

Received September 10, 2019, accepted September 26, 2019, date of publication October 11, 2019, date of current version March 17, 2020.

Digital Object Identifier 10.1109/ACCESS.2019.2946790

Dynamics of Fractional-Order Neural Networks With Discrete and Distributed Delays

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This work was supported in part by the National Natural Science Foundation of China under Grant 615713194, Grant 61873326, Grant 61773004, and Grant 61833005, in part by the Natural Science Foundation of Jiangsu Province of China under Grant BK20181389, and in part by the Key Project of Philosophy and Social Science Research in Colleges and Universities, Jiangsu Province, under Grant 2018SJD1142.

ABSTRACT This paper is concerned with the stability and Hopf bifurcation of fractional-order neural networks with discrete and distributed delays. The novelty of this paper is to take into account the discrete time delay and the distributed time delay for fractional-order systems. By introducing two virtual neurons to the original network, a new four-neuron network only involving discrete delays is formed. The sum of discrete delays is adopted as the bifurcation parameter to demonstrate the existence of Hopf bifurcation. It is found that the critical value of bifurcation can be effectively manipulated by choosing appropriate system parameters and order. Finally, numerical simulations are executed to substantiate the theoretical results and describe the relationships between the parameters and the onset of bifurcation.

INDEX TERMS Stability, Hopf bifurcation, discrete delay, distributed delay, fractional-order neural networks.

I. INTRODUCTION

As is known to all, the neuronal system is a complex nonlinear dynamic system. And the neuron is considered the basic processing unit that has simplicity and simulation. Neurons can generate and transmit actions, encode and decode information, and complete other neural signal processes through the vibration and turbulence process in discharge activity. Once the neuronal system is maladjusted, the physiological mechanism will be abnormal or even chaotic, eventually leading to the emergence of neurological diseases [1]. For example, epilepsy and Parkinson diseases are dynamic neurological disorders. From the perspective of nonlinear dynamical systems, the nature of dynamic disease is related to bifurcations caused by altering the regulatory parameters in the neuronal system. Parkinson disease, for example, is formed by excessive synchronization of the basal ganglia, leading to intensify the local potential field oscillations, and the Hopf bifurcation occurs simultaneously. Therefore, it is extremely important

to seek the specific parameter intervals of bifurcation for the treatment of dynamic diseases.

In the past decade, much investigation on neural networks (NNs) [2]–[6] has sprung up ever since its first simplified model was put forward by Hopfield in [7]. It is now firmly established that NNs can be applied into associative memory, pattern recognition and artificial intelligence. In fact, double-neuron networks exhibit the dynamic behavior in accord with the multi-neuron ones, which can be served as prototypes so as to enhance understanding about the dynamics of complex multi-neuron networks. Furthermore, because the duplex structure consumes so little energy, the physiological brain has been imitated by creating different types of memory-based circuits [8], [9].

Combining the notion of Hopfield NNs in [7], Olien and Bélair [10] investigated the dynamic behavior of a two-neuron system with diverse discrete delays, which can be described as follows

$$\begin{cases} p_1'(t) = -p_1(t) + v_1 f(p_1(t - \tau_1)) + v_1 f(p_2(t - \tau_2)), \\ p_2'(t) = -p_2(t) + v_2 f(p_1(t - \tau_1)) + v_2 f(p_2(t - \tau_2)). \end{cases} \quad (1)$$

The associate editor coordinating the review of this manuscript and approving it for publication was Zhixiang Zou¹.

They analyzed different states with time delays, and revealed that the time delay exerted significant influence on system dynamics. Henceforth, more and more scholars set out to construct artificial neural network models with time delay so as to get close to biological neural networks. Hence, plenty of results about the dynamics of NNs have been obtained [11]–[17]. For example, based on [10], Li and Hu [14] considered the following two-neuron system with discrete and diagonal distributed delays

$$\begin{cases} p_1'(t) = -p_1(t) + v_{11}f\left(\int_{-\infty}^t F(t-s)p_1(s)ds\right) \\ \quad + v_{12}f(p_2(t-\tau)), \\ p_2'(t) = -p_2(t) + v_{21}f(p_1(t-\tau)) \\ \quad + v_{22}f\left(\int_{-\infty}^t F(t-s)p_2(s)ds\right). \end{cases} \quad (2)$$

The direction of Hopf bifurcation and stability of bifurcation periodic solutions were given. Compared with system (2), Karaoğlu *et al.* [15] proposed a more general neural network system that has different discrete delays and activation functions, but with only one distributed delay. Additionally, several studies have developed low dimensional systems into high dimensional ones concerning merely discrete delays [16], [17]. However, these works only dealt with integer-order models of neural networks.

With the development and applications of the fractional calculus, researchers have come to realize that the superiority of fractional-order derivatives. On the one hand, fractional derivatives can describe the memory and hereditary effect of different processes. On the other hand, the order is variable, which means that fractional-order systems have unlimited memory. Thus, fractional-order calculus have been widely studied in diverse disciplines, including engineering [18], [19], physics [20], [21], biology [22]–[24], economics [25], [26], control [27], electromagnetism [28] and so on. However, the fractional-order systems are more complex than integer-order one in terms of properties and analytical processes.

Recently, fractional calculus has been applied in various fields, such as fractional-order genetic regulatory networks [29], fractional-order congestion control algorithms [30], fractional-order predator-prey models [31], [32], and fractional-order neural networks [33]–[43]. Combining with fractional calculus, the fractional-order neural networks (FNNs) will better reflect the memory and genetic characteristics. In [33] and [34], Kaslik and Sivasundaram proposed fractional-order neural networks of Hopfield type with different structures. What's more, a variable-order fractional operator was introduced into NNs in [35]. Also, Song and Cao [36] investigated the FNNs and provided the existence and uniqueness of the nontrivial solution. Moreover, the authors analyzed a category of complex-valued FNNs with hub and ring structured, and presented the conditions of Hopf bifurcation [37]. However, previous works ignored the time delay [33]–[37]. Yang *et al.* [38] discussed the stability of FNNs without and with discrete delay respectively, and established an LMI-based uniform

stability condition. Furthermore, Xu *et al.* [39] considered a double-neuron FNNs with two discrete delays, discussed four possible cases with delays, and finally revealed the effects of different time delays on the stability of networks. Chen *et al.* [40] discussed a category of FNNs, but only dealt with discrete delay. It should be noted that only discrete delay was considered in [38]–[40]. Different from previous studies, we will take into account a fractional-order neural system with distributed and discrete delays. As a consequence, our neural system not only embodies the genetic and memory characteristics of neural networks, but also reflects the imbalance of delay during information transmission.

This paper's main highlights can be listed as follows:

(1) Since fractional derivatives characterize the memory fetures more accurately, we have developed the integer-order neural network model with mixed delays into the fractional-order case to match for the practical biological neural networks more appropriately.

(2) Although there have been some results on the dynamics of stability and bifurcation for delayed fractional-order neural networks, only discrete delays are considered. If distributed delays are added, the imbalance of delays should be discussed in transmitting information. In this paper, both discrete delays and distributed delays are taken into account at the same time in fractional-order neural networks.

(3) By the coordinate transformation, we can convert the original neural network with mixed delays into an equivalent system involving only discrete delays, which eliminates the distributed delay terms. However, the two-dimensional system with order α becomes a four-dimensional system with two different orders, which makes the derivation more interesting.

(4) The influences of mixed delays and order on dynamical behaviors of fractional-order neural networks have been investigated. It is found that the critical value can be effectively manipulated by adopting appropriate system parameters and order.

The paper is arranged as follows: In Section II, several preliminaries are put forward. In Section III, the fractional-order double-neuron model is presented. In Section IV, we arrive at the sufficient conditions of Hopf bifurcation. The validity of theoretical results are verified by simulation in Section V. Finally, Section VI summarizes the paper and indicates the future research.

II. PRELIMINARIES

In this section, the definition of the Caputo derivative and the stability of n -dimensional linear fractional differential system are introduced.

There are various definitions of fractional derivatives, of which the commonly adopted are the Riemann-Liouville definition, the Grünwald-Letnikov definition and the Caputo definition. The definition of Caputo derivatives has the superiority in not limiting the initial conditions and making fractional-order systems simpler after the Laplace transform.

Thus, a lot of papers adopt the Caputo fractional derivative [32], [36], [38], [44], [45].

Definition 1 ([46]): The Caputo fractional derivative is defined by

$${}_0^C D_t^\phi f(t) = \frac{1}{\Gamma(n-\phi)} \int_0^t (t-\tau)^{n-\phi-1} f^{(n)}(\tau) d\tau. \quad (3)$$

where $\phi \geq 0, \phi < n \leq \phi + 1, n \in \mathbb{Z}^+$ and the Gamma function $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$. ϕ denotes the value of the fractional order and $\phi \in (0, 1]$.

The n -dimensional linear fractional differential system is

$$\begin{aligned} \frac{{}_0^C D_t^{\phi_1} p_1(t)}{{}_0^C D_t^{\phi_1} t} &= v_{11}p_1(t) + v_{12}p_2(t) + \dots + v_{1n}p_n(t), \\ \frac{{}_0^C D_t^{\phi_2} p_2(t)}{{}_0^C D_t^{\phi_2} t} &= v_{21}p_1(t) + v_{22}p_2(t) + \dots + v_{2n}p_n(t), \\ &\dots \\ \frac{{}_0^C D_t^{\phi_n} p_n(t)}{{}_0^C D_t^{\phi_n} t} &= v_{n1}p_1(t) + v_{n2}p_2(t) + \dots + v_{nn}p_n(t), \end{aligned} \quad (4)$$

where the order ϕ_i are rational numbers and $\phi_i \in (0, 1]$, for $i = 1, 2, \dots, n$.

The characteristic equation of model (4) is as follows

$$\det \begin{pmatrix} s^{\phi_1} - v_{11} & -v_{12} & \dots & -v_{1n} \\ -v_{21} & s^{\phi_2} - v_{22} & \dots & -v_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -v_{n1} & -v_{n2} & \dots & s^{\phi_n} - v_{nn} \end{pmatrix} = 0. \quad (5)$$

Let Q be the lowest common multiple of q_i of ϕ_i , where $\phi_i = \frac{q_i}{q_i}, \sigma_i, q_i \in \mathbb{N}^+, (\sigma_i, q_i) = 1, \text{ for } i = 1, 2, \dots, n$.

Lemma 1 ([47]): Only when all the roots ϖs of the equation

$$\det \begin{pmatrix} \varpi^{Q\phi_1} - v_{11} & -v_{12} & \dots & -v_{1n} \\ -v_{21} & \varpi^{Q\phi_2} - v_{22} & \dots & -v_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -v_{n1} & -v_{n2} & \dots & \varpi^{Q\phi_n} - v_{nn} \end{pmatrix} = 0, \quad (6)$$

satisfy $|\arg(\varpi)| > \frac{\pi}{2Q}$, the zero solution of system (4) is locally asymptotically stable.

Remark 1: The stability mentioned in this paper is the asymptotic stability.

III. THE MATHEMATICAL MODEL

In order to better describe the dynamic system, we make appropriate improvements based on previous research. More specifically speaking, a double-neuron network with mixed delays is put forward as follows:

$$\begin{cases} {}_0^C D_t^\alpha p_1(t) = -p_1(t) + v_{11}f\left(\int_{-\infty}^t F(t-s)p_1(s)ds\right) \\ \quad + v_{12}f(p_2(t-\tau_2)), \\ {}_0^C D_t^\alpha p_2(t) = -p_2(t) + v_{21}f(p_1(t-\tau_1)) \\ \quad + v_{22}f\left(\int_{-\infty}^t F(t-s)p_2(s)ds\right), \end{cases} \quad (7)$$

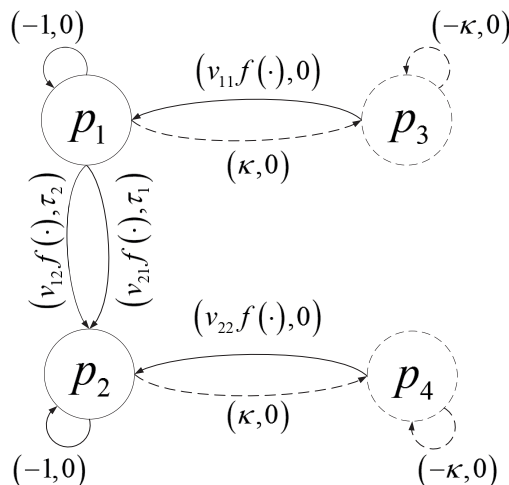


FIGURE 1. Architecture of system (8).

where p_i ($i = 1, 2$) represents the state of the i th neuron at time t and v_{ij} ($i = 1, 2$ and $j = 1, 2$) denotes real constants. The order α is a rational number. Moreover, the delay kernel function $F(\cdot)$ is said to be non-negative when $\int_0^\infty F(s)ds = 1$. $F(\cdot)$ represents the impact of past memory on current dynamics and its form is as follows:

$$F(s) = \kappa^{n+1} \frac{s^n e^{-\kappa s}}{n!}, \quad n = 0, 1, 2, \dots,$$

where κ is the decline rate of the effects of past memories and $\kappa > 0$. The weak delay kernel with $n = 0$ is considered and presented as

$$F(s) = \kappa e^{-\kappa s}.$$

Remark 2: Evidently, system (7) can be converted to system (2) when $\alpha = 1$ and $\tau_1 = \tau_2$. Therefore, system (2) discussed in [14] is a special situation of system (7) proposed in this paper.

In fact, $f(\cdot)$ is an activation function and we make the following assumption:

(H1) $f(0) = 0$ and $f \in C^3$,

where C^3 is the set of third-order differentiable functions.

For simplicity, we introduce two virtual neurons, as follows

$$\begin{aligned} p_3(t) &= \int_{-\infty}^t F(t-s)p_1(s)ds \\ p_4(t) &= \int_{-\infty}^t F(t-s)p_2(s)ds \end{aligned}$$

Then system (7) turns into

$$\begin{cases} {}_0^C D_t^\alpha p_1(t) = -p_1(t) + v_{11}f(p_3(t)) + v_{12}f(p_2(t-\tau_2)), \\ {}_0^C D_t^\alpha p_2(t) = -p_2(t) + v_{21}f(p_1(t-\tau_1)) + v_{22}f(p_4(t)), \\ p_3'(t) = -\kappa p_3(t) + \kappa p_1(t), \\ p_4'(t) = -\kappa p_4(t) + \kappa p_2(t). \end{cases} \quad (8)$$

Remark 3: By introducing two virtual neurons p_3 and p_4 , the new four-neuron system (8) is formed only involving the discrete delays. Fig. 1 reveals the architecture of (8).

Remark 4: System (8) is an incommensurate-order neural model, which shares the property of containing two integer-order differential equations and two fractional differential equations. Compared with the integral-order neural models in [14], [15] and the commensurate-order neural networks in [36], [38], the presence of incommensurate order usually makes the analytical work more challenging.

From (H1), we obtain that the origin $O(0, 0)$ is an equilibrium point of system (7). Thus, $O(0, 0, 0, 0)$ is the equilibrium point of system (8). We assume that the origin is an isolated equilibrium point for simplicity. This paper only takes into account the local dynamics near the origin.

Linearizing (8) at the origin $O(0, 0, 0, 0)$ is

$$\begin{cases} {}^C_0D_t^\alpha p_1(t) = -p_1(t) + \beta_{11}p_3(t) + \beta_{12}p_2(t - \tau_2), \\ {}^C_0D_t^\alpha p_2(t) = -p_2(t) + \beta_{21}p_1(t - \tau_1) + \beta_{22}p_4(t), \\ p_3'(t) = -\kappa p_3(t) + \kappa p_1(t), \\ p_4'(t) = -\kappa p_4(t) + \kappa p_2(t), \end{cases} \quad (9)$$

where $\beta_{ij} = v_{ij}f'(0)$, $i = 1, 2, j = 1, 2$.

The characteristic equation of (9) is

$$\det \begin{pmatrix} s^\alpha + 1 & -\beta_{12}e^{-s\tau_2} & -\beta_{11} & 0 \\ -\beta_{21}e^{-s\tau_1} & s^\alpha + 1 & 0 & -\beta_{22} \\ -\kappa & 0 & s + \kappa & 0 \\ 0 & -\kappa & 0 & s + \kappa \end{pmatrix} = 0, \quad (10)$$

namely,

$$[(s^\alpha + 1)(s + \kappa) - \beta_{11}\kappa][(s^\alpha + 1)(s + \kappa) - \beta_{22}\kappa] - \beta_{12}\beta_{21}(s + \kappa)^2 e^{-s\tau} = 0, \quad (11)$$

where $\tau = \tau_1 + \tau_2$.

IV. LOCAL STABILITY AND HOPF BIFURCATION

Assume that the rational number $\alpha = \frac{m}{n}$, where $m, n \in Z^+$, $(m, n) = 1$, and Z^+ represents the set of positive integers.

Let $\lambda = s^{\frac{1}{n}}$. According to Lemma 1, the following theorem can be summarized directly.

Theorem 1: System (8) with $\tau = 0$ is Lyapunov locally asymptotically stable at the origin $O(0, 0, 0, 0)$ if all the roots of

$$\det \begin{pmatrix} \lambda^m + 1 & -\beta_{12} & -\beta_{11} & 0 \\ -\beta_{21} & \lambda^m + 1 & 0 & -\beta_{22} \\ -\kappa & 0 & \lambda^n + \kappa & 0 \\ 0 & -\kappa & 0 & \lambda^n + \kappa \end{pmatrix} = 0. \quad (12)$$

satisfies $|\arg(\lambda)| > \frac{\pi}{2n}$.

Proof: Clearly, (12) is the characteristic equation of system (8) when $\tau = 0$. The conclusion follows directly from Lemma 1. ■

Let $s = i\omega$ ($\omega > 0$) be a root of (11). Separating the real and imaginary parts leads to

$$\begin{cases} \cos \omega\tau = \frac{1}{\beta_{12}\beta_{21}\eta^2}(A_{11}\eta^2 + A_{12}\eta + A_{13}), \\ \sin \omega\tau = -\frac{1}{\beta_{12}\beta_{21}\eta^2}(A_{21}\eta^2 + A_{22}\eta + A_{23}), \end{cases} \quad (13)$$

where

$$\begin{aligned} \eta &= \kappa^2 + \omega^2, \\ A_{11} &= \omega^{2\alpha} \cos \alpha\pi + 2\omega^\alpha \cos \frac{\alpha\pi}{2} + 1, \\ A_{12} &= -\kappa(\beta_{11} + \beta_{22})(\omega^{\alpha+1} \sin \frac{\alpha\pi}{2} + \kappa\omega^\alpha \cos \frac{\alpha\pi}{2} + \kappa), \\ A_{13} &= \beta_{11}\beta_{22}\kappa^2(\kappa^2 - \omega^2), \\ A_{21} &= \omega^{2\alpha} \sin \alpha\pi + 2\omega^\alpha \sin \frac{\alpha\pi}{2}, \\ A_{22} &= -\kappa(\beta_{11} + \beta_{22})(-\omega^{\alpha+1} \cos \frac{\alpha\pi}{2} + \kappa\omega^\alpha \sin \frac{\alpha\pi}{2} - \omega), \\ A_{23} &= -2\beta_{11}\beta_{22}\kappa^3\omega. \end{aligned}$$

Applying $\sin^2 \omega\tau + \cos^2 \omega\tau = 1$ to (13), it results in

$$G(\omega) = 0, \quad (14)$$

where

$$\begin{aligned} G(\omega) &= A_{11}^2\eta^4 + A_{12}^2\eta^2 + A_{13}^2 + 2A_{11}A_{12}\eta^3 + 2A_{11}A_{13}\eta^2 \\ &\quad + 2A_{12}A_{13}\eta + A_{21}^2\eta^4 + A_{22}^2\eta^2 + A_{23}^2 + 2A_{21}A_{22}\eta^3 \\ &\quad + 2A_{21}A_{23}\eta^2 + 2A_{22}A_{23}\eta - \beta_{12}^2\beta_{21}^2\eta^4. \end{aligned}$$

Now, we make the following assumption:

$$(H2) \quad (\beta_{11} + \beta_{22} - \beta_{11}\beta_{22} - 1)^2 < \beta_{12}^2\beta_{21}^2.$$

Lemma 2: If (H2) holds, (11) has at least a pair of purely imaginary roots.

Proof: It is easy to see that the highest order term of $G(\omega)$ is $\omega^{4\alpha+8}$, and

$$G(0) = [(\beta_{11} + \beta_{22} - \beta_{11}\beta_{22} - 1)^2 - \beta_{12}^2\beta_{21}^2]\kappa^8.$$

If (H2) holds, then $G(0) < 0$. Together with $\lim_{\omega \rightarrow +\infty} G(\omega) = +\infty$, (14) has at least one positive root. Namely, (11) has at least a pair of purely imaginary roots. ■

Since the powers of (14) are rational numbers, it is hard to solve the solutions directly. Thus, we may convert (14) into a polynomial equation by a coordinate transformation. Let $z = \omega^{\frac{1}{n}}$, (14) turns into

$$H(z) = 0, \quad (15)$$

where

$$\begin{aligned} H(z) &= B_{11}^2\vartheta^4 + B_{12}^2\vartheta^2 + B_{13}^2 + 2B_{11}B_{12}\vartheta^3 + 2B_{11}B_{13}\vartheta^2 \\ &\quad + 2B_{12}B_{13}\vartheta + B_{21}^2\vartheta^4 + B_{22}^2\vartheta^2 + B_{23}^2 + 2B_{21}B_{22}\vartheta^3 \\ &\quad + 2B_{21}B_{23}\vartheta^2 + 2B_{22}B_{23}\vartheta - \beta_{12}^2\beta_{21}^2\vartheta^4, \end{aligned}$$

in which

$$\begin{aligned} \vartheta &= \kappa^2 + z^{2n}, \\ B_{11} &= z^{2m} \cos \frac{m\pi}{n} + 2z^m \cos \frac{m\pi}{2n} + 1, \\ B_{12} &= \kappa(\beta_{11} + \beta_{22})(z^{m+n} \sin \frac{m\pi}{2n} + \kappa z^m \cos \frac{m\pi}{2n} + \kappa), \\ B_{13} &= \beta_{11}\beta_{22}\kappa^2(\kappa^2 - z^{2n}), \\ B_{21} &= z^{2m} \sin \frac{m\pi}{n} + 2z^m \sin \frac{m\pi}{2n}, \end{aligned}$$

$$B_{22} = -\kappa(\beta_{11} + \beta_{22})(-z^{m+n} \cos \frac{m\pi}{2n} + \kappa z^m \sin \frac{m\pi}{2n} - z^n),$$

$$B_{23} = -2\beta_{11}\beta_{22}\kappa^3 z^n.$$

$$\Phi = \frac{\Phi}{\beta_{12}\beta_{21}\omega_0^2(\kappa^2 + \omega_0^2)^2}.$$

Remark 5: Compared with (14), (15) is more tractable to determine the distribution of roots. As long as (15) owns the positive root z_k , (14) has the positive root $\omega_k = z_k^n$.

Assuming that (15) has l positive roots $z_k, k = 1, 2, \dots, l$. From (13), we obtain

$$\tau_k^{(j)} = \frac{1}{\omega_k} \left\{ \arccos \frac{A_{11}\eta^2 + A_{12}\eta + A_{13}}{\beta_{12}\beta_{21}\eta^2} + 2j\pi \right\},$$

$$k = 1, 2, \dots, l; \quad j = 0, 1, 2, \dots \quad (16)$$

Define the bifurcation point

$$\tau_0 = \tau_{k0}^{(0)} = \min_{k \in \{1, 2, \dots, l\}} \{\tau_k^{(0)}\}, \quad (17)$$

and $\omega_0 = \omega_{k0}$.

Now, we make the following assumption:

$$(H3) \quad \Phi > 0,$$

where

$$\Phi = r_{11}r_{21}r_{31} - r_{12}r_{22}r_{31} - r_{11}r_{22}r_{32} - r_{12}r_{21}r_{32}$$

$$- 2\beta_{12}\beta_{21}\omega_0^2(\kappa^2 + \omega_0^2),$$

$$r_{11} = (\alpha + 1)\omega_0^\alpha \cos \frac{\alpha\pi}{2} + \alpha\kappa\omega_0^{\alpha-1} \sin \frac{\alpha\pi}{2} + 1,$$

$$r_{12} = (\alpha + 1)\omega_0^\alpha \sin \frac{\alpha\pi}{2} - \alpha\kappa\omega_0^{\alpha-1} \cos \frac{\alpha\pi}{2},$$

$$r_{21} = -2\omega_0^{\alpha+1} \sin \frac{\alpha\pi}{2} + 2\kappa\omega_0^\alpha \cos \frac{\alpha\pi}{2} + (2 - \beta_{11} - \beta_{22})\kappa,$$

$$r_{22} = 2\omega_0^{\alpha+1} \cos \frac{\alpha\pi}{2} + 2\kappa^2\omega_0^\alpha \sin \frac{\alpha\pi}{2} + 2\omega_0,$$

$$r_{31} = 2\kappa\omega_0^2 \cos \omega_0\tau_0 - (\kappa^2 - \omega_0^2)\omega \sin \omega_0\tau_0,$$

$$r_{32} = 2\kappa\omega_0^2 \sin \omega_0\tau_0 + (\kappa^2 - \omega_0^2)\omega \cos \omega_0\tau_0.$$

Lemma 3: Let $s(\tau) = \zeta(\tau) + i\omega(\tau)$ is the root of (11) at $\tau = \tau_0$ satisfying $\zeta(\tau_0) = 0, \omega(\tau_0) = \omega_0$. If (H3) holds, then

$$\text{Re} \left[\frac{ds}{d\tau} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}} > 0.$$

Proof: Differentiating both sides of (11) with respect to τ , we derive that

$$C_1 C_2 \frac{ds}{d\tau} + C_3 [2 \frac{ds}{d\tau} - (s + \kappa)(\tau \frac{ds}{d\tau} + s)] = 0,$$

where

$$C_1 = \alpha s^{\alpha-1}(s + \kappa) + s^\alpha + 1,$$

$$C_2 = 2(s^\alpha + 1)(s + \kappa) - (\beta_{11} + \beta_{22})\kappa,$$

$$C_3 = -\beta_{12}\beta_{21}(s + \kappa)e^{-s\tau}.$$

Then,

$$\left(\frac{ds}{d\tau} \right)_{\substack{\omega=\omega_0 \\ \tau=\tau_0}}^{-1} = \frac{2}{(s + \kappa)s} + \frac{C_1 C_2}{(s + \kappa)s C_3} - \frac{\tau}{s}. \quad (18)$$

Afterwards, we have

$$\text{Re} \left[\frac{ds}{d\tau} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}}^{-1} = \text{Re} \left[\frac{2}{(s + \kappa)s} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}} + \text{Re} \left[\frac{C_1 C_2}{(s + \kappa)s C_3} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}}$$

Thus,

$$\text{sign} \left\{ \text{Re} \left[\frac{ds}{d\tau} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}} \right\} = \text{sign} \left\{ \text{Re} \left[\frac{ds}{d\tau} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}}^{-1} \right\}$$

$$= \text{sign} \left\{ \frac{\Phi}{\beta_{12}\beta_{21}\omega_0^2(\kappa^2 + \omega_0^2)^2} \right\}.$$

In view of (H3), it can be achieved that $\text{Re} \left[\frac{ds}{d\tau} \right]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}} > 0$. ■

Theorem 2: Assuming that (H2) and (H3) hold, and all the roots of (12) satisfies $|\arg(\lambda)| > \frac{\pi}{2n}$. For system (8), the following results hold.

- 1) The equilibrium point $(0, 0, 0, 0)$ of system (8) is locally asymptotically stable for $\tau \in [0, \tau_0)$, and unstable when $\tau > \tau_0$.
- 2) System (8) undergoes a Hopf bifurcation at the equilibrium point $(0, 0, 0, 0)$ when $\tau = \tau_0$.

Proof:

- 1) If all the roots of (12) satisfies $|\arg(\lambda)| > \frac{\pi}{2n}$, then all the roots of (11) with $\tau = 0$ have negative real parts. For another thing, from Lemma 2, we know that if (H2) holds, then (11) has purely imaginary roots $\pm i\omega_k$ when $\tau = \tau_k^{(j)}$. Notice that τ_0 defined in (17) is set as the minimum value for $\tau > 0$ such that (11) has a couple of purely imaginary roots $\pm i\omega_k$ that appear on the imaginary axis. Therefore, all roots of (11) have negative real parts for $\tau \in [0, \tau_0)$, which reveals that system (8) is locally asymptotically stable for $\tau \in [0, \tau_0)$. According to Lemma 3, if (H3) holds, then we have $\text{Re} [ds/d\tau]_{\substack{\omega=\omega_0 \\ \tau=\tau_0}} > 0$. Hence, (11) has at least a pair of roots with positive real parts when $\tau > \tau_0$, which implies that system (8) is unstable when $\tau > \tau_0$.
- 2) The conclusion in Lemma 3 indicates that the transversality condition for the Hopf bifurcation is satisfied under the assumption, so system (8) undergoes a Hopf bifurcation at $\tau = \tau_0$. ■

Remark 6: When $\alpha = 1$ and $\tau_1 = \tau_2$, system (7) is regarded as the integer-order system (2) in [14], which still satisfies Theorems 1 and 2.

V. NUMERICAL SIMULATIONS

In this section, the theoretical results acquired in Sec. IV will be supported by several numerical simulations to certify the accurateness and feasibility.

Example 1: Consider the two-neuron system (7) with $v_{11} = -0.5, v_{12} = -1.8, v_{21} = 1.3, v_{22} = 1.7, \kappa = 1$, and $f(\cdot) = \tanh(\cdot)$, which are adopted in [14].

We set the rational number $\alpha = 0.75$. By (16), we can obtain $\omega_0 = 0.8834$ and $\tau_0 = 2.1976$. Based on Theorem 2, if the discrete delay τ is between 0 and τ_0 , then the equilibrium point $O(0, 0)$ of system (7) is locally asymptotically

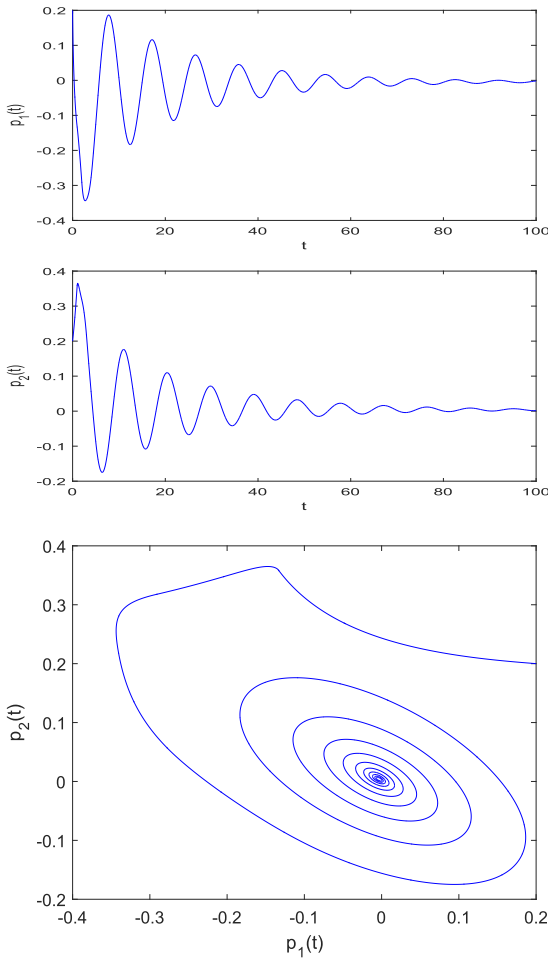


FIGURE 2. Waveform plots and phase portraits of system (7) with $\tau = \tau_1 + \tau_2 = 2 < \tau_0 = 2.1976$.

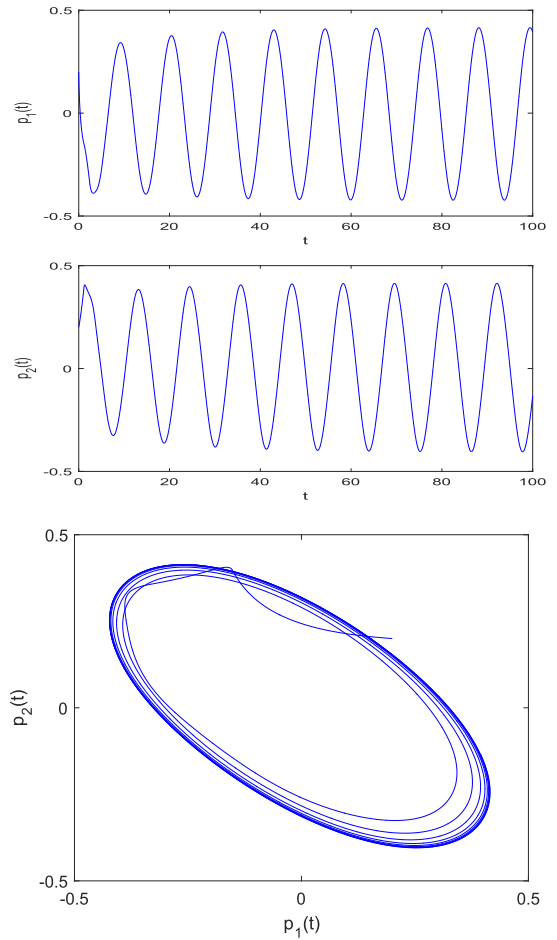


FIGURE 4. Waveform plots and phase portraits of system (7) with $\tau = \tau_1 + \tau_2 = 2.4 > \tau_0 = 2.1976$.

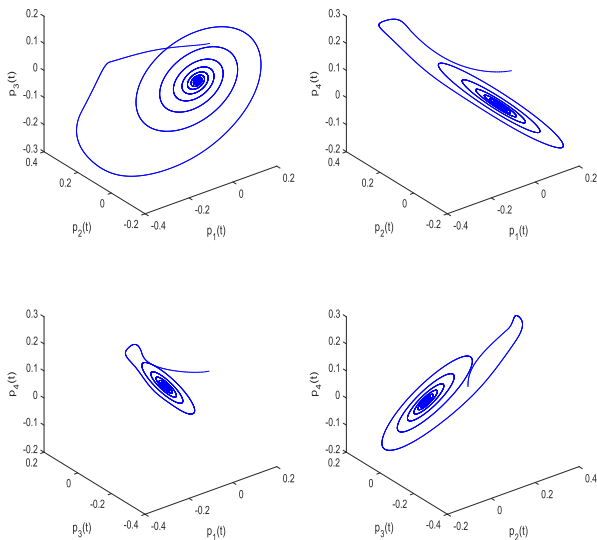


FIGURE 3. Phase portraits of system (8) with $\tau = \tau_1 + \tau_2 = 2 < \tau_0 = 2.1976$.

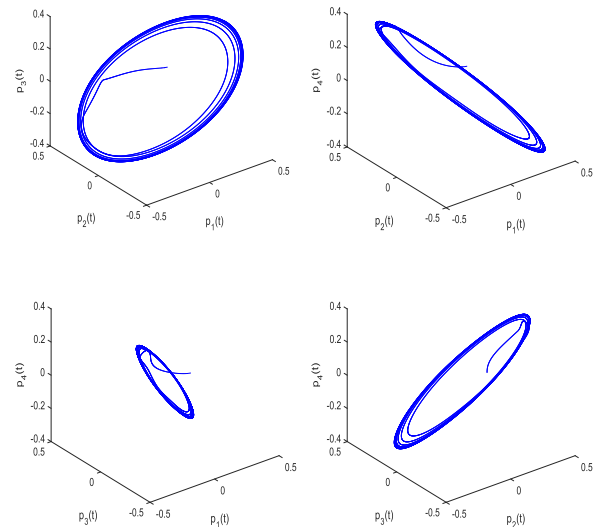


FIGURE 5. Phase portraits of system (8) with $\tau = \tau_1 + \tau_2 = 2.4 > \tau_0 = 2.1976$.

stable. Figs. 2 and 3 show the stable origin of systems (7) and (8) in which $\tau = \tau_1 + \tau_2 = 2 < \tau_0$, respectively. However, system (7) turns out a Hopf bifurcation when τ is

above the critical value τ_0 . Figs. 4 and 5 reveal the periodic oscillation when $\tau = \tau_1 + \tau_2 = 2.4 > \tau_0$.

Next, we will compare with the previous work so as to show our novel method. Setting the rational number $\alpha = 1$,

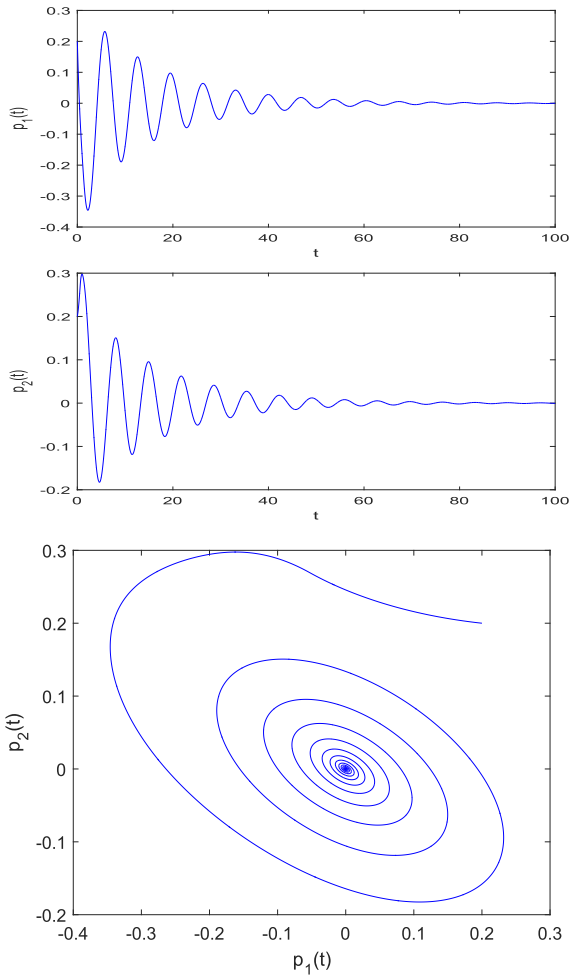


FIGURE 6. Waveform plots and phase portraits of system (2) with $\tau = \tau_1 + \tau_2 = 1.2 < \tau_0 = 1.404$.

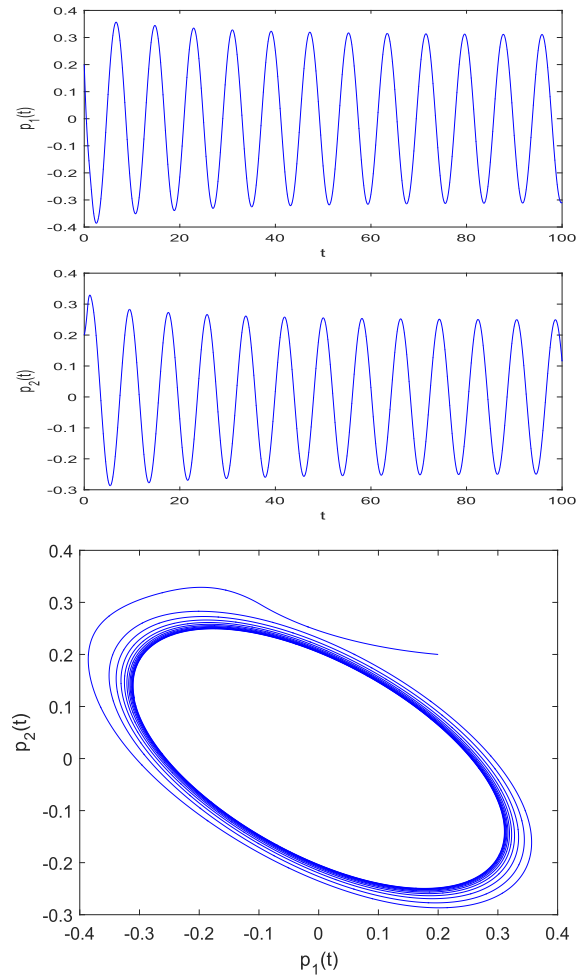


FIGURE 7. Waveform plots and phase portraits of system (2) with $\tau = \tau_1 + \tau_2 = 1.5 > \tau_0 = 1.404$.

system (7) is considered to be the integer-order system which is identical to the system in [14]. By (16), we obtain $\omega_0 = 0.8152$ and $\tau_0 = 1.404$, which is in accordance with the results in [14]. By means of Theorem 2 and Remark 6, the equilibrium point $O(0, 0)$ is locally asymptotically stable for $\tau = \tau_1 + \tau_2 = 1.2 < \tau_0$, which is plotted in Fig. 6. While the equilibrium point $O(0, 0)$ loses stability, Hopf bifurcation occurs when $\tau = \tau_1 + \tau_2 = 1.5 > \tau_0$, which is displayed in Fig. 7.

In what follows, we discuss the relationships between the parameters α , κ and τ_0 for system (7).

Case 1: Fix $\kappa = 1$, then plumb the effect of the order α on the bifurcation point τ_0 for system (7). The order α exhibits a linear relationship with the bifurcation point τ_0 , as shown in Table 1 and Fig. 8. The value of τ_0 rises with the decrease of the order α .

Case 2: Set $\alpha = 0.5, 0.75$ and 1 respectively, then observe the influence of κ on the bifurcation point τ_0 . From Table 2 and Fig. 9, we can get the following information. Firstly, for a fixed value of α , the parameter κ is inversely proportional to bifurcation point τ_0 . Even more intuitively, the value of τ_0

TABLE 1. The influence of α on the value of τ_0 for system (7) with $\kappa = 1$.

Fractional-order parameter α	Bifurcation point τ_0
0.1	4.3340
0.2	4.0316
0.3	3.6911
0.4	3.3406
0.5	2.9967
0.6	2.6670
0.7	2.3516
0.8	2.0444
0.9	1.7340
1	1.4040

decreases as the parameter κ increases. Secondly, for a fixed value of κ , the smaller the order α is, the greater the critical value of τ_0 is.

Remark 7: Fig. 8 clearly shows that the smaller the order is, the larger bifurcation point gets. It is indicated that by contrast with the integer-order model in [14], the presence of fractional order effectively delays the occurrence of Hopf bifurcation for system (7), so the stability region is broadened.

Remark 8: From Fig. 8, it can be also find that the smaller the order is, the greater the bifurcation point is

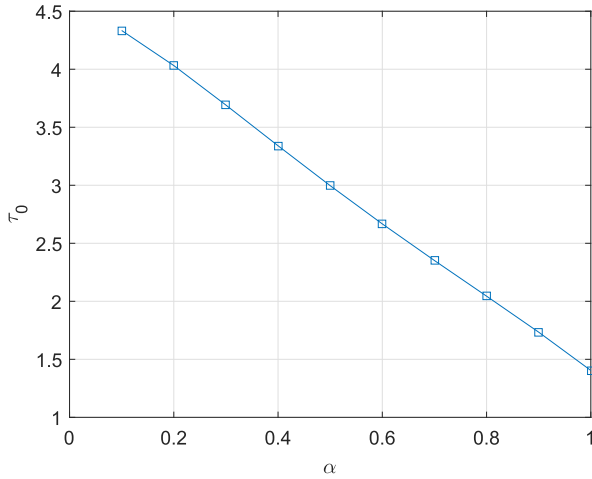


FIGURE 8. The influence of α on the value of τ_0 for system (7) with $\kappa = 1$.

TABLE 2. The influence of κ on the value of τ_0 for system (7) with $\alpha = 0.5, 0.75$ and 1 .

Parameter κ	$\alpha = 0.5, \tau_0$	$\alpha = 0.75, \tau_0$	$\alpha = 1, \tau_0$
1	2.9968	2.1976	1.4040
2	2.0427	1.5255	1.0153
3	1.6830	1.2599	0.8098
4	1.4934	1.1214	0.6973
5	1.376	1.0372	0.6278
6	1.2969	0.9808	0.5809
7	1.2392	0.9405	0.5473
8	1.1958	0.9104	0.5221
9	1.1618	0.8870	0.5026
10	1.1345	0.8684	0.4869

and the larger the stable domain is. This fact is also true not only for fractional-order neural networks [48], [49], but also for the fractional-order genetic regulatory network [29], the fractional-order predator-prey model [31] and the fractional-order SIS epidemic model [45].

Remark 9: It can be seen that the decline rate κ is a parameter in the delay kernel function $F(s)$, which may represent the distributed delay to some extent. Hence, Fig. 9 presents

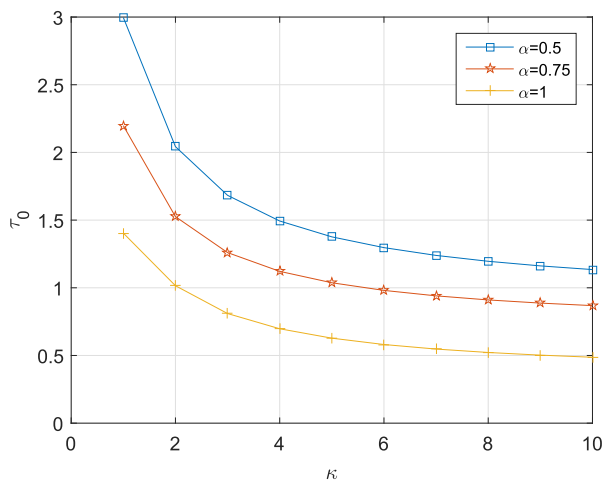


FIGURE 9. The influence of κ on the value of τ_0 for system (7) with $\alpha = 0.5, 0.75$ and 1 .

the influence of distributed delays on the dynamical bifurcation in the fractional-order neural network (7). It is worth noting that the effect of distributed delays on bifurcation of fractional-order systems has not been reported.

Example 2: Consider the two-neuron system (7) with $v_{11} = -2, v_{12} = -1.8, v_{21} = 10, v_{22} = 3.5, \kappa = 1$, and $f(\cdot) = \sin(\cdot)$. When $\alpha = 0.75$, the equivalent system (8) shows chaotic attractors (see Fig. 10).

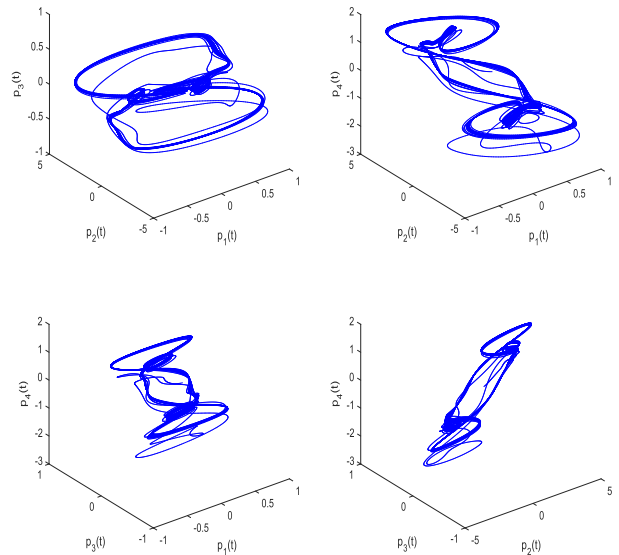


FIGURE 10. Chaotic attractor for $\alpha = 0.75$.

Remark 10: The activation function $f(\cdot) = \tanh(\cdot)$ used in Example 1 is not periodic, while the function $f(\cdot) = \sin(\cdot)$ chosen in Example 2 has the periodicity. Therefore, the selection of activation function is enormously influential for the occurrence of chaos in system (8).

VI. CONCLUSION

This paper studies the dynamic behaviors of a fractional-order neural network with discrete and distributed delays. Firstly, by introducing two virtual neurons, we construct an equivalent four-neuron system that contains two fractional differential equations and two integer differential equations. Next, by adopting the sum of discrete delay as the bifurcation parameter, the sufficient conditions for the stability of the original system have been obtained. Furthermore, the Hopf bifurcation has been ascertained successfully by exploring the derived characteristic equation. Finally, through the numerical simulations, the correctness of the results has been verified and the influence of the order and parameter on the onset of bifurcation has been given. The simulation results show that the neuron system turns out a Hopf bifurcation when the sum of discrete delays reaches a certain critical value. In addition, the order exhibits a linear relationship with the bifurcation point, and diminishing order can postpone the onset of bifurcation. It indicates that the fractional-order system has a larger stability region than the integer-order one.

Chaos plays an important part in the study of system dynamics. In the future, we will devote to probing into chaos in fractional-order neural networks.

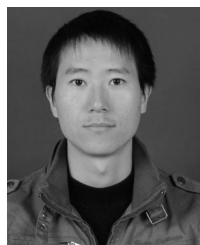
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