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Underdetermined Direction of Arrival Estimation of Non-Circular Signals via Matrix Completion in Nested Array

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ABSTRACT Direction-of-arrival (DOA) estimation using nested linear array ignores the information from repeated sensors, thus involves the problem of losing accuracy. Moreover, non-circular feature of signals is rarely considered and it results in discontinuity of the virtual array, namely, holes appear. This paper first provide an improved data averaging method to increase the accuracy of DOA estimation by fusing data from covariance and elliptic covariance. And then an algorithm based on matrix completion theory in nested array that greatly extends the degrees of freedom (DOF) and is able to find more sources than number of physical sensors is presented. The algorithm reconstructs the covariance matrix of the virtual linear array, which exactly has the same shift invariance as the uniform linear array but a higher aperture. Considering about the missing elements of the virtual array covariance matrix, we apply matrix completion theory to solve the problem. Finally, true DOAs of multiple signals can be obtained through subspace algorithms. Numerical results demonstrate that the proposed algorithms can obtain high accuracy while underdetermined DOA estimation is realized.

INDEX TERMS Direction-of-arrival, nested array, matrix completion theory, data averaging.

I. INTRODUCTION

Direction-of-arrival estimation is a fundamental problem in array processing [1] and is significant in many applications such as MIMO radar [2], [3], mobile communication [4], indoor positioning [5] and underwater acoustics [6]. There are so many subspace-based methods that resolve real DOAs, including multiple signal classification (MUSIC) [7], estimating signal parameters via rotational invariance techniques (ESPRIT) [8] and so on. These algorithms usually need to assume that the number of impinging sources D is less than the number of actual sensors M, which implies that the low rank signal subspace component is necessary to DOA estimation. Recently, several researchers find that this restriction appears to be a result of sub optimal array geometries such as uniform linear array (ULA), viz., uniformly spaced antenna array [9]. Therefore, a new array geometry called nested array [10]–[12] based on non uniform sampling was proposed. Nested array is composed of two or more uniform array and is inherently able to detect $O(M^2)$ impinging sources with only M sensors. [10] proposed a suitable extension of MUSIC algorithm called spatial smoothing MUSIC (SS-MUSIC) to conduct DOA estimation since the data covariance matrix no longer spans the signal subspace when $D > M$. While SS-MUSIC solves the problem of losing low rank component, it neglects communication signals with amplitude shift keying (ASK), binary phase shift keying (BPSK) and unbalance quadrature PSK (UQPSK) modulation which possess non-circular features.

In other words, non-circular signals have a property of ellipse covariance matrix being non-zero, which can double dimensions of array output covariance matrix by combining covariance matrix and ellipse covariance matrix. Hence it can expand the array aperture, increase the degrees of freedom and the accuracy, do which circular signals is not able to. Since most existing algorithms [13], [14] use second order statistics (covariance) to resolve DOAs, we can utilize the non-circular characteristics [15], [16] to obtain much more effective performance. As a result, it always makes sense to concentrate on the non-circular signals and study the impact it brings about in nested array. Consequently, we revise the generation of difference co-array and find that in

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former researches [10], [17]–[19], the repeated sensors were omited. Thus the information from those repeated sensors was not used properly. Most importantly, the co-array based on non-circular signals is no longer consecutive, namely, there are zeros in virtual array covariance matrix.

In order to solve the problem presented above, this paper first prepares a preprocess of data averaging to utilize the information from every single sensor, as is mentioned in [20] which considers circular situation. However, it may not effectively improve the performance using only a single statistic (covariance). Here in this paper we conduct a suitable extension and average the corresponding terms of both covariance data and elliptic covariance data in non-circular circumstance. The DOA estimation method based on date averaging in this paper is called non-circular spatial smoothing MUSIC, short for NC-SS-MUSIC. With the consecutive sensors in virtual array, one can detect the same number of sources with higher accuracy using the preprocessing method.

Moreover, compressive sensing (CS) methods, including sparse bayesian learning and orthogonal matching pursuit (OMP), are becoming a prosperous area of DOA estimation. Recently, several algorithms [21], [22] based on OMP have been proposed and solve the problem of distinguishing between two adjacent sources, improving the performance especially in low signal-to-noise (SNR) ratio. Besides, an algorithm based on compressive sensing [23] was proposed for non-circular signals. However, it can use only consecutive and non-consecutive virtual sensors generated by original sparse array but do nothing to the missing ones. To this end, the paper also proposes a method based on matrix completion theory [24]–[26] which constructs a larger aperture and a higher DOF with fewer sensors while compared with non-sparse arrays.

Considering that JIAN-FENG CAI has developed an algorithm [27] of singular value thresholding, we can estimate zeros in the extended covariance matrix with soft thresholding methods. We first calculate a covariance matrix after averaging the received data from repeated sensors. Then, when impinging sources are non-circular, holes appear and the proposed non-circular singular value thresholding MUSIC (NC-SVT-MUSIC) can be applied to resolve DOAs of high resolution. Compared with NC-SS-MUSIC, the proposed algorithm obtains a more effective performance. Morevoer, it has significantly increased the accuracy and the number of sources available with the non-circular feature.

The paper is organized as follows. In Section 2, a brief introduction to signal model of nested array is presented. Then, difference co-array generation process is studied in Section 3. Section 4 describes the preprocessing method of data fusion and then the proposed algorithm using SVT is presented in detail. The steps are listed subsequently and a direct matrix reconstruction method is also included. In Section 5, computational complexity and freedom degree of proposed algorithm is analyzed and compared with some former methods. Simulation results that demonstrate the efficiency and accuracy are presented in Section 6. Finally, Section 7 makes a conclusion to this paper.

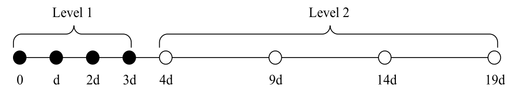


FIGURE 1. A 2 level nested array with 4 sensors in each level.

Throughout the paper, $[\bullet]^*$, $[\bullet]^T$, $[\bullet]^H$ and $|\bullet|$ denote the conjunction, transpose, conjugate transpose and number of elements in $(\bullet)$ respectively.

II. SIGNAL MODEL

Considering the two level nested array composed of M elements, M_1 in level 1 and M_2 in level 2, Figure 1 shows the array geometry with $M_1 = 4$ and $M_2 = 4$. Here $d = \lambda/2$ and λ denotes wavelength of the impinging sources. Suppose that the antenna array receives D narrowband sources from directions $\theta_1, \theta_2, \ldots, \theta_D$. Note that all impinging sources are uncorrelated with each other. The response of the antenna array in the tth time interval can be expressed by

$$\mathbf{X}(t) = \mathbf{A}(\theta)\mathbf{S}(t) + \mathbf{N}(t), \tag{1}$$

where $\mathbf{A}(\theta) \in \mathbb{C}^{M\times D}$ is the array manifold denoted by (2) and (3), $\mathbf{S}(t) \in \mathbb{C}^{D\times 1}$ is the vector form of zero-mean wide sense stationary (WSS) source signals and $\mathbf{N}(t)$ is the additive white Gaussian noise vectors with identical power σ^2.

$$\mathbf{A} = \begin{bmatrix}\mathbf{a}(\theta_1) & \mathbf{a}(\theta_2) & \cdots & \mathbf{a}(\theta_D)\end{bmatrix}, \tag{2}$$

$$\mathbf{a}(\theta_m) = \big[\,1 \quad e^{j2\pi(d/\lambda)sin\theta_m} \quad \cdots \quad e^{j2\pi(M_1+(M_2-1)(M_1+1))(d/\lambda)sin\theta_m}\,\big]^T. \tag{3}$$

To estimate θ with given time samples of source signals, a majority of algorithms utilize second-order statistics of array output data, mainly covariance matrix of the impinging signals.

$$\mathbf{R}_x = E[\mathbf{X}(t)\mathbf{X}^H(t)] = \mathbf{A}(\theta)\mathbf{R}_s\mathbf{A}^H(\theta) + \sigma^2\mathbf{I}, \tag{4}$$

where $\mathbf{R}_s = E[\mathbf{S}(t)\mathbf{S}^H(t)] = diag(\,\sigma_1^2\ \sigma_2^2\ \cdots\ \sigma_D^2\,)$ is the source signal covariance matrix and $\mathbf{I}$ is an unimodular matrix. Besides, the received signal covariance can be defined as

$$\mathbf{R}_{ss} = \mathbf{A}(\theta)\mathbf{R}_s\mathbf{A}^H(\theta). \tag{5}$$

Assuming that the number of source signals is fewer than that of array elements, namely, $D < M$, we call the circumstance as 'overdetermined'. In this regime, signal covariance matrix $\mathbf{R}_{ss} \in \mathbb{C}^{M\times M}$ is low rank(of rank D). To increase the number of detected sources, we were devoted to conduct DOA estimation when $D > M$. In this circumstance called 'underdetermined', the array manifold matrix $\mathbf{A}(\theta)$ is fat and as a result, $\mathbf{R}_{ss}$ is not low rank anymore. Thus classical subspace-based algorithms lose their efficiency of DOA estimation.

As is shown earlier, nested array has higher degrees of freedom, making it possible to estimate more DOAs with fewer physical elements. In terms of circular signals, scholars proposed a great deal of methods for underdetermined DOA estimation using nested array. However, the co-array generated by nested array has holes when it comes to non-circular signals. A complex random variable, x, is recognized as non-circular of the order two, if the elliptic covariance $E[xx]$ is non-zero. For arbitrary signal s, it has form of [28]

$$E[s^2] = \rho e^{j\Phi} E[ss^*] = \rho e^{j\Phi} \sigma_s^2, \tag{6}$$

where ρ and Φ denote the non-circularity rate and phase, respectively. This paper focuses on strictly non-circular signal like BPSK whose non-circularity rate $\rho = 1$. In the next section, we will revise the model based on co-array and concentrate on 'hole' problem.

III. DISCONTINUITY PROBLEM OF CO-ARRAY FOR NON-CIRCULAR SIGNALS

Finding a suitably designed sampling geometry is the first step to perform DOA estimation with high accuracy. It has been shown that a traditional uniform linear array (ULA) is the conventional but poor choice for not dealing with underdetermined estimation problem. Here we utilize structure of a nested array and consider its co-array for non-circular signals, which could identify DOA clearly even when $D > M$.

A. REVISE THE SIGNAL MODEL BASED ON CO-ARRAY

Consider a two-level nested array basically consisted of two ULAs, whose elements are located at $L_1 = \{md, m = 0, 1, \cdots, M_1 - 1\}$ and $L_2 = \{(M_1 + (M_1 + 1)n)d, n = 0, 1, \cdots, M_2 - 1\}$. Also, we have $L = L_1 \cup L_2$. With $\vec{\mathbf{x}}_i$ denoting the position of the ith element, difference co-array [29], [30] set P_v can be defined as

$$P_v = \pm \left\{ \vec{\mathbf{x}}_i - \vec{\mathbf{x}}_j \right\}, \quad \forall i, j = 0, 1, \cdots, M-1. \tag{7}$$

In the set P_v, there are repetitions of those elements thus we define the set P_u which denotes distinct elements of the set P_v. Therefore a virtual uniform linear array (VULA) whose sensors located at positions given by P_u is formed. With up to $M^2/2 + M - 1$ DOFs, a 2 level nested array then is able to resolve underdetermined estimation.

From (7), one can establish a VULA with nested array through a series of methods, in which vectorization approach is the most widely used and convenient one. Now following (4), the vectorized form can be described as

$$\mathbf{z} = vec(\mathbf{R}_x) = (\mathbf{A}^* \circ \mathbf{A})\mathbf{p} + \sigma_n^2 \vec{\mathbf{1}}_n, \tag{8}$$

here $\mathbf{p} = [\sigma_1^2\, \sigma_2^2 \cdots \sigma_D^2]^T$ denotes the power of impinging source signals. $\vec{\mathbf{1}}_n = [\mathbf{w}_1^T\ \mathbf{w}_2^T \cdots \mathbf{w}_D^T]^T$ is a column vector with $\mathbf{w}_i$ being a vector of all zeros except the ith element being 1. Compared with (1), (8) can be denoted as vector form of array output data, though there are lots of repeated elements in it. The equivalent array manifold is given by distinct ones of $\mathbf{A}^* \circ \mathbf{A}$ where $\circ$ denotes the KR product. The locations of sensors in this virtual array are described by the distinct values in (7) above.

According to (7), the VULA has its sensors located from $-M_c d$ to $M_c d$, here $M_c = M^2/4 + M/2 - 1$. The array manifold of such a co-array is then described as

$$\mathbf{A}_v(\Theta) = \begin{bmatrix} \mathbf{a}_v(\theta_1) & \mathbf{a}_v(\theta_2) & \cdots & \mathbf{a}_v(\theta_D) \end{bmatrix}, \tag{9}$$

where $[\mathbf{A}_v(\Theta)]_{m,n} = e^{j2\pi m(d/\lambda) sin\theta_n}$, $-M_c \leq m \leq M_c$, $1 \leq n \leq D$.

Due to transformation from nested array to difference co-array, equivalent signal component composed of the actual sources powers σ_i^2 takes effect like a series of coherent sources, and hence the covariance matrix of VULA is no longer of full rank. [10] proposed a smoothing technique to construct a modified matrix $\mathbf{R}_{smoothed} \in \mathbb{C}^{M_c \times M_c}$. Dividing the co-array into $M_c + 1$ overlapping subarrays, computing those array output covariance matrices and then taking the average of them, we can get

$$\mathbf{R}_{smoothed} = \frac{1}{M_c + 1} \sum_{i=1}^{M_c+1} \mathbf{R}_i, \tag{10}$$

where $\mathbf{R}_i$ denotes covariance matrix of the ith subarray.

Usually, we estimate $\mathbf{R}_x$ with time average instead of statistical average values due to a finite number snapshots, say, T. Thus

$$\hat{\mathbf{R}}_x = \frac{1}{T} \sum_{t=1}^{T} X(t) X^H(t). \tag{11}$$

From (10) and (11), the smoothed covariance matrix can be expressed as

$$\mathbf{R}_{smoothed} = \mathbf{A}_v(\Theta) \hat{\mathbf{R}}_s \mathbf{A}_v^H(\Theta) + \hat{\mathbf{I}}, \tag{12}$$

here $\hat{\mathbf{R}}_s$ is the time average values of source signal, and $\hat{\mathbf{I}}$ is equivalent noise componet, including cross-correlation terms of signal-signal and signal-noise. Noticed that $\mathbf{R}_{sig} = \mathbf{A}_v(\Theta) \hat{\mathbf{R}}_s \mathbf{A}_v^H(\Theta)$ is still a low rank matrix of rank D, just the same as $\mathbf{R}_{ss}$. Thus subspace algorithms could be applied to $\mathbf{R}_{sig}$ as it eliminates the effect of coherence which was discussed earlier.

B. HOLES IN CO-ARRAY FOR NON-CIRCULAR SIGNALS

Co-array transformed from normal nested array is consecutive (i.e., no holes) in the case of circular sources. When non-circular sources impinge on it, the array aperture will be extended and the degree of freedom will also be increased by utilizing the nonzero property of ellipse covariance matrix. Thus the array output signal model will change, and as a result, problem of discontinuity arises.

As is introduced in Section 1, existing angle estimation algorithms mostly use covariance data calculated by (1). Considering that the ellipse covariance matrix of non-circular signals is non-zero, we can extend data dimension by putting array receiving data matrix and its conjugate component together. Consequently, one can 'double' the available sensors, which is denoted by [28]

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{X}^*(t) \end{bmatrix}. \tag{13}$$

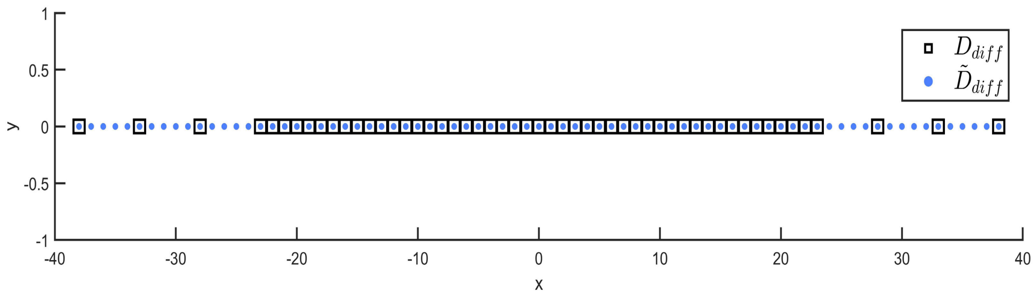


FIGURE 2. Values of sets D_{diff} and $\tilde{D}_{diff}$ when $M_1 = 4$, $M_2 = 4$.

Like (4), the extended covariance matrix can be expressed as [28]

$$\begin{aligned} \mathbf{R}_{nc} &= E\left[\mathbf{Y}(t)\mathbf{Y}^H(t)\right] \\ &= E\left\{\begin{bmatrix}\mathbf{X}(t)\\ \mathbf{X}^*(t)\end{bmatrix}\begin{bmatrix}\mathbf{X}(t) & \mathbf{X}^*(t)\end{bmatrix}\right\} \\ &= E\left\{\begin{bmatrix}\mathbf{X}\mathbf{X}^H & \mathbf{X}\mathbf{X}^T\\ \mathbf{X}^*\mathbf{X}^H & \mathbf{X}^*\mathbf{X}^T\end{bmatrix}\right\}. \end{aligned} \quad (14)$$

From the properties listed earlier, $E[\mathbf{X}\mathbf{X}^T]$ and $E[\mathbf{X}^*\mathbf{X}^H]$ in (14) is non-zero, thus we can increase the available sensors by reconstructing the array received matrix. Similar to (8), vectorize (14) and we can get

$$\bar{\mathbf{z}} = vec(\mathbf{R}_{nc}) = \bar{\mathbf{A}}\mathbf{p} + \sigma_n^2\vec{\mathbf{w}}, \quad (15)$$

where $\bar{\mathbf{A}}$ contains not only all the elements of $\mathbf{A}$, but also sensors located from $-M_{ncs}d$ to $M_{ncs}d$ with $M_{ncs} = M^2/2 + M - 2$, which results from non-circular expansion characteristic. Note that there are several missing values so that $\bar{\mathbf{z}} \in \mathbb{C}^{(M^2/2+3M-3)\times 1}$ and among them the consecutive sensors are located at $-M_{nc}d$ to $M_{nc}d$ and $M_{nc} = M^2/4 + M - 1$. Then, let P_{nc} denotes the sensors location corresponding to $\bar{\mathbf{z}}$.

From (2), (3), (4) and (7), location information included in array manifold leads us to difference co-array definition. Namely, $E[\mathbf{X}\mathbf{X}^H]$ denotes the difference component P_v. Similarly, $E[\mathbf{X}\mathbf{X}^T]$ ($E[\mathbf{X}^*\mathbf{X}^H]$) here denotes the sum component (and its negative) which can be defined as

$$P_s = \pm\left\{\vec{\mathbf{x}}_i + \vec{\mathbf{x}}_j\right\}, \quad \forall i, j = 0, 1, \cdots, M-1, \quad (16)$$

here we also define a set $\bar{P}_u$ that denotes the distinct elements in P_s. Then, the co-array generated by nested array for non-circular signals is exactly described by set P_v and P_s.

To observe more clearly, first we define two set as

$$D_{diff} = \left\{\pm(x_i - x_j) \cup \pm(x_i + x_j) \mid x_i, x_j \in L\right\}, \quad (17)$$

$$\tilde{D}_{diff} = \left\{\pm(x_i - x_j) \cup \pm(x_i + x_j) \mid x_i, x_j \in \tilde{L}\right\}, \quad (18)$$

where $\tilde{L} = \{md, m = 0, 1, \cdots, M_c\}$. In this circumstances, the mth element can be expressed as

$$D_m = \sum_{k=1}^{D} \sigma_k^2 e^{j2\pi(x_m/\lambda)sin_{\theta_k}} = D(x_m) \quad x_m \in L, \quad (19)$$

$$\tilde{D}_m = \sum_{k=1}^{D} \sigma_k^2 e^{j2\pi(x_m/\lambda)sin_{\theta_k}} = \tilde{D}(x_m) \quad x_m \in \tilde{L}. \quad (20)$$

For $M_1 = 4$ and $M_2 = 4$, Figure 2 shows the values in D_{diff} and $\tilde{D}_{diff}$. It indicates that virtual sensors generated in D_{diff} are all included in set $\tilde{D}_{diff}$ while some sensors of $\tilde{D}_{diff}$ are missing in D_{diff}. That is, co-array of nested linear array for non-circular signals is inconsecutive—in other words, holes appear.

Examine Figure 2 carefully, we can either estimate DOAs with $2M_{nc} + 1$ consecutive sensors, or with the whole virtual array possessing sensors in $\tilde{D}(x_m)$, though some of them are missing. Because of shortage in virtual array, the former method has a lower computational complexity than the latter. However, the latter can get high degrees of freedom and array aperture, detecting more sources at the same time. Particularly, the latter estimates $\tilde{D}(x_m)$ with the values of $D(x_m)$ and the missing ones set to zeros tentatively.

IV. PROPOSED ALGORITHMS

It has been found that co-array generated by nested linear array for non-circular signals has holes, which is different from that under circular signal conditions. Considering about it, we first extend a preprocessing method based on data averaging and then propose an approach using matrix completion.

A. CONVENIENT APPROACH BASED ON RECEIVED DATA AVERAGING USING CONSECUTIVE SENSORS

SS-MUSIC, mentioned in [10], works well when covariance matrix is estimated accurately, namely, for large L. However, snapshots number T is always finite in practice, resulting in dramatic deterioration of estimation performance. In this section, we provide a improved DOA estimation approach based on received data averaging to get a more accurate result. From here on, we shall concentrate on the consecutive virtual sensors in D_{diff}.

As is mentioned in [10], there are numerous repeated sensors after vectorize the received covariance. However, scholars are used to choosing data received by one sensor to replace those of all the other repeated virtual sensors. Thus information from those repeated sensors is out of effective utilization undoubtedly. For non-circular signals, both covariance and elliptic covariance can generate plenty of repeated virtual

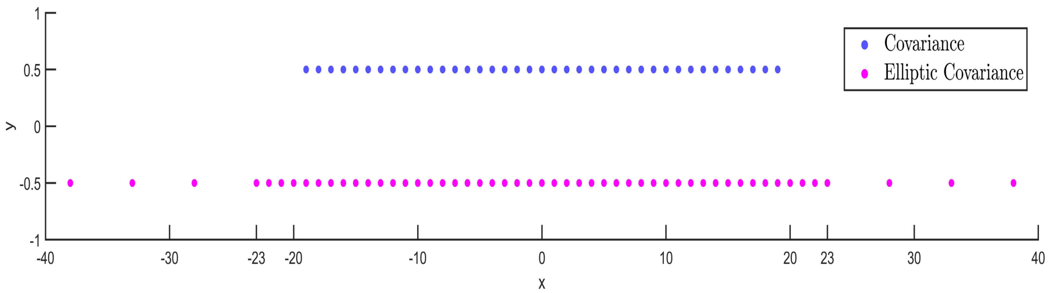


FIGURE 3. Virtual sensors generation based on covariance and elliptic covariance when $M_1 = 4$, $M_2 = 4$.

sensors and most importantly, these two kinds of covariance may refer to sensors at the same position, enabling us to process the data from different statistics so that the accuracy could be increased.

To make full use of all information outputed from antenna array, we modify a preprocessing method mentioned in [20] by fusing data of covariance and elliptic covariance. Consider the consecutive elements under non-circular signal condition, there are 47 consecutive elements in corresponding virtual array, from $-23d$ to $23d$, which is shown in Figure 3. It illustrates that covariance and elliptic covariance generate the same virtual sensors so that it does make sense to average the data from repeated sensors of the two second order statistics.

From Figure 3, we can find that P_v and P_s, which represent the sensor location corresponding to covariance and elliptic covariance, have the same sensors located at $-19d$ to $19d$. The improved averaging method averages the corresponding data from P_v and P_s respectively. As for those only belong to P_s, they are averaged in the range of P_s. As is shown in (15), an inconsecutive data vector is established. Suppose that the consecutive components of $\bar{P}_u$ are described as $\bar{P}'_u$, and the corresponding data vector can be defined as

$$\bar{\mathbf{z}}_1 = \bar{\mathbf{A}}_1\mathbf{p} + \sigma_n^2\vec{\mathbf{w}}_1, \tag{21}$$

where $\bar{\mathbf{A}}_1$ is the matrix sorted to make the ith row corresponding to the sensor location $(-M_{nc} + i)d, i = 0, \ldots, 2M_{nc}$ in difference co-array. And $\vec{\mathbf{w}}_1 \in \mathbb{C}^{(2M_{nc}+1)\times 1}$ is a column vector of all zeros except a 1 at the $(M_{nc} + 1)$th position.

Through averaging the received data by following the approach above, a modified data vector can be obtained. Define a data averaging operator $\mathcal{D}$ and it is presented in detail as

$$\mathcal{D}(\bar{\mathbf{z}}_1) \triangleq \mathcal{D}(\bar{\mathbf{z}}_1(l)), \tag{22}$$

and furthermore,

$$\mathcal{D}(\bar{\mathbf{z}}_1(l)) = \frac{1}{|Q_l|}\sum_{i\in Q_l}\bar{\mathbf{z}}(i), \tag{23}$$

where $\bar{\mathbf{z}}_1(l)$ denotes the lth element of $\bar{\mathbf{z}}_1$, so does $\bar{\mathbf{z}}(i)$. Note that $Q_l = \{i|P_{nc}(i) = \bar{P}'_u(l),\ i = 1, \ldots, 4M^2\}$ is the set consisting indexes of elements in P_{nc} which is equal to $\bar{P}'_u(l)$. Later Numerical results will show that receiving data averaging approach can improve DOA performance effectively.

Follow (10), $\bar{\mathbf{R}}_{smoothed}$ can be calculated and a MUSIC-like null spectrum will be calculated to estimate Θ. Firstly, the SVD of $\bar{\mathbf{R}}_{smoothed}$ is performed as

$$\bar{\mathbf{R}}_{smoothed} = \mathbf{U}\Sigma\mathbf{V}^H. \tag{24}$$

Supposed that $\bar{\mathbf{R}}_{smoothed}$ is of rank r and the singular values are listed in descending order. Denoting the last $M_{nc} + 1 - r$ corresponding columns of $\mathbf{U}$ as $\mathbf{U}_n = [\mathbf{u}_{r+1}\mathbf{u}_{r+2}\cdots\mathbf{u}_{M_{nc}+1}]$. Defining $\bar{\mathbf{a}}(\theta) = \begin{bmatrix}1 \cdots e^{j2\pi(M_{nc}+1)(d/\lambda)sin\theta}\end{bmatrix} \in \mathbb{C}^{(M_{nc}+1)\times 1}$ as the steering vector of the difference co-array of direction θ and hence the MUSIC spectrum is formed as [7]

$$P(\theta) = \frac{\bar{\mathbf{a}}^H(\theta)\bar{\mathbf{a}}(\theta)}{\bar{\mathbf{a}}^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\bar{\mathbf{a}}(\theta)}. \tag{25}$$

The DOA estimates will be given through the peaks locating of null spectrum $P(\theta)$.

B. HYBRID APPROACH BASED ON RECEIVED DATA AVERAGING AND MATRIX COMPLETION

A data preprocessing method and a subspace DOA estimation scheme are introduced in last section and get us better performance. As is shown earlier, non-circular signals will lead to inconsecutive in difference co-array of nested linear array. In this section, we will focus on how to achieve information of those missing virtual sensors so that we can detect more sources than NC-SS-MUSIC provided in last section.

From Figure 2 and 3, sensors in D_{diff} is available, i.e., able to be generated by covariance and elliptic covariance. Thus we can also average the received data from repeated sensors except for the missing sensors. After that the hole problem becomes the key to resolving DOA accurately and is intractable to deal with. Since all elements of D_{diff} are included in set $\tilde{D}_{diff}$, we can estimate $\tilde{D}(x_m)$ with $D(x_m)$. However, the elements of $\tilde{D}_{diff}$ which do not exist in D_{diff} limit us from estimating $\tilde{D}(x_m)$. As a result, we compute $\tilde{D}(x_m)$ with the values of $D(x_m)$ while those missing values are set to zero tentatively. Therefore, we can complete (15) with zeros in position where $\tilde{D}_{diff}$ has elements but D_{diff} does not. Then we get a new equivalent array output vector $\bar{\mathbf{z}}_2$ after data averaging. Note that $\bar{\mathbf{z}}_2 \in \mathbb{C}^{(M^2+2M-3)\times 1}$.

Since a nested linear array is considered in this paper, the toeplitz characteristic can be used to reconstruct a output covariance matrix. Consider about $\bar{\mathbf{z}}_2$ with zero elements in it, we can divide it into two parts on average and take them for toeplitz reconstruction. Suppose that the reconstructed matrix is described as $\bar{\mathbf{R}}_c$ and it contains $M(M/2-1)$ zero elements.

Matrix completion theory has been researched in antenna array especially for coprime array due to its properties that the difference co-array transformed from it has holes naturally [31]. Recently, matrix completion schemes using convex optimization show up frequently. Some of them estimate the missing values in covariance matrix of co-array with the hole values set to zero [32]. In other words, the toeplitz reconstructed matrix $\bar{\mathbf{R}}_c$ is just suitable for it. In matrix completion theory [33], [34], the problem is expressed as

$$\begin{aligned} &min\ rank(\hat{\mathbf{R}}_c) \\ &s.t.\ \hat{\mathbf{R}}_c^{(i,j)} = \bar{\mathbf{R}}_c^{(i,j)}, \quad (i,j) \in \Omega, \end{aligned} \tag{26}$$

where $\hat{\mathbf{R}}_c$ is the unknown matrix of low rank and it can be expressed as

$$\hat{\mathbf{R}}_c = \hat{\mathbf{A}}_c \hat{\mathbf{R}}_s \hat{\mathbf{A}}_c^H + \hat{\mathbf{I}}, \tag{27}$$

here $[\hat{\mathbf{A}}_c]_{m,n} = e^{j2\pi m(d/\lambda)sin\theta_n}$, $0 \le m \le M_{ncs}$, $1 \le n \le D$ denotes a equivalent manifold of virtual uniform array. Candes and Recht proved [33] that almost all low-rank matrices can be recovered exactly from those sampled entries by solving a convex optimization problem, like (26). Here Ω is the location set of all non-zero components in $\bar{\mathbf{R}}_c$, thus we can denote a orthogonal projector P_Ω satisfying that the (i,j)th element of $P_\Omega(\bar{\mathbf{R}}_c)$ is equal to $\bar{\mathbf{R}}_c^{(i,j)}$ if $(i,j) \in \Omega$ and zero otherwise. It is a NP-hard problem to solve the rank minimization, and hence a convex relaxation is undoubtedly necessary. Given that nuclear norm minimization problem is the tightest convex relaxation of the (26), singular value thresholding is a computationally efficiency and fast converging solution to it. With convex relaxation, problem can be converted to [27]

$$\begin{aligned} &min\ \gamma \left\|\hat{\mathbf{R}}_c\right\|_* + \frac{1}{2}\left\|\hat{\mathbf{R}}_c\right\|_F^2 \\ &s.t. \left\|P_\Omega(\hat{\mathbf{R}}_c) - \bar{\mathbf{R}}_c\right\|_F^2 \le \epsilon, \end{aligned} \tag{28}$$

where $\|\hat{\mathbf{R}}_c\|_*$ denotes the nuclear norm of $\hat{\mathbf{R}}_c$, which is the sum of its singular values. $\|\bullet\|_F$ is the Frobenius norm which equals to the square root of the sum of squares of all elements in the matrix it works on and in (28), $\|\hat{\mathbf{R}}_c\|_F^2$ equals to the standard inner product of $\hat{\mathbf{R}}_c$ itself. γ is a constant that the bigger it is, the closer (28) is to (26). Besides, ϵ is a constant depend on noise power. First of all, several definitions will be given in order to set forth the algorithm. $G_\gamma(\bullet)$ denotes the soft threshold operator and is defined as

$$G_\gamma(\hat{\mathbf{R}}_c) = UG_\gamma(\Sigma)V^H, \tag{29}$$

where $\hat{\mathbf{R}}_c = U\Sigma V^H$ denotes the singular value decomposition. Σ is a diagonal matrix composed of λ_i, which represents the ith singular value sorted in descending order. Furthermore,

$$G_\gamma(\Sigma) = diag[(\lambda_1-\gamma)_+\ (\lambda_2-\gamma)_+ \cdots (\lambda_{M^2/2+M-2}-\gamma)_+], \tag{30}$$

here $(\lambda_i-\gamma)_+ = max[0, \lambda_i-\gamma]$. Hence the main steps in algorithm of SVT is tabulated in Table 1.

TABLE 1. The steps of SVT.

Method: Singular value soft thresholding method
1) **Step 1**: Initialize $\hat{\mathbf{R}}_c^0 = \bar{\mathbf{R}}_c$ and set appropriate initial value of coefficients γ, ϵ and step δ.
2) **Step 2**: Compute $\hat{\mathbf{R}}_{sm}^i = G_\gamma(\hat{\mathbf{R}}_c^{i-1})$ when it comes to ith iteration.
3) **Step 3**: Update variable through $\hat{\mathbf{R}}_c^i = \hat{\mathbf{R}}_c^{i-1} + \delta(\bar{\mathbf{R}}_c - G_\gamma(\hat{\mathbf{R}}_{sm}^{i-1}))$.
4) **Step 4**: Let $\hat{\mathbf{R}}_{sm} = \hat{\mathbf{R}}_{sm}^i$ if $\left\|G_\gamma(\hat{\mathbf{R}}_{sm}^i) - \bar{\mathbf{R}}_c\right\|_F^2 \le \epsilon$; otherwise let $i = i+1$ and turn to step 2.

The initial value was considered in [27] and here we initialize them as $\gamma = 20M$, $\delta = 1.03$ and $\epsilon = 10^{-4}$. After that the zero-location value will be recovered and hence we obtain the covariance matrix of completed virtual uniform linear array. Suppose that $\hat{\mathbf{R}}_c^*$ is the optimal solution to SVT.

Generally, we apply the classical MUISC algorithm to find the arrival angle of signals. First of all, SVD of $\hat{\mathbf{R}}_c^*$ is conducted and we can get corresponding singular vectors. Here we assume that the number of impinging sources is known as D. Hence the last $M_{ncs}+1-D$ columns of left singular vector constitute $\mathbf{U}_n$, which spans the noise subspace. The steering vector of difference co-array is $\bar{\mathbf{a}}_c(\theta) = \left[1 \cdots e^{j2\pi(M_{ncs}+1)(d/\lambda)sin\theta}\right] \in \mathbb{C}^{(M_{ncs}+1)\times 1}$ and the MUSIC spectrum is expressed as

$$P(\theta) = \frac{\bar{\mathbf{a}}_c^H(\theta)\bar{\mathbf{a}}_c(\theta)}{\bar{\mathbf{a}}_c^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\bar{\mathbf{a}}_c(\theta)}. \tag{31}$$

It is surely that we need to estimate the signal number first to obtain results of high accuracy. Method [35] can detect more signals than the number of physical sensors together with source number estimation, which is equal to the rank of matrix recovered by the algorithm. However, the proposed algorithm is mainly aimed at estimating the zero components of $\bar{\mathbf{R}}_c$. It can still estimate the number of impinging sources, though, in such a complicated model and it needs to do a further research on it. Similarly, arrival angle will be shown in the form of MUSIC spectrum peak. To sum up, the entire proposed algorithm can be summarized in Table 2 below.

TABLE 2. The steps of proposed algorithm.

Algorithm: DOA estimation based on maximum virtual sensors
1) **Step 1**: Calculate $\hat{\mathbf{R}}_x = \frac{1}{T}\sum_{t=1}^{T} X(t)X^H(t)$ for infinite snapshots T.
2) **Step 2**: Conduct vectorization to $\hat{\mathbf{R}}_x$ and then choose maximum number of virtual sensors despite the discontinuity.
3) **Step 3**: Apply direct toeplitz matrix reconstruction to get $\bar{\mathbf{R}}_c$.
4) **Step 4**: Solve the nuclear norm minimization problem (28) by SVT and obtain recovered matrix $\hat{\mathbf{R}}_c^*$.
5) **Step 5**: Perform singular value decomposition of $\hat{\mathbf{R}}_c^*$ and find $\mathbf{U}_n$ which spans the noise subspace.
6) **Step 6**: Construct the MUSIC spectrum $P(\theta) = \frac{\hat{\mathbf{a}}'^H(\theta)\hat{\mathbf{a}}'(\theta)}{\hat{\mathbf{a}}'^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\hat{\mathbf{a}}'(\theta)}$ and locate the peaks, thus get the estimates of Θ.

In this part we develop a hybrid approach of DOA estimation based on received data averaging and matrix completion, and now we shall present relations to other work.

P. Pal *et al.* developed a gridless method of DOA estimation via low-rank recovery, which considers nuclear norm minimization as well [35]. Thus it is meaningful to make a comparison with it. This paper first raises the discontinuity problem of co-array for non-circular signals in nested array and based on this, a SVT-based matrix completion method is proposed. However, [35] considers hole-free co-array of nested array for circular signals and hence it does not need matrix completion. In their paper, P. Pal *et al.* have the spatial smoothing operator $\mathcal{P}_{smooth}$ acting on covariance matrix while the proposed method reconstructs an equivalent covariance matrix with the missing values set to zeros. The result is that [35] accomplishes the aim of denoising, which is characterized by ϵ and the proposed method estimates those missing values. Of course the matrix $\bar{\mathbf{R}}_c$ is reconstructed of vectors averaged with both covariance and elliptic covariance data.

The array interpolation method for coprime array [36] poses a semidefinite programming scheme to solve nuclear norm minimization problem (P1) and recover correlation information. However, it has a constraint of Hermitian matrix while we do not need. By the way, a Toeplitz reconstruction of steering vectors in antenna array has already ensured this constraint. As is analyzed earlier, the received data vector has been preprocessed with covariance and elliptic covariance information through scheme provided in Section 4, Part A. In terms of optimization, we derive a different way to solve the problem. Note that (28) is closely related to nuclear norm minimization problem so the solution to (28) eventually converges to a matrix that nearly minimizes nuclear norm of $\hat{\mathbf{R}}_c$. Besides, SVT provides a computationally efficiency and fast converging solution to (28) so that we can solve nuclear norm minimization problem under low complexity. Note that the complexity of SVT has been analyzed particularly in [27] and proved to have a fast converging solution so that this paper does not discuss the issue in detail.

V. COMPUTATIONAL COMPLEXITY AND FREEDOM DEGREE ANALYSIS

A. COMPUTATIONAL COMPLEXITY ANALYSIS

In this section we present the complexity of proposed algorithm and compare it with some former schemes like NC-SS-MUSIC and CS-based methods. Specificly, OMP method mentioned in [37], [38].

The complexity of the proposed algorithm is mainly composed of four parts: covariance matrix calculation through time averaging, eigenvalue decomposition, SVT and MUSIC, whose complexities are $O(4TM^2)$, $O(M_{ncs}^3)$, $O(2M_{ncs}^3N_i)$ and $O((M_{ncs}-D)M_{ncs}D_\theta)$ respectively. Here N_i denotes the iteration number and $D_\theta = 180/\Delta\theta + 1$ is the number of spectral points in peak search grid with $\Delta\theta$ represents search step. Therefore the computational complexity of the proposed algorithm is $O(4TM^2+M_{ncs}^3+2M_{ncs}^3N_i+(M_{ncs}-D)M_{ncs}D_\theta)$. The computational efficiency of SVT has been analyzed and proved by numerical results in [27] with comparison to other methods solving nuclear norm minimization problem.

Besides, the SS-MUSIC in [10] can resolve DOA with consecutive sensors so that it do not need to apply SVT. The complexity of SS-MUSIC is $O(TM^2+M_c^3+(M_c-D)M_cD_\theta)$ and for comparison, NC-SS-MUSIC has a complexity of $O(4TM^2+M_{nc}^3+(M_{nc}-D)M_{nc}D_\theta)$. Besides, OMP mentioned in [37] is a method based on CS, whose computational complexity mainly exists in two parts: calculating covariance matrix and iteration calculation. The OMP method has a complexity of $O(TM^2+DM_c^2O_\theta)$, where $O_\theta = 180/\Delta\theta + 1$ denotes redundant dictionary.

It shows that the proposed method can detect more sources than number of sensors without significant increase in complexity from Table 3. This is due to the extension based on non-circular signals and improvement in degrees of freedom, as is shown in (13), (14) and Figure 2. Moreover, the SVT method also has a significant effect on complexity which mainly comes from singular value decomposition. The number of snapshots projects little influence on complexity while the searching step size $\Delta\theta$ exerts a stronger impact. Peak searching and redundant dictionary are all depend on grids, which means that once the step size is small, the complexity becomes really high.

TABLE 3. Complexity comparison of different algorithms.

Algorithms	Complexity
Method in [37]	$O(TM^2+DM_c^2O_\theta)$
SS-MUSIC	$O(TM^2+M_c^3+(M_c-D)M_cD_\theta)$
NC-MUSIC	$O(4TM^2+M_{nc}^3+(M_{nc}-D)M_{nc}D_\theta)$
NC-SVT-MUSIC	$O(4TM^2+M_{ncs}^3+2M_{ncs}^3N_i+(M_{ncs}-D)M_{ncs}D_\theta)$

B. FREEDOM DEGREE ANALYSIS

Degrees of freedom are affected by antenna array manifold and specific algorithm one uses, which are also influencing application performance and source numbers of estimation conversely. The property of high degrees of freedom, which means ability of multiple sources detection, is the key to the proposed algorithm. We consider a two level nested array with M sensors, $M/2$ for each level.

For non-circular signals, the corresponding virtual array has holes and hence we can develop two ways to resolve DOA. First we only make use of the consecutive virtual sensors together with a preprocessing technique of data fusion by averaging, obtaining a freedom degree of $O(M^2/4+M)$. Besides, all the virtual sensors can be fully utilized by completing the missing elements using SVT, thus we can get a freedom degree of $O(M^2/2+M-1)$, which is much higher than conventional methods like SS-MUSIC and NC-SS-MUSIC. The freedom degree comparison among these algorithms is presented below in Figure 4. The proposed algorithm has the highest degrees of freedom and it can detect sources when the number of sources are more than the

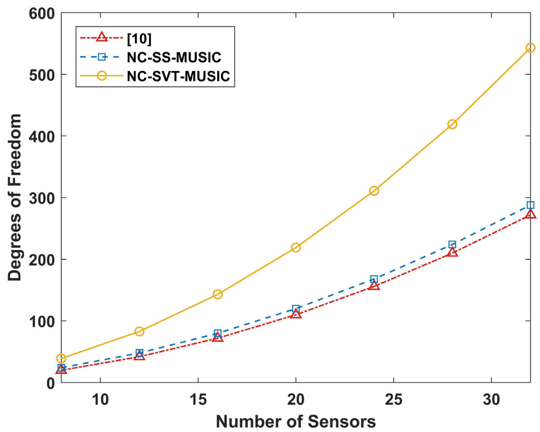


FIGURE 4. Degrees of freedom versus number of sensors.

number of antenna, which is recognized as underdetermined DOA estimation.

VI. SIMULATION RESULTS

This section presents numerical results that illustrate performance of the proposed algorithms compared to methods in [10], [36] and [37]. First we shall define an evaluation criteria called root mean square error (RMSE), which is expressed as

$$RMSE = \sqrt{\frac{1}{KD}\sum_{i=1}^{K}\left\|\Theta - \hat{\Theta}_i\right\|^2}, \tag{32}$$

where Θ and $\hat{\Theta}_i$ denote the real value and the ith value of estimation respectively; K is the number of Monte Carlo simulations. Moreover, we assume that a two level nested array with $M_1 = M_2 = 4$ is under consideration for all simulations below.

Then, a lower bound on the variances of the unbiased estimator called the Cramér-Rao bound (CRB) shall be defined. The CRB describes the smallest mean square error of unbiased estimates of parameters such as DOA, which can be presented as [39]

$$CRB(\theta) = \frac{\sigma^2}{2T}\{Re[(\mathbf{A}'^H\mathbf{P}_A\mathbf{A}') \odot \hat{\mathbf{S}}^T]\}^{-1}, \tag{33}$$

where

$$\mathbf{A}' = \left[\frac{\partial \mathbf{a}(\theta_1)}{\partial\theta_1} \quad \frac{\partial \mathbf{a}(\theta_2)}{\partial\theta_2} \quad \cdots \quad \frac{\partial \mathbf{a}(\theta_D)}{\partial\theta_D}\right], \tag{34}$$

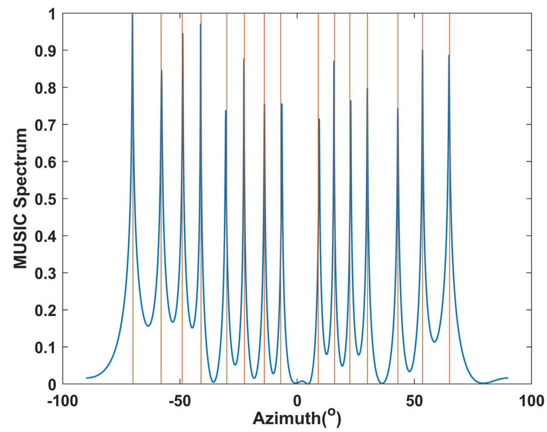
$$\mathbf{P}_A = \mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H, \tag{35}$$

$$\hat{\mathbf{S}} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix}\mathbf{s}_1(t)\\ \mathbf{s}_2(t)\\ \vdots\\ \mathbf{s}_D(t)\end{bmatrix}\begin{bmatrix}\mathbf{s}_1(t)\\ \mathbf{s}_2(t)\\ \vdots\\ \mathbf{s}_D(t)\end{bmatrix}^H. \tag{36}$$

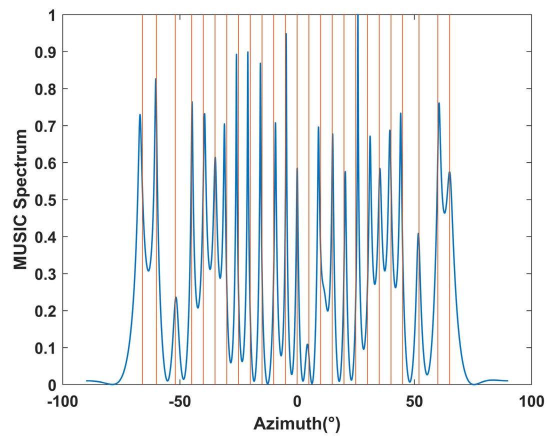
and $\odot$ denotes the Hadamard product. Note that $\mathbf{S}(t) = [\mathbf{s}_1(t)\ \mathbf{s}_2(t) \cdots \mathbf{s}_D(t)]^T$.

A. PERFORMANCE IN UNDERDETERMINED CONDITION ($D > M$)

In this part we concentrate on underdetermined DOA estimation and consider $D = 15$ or $D = 25$ sources while using a two level nested linear array of 8 sensors in total. The SNR is 0 dB, snapshots $T = 500$ and searching step size $\Delta\theta = 0.05°$. The MUSIC spectrum using two proposed methods is presented in Figure 5 (a) and (b) respectively. The two figures demonstrate that the proposed algorithms can correctly estimate the DOA of impinging sources and realize underdetermined DOAs estimation successfully. In addition, matrix completion technique makes it possible to further extend degrees of freedom, namely, to improve detectable sources numbers.



(a)

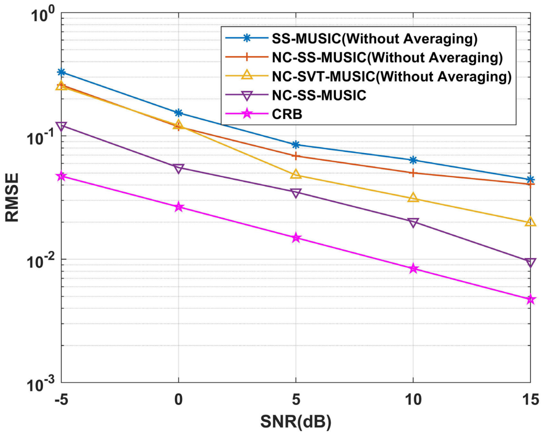


(b)

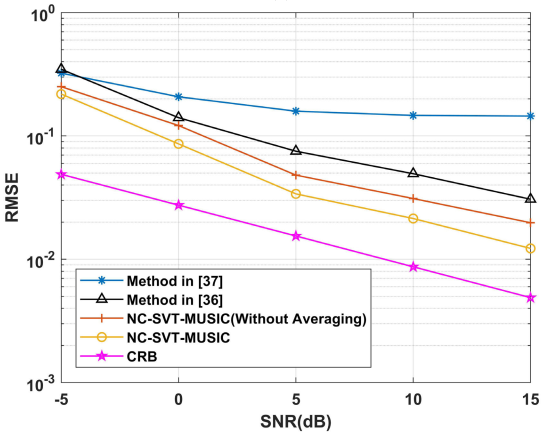
FIGURE 5. MUISC spectrum under the condition of SNR = 0dB (a) 15 sources estimated with maximum consecutive sensors; (b) 25 sources estimated using matrix completion.

B. RMSE COMPARISON WITH BOTH CONSECUTIVE SENSORS AND MATRIX COMPLETION UNDER DIFFERENT SNRs

RMSE performance versus SNR is be studied in this part. We compare the performance of two proposed algorithms together with methods in [36] and [37] in the case of $K = 100$, $T = 500$ and SNRs from -5dB to 15dB with step size 5 dB. For proposed algorithms, we consider two scenarios of $D = 3$ and $D = 5$, in both of which signals are strictly non-circular so that both proposed algorithms, NC-SS-MUSIC and NC-SVT-MUSIC can be applied. The results are presented in Figure 6 and 7.



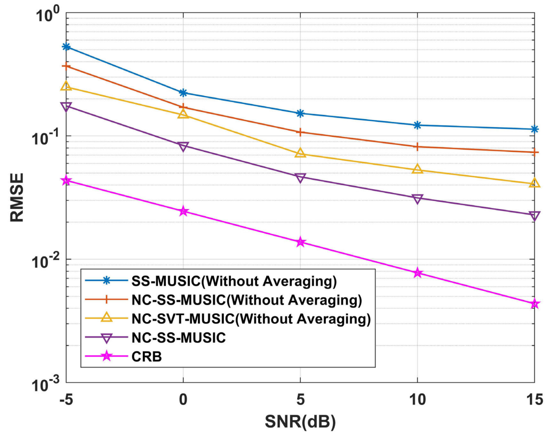
(a)



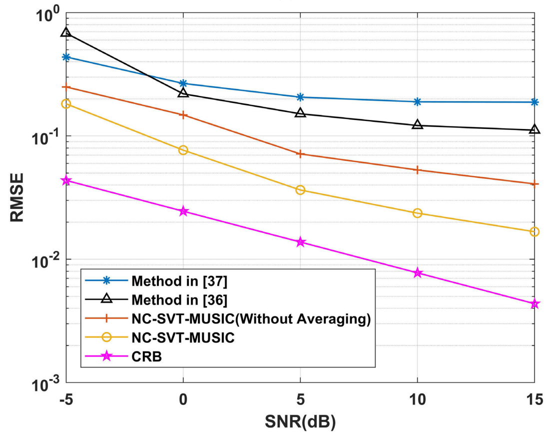
(b)

FIGURE 6. RMSE comparison under different SNRs with $D = 3$, $\Delta\theta = 0.05°$ (a) RMSE performance with maximum consecutive sensors; (b) RMSE performance using matrix completion.

From (a) in Figure 6 and 7, we can find that the accuracy of estimation gets considerable improvement by making full use of the non-circular characteristics. Because, using the non-zero property of elliptic covariance matrix, one can double the received data matrix so that the information available is also doubled. Furthermore, replacing the data from a single virtual sensor with the average of those received data from repeated sensors can get us more accurate results. Figure (a) illustrates that data averaging does improve the performance of DOA estimation. Without using data averaging, we also present the results of proposed NC-SVT-MUSIC in (a), which can detect more signals and attain results as accurate as proposed NC-SS-MUSIC, and even better. It means that SVT method works well and is helpful to improve performance of DOA estimation.



(a)



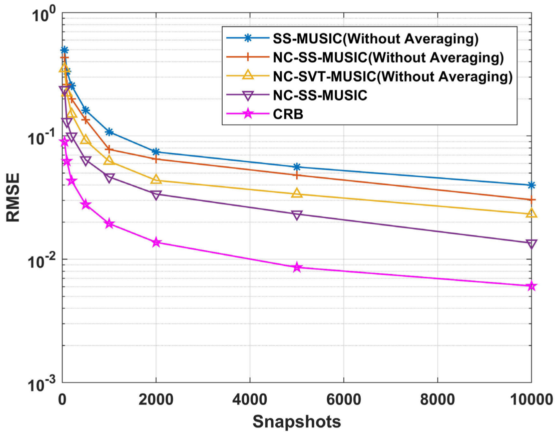
(b)

FIGURE 7. RMSE comparison under different SNRs with $D = 5$, $\Delta\theta = 0.05°$ (a) RMSE performance with maximum consecutive sensors; (b) RMSE performance using matrix completion.

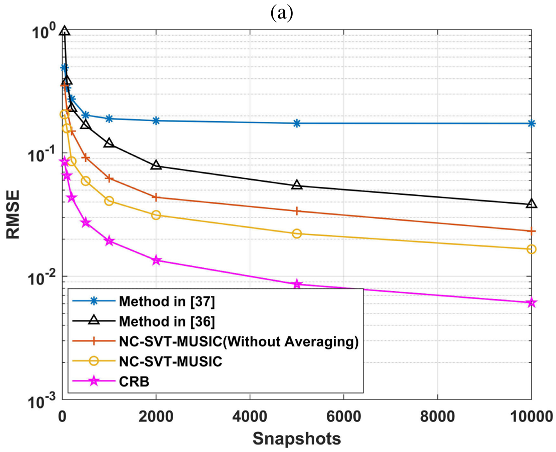
In (b) of Figure 6 and 7, the CS-based method, OMP in [37], obtains good performance when SNR ≤ 0 dB but goes worse gradually with SNR increasing. Once matrix completion theory is applied, the holes in virtual array will be completed. Moreover, the array aperture and degrees of freedom also increase as the number of virtual sensors increases, which means more detectable sources. For comparison, RMSE results of nuclear norm minimization method in [36] are presented in line of triangle. We can find that the proposed NC-SVT-MUSIC attains higher accuracy than method in [36] with lower computational complexity.

C. RMSE COMPARISON WITH BOTH CONSECUTIVE SENSORS AND MATRIX COMPLETION UNDER DIFFERENT NUMBER OF SNAPSHOTS

In simulation 3, we keep SNR $= 0$ dB, Monte Carlo simulations $K = 100$, and varies the number of snapshots to $T = [50, 100, 200, 1000, 2000, 5000, 10000]$ to study the impact of snapshots on DOA estimation. Figure 8 and 9 illustrate the results of 3 and 5 sources with $\Delta\theta = 0.05°$ respectively. The RMSE becomes lower with the increase

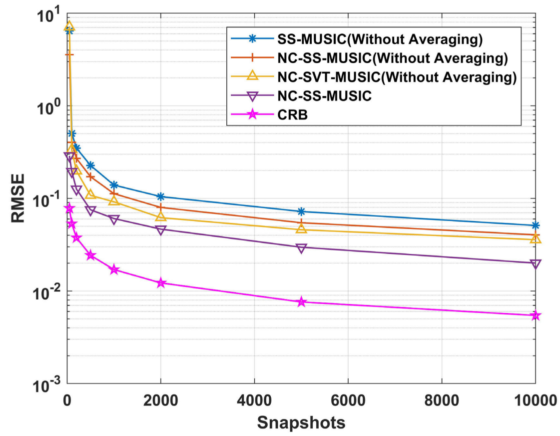


(a)

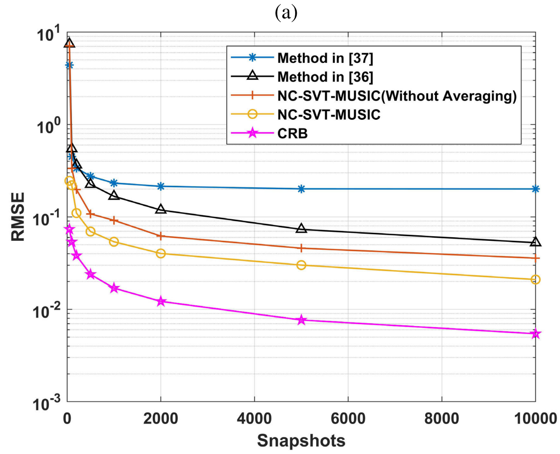


(b)

FIGURE 8. **RMSE comparison under different snapshots with $D = 3$, $\Delta\theta = 0.05°$ (a) RMSE performance with maximum consecutive sensors; (b) RMSE performance using matrix completion.**



(a)



(b)

FIGURE 9. **RMSE comparison under different snapshots with $D = 5$, $\Delta\theta = 0.05°$ (a) RMSE performance with maximum consecutive sensors; (b) RMSE performance using matrix completion.**

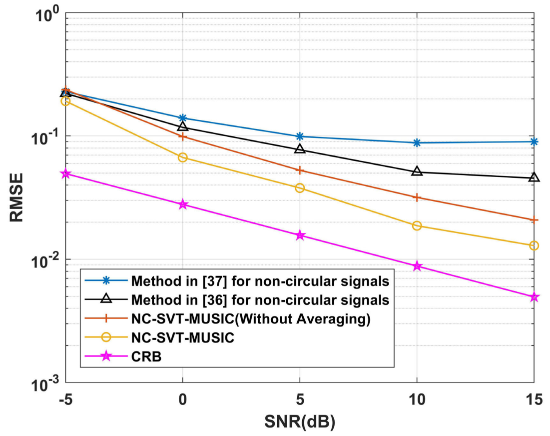
of number of snapshots, and basically keep unchanged once $T > 500$. Analyze the four pictures below carefully, and we find that NC-SS-MUSIC is more accurate than SS-MUSIC, NC-SVT-MUSIC also performs better than methods in [36] and [37]. It also illustrates that the method after process of data averaging is more effective than that does not according to line of plus sign and circle in (b) of Figure 8 and 9. From simulation 2 and 3, we can find that NC-SVT-MUSIC and NC-SS-MUSIC have similar accuracy. To sum up, NC-SS-MUSIC can be used for more accurate results than SS-MUSIC with high speed while NC-SVT-MUSIC can be used for detecting more sources than number of sensors with a higher complexity. Though it is lower than other nuclear norm minimization methods. The other conclusions are also similar to those in simulation 2.

D. RMSE COMPARISON WITH OTHER METHODS FOR NON-CIRCULAR SIGNALS

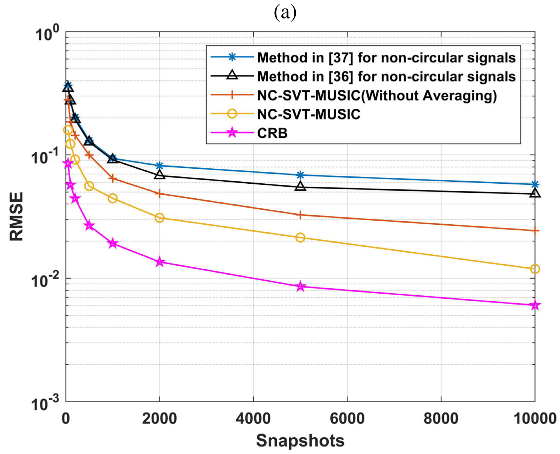
In former simulations, it has been proved that the proposed NC-SVT-MUSIC outperforms methods in [36] and [37]. However, the latter two methods were proposed for circular signals and in this part, we set $D = 3$ strictly non-circular signals to study the performance under different SNRs and snapshots. Similarly, Monte Carlo simulations $K = 100$. Figure 10 shows the results of DOA estimation. It shows that NC-SVT-MUSIC has better performance compared to methods [36] and [37] even in scenario of non-circular signals.

E. RMSE COMPARISON UNDER DIFFERENT NUMBER OF SENSORS

The number of sensors can affect the virtual array aperture and freedom degree, thus having an important influence on RMSE performance. In this simulation, we set SNR = 0dB, snapshots $T = 500$, Monte Carlo simulations $K = 100$ and varies the number of sensors to $M = [4, 8, 12, 16, 20]$. Note that each level of nested array has $M/2$ sensors and there are 3 sources impinging on antenna array. Figure 11 shows the result of RMSE with $\Delta\theta = 0.05°$. It is clear that as the number of sensors increase, the RMSE decreases. When there are 20 sensors in nested linear array, it can obtain a high accuracy, for instance, RMSE less than 10^{-2}.



(a)



(b)

FIGURE 10. **RMSE comparison with other methods for non-circular signals with $D = 3$, $\Delta\theta = 0.05°$ (a) RMSE performance under different SNRs; (b) RMSE performance under different snapshots.**

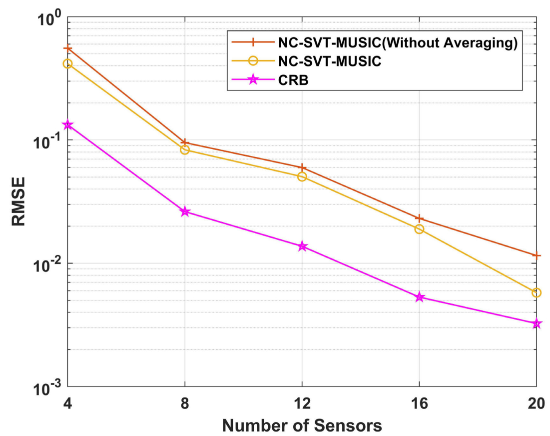


FIGURE 11. **RMSE comparison under different number of sensors.**

VII. CONCLUSION

The paper has presented an underdetermined DOA estimation technique using a two-level nested linear array. After a brief introduction to the model and associated problems, we analyze the information loss problem and the hole problem of nested array for non-circular signals carefully and give the theoretical analysis. Through received data averaging and matrix completion theory, the proposed algorithms solve the problems of existing approaches, namely the inability to find more sources than number of physical sensors and the effective and accurate DOA estimation with low complexity in this circumstance. Through a series of simulation experiments, we demonstrate that the received data averaging method can improve the performance of DOA estimation effectively and the proposed hybrid approach using matrix completion theory achieves better results compared with [36] and [37] while it can estimate more sources than number of sensors.

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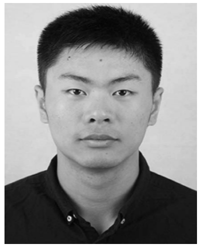

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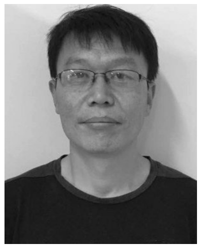

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