], [17], [23] put forward different DTRSs on uncertain environments;

However, there are also many scholars researching 3WD from other perspectives. In view of decision makers have different risk preferences, Li and Zhou [26] presented optimistic, pessimistic and neutral decision models, furthermore studied decision rules of different risk preference. Zhan *et al.* [52], [55] investigated 3WDs in different models with multi-attribute decision-making; Jia and Liu [27] brought forward a new decision-making model based on 3WD; Min *et al.* [30] researched cost sensitive 3WD and three-way recommendation problem; Li *et al.* [10], [22] researched the multi-granulation DTRS method in distributed *fc*-decision ISs, and further discussed 3WD method of a fuzzy condition decision IS and its application in credit card evaluation. Liu [8] advanced a multiple attribute group decision making approach. Wang *et al.* [56] gave a 3WD method based on Gaussian kernel in a hybrid IS with images and applied this method in medical diagnosis.

In practical applications, Liu et al. summarized the application of 3WD in [2], [3], [15], [25], [41], [47], [54].

An IS, was introduced by Pawlak, which is the concern of rough sets. A multitude of applications [4], [7], [28], [34]–[36], [42] of rough sets are involved in ISs.

From what have been discussed above, the determination of loss functions of 3WD are closely related to decision-makers. Most of the existing researches use single decision-maker to evaluate loss functions. Faced with complex decision-making environment, such as limited domain knowledge, tight deadlines, limited budgets and so on, it may be difficult for a single decision maker to make reasonable decisions [20], [43], [49]. Group decision-making can solve these problems and provide an effective evaluation method for loss functions [11]. It can pool the wisdom of experts in different fields to effectively deal with the problem of risk decision-making. Until now, three-way group decisions based on CDISs hasn't been investigated. This paper devoted to research three-way group decisions based on a CDIS and its' application on the problem of position competition. The main contributions of our work are displayed as follows by comparing with the existing studies.

(a) On the basis of the definition of similarity relation, similarity classes are obtained to constructed a CDIS.

(b) Because that risks are uncertain, loss functions are expressed as interval numbers through group decision-making.

(c) In light of the idea of DTRSs, three-way group decision in a CDIS is given and an example of position competition is applied to support our proposed decision-making method.

And the related work of our investigation are displayed in FIGURE 1.

This article is organised as follows. Section 2 retrospects the essential notions of binary relations, rough sets, CDISs and interval-valued numbers. Section 3 reviews DTRS and give a certain ranking method to generated 3WD. Section 4 presents a 3WD method in CDISs. Section 5 gives an application of position competition to explain the flexibility of our presented method. Section 6 discusses and concludes This paper.

II. PRELIMINARIES

Some essential notions of binary relations, rough sets, CDISs and interval-valued numbers are retrospected in this section.

In this paper, U is a finite set, 2^U indicates the collection of all subsets of U, and $|X|$ represents the cardinality of $X \in 2^U$.

Put

$$U = \{u_1, u_2, \cdots, u_n\}$$

A. BINARY RELATIONS AND ROUGH SETS

If $R \subseteq U^2$ (i.e., $U \times U$), then R is addressed as a binary relation on U. Suppose $(u, v) \in R$. Then we write it as uRv.

Given $R \subseteq U^2$. Suppose that R meets the following conditions:

(1) Reflexivity: $\forall\, u \in U$, uRu;

(2) Symmetry: $\forall\, u, v \in U$, $uRv \Rightarrow vRu$;

(3) Transitivity: $\forall\, u, v, w \in U$, uRv and $vRw \Rightarrow uRw$.

Then we call R is an equivalence relation on U.

If R satisfies reflexivity and transitivity, then R is addressed as a similarity relation on U.

Assume that R is an equivalence relation on U. $\forall\, u \in U$, the equivalence class including u is expressed as

$$[u]_R = \{v \in U : uRv\}.$$

The collection of all equivalence classes of R can induce a quotient set, denote

$$U/R = \{[u]_R : u \in U\}.$$

Definition 1 [31]: Suppose that R is an equivalence relation on U. Then the ordered pair (U, R) is said to be a Pawlak approximation space. The lower and upper approximations of $X \in 2^U$ are represented as

$$R_*(X) = \{u \in U : [u]_R \subseteq X\},$$
$$R^*(X) = \{u \in U : [u]_R \cap X \neq \emptyset\}.$$

$\forall\, X \in 2^U$, the positive, boundary and negative regions of X are defined, respectively, as

$$POS(X) = R_*(X),$$
$$BND(X) = R^*(X) - R_*(X),$$
$$NEG(X) = U - R^*(X).$$

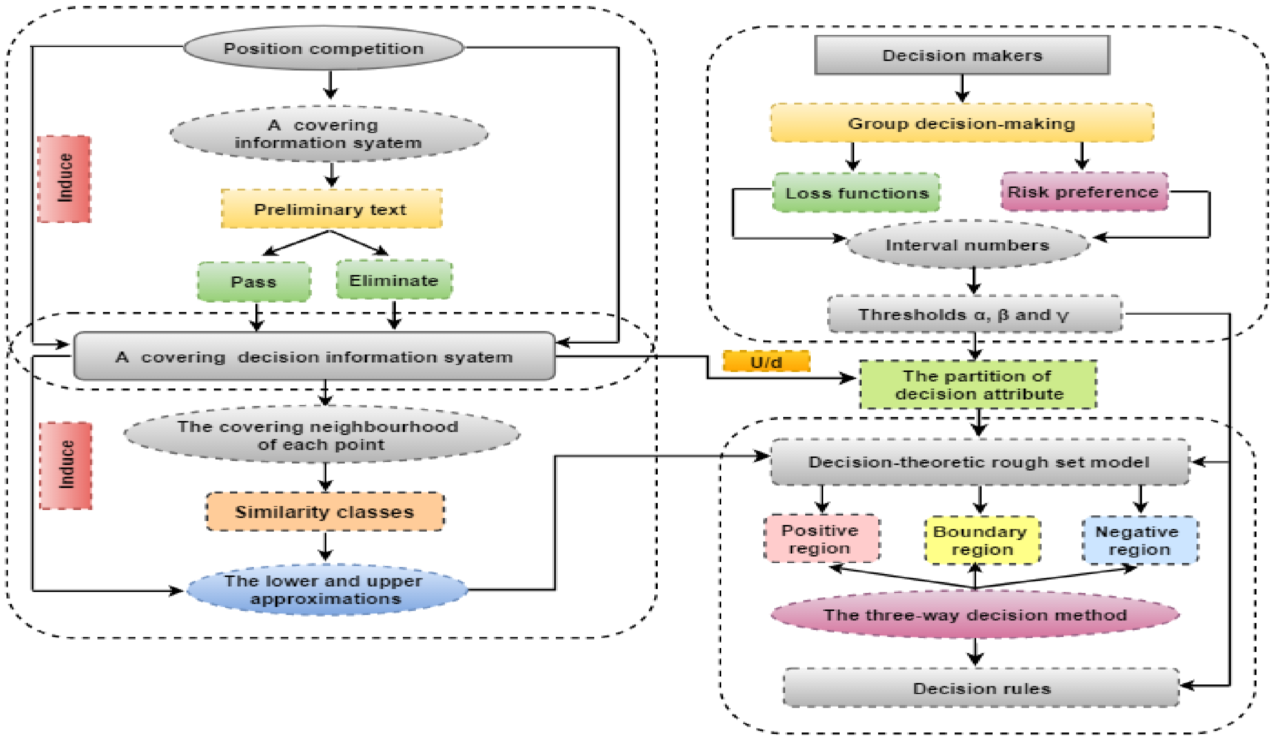


FIGURE 1. **Flow chart of decision process.**

Obviously,

$$BND(X) = \{u \in U : [u]_R \cap X \neq \emptyset \wedge [u]_R \nsubseteq X\},$$
$$NEG(X) = \{u \in U : [u]_R \cap X \neq \emptyset\}.$$

B. CDISs

In an IIS, for any attribute can determine a similarity relation and each this relation can induce a quotient set. The collection of all quotient sets can be seen as coverings. Thus, an IIS is able to be induced a CDIS.

Definition 2 [31]: Suppose that U is a finite set of objects. Assume that A expresses a finite set of attributes. Then the ordered pair (U, A) is referred to as an information system (IS), if every attribute $a \in A$ is able to decide a function $a : U \to V_a$, where $V_a = \{a(u) : u \in U\}$.

Definition 3 [7]: Suppose that (U, A) is an IS. Then (U, A) is said to be an incomplete information system (IIS), if there exist $u \in U$ and $a \in A$ such that $a(u)$ is unknown.

Usually, the unknown value is denoted as $*$.

For any subset $P \subseteq A$, the similarity relation $SIM(P)$ is defined as $SIM(P) = \{(u, v) \in U \times U : \forall a \in P, a(u) = *or a(v) = *or a(u) = a(v)\}$.

Specially, $\forall a \in A$, $SIM(\{a\}) = \{(u, v) \in U \times U : a(u) = *or a(v) = *or a(u) = a(v)\}$.

Definition 4 [51]: Given $\mathscr{C} \subseteq 2^U$. If for any $C \in \mathscr{C}$, $C \neq \emptyset$ and $\bigcup_{C \in \mathscr{C}} C = U$, then $\mathscr{C}$ is referred to as a covering on U.

In our article, we write as $C(U)$ to stand for the collection of all coverings of U.

Definition 5 [38]: Let $\mathscr{C}$ be a covering on U. Then for each $u \in U$, define

$$\mathscr{C}_u = \cap\{C \in \mathscr{C} : u \in C\}.$$

Proposition 6 [38]: Suppose that $\mathscr{C}$ is a covering on U. Then it satisfies the following properties.

(1) $\forall u \in U$, $u \in \mathscr{C}_u$;

(2) $\forall u, v, w \in U$, $u \in \mathscr{C}_v$, $v \in \mathscr{C}_w$ imply $u \in \mathscr{C}_w$.

Definition 7 [38]: Given $\Delta \subseteq C(U)$. Then (U, Δ) is framed as a covering information system (CIS).

Based on an IIS, one can induce a CIS.

Suppose that (U, A) is an IIS. Given $A = \{a_1, a_2, \cdots, a_s\}$. Set

$$\mathscr{C}_i = U/SIM(\{a_i\}) \quad (i = 1, 2, \cdots, s),$$
$$\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s\}.$$

Then (U, Δ) is alluded to as a CIS induced by (U, A).

Definition 8 [39]: Let $\Delta \subseteq C(U)$. Assume that d is a decision attribute which decides a function $d : U \to V_d$ where $V_d = \{d(x) : x \in U\}$. Then $(U, \Delta, \{d\})$ is called a covering decision information system (CDIS).

Let $(U, \Delta, \{d\})$ be a CDIS. Then, the equivalence relation $ind(\{d\})$ can be defined as

$$ind(\{d\}) = \{(u_i, u_j) : d(u_i) = d(u_j)\}.$$

Denote

$$[u_i]_{ind(\{d\})} = \{u_j : (u_i, u_j) \in ind(\{d\})\},$$
$$U/ind(\{d\}) = \{[u_i]_{ind(\{d\})} : u_i \in U\}.$$

In general, $[u_i]_{ind(\{d\})}$ and $U/ind(\{d\})$ are simply denoted by $[u_i]_d$ and $U/\{d\}$, respectively.

In our article, put

$$\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s\}.$$

Definition 9 [9]: Let $(U, \Delta, \{d\})$ be a CDIS. For each $u \in U$,

$$\Delta_u = \bigcap_{i=1}^{s} (\mathscr{C}_i)_u.$$

Then Δ_u is referred to as the neighbourhood of the point u relative to Δ.

Obviously, if $\Delta = \{\mathscr{C}\}$, then $\Delta_u = \mathscr{C}_u$.

Given $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s\}$. Define

$$uR_\Delta v \Leftrightarrow v \in \Delta_u,$$

i.e., $uR_\Delta v \Leftrightarrow \forall\, i, v \in (\mathscr{C}_i)_u$.

By Proposition 6, R_Δ is a similarity relation.

Definition 10 [9]: Let $(U, \Delta, \{d\})$ be a CDIS. For any $u \in U$, denote

$$[u]_{R_\Delta} = \{v \in U : uR_\Delta v\}.$$

Then $[u]_{R_\Delta}$ is called the similarity class of u in $(U, \Delta, \{d\})$.

Clearly,

$$\Delta_u = [u]_{R_\Delta}.$$

The lower and upper approximations of $X \in 2^U$ relative to Δ are defined, respectively, as

$$\underline{\Delta}(X) = \{u \in U : \Delta_u \subseteq X\},$$
$$\overline{\Delta}(X) = \{u \in U : \Delta_u \cap X \neq \emptyset\}.$$

C. INTERVAL-VALUED NUMBERS

In this section, we utilize interval-valued numbers to acquire the loss functions. So we recall some basis concepts and properties of interval-valued numbers.

Let

$$[R] = \{m = [m^-, m^+] : m^-, m^+ \in R,\ m^- \leq m^+\}.$$

$\forall\, m \in R$, denote $\overline{m} = [m, m]$.

Definition 11 [24]: $\forall\, m, n \in [R]$ and $k \geq 0$, define

(1) $m = n \Longleftrightarrow m^- = n^-, m^+ = n^+$;
(2) $m \leq n \Longleftrightarrow m^- \leq n^-, m^+ \leq n^+$; $m < n \Longleftrightarrow m \leq n,\ m \neq n$;
(3) $m + n = [m^- + n^-, m^+ + n^+]$;
(4) $km = [km^-, km^+]$.

Example 12: Pick $k = 3$, $m = [3, 4]$ and $n = [7, 9]$. Then
(1) $m + n = [3 + 7, 4 + 9] = [10, 13]$;
(2) $km = 3[3, 4] = [9, 12]$.

Definition 13 [24]: Let $m \in [R]$ and $\mu \in [0, 1]$. Define

$$m_\mu = (1 - \mu)m^- + \mu m^+.$$

Then m_μ is called μ-dot of the interval number m.

Theorem 14: Suppose $m, n \in [R]$. Then the following properties hold:

(1) $m_0 = m^-,\ m_1 = m^+,\ m_{0.5} = \frac{m^- + m^+}{2}$.
(2) $\forall\, \mu \in [0, 1],\ m_\mu \in m$;
(3) If $0 \leq \mu_1 \leq \mu_2 \leq 1$, then $m_{\mu_1} \leq m_{\mu_2}$;
(4) $m = n \Rightarrow \forall\, \mu \in [0, 1],\ m_\mu = n_\mu$;
(5) $(km)_\mu = km_\mu\ (k \geq 0)$;
(6) $(m + n)_\mu = m_\mu + n_\mu$.

Proof: (1), (2) and (4) are obvious.

(3) By Definition 13,

$$m_{\mu_1} = (1 - \mu_1)m^- + \mu_1 m^+ = m^- + \mu_1(m^- + m^+),$$
$$m_{\mu_2} = (1 - \mu_2)m^- + \mu_2 m^+ = m^- + \mu_2(m^- + m^+).$$

Since $0 \leq \mu_1 \leq \mu_2 \leq 1$, we have $m_{\mu_1} \leq m_{\mu_2}$.

(5) Since $k \geq 0$, by Definition 13, we have

$$\begin{aligned}(km)_\mu &= (1 - \mu)(km^-) + \mu_1(km^+)\\ &= k[(1 - \mu)m^- + \mu m^+)] = km_\mu.\end{aligned}$$

(6) By Definition 13, we have

$$\begin{aligned}(m + n)_\mu &= (1 - u)(m^- + n^-) + u(m^+ + n^+)\\ &= (1 - \mu)m^- + (1 - \mu)n^- + \mu m^+ + \mu n^+)\\ &= [(1 - \mu)m^- + \mu m^+] + [(1 - \mu)n^- + \mu n^+]\\ &= m_\mu + n_\mu.\end{aligned}$$

□

Theorem 15 [24]: Given $\mu \in [0, 1]$. Define

$$m \preccurlyeq_\mu n \Leftrightarrow m_\mu \leq n_\mu.$$

Then $\preccurlyeq_\mu$ is m similarity relation on $[R]$.

Define

$$m \prec_\mu n \Leftrightarrow m_\mu \leq n_\mu \Leftrightarrow m \preccurlyeq_\mu n \text{ and } m \neq n.$$

Specially,

(1) $m \preccurlyeq_0 n \Leftrightarrow m^- \leq n^-$, "$\preccurlyeq_0$" can be applied to conservative decision;
(2) $m \preccurlyeq_{0.5} n \Leftrightarrow \frac{m^- + m^+}{2} \leq \frac{n^- + n^+}{2}$, "$\preccurlyeq_{0.5}$" can be applied to neutral decision;
(3) $m \preccurlyeq_1 n \Leftrightarrow m^+ \leq n^+$, "$\preccurlyeq_1$" can be applied to risky decision.

III. THE 3WD IN LIGHT OF GROUP DECISION-MAKING

The 3WD is a method on account of human cognitive process. As Yao stated in [45], [46], the two key researches of 3WD are focus on conditional probability and the threshold pair (α, β).

A. THE 3WD-BASED DECISION-THEORETIC ROUGH SETS (DTRSs)

In this subsection, we recall the 3WD is generated from DTRSs.

Below, we can construct a 3WD method in a CDIS. Given that $(U, \Delta, \{d\})$ is a CDIS. For any $X \in 2^U$, two states are denoted by X and X^c that indicate an object belongs to X and an object does not belong to X, respectively. In this paper, it's worth noting that X not only is a subset of U but also expresses a state set. Three actions are denoted by a_P, a_B and a_N which mean accepting something, deferment decision and rejecting something, respectively. We can apply a named "cost table" to show loss values during decision-making process. As is known to all, loss values in DTRSs are exact real numbers. Then, the loss function of three actions in two states is presented by the following:

TABLE 1. The loss functions of three actions in two states.

	X	X^c
a_P	λ_X^P	$\lambda_{X^c}^P$
a_B	λ_X^B	$\lambda_{X^c}^B$
a_N	λ_X^N	$\lambda_{X^c}^N$

From TABLE 1, the symbol $\lambda_\bullet^\diamond$ ($\diamond = P, B, N$; $\bullet = X, X^c$) means the losses of taking correspondingly actions. Where λ_X^P, λ_X^B and λ_X^N express the losses of taking correspondingly actions, respectively, when an object belongs to X; λ_X^P, $\lambda_{X^c}^B$ and $\lambda_{X^c}^N$ indicate the losses for taking the same actions, when an object belongs to X^c.

Clearly, loss functions in a CDIS satisfy the following conditions:

$$\lambda_X^P \le \lambda_X^B \le \lambda_X^N;\quad \lambda_{X^c}^N \le \lambda_{X^c}^B \le \lambda_{X^c}^P. \tag{III.1}$$

Taking the individual actions contacts with the expectation cost $R(a_i|[u]_R)(i = P, B, N)$ in a CDIS can be represented as

$$\begin{aligned} R(a_P|[u]_R) &= \lambda_X^P P(X|[u]_R) + \lambda_{X^c}^P P(X^c|[u]_R),\\ R(a_B|[u]_R) &= \lambda_X^B P(X|[u]_R) + \lambda_{X^c}^B P(X^c|[u]_R),\\ R(a_N|[u]_R) &= \lambda_X^N P(X|[u]_R) + \lambda_{X^c}^N P(X^c|[u]_R). \end{aligned}$$

Since $P(X|[u]_R)+P(X^c|[u]_R) = 1$, on account of Bayesian decision criterion, the above-mentioned conditions can be written as

($P1$) $R(a_P|[u]_R) \le R(a_B|[u]_R)$ and $R(a_P|[u]_R) \le R(a_N|[u]_R)$ imply $u \in POS(X)$;

($B1$) $R(a_B|[u]_R) \le R(a_P|[u]_R)$ and $R(a_B|[u]_R) \le R(a_N|[u]_R)$ imply $u \in BND(X)$;

($N1$) $R(a_N|[u]_R) \le R(a_P|[u]_R)$ and $R(a_N|[u]_R) \le R(a_B|[u]_R)$ imply $u \in NEG(X)$.

Below, we simplify the conditions $(P1) - (N1)$. First part of the condition ($P1$) is denoted as

$$R(a_P|[u]_R) \le R(a_B|[u]_R).$$

Then

$$\lambda_X^P P(X|[u]_R) + \lambda_{X^c}^P P(X^c|[u]_R) \le \lambda_X^B P(X|[u]_R) + \lambda_{X^c}^B P(X^c|[u]_R).$$

Thus

$$P(X|[u]_R) \ge \frac{\lambda_{X^c}^P - \lambda_{X^c}^B}{(\lambda_{X^c}^P - \lambda_{X^c}^B) + (\lambda_X^B - \lambda_X^P)}.$$

Second part of the condition ($P1$) is denoted as

$$R(a_P|[u]_R) \le R(a_N|[u]_R).$$

Then

$$\lambda_X^P P(X|[u]_R) + \lambda_{X^c}^P (1 - P(X|[u]_R)) \le \lambda_X^B P(X|[u]_R) + \lambda_{X^c}^B (1 - P(X|[u]_R)).$$

Thus

$$P(X|[u]_R) \ge \frac{\lambda_{X^c}^P - \lambda_{X^c}^N}{(\lambda_{X^c}^P - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^P)}.$$

Similarly, one may adjust the expression of the conditions ($B1$) and ($N1$).

According to the foreshadowing of the calculation results, let

$$\begin{aligned} \alpha &= \frac{\lambda_{X^c}^P - \lambda_{X^c}^B}{(\lambda_{X^c}^P - \lambda_{X^c}^B) + (\lambda_X^B - \lambda_X^P)},\\ \beta &= \frac{\lambda_{X^c}^B - \lambda_{X^c}^N}{(\lambda_{X^c}^B - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^B)},\\ \gamma &= \frac{\lambda_{X^c}^P - \lambda_{X^c}^N}{(\lambda_{X^c}^P - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^P)}. \end{aligned}$$

Furthermore, the condition ($B1$) shows $\alpha > \beta$, that is

$$\begin{aligned} \alpha &= \frac{\lambda_{X^c}^P - \lambda_{X^c}^B}{(\lambda_{X^c}^P - \lambda_{X^c}^B) + (\lambda_X^B - \lambda_X^P)}\\ > \beta &= \frac{\lambda_{X^c}^B - \lambda_{X^c}^N}{(\lambda_{X^c}^B - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^B)}. \end{aligned}$$

Then

$$\frac{1}{1 + \frac{\lambda_X^B - \lambda_X^P}{\lambda_{X^c}^P - \lambda_{X^c}^B}} > \frac{1}{1 + \frac{\lambda_X^N - \lambda_X^B}{\lambda_{X^c}^B - \lambda_{X^c}^N}}.$$

Thus

$$\frac{\lambda_X^B - \lambda_X^P}{\lambda_{X^c}^P - \lambda_{X^c}^B} < \frac{\lambda_X^N - \lambda_X^B}{\lambda_{X^c}^B - \lambda_{X^c}^N}.$$

From this inequality, we have

$$\begin{aligned} \frac{\lambda_X^B - \lambda_X^P}{\lambda_{X^c}^P - \lambda_{X^c}^B} &< \frac{(\lambda_X^B - \lambda_X^P) + (\lambda_X^N - \lambda_X^B)}{(\lambda_{X^c}^P - \lambda_{X^c}^B) + (\lambda_{X^c}^B - \lambda_{X^c}^N)}\\ &= \frac{\lambda_X^N - \lambda_X^P}{\lambda_{X^c}^P - \lambda_{X^c}^N} < \frac{\lambda_X^N - \lambda_X^B}{\lambda_{X^c}^B - \lambda_{X^c}^N}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\le \frac{\lambda_{X^c}^B - \lambda_{X^c}^N}{(\lambda_{X^c}^P - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^B)}\\ &< \frac{\lambda_{X^c}^P - \lambda_{X^c}^N}{(\lambda_{X^c}^P - \lambda_{X^c}^N) + (\lambda_X^N - \lambda_X^P)}\\ &< \frac{\lambda_{X^c}^P - \lambda_{X^c}^B}{(\lambda_{X^c}^P - \lambda_{X^c}^B) + (\lambda_X^B - \lambda_X^P)}\\ &\le 1. \end{aligned}$$

This suggests $0 \le \beta < \gamma < \alpha \le 1$. Then, we can otain

$$\begin{aligned} POS^{(\alpha,\beta)}(X) &= \{u \in U : P(X|[u]_R) \ge \alpha\},\\ BND^{(\alpha,\beta)}(X) &= \{u \in U : \beta < P(X|[u]_R) < \alpha\},\\ NEG^{(\alpha,\beta)}(X) &= \{u \in U : P(X|[u]_R) \le \beta\}. \end{aligned}$$

Especially, if $\alpha < \beta$, we set "$\alpha = \beta = \gamma$", then the 3WD transformed into 2WD. It is able to be rewritten as

$$POS^{(\gamma,\gamma)}(X) = \{u \in U : P(X|[u]_R) \geq \gamma\},$$
$$NEG^{(\gamma,\gamma)}(X) = \{u \in U : P(X|[u]_R) < \gamma\}.$$

B. LOSS FUNCTIONS BASED ON GROUP DECISION-MAKING

As described in subsection 3.1, the 3WD-based DTRSs mainly concentrates upon the single decision making. In light of DTRSs and the results provided in [20], this subsection we will extend to group decision-making [19]. Under the background of group decision-making, how to obtain loss functions is a meaningful research topic. Liu et al. [21] take advantage of interval numbers to obtain loss functions $\lambda_{\bullet}^{\diamond}$ ($\diamond = P, B, N; \bullet = X, X^c$). Interval number, as an extended form of single value, is used to measure uncertain or inaccurate problem. It accords with the fuzziness of human thinking and insufficient information in practical decision-making. In the following, we describe the process to obtain interval number loss functions from the angle of group decision-making.

On the basic of idea of 3WD, DTRS utilize two states $\Omega = \{X, \neg X\}$ and three actions $\mathcal{A} = \{P, B, N\}$ to depict the decision-making process. During practical decision-making process, there are many experts being invited to evaluate loss functions. Suppose that $E = \{e_1, e_2, \cdots, e_m\}$ is a set of experts and $W = \{w_{e_1}, w_{e_2}, \cdots, w_{e_m}\}^T$ is a weight vector of experts, where $\sum_{t=1}^{m} w_l = 1$ and $w_l \geqslant 0$. As one can see the loss functions of TABLE 1, all experts should be assigned the results of loss functions that are represented by TABLE 2.

TABLE 2. The results of loss functions from all experts.

	λ_X^P	$\lambda_{X^c}^P$	λ_X^B	$\lambda_{X^c}^B$	λ_X^N	$\lambda_{X^c}^N$
e_1	$(\lambda_X^P)^1$	$(\lambda_{X^c}^P)^1$	$(\lambda_X^B)^1$	$(\lambda_{X^c}^B)^1$	$(\lambda_X^N)^1$	$(\lambda_{X^c}^N)^1$
e_2	$(\lambda_X^P)^2$	$(\lambda_{X^c}^P)^2$	$(\lambda_X^B)^2$	$(\lambda_{X^c}^B)^2$	$(\lambda_X^N)^2$	$(\lambda_{X^c}^N)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_l	$(\lambda_X^P)^l$	$(\lambda_{X^c}^P)^l$	$(\lambda_X^B)^l$	$(\lambda_{X^c}^B)^l$	$(\lambda_X^N)^l$	$(\lambda_{X^c}^N)^l$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_m	$(\lambda_X^P)^m$	$(\lambda_{X^c}^P)^m$	$(\lambda_X^B)^m$	$(\lambda_{X^c}^B)^m$	$(\lambda_X^N)^m$	$(\lambda_{X^c}^N)^m$

From the results of TABLE 2, the value of loss function $\lambda_{\bullet}^{\diamond}$ is evaluated by the expert e_l ($1 \leq l \leq m$) is $(\lambda_{\bullet}^{\diamond})^l$ ($\diamond = P, B, N; \bullet = X, X^c$). The value of loss function $(\lambda_{\bullet}^{\diamond})^l$ also satisfy the following conditions:

$$(\lambda_X^P)^l \leq (\lambda_X^B)^l \leq (\lambda_X^N)^l;$$
$$(\lambda_{X^c}^N)^l \leq (\lambda_{X^c}^B)^l \leq (\lambda_{X^c}^P)^l.$$

Each loss function in TABLE 1 has multiple evaluation results in group decision-making. For example, loss function λ_X^P has multiple evaluation results of $(\lambda_X^P)^1, (\lambda_X^P)^2, \cdots, (\lambda_X^P)^l, \cdots, (\lambda_X^P)^m$ (see TABLE 2). In this case, in order to obtain the evaluation results of all experts and improve the effectiveness of evaluation in group decision-making, we have to aggregate all evaluation results to determine the loss function. Namely, we need to extract features from datasets to improve the inconsistency of group decision-making [43]. The principle of justifiable granularity can satisfy our demands. It supports a method of searching for information granule with numerical features [33]. The information granularity of loss functions in group decision-making is shown in FIGURE 2.

On the basis of these ideas, we able to employ the principle of justifiable granularity to achieve information granule of every loss function with interval number. We consider the loss function $\lambda_{\bullet}^{\diamond}$, the evaluation results of all experts are acquired in the following:

$$(\lambda_{\bullet}^{\diamond})^1, (\lambda_{\bullet}^{\diamond})^2, \cdots, (\lambda_{\bullet}^{\diamond})^l, \cdots, (\lambda_{\bullet}^{\diamond})^m.$$

In the actual decision-making process, we need to consider the weight of all experts, i.e., $\omega_{e_1}, \omega_{e_2}, \cdots, \omega_{e_l}, \cdots, \omega_{e_m}$.

For the loss function $\lambda_{\bullet}^{\diamond}$, we merge the same values and sort them in ascending order:

$$(\lambda_{\bullet}^{\diamond})^{\sigma(1)}, (\lambda_{\bullet}^{\diamond})^{\sigma(2)}, \cdots, (\lambda_{\bullet}^{\diamond})^{\sigma(l)}, \cdots, (\lambda_{\bullet}^{\diamond})^{\sigma(m')}.$$

where $(\lambda_{\bullet}^{\diamond})^{\sigma(l)}$ expresses the lth value of all experts' evaluation results and $1 \leq l \leq m' \leq m$. In the meantime, we get their corresponding weight:

$$(\omega_{\bullet}^{\diamond})^{\sigma(1)}, (\omega_{\bullet}^{\diamond})^{\sigma(2)}, \cdots, (\omega_{\bullet}^{\diamond})^{\sigma(l)}, \cdots, (\omega_{\bullet}^{\diamond})^{\sigma(m')}.$$

Considering weights of experts [43], we usually use mean value to depict the result of group decision-making, which is computed as:

$$m_{\bullet}^{\diamond} = \sum_{t=1}^{m'} (\lambda_{\bullet}^{\diamond})^{\sigma(t)} (\omega_{\bullet}^{\diamond})^{\sigma(t)}.$$

For the sake of obtaining majority suggestions of experts, we need to determine its lower and upper bound of $\lambda_{\bullet}^{\diamond}$. We mainly focus on the implementation of interval information granularity with evaluation results, which is considered as an optimization problem [32], [40]. According to the results reported in [32], [40], the optimization functions of $l_{\bullet}^{\diamond}$ and $u_{\bullet}^{\diamond}$, denoted by $V(l_{\bullet}^{\diamond})$ and $V(u_{\bullet}^{\diamond})$, respectively, are calculated as:

$$V(l_{\bullet}^{\diamond}) = \exp^{(-\varepsilon|m_{\bullet}^{\diamond} - l_{\bullet}^{\diamond}|)} * \sum_{l_{\bullet}^{\diamond} \leq (\lambda_{\bullet}^{\diamond})^{\sigma(t)} \leq m_{\bullet}^{\diamond}} (\omega_{\bullet}^{\diamond})^{\sigma(t)}, \qquad \text{(III.2)}$$

$$V(u_{\bullet}^{\diamond}) = \exp^{(-\varepsilon|m_{\bullet}^{\diamond} - u_{\bullet}^{\diamond}|)} * \sum_{m_{\bullet}^{\diamond} \leq (\lambda_{\bullet}^{\diamond})^{\sigma(t)} \leq u_{\bullet}^{\diamond}} (\omega_{\bullet}^{\diamond})^{\sigma(t)}. \qquad \text{(III.3)}$$

where $l_{\bullet}^{\diamond}$ and $u_{\bullet}^{\diamond}$ indicate the lower and upper bound of $(\omega_{\bullet}^{\diamond})^{\sigma(t)}$ ($1 \leq t \leq m'$). ε is a positive parameter, which provides flexibility for generating information granule.

When ε is a constant, the optimized result of $\lambda_{\bullet}^{\diamond}$ with an interval is written as $\lambda_{\bullet}^{\diamond} = [(\lambda_{\bullet}^{\diamond})^{-}, (\lambda_{\bullet}^{\diamond})^{+}]$ by optimizing (3.2) and (3.3), namely,

$$(\lambda_{\bullet}^{\diamond})^{-} = \arg\max_{l_{\bullet}^{\diamond} \leq m_{\bullet}^{\diamond}} V(l_{\bullet}^{\diamond}), \qquad \text{(III.4)}$$

$$(\lambda_{\bullet}^{\diamond})^{+} = \operatorname{argmax}_{m_{\bullet}^{\diamond} \leq u_{\bullet}^{\diamond}} V(u_{\bullet}^{\diamond}). \qquad \text{(III.5)}$$

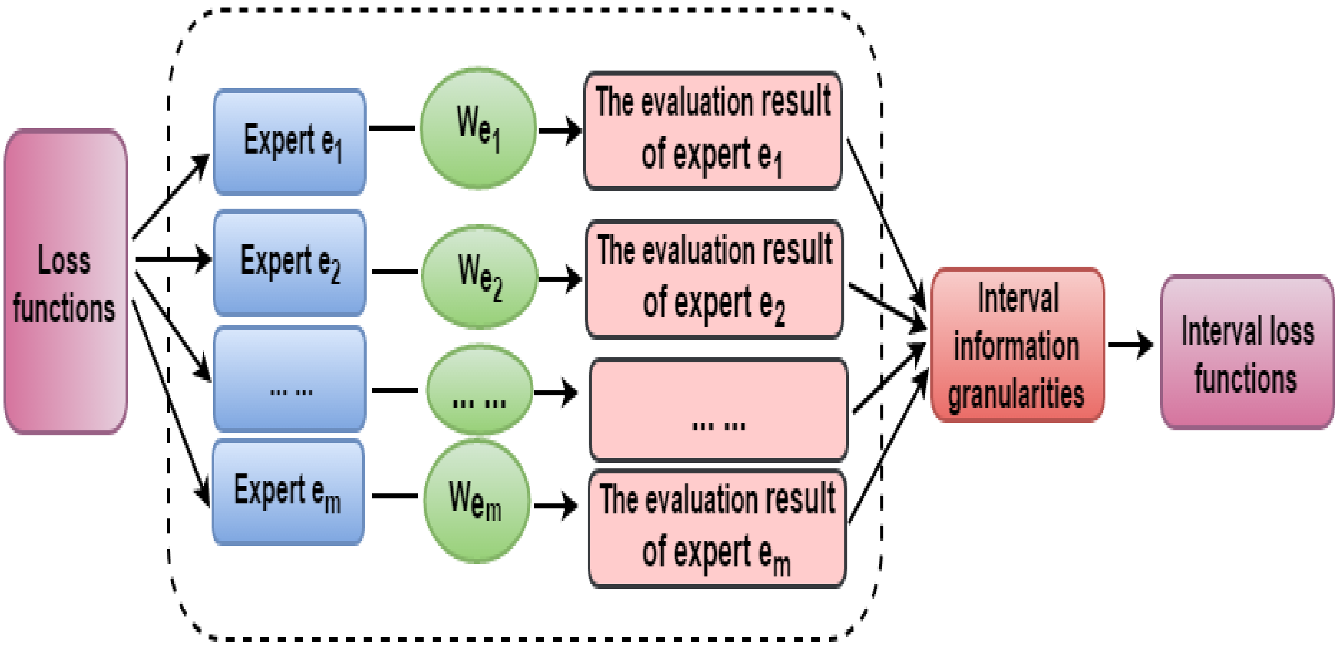


FIGURE 2. The generation process of interval loss functions.

C. 3WD INDUCED BY INTERVAL LOSS FUNCTION OF GROUP-DECISION MAKING

As is known to all, 3WD procedure is mainly dependent on the conditional probability $P(X|[u]_R)$ and the threshold pair (α, β), where $P(X|[u]_R)$ is related to the equivalent class $[u]_R$, The pair of (α, β) are dependent on loss functions $\lambda_\bullet^\diamond$ ($\diamond = P, B, N$; $\bullet = X, X^c$). In the following discussions, we investigate 3WD derived from the δ rank method. It can convert an intervals to real numbers, thus the formula conversion process is the key step to construct the 3WD.

Definition 16 [21]: Let $\lambda \in [R]$ and $\delta \in [0, 1]$. Define

$$f_\delta(\lambda) = \lambda_\delta,$$

where $\lambda_\delta = (1-\delta)\lambda^- + \delta\lambda^+$. Then $f_\delta(\lambda)$ is called the transformed outcome of λ with respect to δ, and the threshold δ reflects the risk attitude of decision makers.

In practical application, those risk-averters may seek a higher δ to lessen the probability of making mistakes. Inversely, those risk-lovers may choose a lower δ to pursue high risks and yields.

Particularly, By Definition 16, three special decisions are expressed as follows:

(1) Conservative decision: If $\delta = 0$, then $f_\delta(\lambda) = \lambda^-$, the risk-averter selects the minimum of $f_\delta(\lambda)$;
(2) Neutral decision: If $\delta = 0.5$, then $f_\delta(\lambda) = \frac{\lambda^+ + \lambda^-}{2}$, the decision maker adopts the standpoint of a risk neutral;
(3) Risky decision: If $\delta = 1$, then $f_\delta(\lambda) = \lambda^+$, the risk-lover selects the maximum of $f_\delta(\lambda)$.

Proposition 17: On the basis of Definition 16, the following properties hold.

(1) $f_\delta(\lambda_X^P) \le f_\delta(\lambda_X^B) < f_\delta(\lambda_X^N)$;
(2) $f_\delta(\lambda_{X^c}^N) \le f_\delta(\lambda_{X^c}^B) < f_\delta(\lambda_X^P)$.

Proof: Obviously. □

The δ ranking method is a typical approach by we chosen which can describe 3WD. According to the previous definitions and conclusions, decision rules $(P1)-(N1)$ may further write as

$(P2)$ If

$$f_\delta(R(a_P|[u]_R)) \le f_\delta(R(a_B|[u]_R)),$$
$$f_\delta(R(a_P|[u]_R)) \le f_\delta(R(a_N|[u]_R)),$$

then $[u]_R \in POS(X)$;

$(B2)$ If

$$f_\delta(R(a_B|[u]_R)) \le f_\delta(R(a_P|[u]_R)),$$
$$f_\delta(R(a_B|[u]_R)) \le f_\delta(R(a_N|[u]_R)),$$

then $[u]_R \in BND(X)$,

$(N2)$ If

$$f_\delta(R(a_N|[u]_R)) \le f_\delta(R(a_P|[u]_R)),$$
$$f_\delta(R(a_N|[u]_R)) \le f_\delta(R(a_B|[u]_R)),$$

then $[u]_R \in NEG(X)$,
where,

$$\begin{aligned} &f_\delta(R(a_P|[u]_R)) \\ &= (1-\delta)(\lambda_X^P)^- P(X|[u]_R) + \delta(\lambda_X^P)^+ P(X|[u]_R) \\ &\quad + (1-\delta)(\lambda_{X^c}^P)^- P(X^c|[u]_R) + \delta(\lambda_{X^c}^P)^+ P(X^c|[u]_R) \\ &= P(X|[u]_R)((1-\delta)(\lambda_X^P)^- + \delta(\lambda_X^P)^+)) \\ &\quad + (1-P(X|[u]_R))((1-\delta)(\lambda_X^P)^- + \delta(\lambda_X^P)^+) \\ &= P(X|[u]_R)f_\delta(\lambda_X^P) + (1-P(X|[u]_R))f_\delta(\lambda_X^P), \end{aligned}$$

$$\begin{aligned} &f_\delta(R(a_B|[u]_R)) \\ &\quad= (1-\delta)(\lambda_X^B)^- P(X|[u]_R) + \delta(\lambda_X^B)^+ P(X|[u]_R) \\ &\qquad + (1-\delta)(\lambda_{X^c}^B)^- P(X^c|[u]_R) + \delta(\lambda_{X^c}^B)^+ P(X^c|[u]_R) \\ &\quad= P(X|[u]_R)((1-\delta)(\lambda_X^B)^- + \delta(\lambda_X^B)^+)) \\ &\qquad + (1-P(X|[u]_R))((1-\delta)(\lambda_{X^c}^B)^- + \delta(\lambda_{X^c}^B)^+) \\ &\quad= P(X|[u]_R) f_\delta(\lambda_X^B) + (1-P(X|[u]_R)) f_\delta(\lambda_{X^c}^B), \\ &f_\delta(R(a_N|[u]_R)) \\ &\quad= (1-\delta)(\lambda_X^N)^- P(X|[u]_R) + \delta(\lambda_X^N)^+ P(X|[u]_R) \\ &\qquad + (\lambda_{X^c}^N)^- P(X^c|[u]_R) + (\lambda_{X^c}^N)^+ P(X^c|[u]_R) \\ &\quad= P(X|[u]_R)((1-\delta)(\lambda_X^N)^- + \delta(\lambda_X^N)^+) \\ &\qquad + (1-P(X|[u]_R))((1-\delta)(\lambda_{X^c}^N)^- + \delta(\lambda_{X^c}^N)^+) \\ &\quad= P(X|[u]_R) f_\delta(\lambda_X^N) + (1-P(X|[u]_R)) f_\delta(\lambda_{X^c}^N). \end{aligned}$$

By Proposition 17, the decision rules can be simplified as follows:

($P2'$) $P(X|[u]_R) \geq \alpha$ and $P(X|[u]_R) \geq \gamma$ imply $u \in POS(X)$;
($B2'$) $P(X|[u]_R) \leq \alpha$ and $P(X|[u]_R) \geq \beta$ imply $u \in BNX(X)$;
($N2'$) $P(X|[u]_R) \leq \beta$ and $P(X|[u]_R) \leq \gamma$ imply $u \in NEG(X)$.

Then, three thresholds α, β, γ are provided by

$$\begin{aligned} \alpha &= \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P))}, \\ \beta &= \frac{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B))}, \\ \gamma &= \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^P))}. \end{aligned}$$

Moreover, so as to well-defined boundary region, the condition of ($B2'$) indicates $\alpha > \beta$, that is

$$\begin{aligned} &\frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P))} \\ &\quad > \frac{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B))}. \end{aligned}$$

Then

$$\frac{1}{1 + \frac{f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P)}{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}} > \frac{1}{1 + \frac{f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B)}{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}}.$$

Thus

$$\frac{f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P)}{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)} < \frac{f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B)}{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}.$$

From this inequality, we have

$$\begin{aligned} &\frac{f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P)}{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)} \\ &\quad < \frac{(f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P) + f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B)}{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N))} \\ &\quad = \frac{f_\delta(\lambda_X^N) - f_\delta(\lambda_X^P)}{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)} < \frac{f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B)}{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}. \end{aligned}$$

When $0 \leq \beta < \gamma < \alpha \leq 1$, the decision rules can also be written in the following:

($P2''$) $P(X|[u]_R) \geq \alpha$ implies $u \in POS^{(\alpha,\beta)}(X)$;
($B2''$) $\beta < P(X|[u]_R) < \alpha$ implies $u \in BNX^{(\alpha,\beta)}(X)$;
($N2''$) $P(X|[u]_R) \leq \beta$ implies $u \in NEG^{(\alpha,\beta)}(X)$.

When $0 \leq \beta = \gamma = \alpha \leq 1$, the decision rules can also be written in the following:

($P2'''$) $P(X|[u]_R) \geq \gamma$ implies $u \in POS^{(\gamma,\gamma)}(X)$;
($N2'''$) $P(X|[u]_R) < \gamma$ implies $u \in NEG^{(\gamma,\gamma)}(X)$.

IV. THREE-WAY GROUP DECISIONS IN A CDIS

We have already mentioned that the threshold pair (α, β) is one of the most important researches of 3WD. In Section 3, we have give the approach to obtain loss functions by means of group-decision making. On the basis of the idea of DTRSs, we can structure a three-way group decision method in a CDIS.

Definition 18: Suppose that $(U, \Delta, \{d\})$ is a CDIS. Given a pair thresholds (α, β) with $0 \leq \beta < \alpha \leq 1$. The (α, β)-lower and (α, β)-upper approximations of $D \in U/\{d\}$, denoted by $\underline{(R_\Delta)}^{(\alpha,\beta)}(D)$ and $\overline{(R_\Delta)}^{(\alpha,\beta)}(D)$, respectively, are defined as

$$\begin{aligned} \underline{(R_\Delta)}^{(\alpha,\beta)}(D) &= \{u \in U : P(D|\Delta_u) \geq \alpha\}, \\ \overline{(R_\Delta)}^{(\alpha,\beta)}(D) &= \{u \in U : P(D|\Delta_u) > \beta\}. \end{aligned}$$

Theorem 19: Let $(U, \Delta, \{d\})$ be a CDIS. Suppose $D, D_1, D_2, \in U/ind(\{d\})$, Given $0 \leq \beta < \alpha \leq 1$. Then the following properties hold.

(1) $\underline{(R_\Delta)}^{(\alpha,\beta)}(D) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D)$.

(2) $\underline{(R_\Delta)}^{(\alpha,\beta)}(D) =\sim \overline{(R_\Delta)}^{(1-\alpha,\beta)}(\sim D)$;

$$\overline{(R_\Delta)}^{(\alpha,\beta)}(D) =\sim \underline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D).$$

(3)

$$\begin{aligned} \underline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2) &\supseteq \underline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \cup \underline{(R_\Delta)}^{(\alpha,\beta)}(D_2); \\ n\overline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2) &\supseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \cup \overline{(R_\Delta)}^{(\alpha,\beta)}(D_2). \end{aligned}$$

(4) *If $0 \leq \beta < \alpha_1 \leq \alpha_2 \leq 1$, then*

$$\underline{(R_\Delta)}^{(\alpha_2,\beta)}(D) \subseteq \underline{(R_\Delta)}^{(\alpha_1,\beta)}(D).$$

(5) *If $0 \leq \beta_1 \leq \beta_2 < \alpha \leq 1$, then*

$$\overline{(R_\Delta)}^{(\alpha,\beta_2)}(D) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta_1)}(D).$$

Proof: (1) Suppose $u \in \underline{(R_\Delta)}^{(\alpha,\beta)}(D)$. By Definition 18, we have $P(D|\Delta_u) \geq \alpha$. Since $0 \leq \beta < \alpha \leq 1$, then $P(D|\Delta_u) > \beta$. So,

$$u \in \{u \in U : P(D|\Delta_u) > \beta\}.$$

Thus, $u \in \overline{(R_\Delta)}^{(\alpha,\beta)}(D)$. Hence

$$\underline{(R_\Delta)}^{(\alpha,\beta)}(D) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D).$$

(2) (a) Suppose $u \in \underline{(R_\Delta)}^{(\alpha,\beta)}(D)$. By Definition 18, we have $P(D|\Delta_u) \geq \alpha$. Then,

$$P(\sim D|\Delta_u) \leq 1 - \alpha.$$

So, $u \notin \{u \in U | P(\sim D|\Delta_u) \leq 1-\alpha\}$. Thus, $u \notin \overline{(R_\Delta)}^{(1-\alpha,\beta)}(\sim D)$. Hence,

$$u \in \sim \overline{(R_\Delta)}^{(1-\alpha,\beta)}(\sim D).$$

Conversely, $u \in \sim \overline{(R_\Delta)}^{(\alpha,\beta)}(\sim D)$. Then, $u \notin \overline{(R_\Delta)}^{(1-\alpha,\beta)}(\sim D)$. By Definition 18, we have

$$P(\sim D|\Delta_u) \leq 1-\alpha.$$

So, $P(D|\Delta_u) \geq \alpha$. This implies $u \in \{u \in U | P(D|\Delta_u) \leq \alpha\}$. Thus, $u \in \underline{(R_\Delta)}^{(\alpha,\beta)}(D)$.

Hence,

$$\underline{(R_\Delta)}^{(\alpha,\beta)}(D) = \sim \overline{(R_\Delta)}^{(1-\alpha,\beta)}(\sim D).$$

(b) Suppose $u \in \overline{(R_\Delta)}^{(\alpha,\beta)}(D)$. By Definition 18, we have $P(D|\Delta_u) > \beta$. Then,

$$P(\sim D|\Delta_u) \leq 1-\beta.$$

So, $u \notin \{u \in U | P(\sim D|\Delta_u) \leq 1-\beta\}$. Thus, $u \notin \overline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D)$. Hence,

$$u \in \sim \overline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D).$$

Conversely, $u \in \sim \overline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D)$. Then, $u \notin \overline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D)$. By Definition 18, we have

$$P(\sim D|\Delta_u) \leq 1-\beta.$$

So, $P(D|\Delta_u) > \beta$. This implies $u \in \{u \in U | P(D|\Delta_u) > \beta\}$. Thus, $u \in \overline{(R_\Delta)}^{(\alpha,\beta)}(D)$.

Hence,

$$\overline{(R_\Delta)}^{(\alpha,\beta)}(D) = \sim \underline{(R_\Delta)}^{(\alpha,1-\beta)}(\sim D).$$

(3) (a) Since $D_1 \subseteq D_1 \cup D_2$, we have

$$\underline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \subseteq \underline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2),$$
$$\underline{(R_\Delta)}^{(\alpha,\beta)}(D_2) \subseteq \underline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2).$$

Thus,

$$\underline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \cup \underline{(R_\Delta)}^{(\alpha,\beta)}(D_2) \subseteq \underline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2).$$

(b) Since $D_1 \subseteq D_1 \cup D_2$, we have

$$\overline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2),$$
$$\overline{(R_\Delta)}^{(\alpha,\beta)}(D_2) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2).$$

Thus,

$$\overline{(R_\Delta)}^{(\alpha,\beta)}(D_1) \cup \overline{(R_\Delta)}^{(\alpha,\beta)}(D_2) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta)}(D_1 \cup D_2).$$

(4) Suppose $u \in \underline{(R_\Delta)}^{(\alpha_2,\beta)}(D)$. By Definition 18, we have $P(D|\Delta_u) \geq \alpha_2$. Since $0 \leq \beta < \alpha_1 \leq \alpha_2 \leq 1$, then $P(D|\Delta_u) \geq \alpha_1$. So, $u \in \underline{(R_\Delta)}^{(\alpha_1,\beta)}(D)$.

Thus,

$$\underline{(R_\Delta)}^{(\alpha_2,\beta)}(D) \subseteq \underline{(R_\Delta)}^{(\alpha_1,\beta)}(D).$$

(5) Suppose $u \in \overline{(R_\Delta)}^{(\alpha,\beta_2)}(D)$. By Definition 18, we have $P(D|\Delta_u) > \beta_2$. Since $0 \leq \beta_1 \leq \beta_2 < \alpha \leq 1$, then $P(D|\Delta_u) > \beta_1$. So, $u \in \overline{(R_\Delta)}^{(\alpha,\beta_1)}(D)$.

Thus,

$$\overline{(R_\Delta)}^{(\alpha,\beta_2)}(D) \subseteq \overline{(R_\Delta)}^{(\alpha,\beta_1)}(D).$$

□

On the basis of above DTRS model, $\forall\ D \in U/\{d\}$, U can be divided into positive, boundary and negative region of D, denoted by $POS^{(\alpha,\beta)}(D)$, $BND^{(\alpha,\beta)}(D)$, $NEG^{(\alpha,\beta)}(D)$, respectively, are given as follows:

$$POS^{(\alpha,\beta)}(D) = \underline{(R_\Delta)}^{(\alpha,\beta)}(D),$$
$$BND^{(\alpha,\beta)}(D) = \overline{(R_\Delta)}^{(\alpha,\beta)}(D) - \underline{(R_\Delta)}^{(\alpha,\beta)}(D),$$
$$NEG^{(\alpha,\beta)}(D) = U - \overline{(R_\Delta)}^{(\alpha,\beta)}(D).$$

Obviously,

$$BND^{(\alpha,\beta)}(D) = \{u \in U : \beta < P(D|\Delta_u) < \alpha\},$$
$$NEG^{(\alpha,\beta)}(D) = \{u \in U :\ P(D|\Delta_u) \leq \beta\}.$$

According to the idea of 3WD, the decision rules of $D \in U/\{d\}$ can be written as

(P) : if $P(D|\Delta_u) \geq \alpha$, then $u \in POS^{(\alpha,\beta)}(D)$;
(N) : if $P(D|\Delta_u) \leq \beta$, then $u \in NEG^{(\alpha,\beta)}(D)$;
(B) : if $\beta < P(D|\Delta_u) < \alpha$, then $u \in BND^{(\alpha,\beta)}(D)$.

In the following, we give the detailed step-wise procedure as an algorithm on 3WD based on group decision-making of a CDIS.

Input: A CDIS $(U, \Delta, \{d\})$, a positive parameter ε, a threshold δ, loss functions $(\lambda_\bullet^\diamond)^1$,$(\lambda_\bullet^\diamond)^2$,$\cdots$, $(\lambda_\bullet^\diamond)^m$ ($\diamond = P, B, N$; $\bullet = X, X^c$) of each object and the corresponding weight $\omega_{e_1}, \omega_{e_2}, \cdots, \omega_{e_m}$.

Output: The three-way group decision rules.

Step 1. For a given positive parameter ε, we achieve interval-valued information granule for each loss function in light of (3.2)-(3.5), i.e., $\lambda_X^P = [(\lambda_X^P)^-, (\lambda_X^P)^+]$, $\lambda_X^B = [(\lambda_X^B)^-, (\lambda_X^B)^+]$, $\lambda_X^N = [(\lambda_X^N)^-, (\lambda_X^N)^+]$, $\lambda_{X^c}^N = [(\lambda_{X^c}^N)^-, \lambda_{X^c}^N)^+]$, $\lambda_{X^c}^B = [(\lambda_{X^c}^B)^-, \lambda_{X^c}^B)^+]$, $\lambda_{X^c}^P = [(\lambda_{X^c}^P)^-, \lambda_{X^c}^P)^+]$.

Step 2. For a given threshold δ, compute $f_\delta(\lambda_\bullet^\diamond) = (1-\delta)(\lambda_\bullet^\diamond)^- + \delta(\lambda_\bullet^\diamond)^+$.

Step 3. Calculate thresholds α, β and γ,

$$\alpha = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P))},$$
$$\beta = \frac{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B))},$$
$$\gamma = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^P))}.$$

Step 4. For $u_i \in U$, obtain the similarity class Δ_{u_i};

Step 5. For $u_i \in U$ and $D \in U/\{d\}$, compute $P(D|\Delta_{u_i})$;

Step 6. Based on the results of **Step 3**, for any $D \in U/\{d\}$, obtain $POS^{(\alpha,\beta)}(D)$, $BND^{(\alpha,\beta)}(D)$, $NEG^{(\alpha,\beta)}(D)$;

Step 7. For any $D \in U/\{d\}$, give the three-way group decision rules of D.

Algorithm 1 An Algorithm on 3WD Based on Group Decision-Making of a CDIS

Input: A CDIS $(U, \Delta, \{d\})$, a positive parameter ε, a threshold δ, loss functions $(\lambda_{\bullet}^{\diamond})^1, (\lambda_{\bullet}^{\diamond})^2, \cdots, (\lambda_{\bullet}^{\diamond})^m$ ($\diamond = P, B, N$; $\bullet = X, X^c$) of each object and the corresponding weight $\omega_{e_1}, \omega_{e_2}, \cdots, \omega_{e_m}$.

Output: The three-way group decision rules;

for $\varepsilon > 0$, **do**

  Based on (3.2)-(3.5), we calculate $\lambda_{\bullet}^{\diamond} = [(\lambda_{\bullet}^{\diamond})^-, (\lambda_{\bullet}^{\diamond})^+]$.

end

for $0 \le \delta \le 1$, **do**

  compute $f_\delta(\lambda_{\bullet}^{\diamond}) = (1-\delta)(\lambda_{\bullet}^{\diamond})^- + \delta(\lambda_{\bullet}^{\diamond})^+$, then calculate

$$\alpha = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P))},$$

$$\beta = \frac{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B))},$$

$$\gamma = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^P))}.$$

end

for $u_i \in U$, **do**

  obtain the similarity class Δ_{u_i}.

end

for $u \in U$ *and* $D \in U/\{d\}$ **do**

  Calculate $P(D|\Delta_{u_i})$.

  if $P(D|\Delta_{u_i}) \ge \alpha$ **then**

    then $u_i \in POS^{(\alpha,\beta)}(D)$;

  end

  if $\beta < P(D|\Delta_{u_i}) < \alpha$ **then**

    then $u_i \in BND^{(\alpha,\beta)}(D)$;

  end

  if $P(D|\Delta_{u_i}) \le \beta$ **then**

    then $u_i \in NEG^{(\alpha,\beta)}(D)$.

  end

end

V. AN ILLUSTRATIVE EXAMPLE

In this section, we demonstrate the feasibility of our proposed method in a CDIS by an example.

Example 20: We will describe the process of 3WD in a CDIS by an example of interviewees' position competition.

Suppose that $U = \{u_1, u_2, \cdots, u_{10}\}$ *is a set of ten interviewees compete for a job position. Assume that* $E = \{Education, Weight, Height, Ability\}$ *is a set of four characteristics to evaluate these interviewees whose values are given as follows*:

"Education" = {high, middle, low};

"Weight" = {heavy, average};

"Height" = {tall, average, short};

"Ability" = {very strong, strong, normal, weak}.

We have four interviewers $E = \{A, B, C, D\}$ *to evaluate these interviewees through their performance. It is possible that the interviewers' evaluation results may not be the same for the same interviewee. But the evaluation results given by these interviewers are the same importance. If we want to combine these evaluation results without losing information, then we should union the evaluation results given by each interviewer. The interviewers' evaluation results for each characteristic are listed as follows.*

For "Education", A: high=$\{u_1, u_2, u_3, u_8, u_9, u_{10}\}$*, middle =*$\{u_4, u_6\}$*, low=*$\{u_5, u_7\}$*;*

B: high=$\{u_1, u_2, u_6, u_8, u_9, u_{10}\}$*, middle=*$\{u_4, u_7\}$*, low=*$\{u_3, u_5\}$*;*

C: high=$\{u_1, u_2, u_6, u_8, u_9, u_{10}\}$*, middle=*$\{u_3, u_7\}$*, low=*$\{u_4, u_5\}$*;*

D: high=$\{u_1, u_2, u_4, u_7, u_8, u_9, u_{10}\}$*, middle=*$\{u_3\}$*, low=*$\{u_5, u_6\}$*.*

For "Weight",

A: heavy=$\{u_1, u_2, u_6, u_8, u_9, u_{10}\}$*, average=*$\{u_3, u_4, u_5, u_7\}$*;*

B: heavy=$\{u_1, u_3, u_6, u_8, u_{10}\}$*, average=*$\{u_3, u_4, u_5, u_7, u_9\}$*;*

C: heavy=$\{u_1, u_2, u_8, u_9, u_{10}\}$*, average=*$\{u_3, u_4, u_5, u_6, u_7\}$*;*

D: heavy=$\{u_1, u_3, u_8, u_{10}\}$*, average=*$\{u_2, u_4, u_5, u_6, u_7, u_9\}$*;*

For "Height",

A: tall=$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$*, average=*$\{u_8, u_9\}$*, short=*$\{u_{10}\}$*;*

B: tall=$\{u_1, u_2, u_3, u_4, u_5, u_6\}$*, average=*$\{u_7, u_8, u_9\}$*, short=*$\{u_{10}\}$*;*

C: tall=$\{u_1, u_2, u_3, u_4, u_5, u_6\}$*, average=*$\{u_7, u_8, u_9\}$*, short=*$\{u_{10}\}$*;*

D: tall=$\{u_1, u_2, u_3, u_4, u_5, u_7\}$*, average=*$\{u_6, u_8, u_9\}$*, short=*$\{u_{10}\}$*.*

For "Ability",

A: very strong=$\{u_1, u_2, u_3\}$*, strong=*$\{u_4, u_5, u_7\}$*, normal=*$\{u_8, u_9, u_{10}\}$*, weak=*$\{u_6\}$*;*

B: very strong=$\{u_1, u_2, u_3\}$*, strong=*$\{u_4, u_5\}$*, normal=*$\{u_6, u_8, u_{10}\}$*, weak=*$\{u_7, u_9\}$*;*

C: very strong=$\{u_1, u_3\}$*, strong=*$\{u_2, u_4, u_5, u_6\}$*, normal=*$\{u_8, u_9, u_{10}\}$*, weak=*$\{u_7\}$*;*

D: very strong=$\{u_1, u_2, u_6\}$*, strong=*$\{u_3, u_4, u_5\}$*, normal=*$\{u_8, u_{10}\}$*, weak=*$\{u_7, u_9\}$*.*

Since these interviewers' evaluation results are equal importance, we should consider all results. For "Education", the covering $\mathscr{C}_1$ *is obtained to describe the evaluation results of ten interviewees.*

$\mathscr{C}_1 = \{\{u_1, u_2, u_3, u_8, u_9, u_{10}\} \cup \{u_1, u_2, u_6, u_8, u_9, u_{10}\} \cup \{u_1, u_2, u_6, u_8, u_9, u_{10}\} \cup \{u_1, u_2, u_4, u_7, u_8, u_9, u_{10}\}, \{u_4, u_6\} \cup \{u_4, u_7\} \cup \{u_3, u_7\} \cup \{u_3\}, \{u_5, u_7\} \cup \{u_3, u_5\} \cup \{u_4, u_5\} \cup \{u_5, u_6\}\} = \{\{u_1, u_2, u_3, u_4, u_6, u_7, u_8, u_9, u_{10}\}, \{u_3, u_4, u_6, u_7\}, \{u_3, u_4, u_5, u_6, u_7\}\}$*;*

Similarly, for "Weight", "Height" and "Ability", the coverings $\mathscr{C}_2$*,* $\mathscr{C}_3$ *and* $\mathscr{C}_4$ *are obtained to describe the evaluation results of ten interviewees, respectively.*

TABLE 3. Compute the similarity classes Δ_{u_i}.

$(\mathscr{C}_j)_{u_i}$	1	2	3	4	Δ_{u_i}
1	$U-\{u_5\}$	$U-\{u_4,u_5,u_7\}$	$U-\{u_8,u_9,u_{10}\}$	$\{u_1,u_2,u_3,u_6\}$	$\{u_1,u_2,u_3,u_6\}$
2	$U-\{u_5\}$	$\{u_2,u_3,u_6,u_9\}$	$U-\{u_8,u_9,u_{10}\}$	$\{u_2,u_3,u_6\}$	$\{u_2,u_3,u_6\}$
3	$\{u_3,u_4,u_6,u_7\}$	$\{u_2,u_3,u_6,u_9\}$	$U-\{u_8,u_9,u_{10}\}$	$\{u_2,u_3,u_6\}$	$\{u_3,u_6\}$
4	$\{u_3,u_4,u_6,u_7\}$	$U-\{u_1,u_8,u_{10}\}$	$U-\{u_8,u_9,u_{10}\}$	$\{u_2,u_3,u_4,u_5,u_6\}$	$\{u_3,u_4,u_6\}$
5	$\{u_3,u_4,u_5,u_6,u_7\}$	$U-\{u_1,u_8,u_{10}\}$	$U-\{u_8,u_9,u_{10}\}$	$\{u_2,u_3,u_4,u_5,u_6\}$	$\{u_3,u_4,u_5,u_6\}$
6	$\{u_3,u_4,u_6,u_7\}$	$\{u_2,u_3,u_6,u_9\}$	$\{u_6,u_7\}$	$\{u_6\}$	$\{u_6\}$
7	$\{u_3,u_4,u_6,u_7\}$	$U-\{u_1,u_8,u_{10}\}$	$\{u_6,u_7\}$	$\{u_6,u_7\}$	$\{u_6,u_7\}$
8	$U-\{u_5\}$	$U-\{u_4,u_5,u_7\}$	$\{u_6,u_7,u_8,u_9\}$	$\{u_6,u_8,u_9\}$	$\{u_6,u_8,u_9\}$
9	$U-\{u_5\}$	$\{u_2,u_3,u_6,u_9\}$	$\{u_6,u_7,u_8,u_9\}$	$\{u_6,u_9\}$	$\{u_6,u_9\}$
10	$U-\{u_5\}$	$U-\{u_4,u_5,u_7\}$	$\{u_{10}\}$	$\{u_6,u_8,u_9,u_{10}\}$	$\{u_{10}\}$

$\mathscr{C}_2 = \{\{u_1, u_2, u_6, u_8, u_9, u_{10}\} \cup \{u_1, u_3, u_6, u_8, u_{10}\} \cup \{u_1, u_2, u_8, u_9, u_{10}\} \cup \{u_1, u_3, u_8, u_{10}\}, \{u_3, u_4, u_5, u_7\} \cup \{u_3, u_4, u_5, u_7, u_9\} \cup \{u_3, u_4, u_5, u_6, u_7\} \cup \{u_2, u_4, u_5, u_6, u_7, u_9\}\} = \{\{u_1, u_2, u_3, u_6, u_8, u_9, u_{10}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_9\}\}$;

$\mathscr{C}_3 = \{\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\} \cup \{u_1, u_2, u_3, u_4, u_5, u_6\} \cup \{u_1, u_2, u_3, u_4, u_5, u_6\} \cup \{u_1, u_2, u_3, u_4, u_5, u_7\}, \{u_8, u_9\} \cup \{u_7, u_8, u_9\} \cup \{u_7, u_8, u_9\} \cup \{u_6, u_8, u_9\}, \{u_{10}\} \cup \{u_{10}\} \cup \{u_{10}\} \cup \{u_{10}\}\} = \{\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \{u_6, u_7, u_8, u_9\}, \{u_{10}\}\}$;

$\mathscr{C}_4 = \{\{u_1, u_2, u_3\} \cup \{u_1, u_2, u_3\} \cup \{u_1, u_3\} \cup \{u_1, u_2, u_6\}, \{u_4, u_5, u_7\} \cup \{u_4, u_5\} \cup \{u_2, u_4, u_5, u_6\} \cup \{u_3, u_4, u_5\}, \{u_8, u_9, u_{10}\} \cup \{u_6, u_8, u_{10}\} \cup \{u_8, u_9, u_{10}\} \cup \{u_8, u_{10}\}, \{u_6\} \cup \{u_7, u_9\} \cup \{u_7\} \cup \{u_7, u_9\}\} = \{\{u_1, u_2, u_3, u_6\}, \{u_2, u_3, u_4, u_5, u_6, u_7\}, \{u_6, u_8, u_9, u_{10}\}, \{u_6, u_7, u_9\}\}$.

Let d denotes the decision attribute which determines an information function $d : U \to V_d$, where $V_d = \{pass, eliminate\}$.

Pick

$$d(u_1) = d(u_2) = d(u_3) = d(u_6) = d(u_7) = pass,$$
$$d(u_4) = d(u_5) = d(u_8) = d(u_9) = d(u_{10}) = eliminate.$$

Then, $U/\{d\} = \{D_1, D_2\}$, where

$D_1 = \{u_1, u_2, u_3, u_6, u_7\}$ is a set of interviewees which may be passed;

$D_2 = \{u_4, u_5, u_8, u_9, u_{10}\}$ is a set of interviewees which may be eliminated.

Put

$$\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}.$$

Then $(U, \Delta, \{d\})$ is a CDIS.

Below, we will describe in detail the process of 3WD in $(U, \Delta, \{d\})$.

Step 1: Compute the similarity classes Δ_{u_i} ($i = 1, 2, \cdots, 10$) of $(U, \Delta, \{d\})$ (see TABLE 3).

Step 2: The conditional probability $P(D_j|\Delta_{u_i})$ ($i = 1, 2, \cdots, 10$, $j = 1, 2$) can be calculated in TABLE 4.

Step 3: Based on the idea of DTRS, the problem of interviewees competing for a job can make use of a set of two states and a set of three actions to describe the decision process. For each interviewee u, the state set is denoted by $\Omega = \{X, X^c\}$, which indicates an interviewee has a good performance or not, respectively. The action set of for each interviewee is written as $A = \{a_P, a_B, a_N\}$, in which a_P, a_B, a_N expresses the actions of being passed, being delayed decision and being eliminated, respectively. $\lambda_\bullet^\diamond$ ($\diamond = P, B, N$; $\bullet = X, X^c$) expresses the loss when one adopts exact action with it's homologous state. The loss function value for each interviewee is carefully estimated by interviewers. However, even an experienced interviewer may make errors when he determines the loss function value. One may use interval numbers to better express the loss functions. Thus, λ_X^P, λ_X^B and λ_X^N denote the losses for adopting actions of a_P, a_B and a_N, respectively, when an interviewee has a good performance. Similarly, $\lambda_{X^c}^P$, $\lambda_{X^c}^B$ and $\lambda_{X^c}^N$ denote the losses for adopting the corresponding actions when an interviewee doesn't have a good performance.

TABLE 4. Calculate the conditional probability $P(D_j|\Delta_{u_i})$.

$P(D_j\|\Delta_{u_i})$	1	2
1	1	0
2	1	0
3	1	0
4	0.6667	0.3333
5	0.5	0.5
6	1	0
7	1	0
8	0.3333	0.6667
9	0.5	0.5
10	0	1

In order to make a reasonable decision, these four interviewers give the evaluation results of loss functions for each interviewee. The set of interviewers is $E = \{A, B, C, D\}$, the corresponding weight vector of interviewers is $\{\omega_A, \omega_B, \omega_C, \omega_D\}^T = \{0.4, 0.25, 0.25, 0.1\}^T$. Based on TABLE 1, loss functions from interviewers of each interviewee are summarized in TABLEs 5-6.

On the basis of loss functions from TABLEs 5-6, we can calculate mean value for $\lambda_\bullet^\diamond$ of group decision-making. Taking λ_X^P as an example, for the interviewee u_{10}, the evaluation results of $(\lambda_X^B)(u_{10})$ are obtained from four interviewers as follows:

$$(\lambda_X^B)^1(u_{10}) = 22u, \quad (\lambda_X^B)^2(u_{10}) = 8u,$$
$$(\lambda_X^B)^3(u_{10}) = 13u, \quad (\lambda_X^B)^4(u_{10}) = 13u.$$

TABLE 5. Loss functions determined by interviewers for each interviewee.

(U, E)	A						B					
	$(\lambda_X^P)^A$	$(\lambda_{X^c}^P)^A$	$(\lambda_X^B)^A$	$(\lambda_{X^c}^B)^A$	$(\lambda_X^N)^A$	$(\lambda_{X^c}^N)^A$	$(\lambda_X^P)^B$	$(\lambda_{X^c}^P)^B$	$(\lambda_X^B)^B$	$(\lambda_{X^c}^B)^B$	$(\lambda_X^N)^B$	$(\lambda_{X^c}^N)^B$
u_1	1u	3u	8u	30u	6u	0.5u	1.5u	4u	10u	26u	5u	0.6u
u_2	2.5u	8u	13u	25u	9u	0u	2u	7u	13u	27u	8u	1u
u_3	1.5u	2u	11u	17u	9u	0.6u	2u	7u	16u	25u	6u	1u
u_4	0u	5u	12u	24u	12u	2u	1u	12u	28u	24u	11u	2u
u_5	1u	4u	15u	21u	10u	0u	2u	9u	14u	30u	9u	1.2u
u_6	0.7u	16u	25u	30u	13u	1.4u	0.8u	13u	15u	33u	7u	1.2u
u_7	1.2u	9u	17u	25u	10u	1u	1.5u	10.2u	20u	22u	9u	1.4u
u_8	0.5u	10.2u	14u	18u	9.5u	1u	1.5u	10.2u	20u	20u	5u	1u
u_9	0.6u	17u	23u	28u	8u	0.7u	1.2u	14u	22u	30u	4u	1.5u
u_{10}	0.8u	22u	32u	35u	7u	2u	1u	8u	14u	24u	8u	2u

Note: u is a unit cost determined by interviewers

TABLE 6. Loss functions determined by interviewers for each interviewee.

(U, E)	C						D					
	$(\lambda_X^P)^C$	$(\lambda_{X^c}^P)^C$	$(\lambda_X^B)^C$	$(\lambda_{X^c}^B)^C$	$(\lambda_X^N)^C$	$(\lambda_{X^c}^N)^C$	$(\lambda_X^P)^D$	$(\lambda_{X^c}^P)^D$	$(\lambda_X^B)^D$	$(\lambda_{X^c}^B)^D$	$(\lambda_X^N)^D$	$(\lambda_{X^c}^N)^D$
u_1	2u	4u	7u	32u	6u	0.8u	1.3u	3u	7u	21u	9.2u	1u
u_2	2.3u	10u	15u	30u	8u	1.5u	1.2u	6u	12u	24u	8.5u	1.5u
u_3	2.3u	11u	13u	28u	8u	1.5u	1.6u	6u	10u	30u	8.2u	0.8u
u_4	0u	5u	15u	31u	18u	3u	1u	10.1u	15u	20u	9.2u	0.5u
u_5	1.5u	10.1u	15u	20u	9.2u	0.5u	2u	14.2u	20u	30u	9u	1.2u
u_6	0.9u	10u	16u	26u	11u	0.4u	2.1u	16u	18u	32u	7u	1.3u
u_7	2u	14.2u	40u	21u	9u	1.3u	2.5u	15u	23u	27u	8u	1.5u
u_8	2u	14.2u	23u	21u	10u	1.2u	2.5u	11u	21u	23u	7u	2u
u_9	1u	18u	32u	27u	16u	1.4u	2.2u	5u	16u	22u	7.5u	0.7u
u_{10}	0u	13u	22u	35u	15u	2u	1u	13u	20u	26u	6u	0.5u

Note: u is a unit cost determined by interviewers

TABLE 7. The interval-valued information granules for each interviewee when $\varepsilon = 0.3$.

U	λ_X^P	λ_X^B	λ_X^N	$\lambda_{X^c}^P$	$\lambda_{X^c}^B$	$\lambda_{X^c}^N$
u_1	[1u,2u]	[3u,4u]	[7u,10u]	[21u,30u]	[5u,9.2u]	[0.5u,1u]
u_2	[1.2u,2.5u]	[8u,10u]	[13u,15u]	[25u,27u]	[8u,9u]	[0u,1.5u]
u_3	[1.5u,2.3u]	[2u,7u]	[11u,13u]	[17u,25u]	[6u,9u]	[0.6u,1.5u]
u_4	[0u,1u]	[5u,12u]	[15u,28u]	[24u,31u]	[11u,18u]	[2u,3u]
u_5	[1u,2u]	[4u,10.1u]	[14u,20u]	[20u,30u]	[9u,10u]	[0u,1.2u]
u_6	[0.7u,2.1u]	[10u,16u]	[15u,25u]	[26u,30u]	[7u,13u]	[0.4u,1.4u]
u_7	[1.2u,2.5u]	[9u,14.2u]	[20u,40u]	[21u,25u]	[9u,10u]	[1u,1.5u]
u_8	[0.5u,2.5u]	[10u,14.2u]	[14u,21u]	[18u,21u]	[5u,10u]	[1u,2u]
u_9	[0.6u,2.2u]	[14u,18u]	[23u,32u]	[27u,28u]	[7.5u,16u]	[0.7u,1.5u]
u_{10}	[0u,1u]	[13u,22u]	[20u,32u]	[24u,35u]	[7u,15u]	[0.5u,2u]

And the corresponding weight for four interviewers are given in the following:

$$\omega_A = 0.4, \quad \omega_B = 0.25, \ \omega_C = 0.25, \ \omega_D = 0.1.$$

We need to merge the same values and sort them in ascending order:

$$(\lambda_X^B)^{\sigma(1)}(u_{10}) = 8u, \quad (\lambda_X^B)^{\sigma(2)}(u_{10}) = 13u,$$

$$(\lambda_X^B)^{\sigma(3)}(u_{10}) = 22u.$$

We also get the corresponding weight for loss function after sorting them in ascending order as follows:

$$(\omega_X^B)^{\sigma(1)} = 0.25, \quad (\omega_X^B)^{\sigma(2)} = 0.35, \ (\omega_X^B)^{\sigma(3)} = 0.4.$$

Thus, the mean value is $(m_X^B)(u_{10}) = 15.35u$.

According to the result of $(m_X^B)(u_{10})$*, we achieve interval-valued information granule for the interviewee* u_{10} *in light of (3.2)-(3.5), i.e.,* $(\lambda_X^B)(u_{10}) = [13u, 22u]$.

Homoplastically, we can get interval-valued information granules for all interviewees in TABLE 7.

Here, $(\lambda_X^P)(u_1) = [1u, 2u]$ expresses that the loss function values under group decision-making are between $1u$ and $2u$ for taking the action of being passed when the interviewee u_1 has a good performance; $(\lambda_X^B)(u_2) = [8u, 10u]$ means that the loss function values under group decision-making are between $8u$ and $10u$ for taking the action of being delayed decision when the interviewee u_2 has a good performance; $(\lambda_X^N)(u_3) = [11u, 13u]$ shows that the loss function values under group decision-making are between $11u$ and $13u$ for taking the action of being eliminated when the interviewee u_3 has a good performance.

Analogously, $(\lambda_{X^c}^P)(u_8) = [18u, 21u]$ expresses that the loss function values under group decision-making are between $18u$ and $21u$ for taking the action of being passed when the interviewee u_8 doesn't have a good performance; $(\lambda_{X^c}^B)(u_9) = [7.5u, 16u]$ means that the loss function values under group decision-making are between $7.5u$ and $16u$ for taking the action of being delayed decision when the interviewee u_9 doesn't have a good performance; $(\lambda_{X^c}^N)(u_{10}) = [0.5u, 2u]$ shows that the loss values under group decision-making are between $0.5u$ and $2u$ taking the action of being eliminated when the interviewee u_{10} doesn't have a good performance.

Interval-valued information granules for each interviewee are clearly satisfy the following conditions:

$$\begin{aligned}
(\lambda_X^P)^-(u_i) &< (\lambda_X^B)^-(u_i) < (\lambda_X^N)^-(u_i),\\
(\lambda_X^P)^+(u_i) &< (\lambda_X^B)^+(u_i) < (\lambda_X^N)^+(u_i);\\
(\lambda_{X^c}^P)^-(u_i) &< (\lambda_{X^c}^B)^-(u_i) < (\lambda_{X^c}^N)^-(u_i),\\
(\lambda_{X^c}^P)^+(u_i) &< (\lambda_{X^c}^B)^+(u_i) < (\lambda_{X^c}^N)^+(u_i).
\end{aligned}$$

Step 4: Suppose interviewers are different risk-takers and given $\delta = 0, 0.25, 0.5, 0.75, 1$, we compute three thresholds α, β, γ for all interviewees. TABLEs 5-9 list the homologous calculating results, and under-lined values represent the invalid values.

Step 5: The key step of 3WD in the $(U, \Delta, \{d\})$ is based on $P(D_j|\Delta_{u_i})$ $(i = 1, 2, \cdots, 10,\ j = 1, 2)$ and δ. In 3WD in the CDIS, we are able to reckon three thresholds α, β, γ, which are closely related to δ, so we can reflect the change of three thresholds for each interviewee, as shown in FIGUREs 2-6.

Step 6: Given $\delta = 0, 0.25, 0.5, 0.75, 1$. The rules can be generated by comparing conditional probability $P(D_j|\Delta_{u_i})$ $(i = 1, 2, \cdots, 10,\ j = 1, 2)$ and three thresholds in TABLEs 7-11. The decision results of each interviewee are listed in TABLEs 12-13, and the under-lined values in TABLEs 12-13 express invalid decision results.

Step 7: The detailed step-wise procedure as an algorithm on 3WD based on group decision-making of $(U, \Delta, \{d\})$ are given in the following:

Input: *A $(U, \Delta, \{d\})$, $\varepsilon = 0.3$, $\delta = 0, 0.25, 0.5, 0.75, 1$, loss functions $(\lambda_\bullet^\diamond)^A$, $(\lambda_\bullet^\diamond)^B$, $(\lambda_\bullet^\diamond)^C$, $(\lambda_\bullet^\diamond)^D$ $(\diamond = P, B, N; \bullet = X, X^c)$ for each interviewee and the corresponding weight vector of interviewers is $\{\omega_A, \omega_B, \omega_C, \omega_D\}^T = \{0.4, 0.25, 0.25, 0.1\}^T$.*

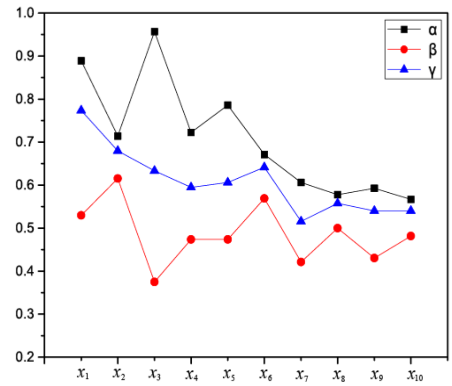


FIGURE 3. $\delta = 0$.

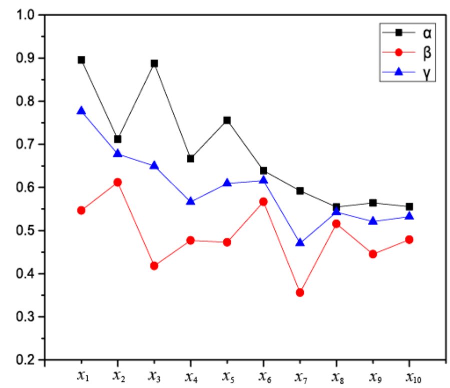


FIGURE 4. $\delta = 0.25$.

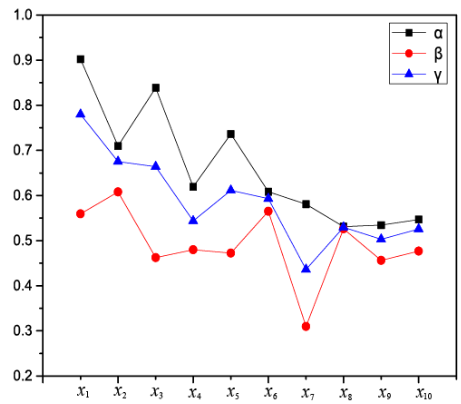


FIGURE 5. $\delta = 0.5$.

Output: *The three-way group decision rules.*

1. *For any $u_i \in U$, obtain Δ_{u_i} (see TABLE 3);*

2. *For any $u_i \in U$ and $D_j \in U/\{d\}$, compute the conditional probability $P(D_j|\Delta_{u_i})$ (see TABLE 4);*

3. *For $\varepsilon = 0.3$, based on TABLEs 5-6 and formulas (3.2)-(3.5), we calculate $\lambda_\bullet^\diamond = [(\lambda_\bullet^\diamond)^-, (\lambda_\bullet^\diamond)^+]$ (see TABLE 7), i.e., $\lambda_X^P = [(\lambda_X^P)^-, (\lambda_X^P)^+]$, $\lambda_X^B = [(\lambda_X^B)^-, (\lambda_X^B)^+]$, $\lambda_X^N = [(\lambda_X^N)^-, (\lambda_X^N)^+]$, $\lambda_{X^c}^N = [(\lambda_{X^c}^N)^-, \lambda_{X^c}^N)^+]$, $\lambda_{X^c}^B = [(\lambda_{X^c}^B)^-, \lambda_{X^c}^B)^+]$, $\lambda_{X^c}^P = [(\lambda_{X^c}^P)^-, \lambda_{X^c}^P)^+]$;*

TABLE 8. The values of α, β, γ when $\delta = 0$.

U	$f_0(\lambda_X^P)$	$f_0(\lambda_X^B)$	$f_0(\lambda_X^N)$	$f_0(\lambda_{X^c}^N)$	$f_0(\lambda_{X^c}^B)$	$f_0(\lambda_{X^c}^P)$	α	β	γ
u_1	1	3	7	21	5	0.5	0.8889	0.5294	0.7736
u_2	1.2	8	13	25	8	0	0.7143	0.6154	0.6793
u_3	1.5	2	11	17	6	0.6	0.9565	0.3750	0.6332
u_4	0	5	15	24	11	2	0.7222	0.4737	0.5946
u_5	1	4	14	20	9	0	0.7857	0.4737	0.6061
u_6	0.7	10	15	26	7	0.4	0.6714	0.5690	0.6416
u_7	1.2	9	20	21	9	1	0.6061	0.4211	0.5155
u_8	0.5	10	14	18	5	1	0.5778	0.5000	0.5574
u_9	0.6	14	23	27	7.5	0.7	0.5927	0.4304	0.5400
u_{10}	0	13	20	24	7	0.5	0.5667	0.4815	0.5402

TABLE 9. The values of α, β, γ when $\delta = 0.25$.

U	$f_{0.25}(\lambda_X^P)$	$f_{0.25}(\lambda_X^B)$	$f_{0.25}(\lambda_X^N)$	$f_{0.25}(\lambda_{X^c}^N)$	$f_{0.25}(\lambda_{X^c}^B)$	$f_{0.25}(\lambda_{X^c}^P)$	α	β	γ
u_1	1.25	3.25	7.75	23.25	6.05	0.625	0.8958	0.5466	0.7768
u_2	1.525	8.5	13.5	25.5	8.25	0.375	0.7121	0.6117	0.6772
u_3	1.7	3.25	11.5	19	6.75	0.825	0.8877	0.4180	0.6497
u_4	0.25	6.75	18.25	25.75	12.75	2.25	0.6667	0.4773	0.5663
u_5	1.25	5.525	15.5	22.5	9.25	0.3	0.7561	0.4729	0.6091
u_6	1.05	11.5	17.5	27	8.5	0.65	0.6390	0.5668	0.6157
u_7	1.525	10.3	25	22	9.25	1.125	0.5923	0.3560	0.4707
u_8	1	11.05	15.75	18.75	6.25	1.25	0.5543	0.5155	0.5426
u_9	1	15	25.25	27.25	9.125	0.9	0.5642	0.4452	0.5208
u_{10}	0.25	15.25	23	26.75	8	0.875	0.5556	0.4790	0.5321

TABLE 10. The values of α, β, γ when $\delta = 0.5$.

U	$f_{0.5}(\lambda_X^P)$	$f_{0.5}(\lambda_X^B)$	$f_{0.5}(\lambda_X^N)$	$f_{0.5}(\lambda_{X^c}^N)$	$f_{0.5}(\lambda_{X^c}^B)$	$f_{0.5}(\lambda_{X^c}^P)$	α	β	γ
u_1	1.5	3.5	8.5	25.5	7.1	0.75	0.9020	0.5595	0.7795
u_2	1.85	9	14	26	8.5	0.75	0.7099	0.6078	0.6751
u_3	1.9	4.5	12	21	7.5	1.05	0.8385	0.4624	0.6639
u_4	0.5	8.5	21.5	27.5	14.5	2.5	0.6190	0.4800	0.5435
u_5	1.5	7.05	17	25	9.5	0.6	0.7363	0.4721	0.6115
u_6	1.4	13	20	28	10	0.9	0.6081	0.5652	0.5930
u_7	1.85	11.6	30	23	9.5	1.25	0.5806	0.3096	0.4359
u_8	1.5	12.1	17.5	19.5	7.5	1.5	0.5310	0.5263	0.5294
u_9	1.4	16	27.5	27.5	10.75	1.1	0.5343	0.4563	0.5029
u_{10}	0.5	17.5	26	29.5	9	1.25	0.5467	0.4769	0.5256

TABLE 11. The values of α, β, γ when $\delta = 0.75$.

U	$f_{0.75}(\lambda_X^P)$	$f_{0.75}(\lambda_X^B)$	$f_{0.75}(\lambda_X^N)$	$f_{0.75}(\lambda_{X^c}^N)$	$f_{0.75}(\lambda_{X^c}^B)$	$f_{0.75}(\lambda_{X^c}^P)$	α	β	γ
u_1	1.75	3.75	9.25	27.75	8.15	0.875	0.9074	0.5695	0.7818
u_2	2.175	9.5	14.5	26.5	8.75	1.125	0.7079	0.6040	0.6731
u_3	2.1	5.75	12.5	23	8.25	1.275	0.8016	0.5082	0.6763
u_4	0.75	10.25	24.75	29.25	16.25	2.75	0.5778	0.4821	0.5248
u_5	1.75	8.575	18.5	27.5	9.75	0.9	0.7223	0.4714	0.6136
u_6	1.75	14.5	22.5	29	11.5	1.15	0.5785	0.5640	0.5730
u_7	2.175	12.9	35	24	9.75	1.375	0.5706	0.2748	0.4080
u_8	2	13.15	19.25	20.25	8.75	1.75	0.5077	0.5344	0.5175
u_9	1.8	17	29.75	27.75	12.375	1.3	0.5029	0.4648	0.4862
u_{10}	0.75	19.75	29	32.25	10	1.625	0.5394	0.4752	0.5202

4. *Pick* $\delta = 0, 0.25, 0.5, 0.75, 1$, *compute* $f_\delta(\lambda_\bullet^\diamond) = (1-\delta)(\lambda_\bullet^\diamond)^- + \delta(\lambda_\bullet^\diamond)^+$ *(see TABLEs 7-11);*

5. *Calculate corresponding thresholds* α, β *and* γ *for each interviewee,*

$$\alpha = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^B)) + (f_\delta(\lambda_X^B) - f_\delta(\lambda_X^P))},$$

$$\beta = \frac{f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_{X^c}^B) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^B))},$$

$$\gamma = \frac{f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)}{(f_\delta(\lambda_X^P) - f_\delta(\lambda_{X^c}^N)) + (f_\delta(\lambda_X^N) - f_\delta(\lambda_X^P))};$$

6. *For any* D_j, *based on* α, β, γ *in Step 5, obtain* $POS^{(\alpha,\beta)}(D_j)$, $NEG^{(\alpha,\beta)}(D_j)$, $BND^{(\alpha,\beta)}(D_j)$;

7. *For any* D_j, *give the three-way group decision rules of* D_j *(see TABLEs 12-13).*

For an example, we analysis two special cases ($\delta = 0, 0.5$) *to display the 3WD of* $D \in U/\{d\}$ *in the following:*

TABLE 12. The values of α, β, γ when $\delta = 1$.

U	$f_1(\lambda_X^P)$	$f_1(\lambda_X^B)$	$f_1(\lambda_X^N)$	$f_1(\lambda_{X^c}^N)$	$f_1(\lambda_{X^c}^B)$	$f_1(\lambda_{X^c}^P)$	α	β	γ
u_1	2	4	10	30	9.2	1	0.9123	0.5775	0.7838
u_2	2.5	10	15	27	9	1.5	0.7059	0.6000	0.6711
u_3	2.3	7	13	25	9	1.5	0.7729	0.5556	0.6871
u_4	1	12	28	31	18	3	0.5417	0.4839	0.5091
u_5	2	10.1	20	30	10	1.2	0.7117	0.4706	0.6154
u_6	2.1	16	25	30	13	1.4	0.5502	0.5631	0.5553
u_7	2.5	14.2	40	25	10	1.5	0.5618	0.2478	0.3852
u_8	2.5	14.2	21	21	10	2	0.4846	0.5405	0.5067
u_9	2.2	18	32	28	14	1.5	0.4698	0.4717	0.4707
u_{10}	1	22	32	35	11	2	0.5333	0.4737	0.5156

TABLE 13. The three-way group decision rules of D_1 when δ change.

	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1$
$POS^{(\alpha,\beta)}(D_1)$	$\{u_1, u_2, u_3, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, \underline{u_6}, u_7\}$
$NEG^{(\alpha,\beta)}(D_1)$	$\{u_4, u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5\}$
$BND^{(\alpha,\beta)}(D_1)$	$\{u_8, u_{10}\}$	$\{u_8, u_{10}\}$	$\{u_8, u_{10}\}$	$\{\underline{u_8}, u_{10}\}$	$\{\underline{u_8}, \underline{u_9}, u_{10}\}$

TABLE 14. The three-way group decision rules of D_2 when δ change.

	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1$
$POS^{(\alpha,\beta)}(D_2)$	$\{u_8, u_{10}\}$	$\{u_8, u_{10}\}$	$\{u_8, u_{10}\}$	$\{\underline{u_8}, u_{10}\}$	$\{\underline{u_8}, \underline{u_9}, u_{10}\}$
$NEG^{(\alpha,\beta)}(D_2)$	$\{u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5, u_9\}$	$\{u_5\}$
$BND^{(\alpha,\beta)}(D_2)$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, u_6, u_7\}$	$\{u_1, u_2, u_3, u_4, \underline{u_6}, u_7\}$

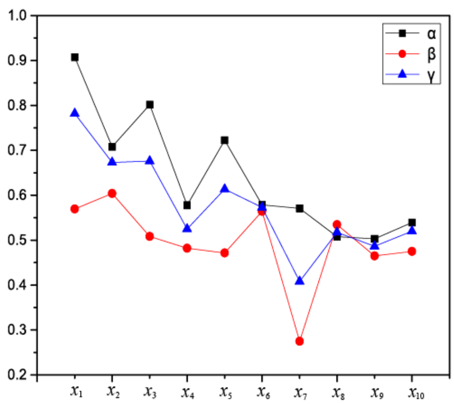


FIGURE 6. $\delta = 0.75$.

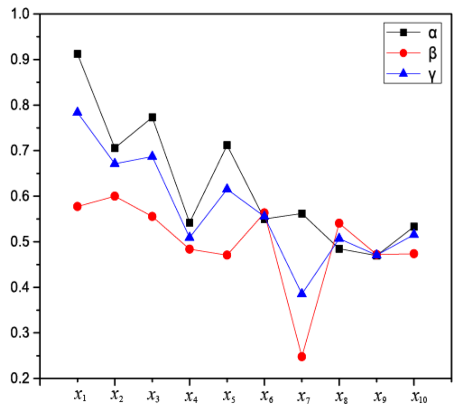


FIGURE 7. $\delta = 1$.

(1) *Pick* $\delta = 0$.

We obtain the positive, negative and boundary regions of D_1 *from TABLE 12 as follows*:

$$POS^{(\alpha,\beta)}(D_1) = \{u_1, u_2, u_3, u_6, u_7\},$$
$$NEG^{(\alpha,\beta)}(D_1) = \{u_8, u_{10}\},$$
$$BND^{(\alpha,\beta)}(D_1) = \{u_4, u_5, u_9\}.$$

The results imply that interviewees u_1, u_2, u_3, u_6, u_7 *should be passed, interviewees* u_8, u_{10} *should be eliminated, interviewees* u_4, u_5, u_9 *should be further evaluated.*

(2) *Pick* $\delta = 0.5$.

We obtain the positive, negative and boundary regions of D_2 *from TABLE 13 as follows*:

$$POS^{(\alpha,\beta)}(D_2) = \{u_8, u_{10}\},$$
$$NEG^{(\alpha,\beta)}(D_2) = \{u_1, u_2, u_3, u_4, u_6, u_7\},$$
$$BND^{(\alpha,\beta)}(D_2) = \{u_5, u_9\}.$$

The results imply that interviewees u_8 *and* u_{10} *should be eliminated, interviewees* u_1, u_2, u_3, u_4, u_6, u_7 *should be passed, interviewee* u_5, u_9 *should be further evaluated.*

VI. CONCLUSION

In this paper, the similarity classes induced by this CDIS have been obtained by neighbourhood of the point. On the light of idea of DTRS, a method of three-way group decisions in this CDIS has been presented. An example of position competition has been displayed to explain feasibility of our proposed method. In future work, we will employ other methods to research decision problems and use big data to analyze three-way group decisions in a CDIS.

Compliance with ethical standards

Conflict of interest: All authors declare that there is no conflict of interests regarding the publication of this manuscript.

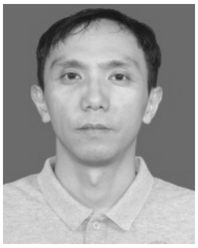
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

REFERENCES

[1] K. Agbodah, "The determination of three-way decisions with decision-theoretic rough sets considering the loss function evaluated by multiple experts," *Granular Comput.*, vol. 4, no. 2, pp. 285–297, Apr. 2019.
[2] A. R. Abbas and L. Juan, "Supporting E–learning system with modified Bayesian rough set model," in *Proc. 6th Int. Symp. Neural Netw.*, 2009, pp. 192–200.
[3] R. Goudey, "Do statistical inferences allowing three alternative decision give better feedback for environmentally precautionary decision-making," *J. Environ. Manage.*, vol. 85, pp. 338–344, 2007.
[4] S. Greco, M. Inuiguchi, and R. Slowinski, "Fuzzy rough sets and multiple–premise gradual decision rules," *Int. J. Approx. Reasoning*, vol. 41, pp. 179–211, Feb. 2006.
[5] S. Greco, B. Matarazzo, and R. Słowiński, "Parameterized rough set model using rough membership and Bayesian confirmation measures," *Int. J. Approx. Reasoning*, vol. 49, no. 2, pp. 285–300, Oct. 2008.
[6] J. P. Herbert and J. Yao, "Analysis of data–driven parameters in game–theoretic rough sets," in *Proc. Int. Conf. Rough Sets Knowl. Technol.*, New York, NY, USA, Springer-Verlag, 2011, pp. 447–456.
[7] M. Kryszkiewicz, "Rules in incomplete information systems," *Inf. Sci.*, vol. 113, nos. 3–4, pp. 271–292, Feb. 1999.
[8] P. Liu, "Multiple attribute group decision making method based on interval-valued intuitionistic fuzzy power Heronian aggregation operators," *Comput. Ind. Eng.*, vol. 108, pp. 199–212, Jun. 2017.
[9] Z. Li, D. Huang, X. Liu, N. Xie, and G. Zhang, "Information structures in a covering information system," *Inf. Sci.*, vol. 507, pp. 449–471, Jan. 2020, doi: 10.1016/j.ins.2018.09.048.
[10] Z. Li and D. Huang, "A three-way decision method in a fuzzy condition decision information system and its application in credit card evaluation," *Granular Comput.*, May 2019, doi: 10.1007/s41066-019-00172-8.
[11] D. Liang and D. Liu, "Systematic studies on three-way decisions with interval-valued decision-theoretic rough sets," *Inf. Sci.*, vol. 276, pp. 186–203, Aug. 2014.
[12] D. Liang and D. Liu, "Deriving three-way decisions from intuitionistic fuzzy decision-theoretic rough sets," *Inf. Sci.*, vol. 300, pp. 28–48, Apr. 2015.
[13] D. C. Liang and D. Liu, "A novel risk decision-making based on decision-theoretic rough sets under hesitant fuzzy information," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 2, pp. 237–247, Apr. 2015.
[14] D. Liu and D. Liang, "Three-way decisions in ordered decision system," *Knowl.-Based Syst.*, vol. 137, pp. 182–195, Dec. 2017.
[15] D. Liu, T. Li, and D. Liang, "Three–way government decision analysis with decision–theoretic rough sets," *Fuzziness Knowl.-Based Syst.*, vol. 20, pp. 119–132, Jun. 2012.
[16] D. Liu, T.-R. Li, and H.-X. Li, "Interval-valued decision-theoretic rough sets," *Comput. Sci.*, vol. 39, no. 7, pp. 178–181, 2012.
[17] D. Liu, T. R. Li, and D. C. Liang, "Three-way decisions instochastic decision-theoretic rough sets," in *Transactions on Rough Sets XVIII*, vol. 18. New York, NY, USA: Springer-Verlag, 2014, pp. 110–130.
[18] D. Liu, T. Li, and D. Liang, "Incorporating logistic regression to decision-theoretic rough sets for classifications," *Int. J. Approx. Reasoning*, vol. 55, no. 1, pp. 197–210, Jan. 2014.
[19] D. Liang, D. Liu, and A. Kobina, "Three-way group decisions with decision-theoretic rough sets," *Inf. Sci.*, vol. 345, pp. 46–64, Jun. 2016.
[20] D. Liang, D. Liu, W. Pedrycz, and P. Hu, "Triangular fuzzy decision-theoretic rough sets," *Int. J. Approx. Reasoning*, vol. 54, no. 8, pp. 1087–1106, Oct. 2013.
[21] D. Liu, D. Liang, and C. Wang, "A novel three-way decision model based on incomplete information system," *Knowl.-Based Syst.*, vol. 91, pp. 32–45, Jan. 2016.
[22] Z. W. Li, X. Liu, G. Zhang, N. Xie, and S. Wang, "A multi-granulation decision-theoretic rough set method for distributed *fc*-decision information systems: An application in medical diagnosis," *Appl. Soft Comput.*, vol. 56, pp. 233–244, Jul. 2017.
[23] D. Liang, W. Pedrycz, D. Liu, and P. Hu, "Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making," *Appl. Soft Comput.*, vol. 29, pp. 256–269, Apr. 2015.
[24] J. S. Liu, X. Z. Wang, and B. Y. Zhang, "The ranking of interval numbers," *J. Eng. Math.*, vol. 18, no. 4, pp. 103–109, 2001.
[25] D. Liu, Y. Yao, and T. Li, "Three-way investment decisions with decision-theoretic rough sets," *Int. J. Comput. Intell. Syst.*, vol. 4, no. 1, p. 66, 2011.
[26] H. X. Li and X. Z. Zhou, "Risk decision making based on decision-theoretic rough set: A three-way view decision model," *Int. J. Comput. Intell.*, vol. 4, no. 1, pp. 1–11, 2011.
[27] F. Jia and P. Liu, "A novel three-way decision model under multiple-criteria environment," *Inf. Sci.*, vol. 471, pp. 29–51, Jan. 2019.
[28] P. Maji, "Rough hypercuboid approach for feature selection in approximation spaces," *IEEE Trans. Knowl. Data Eng.*, vol. 26, no. 1, pp. 16–29, Jan. 2014.
[29] P. Mandal and A. S. Ranadive, "Multi-granulation interval-valued fuzzy probabilistic rough sets and their corresponding three-way decisions based on interval-valued fuzzy preference relations," *Granular Comput.*, vol. 4, no. 1, pp. 89–108, Jan. 2019.
[30] F. Min, H. He, Y. Qian, and W. Zhu, "Test-cost-sensitive attribute reduction," *Inf. Sci.*, vol. 181, no. 22, pp. 4928–4942, Nov. 2011.
[31] Z. Pawlak, "Rough sets," *Int. J. Comput. Inf. Sci.*, vol. 11, pp. 341–356, Oct. 1982.
[32] W. Pedrycz, "Allocation of information granularity in optimization and decision-making models: Towards building the foundations of Granular computing," *Eur.J. Oper. Res.*, vol. 232, no. 1, pp. 137–145, Jan. 2014.
[33] W. Pedrycz and W. Homenda, "Building the fundamentals of granular computing: A principle of justifiable granularity," *Appl. Soft Comput.*, vol. 13, no. 10, pp. 4209–4218, Oct. 2013.
[34] A. M. Radzikowska and E. E. Kerre, "A comparative study of fuzzy rough sets," *Fuzzy Sets Syst.*, vol. 126, no. 2, pp. 137–155, Mar. 2002.
[35] R. W. Swiniarski and A. Skowron, "Rough set methods in feature selection and recognition," *Pattern Recognit. Lett.*, vol. 24, pp. 833–849, Mar. 2003.
[36] R. Slowinski and D. Vanderpooten, "A generalized definition of rough approximations based on setilarity," *IEEE Trans. Knowl. Data Eng.*, vol. 12, no. 2, pp. 331–336, Mar./Apr. 2000.
[37] C. Wang, D. Chen, C. Wu, and Q. Hu, "Data compression with homomorphism in covering information systems," *Int. J. Approx. Reasoning*, vol. 52, no. 4, pp. 519–525, Jun. 2011.
[38] C. Wang, D. Chen, B. Sun, and Q. Hu, "Communication between information systems with covering based rough sets," *Inf. Sci.*, vol. 216, pp. 17–33, Dec. 2012.
[39] C. Wang, Q. He, D. Chen, and Q. Hu, "A novel method for attribute reduction of covering decision systems," *Inf. Sci.*, vol. 254, pp. 181–196, Jan. 2014.
[40] L. Wang, X. Liu, and W. Pedrycz, "Effective intervals determined by information granules to improve forecasting in fuzzy time series," *Expert Syst. With Appl.*, vol. 40, no. 14, pp. 5673–5679, Oct. 2013.
[41] P. W. Woodward and J. C. Naylor, "An application of Bayesian methods in SPC," *J. Roy. Stat. Soc. Ser. D, Statistician*, vol. 42, no. 4, pp. 461–469, 1993.
[42] X. Wang, E. C. Tsang, S. Zhao, D. Chen, and D. S. Yeung, "Learning fuzzy rules from fuzzy samples based on rough set technique," *Inf. Sci.*, vol. 177, no. 20, pp. 4493–4514, Oct. 2007.
[43] Z. Xu, "Au automatic approach to reaching consensus in multiple attribute group decision making," *Comput. Ind. Eng.*, vol. 56, pp. 1369–1374, May 2009.
[44] Y. Y. Yao, "Three-way decision: An interpretation of rules in rough set theory," in *Rough Sets Knowledge Technology* (Lecture Notes Computer Science), vol. 5589. New York, NY, USA: Springer-Verlag, 2009, pp. 642–649.
[45] Y. Yao, "Three-way decisions with probabilistic rough sets," *Inf. Sci.*, vol. 180, no. 3, pp. 341–353, Feb. 2010.
[46] Y. Yao, "The superiority of three-way decisions in probabilistic rough set models," *Inf. Sci.*, vol. 181, no. 6, pp. 1080–1096, Mar. 2011.
[47] J. Yao and J. P. Herbert, "Web–based support systems with rough set analysis," in *Proc. Int. Conf. Rough Set Emerg. Intell. Syst. Paradigms*, vol. 4585, 2007, pp. 360–370.
[48] H. Yu, Z. Liu, and G. Wang, "An automatic method to determine the number of clusters using decision-theoretic rough set," *Int. J. Approx. Reasoning*, vol. 55, no. 1, pp. 101–115, Jan. 2014.
[49] X. Yang and J. Yao, "Modelling multi-agent three-way decisions with decision-theoretic rough sets," *Fundamenta Informaticae*, vol. 115, nos. 2–3, pp. 157–171, 2012.
[50] Y. Y. Yao and B. Zhou, "Naive Bayesian rough sets," in *Rough Set Knowledge Technology*. vol. 6401, 2010, pp. 719–726.

[51] W. Zakowski, "Approximations in the space (U,$\prod$)," *Demonstratio Math.*, vol. 16, pp. 761–769, Jul. 1983.
[52] J. M. Zhan, B. Z. Sun, and J. C. R. Alcantud, "Covering based multigranulation (I, T)-Fuzzy rough set models and applications in multi-attribute group decision-making," *Inf. Sci.*, vol. 476, pp. 290–318, Feb. 2019.
[53] W. X. Zhang, W. Z. Wu, J. Y. Liang, and D. Y. Li, *Rough Set Theory and Methods*. Beijing, China: Scientific Publishers, 2001.
[54] B. Zhou, Y. Yao, and J. Luo, "Cost-sensitive three-way email spam filtering," *J. Intell. Inf. Syst.*, vol. 42, no. 1, pp. 19–45, Feb. 2014.
[55] K. Zhang, J. M. Zhan, W. Z. Wu, and J. C. R. Alcantud, "Fuzzy β-covering based (I, T)-fuzzy rough set models and applications to multi-attribute decision-making," *Comput. Ind. Eng.*, vol. 128, pp. 605–621, Feb. 2019.
[56] P. Wang, P. Zhang, and Z. Li, "A three-way decision method based on Gaussian Kernel in a hybrid information system with images: An application in medical diagnosis," *Appl. Soft Comput.*, vol. 77, pp. 734–749, Apr. 2019.

XIAOLIANG XIE received the M.Sc. degree in mathematics from Hunan Normal University, Changsha, China, in 1998, and the Ph.D. degree in mathematics from Central South University, Changsha, in 2010. He is currently a Professor with the School of Mathematics and Statistics, Hunan University of Commerce. His research interests include optimization theory, rough set theory, and information systems.



JIALI HE received the M.Sc. degree in computer from the Guangxi University for Nationalities, Nanning, China, in 2008. He is currently an Associate Professor with the School of Mathematics and Statistics, Yulin Normal University, Yulin, China. His research interests include rough set theory, soft computing, and information systems.

DAN HUANG received the M.Sc. degree in mathematics from the Guangxi University for Nationalities, Nanning, China, in 2019. She is currently a Teacher with the School of Mathematics and Statistics, Baise University, Baise, China. Her research interests include rough set theory, fuzzy set theory, and information systems.