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Italian Reinforcement Number in Graphs

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ABSTRACT An Italian dominating function (IDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to at least two vertices assigned 1 under f. The weight of an IDF f is the value $\sum_{v\in V} f(v)$. The Italian domination number of a graph G is the minimum weight of an IDF on G. The Italian reinforcement number of a graph is the minimum number of edges that have to be added to the graph in order to decrease the Italian domination number. In this paper, we initiate the study of Italian reinforcement number and we present some sharp upper bounds for this parameter. In particular, we determine the exact Italian reinforcement numbers of some classes of graphs.

INDEX TERMS Italian domination number, Italian reinforcement number, Cartesian product.

I. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$. The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. The maximum degree among all vertices of G is denoted by $\Delta(G)$. For a set $S \subseteq V(G)$ and a vertex $v \in S$, the S*-private neighborhood* of v, denoted by $\mathrm{pn}_G(v, S)$, consists of all vertices u such that $N[u] \cap S = \{v\}$. If the graph G is clear from the context, then we will simply write $N(v)$, $N[v]$, $N(S)$, $d(v)$, Δ and $\mathrm{pn}(v, S)$ rather than $N_G(v)$, $N_G[v]$, $N_G(S)$, $d_G(v)$, $\Delta(G)$ and $\mathrm{pn}_G(v, S)$, respectively.

We write C_n for the cycle of length n, P_n for the path of order n, K_n for the complete graph of order n and $K_{n_1,n_2,\ldots,n_t}$ for the complete t-partite graph with t partite sets of cardinality $n_1, n_2, \ldots, n_t$ $(t \geq 2)$. A *star* of order $n \geq 2$ is the complete bipartite graph $K_{1,n-1}$. We call the *center* of a star to be a vertex of maximum degree. The *corona graph* $H \circ K_1$ of a graph H is the graph obtained from H by attaching one pendent edge at each vertex of H. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. The *complement* of a graph G is the graph $\overline{G}$, where $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. For a subset S of vertices of a graph G and a real-valued function $f : V(G) \to \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$.

A *dominating set* S in a graph G is a set of vertices of G such that each vertex not in S is adjacent to a vertex of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. Kok and Mynhardt [1] introduced the *reinforcement number* $r(G)$ of a graph G as the minimum number of edges that have to be added to the graph in order to decrease the domination number. Since the domination number of every graph G is at least 1, by convention Kok and Mynhardt defined $r(G) = 0$ if $\gamma(G) = 1$. This concept of the reinforcement number in a graph was further considered for several domination variants, including total domination, Roman domination and rainbow domination. See, for example, [2]–[9], and elsewhere.

As a new variant of the domination, Italian domination was introduced in [10], where it was called Roman {2}-domination. An *Italian dominating function* (IDF) on a graph G is defined as a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for every vertex $v \in V(G)$ with $f(v) = 0, f(N(v)) \geq 2$, that is, either there is a vertex $u \in N(v)$ with $f(u) = 2$, or at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The *weight* of an IDF f is the value $\omega(f) = f(V(G))$. The *Italian domination number* of a graph G, denoted by $\gamma_I(G)$, is the minimum weight of an IDF on G.

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An IDF on G with weight $\gamma_I(G)$ is called a *$\gamma_I(G)$-function*. For a sake of simplicity, an IDF f on G will be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. For some advance we refer the reader to [11]–[14].

In this paper, we extend the idea of reinforcement number to Italian domination as follows: For a graph G, a subset F of $E(\overline{G})$ is an *Italian reinforcement set* (*IR-set*) of G if $\gamma_I(G + F) < \gamma_I(G)$. The *Italian reinforcement number* of a graph G, denoted by $r_I(G)$, is the minimum size of an IR-set of G. An IR-set F of G is called a *$r_I(G)$-set* if $|F| = r_I(G)$. Observe that if $\gamma_I(G) \in \{1, 2\}$, then addition of edges does not reduce the Italian domination number. We define $r_I(G) = 0$ if $\gamma_I(G) \in \{1, 2\}$. Thus we always assume that when we discuss $r_I(G)$, all graphs involved satisfy $\gamma_I(G) \geq 3$.

Our purpose in this paper is to initiate the study of Italian reinforcement number in graphs. We derive some sharp upper bounds on the Italian reinforcement number and we also determine exact values of Italian reinforcement number of some classes of graphs.

II. PROPERTIES AND UPPER BOUNDS

Our aim in this section is to present basic properties of the Italian reinforcement number and derive some sharp upper bounds for this parameter. We start with a fundamental lemma that will be used in the proof of some results.

Lemma 1: For any graph G with $\gamma_I(G) \geq 3$, let F be an $r_I(G)$-set and let f be a $\gamma_I(G+F)$-function. Then the following hold:

(*i*) *For each edge $v_1v_2 \in F$, there exists an integer $i \in \{1, 2\}$ such that $f(v_i) = 0$ and $f(v_{3-i}) \neq 0$.*

(*ii*) *$\gamma_I(G + F) = \gamma_I(G) - 1$.*

Proof: (*i*) Suppose, to the contrary, that there exists an edge $v_1v_2 \in F$ such that $f(v_i) \neq 0$ for each $i \in \{1, 2\}$ or $f(v_1) = f(v_2) = 0$. Observe that f is an IDF on $G + (F\backslash\{v_1v_2\})$, and so $F\backslash\{v_1v_2\}$ is an IR-set of G, implying that $r_I(G) \leq |F\backslash\{v_1v_2\}| = |F| - 1$, a contradiction. So, (*i*) holds.

(*ii*) Since F is an $r_I(G)$-set, $\gamma_I(G + F) \leq \gamma_I(G) - 1$. Suppose, to the contrary, that $\gamma_I(G + F) \leq \gamma_I(G) - 2$. Let $v_1v_2 \in F$. By (*i*), we may assume that $f(v_1) = 0$ and $f(v_2) \neq 0$. Then the function g defined by $g(v_1) = 1$ and $g(x) = f(x)$ otherwise, is an IDF on $G + (F\backslash\{v_1v_2\})$ with $\omega(g) = \omega(f) + 1 \leq \gamma_I(G) - 1$, and so $F\backslash\{v_1v_2\}$ is an IR-set of G, implying that $r_I(G) \leq |F\backslash\{v_1v_2\}| = |F| - 1$, a contradiction. As a result, we have $\gamma_I(G + F) = \gamma_I(G) - 1$ and so (*ii*) also holds. □

We now provide a characterization of all the graphs G with $r_I(G) = 1$, which will be useful in many of the results of this paper.

Theorem 1: For any graph G with $\gamma_I(G) \geq 3$, $r_I(G) = 1$ if and only if there exist a $\gamma_I(G)$-function $f = (V_0^f, V_1^f, V_2^f)$ and a vertex $v \in V_1^f$ satisfying one of the following conditions:

(*i*) *$f(N(v)) = 1$ and $f(N(x)\backslash\{v\}) \geq 2$ for each $x \in N(v) \cap V_0^f$.*

(*ii*) *$f(N(v)) = 0$, $f(N(x)\backslash\{v\}) \geq 2$ for each $x \in N(v)$ and $V_2^f \neq \emptyset$.*

Proof: Suppose that (*i*) holds. Since $f(N(v)) = 1$, we may assume that $N(v) \cap V_1^f = \{u\}$. Since $\omega(f) = \gamma_I(G) \geq 3$, there exists some vertex $w \in (V_1^f \cup V_2^f)\backslash N[v]$. Moreover, since $uv \in E(G)$ and $f(N(x)\backslash\{v\}) \geq 2$ for each $x \in N(v) \cap V_0^f$, we have that the function $g = (V_0^f \cup \{v\}, V_1^f\backslash\{v\}, V_2^f)$ is an IDF on $G + vw$ with $\omega(g) = \omega(f) - 1$, and so $\{vw\}$ is an IR-set of G, implying that $r_I(G) = 1$. Now suppose that (*ii*) holds and let $u \in V_2^f$. Since $f(N(v)) = 0$, $uv \notin E(G)$. Furthermore, since $f(N(x)\backslash\{v\}) \geq 2$ for each $x \in N(v)$, we have that the function g defined earlier is also an IDF on $G + uv$ with $\omega(g) = \omega(f) - 1$, and so $\{uv\}$ is an IR-set of G, implying that $r_I(G) = 1$.

Conversely, suppose that $r_I(G) = 1$. Let $\{uv\}$ be an $r_I(G)$-set and let $h = (V_0^h, V_1^h, V_2^h)$ be a $\gamma_I(G+uv)$-function. By Lemma 1(*i*), we may assume that $h(u) \neq 0$ and $h(v) = 0$. It is easy to check that the function $f = (V_0^h\backslash\{v\}, V_1^h\cup\{v\}, V_2^h)$ is an IDF on G. By Lemma 1(*ii*), we have $\omega(f) = \omega(h)+1 = (\gamma_I(G) - 1) + 1 = \gamma_I(G)$, implying that f is a $\gamma_I(G)$-function. If $f(N(v)) \geq 2$, then $h(N(v)) = f(N(v)) \geq 2$ and hence h is an IDF on G, implying that $\gamma_I(G) \leq \omega(h) = \gamma_I(G + uv)$, a contradiction to Lemma 1(*ii*). Therefore, we may assume that $f(N(v)) \leq 1$.

Since $h(v) = 0$ and h is a $\gamma_I(G + uv)$-function, we get $f(N(x)\backslash\{v\}) = h(N(x)\backslash\{v\}) \geq 2$ for each $x \in N(v) \cap V_0^f$. If $f(N(v)) = 1$, then (*i*) holds. Suppose now that $f(N(v)) = 0$. Obviously, $h(N(v)) = f(N(v)) = 0$. Moreover, since $h(v) = 0$ and u is adjacent to v in $G + uv$, we have $f(u) = h(u) = 2$, implying that (*ii*) is true.

The proof is completed. □

Theorem 2: Let G be a graph with $\gamma_I(G) \geq 3$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_I(G)$-function. Then

(*i*) *For any vertex $v \in V_1$, $r_I(G) \leq |(N(v)\backslash pn(v, V_1)) \cap V_0| + 2$.*

(*ii*) *For any vertex $v \in V_2$, $r_I(G) \leq |pn(v, V_2) \cap V_0|$.*

Proof:

(*i*) Let v be any vertex of V_1. Suppose now that $|V_2| \geq 1$. Let u be a vertex of V_2 and let $F = (\{ux : x \in (N(v)\backslash pn(v, V_1)) \cap V_0\} \cup \{uv\})\backslash E(G)$. Note that $pn(v, V_1) \cap V_0 \subseteq N(V_2)$. Thus the function $g_1 = (V_0 \cup \{v\}, V_1\backslash\{v\}, V_2)$ is an IDF on $G + F$ with $\omega(g_1) = \omega(f) - 1$. Therefore F is an IR-set of G and so

$$
\begin{aligned}
r_I(G) &\leq |F| \\
&= |(\{ux : x \in (N(v)\backslash pn(v, V_1)) \cap V_0\} \cup \{uv\})\backslash E(G)| \\
&\leq |(N(v)\backslash pn(v, V_1)) \cap V_0| + 1.
\end{aligned}
$$

Suppose next that $|V_2| = 0$. Clearly $pn(v, V_1) \cap V_0 = \emptyset$ and hence $(N(v)\backslash pn(v, V_1)) \cap V_0 = N(v) \cap V_0$. Since $\gamma_I(G) \geq 3$, there exist two vertices u and w different from v in V_1. Let $X = (N(v) \cap V_0) \cap N(u)$, $Y = (N(v) \cap V_0)\backslash N(u)$ and let $F = (\{vu, vw\} \cup \{wx : x \in X\} \cup \{ux : x \in Y\})\backslash E(G)$. It is easy to verify that the function g_1 defined earlier is an IDF on $G + F$ with $\omega(g_1) = \omega(f) - 1$. Therefore F is an IR-set of G

and so

$$\begin{aligned} r_I(G) &\le |F| \\ &= |(\{vu, vw\} \cup \{wx : x \in X\} \cup \{ux : x \in Y\})\backslash E(G)| \\ &\le |\{wx : x \in X\} \cup \{ux : x \in Y\}| + 2 \\ &= |(N(v) \cap V_0)| + 2. \\ &= |(N(v)\backslash pn(v, V_1)) \cap V_0| + 2. \end{aligned}$$

Thus (i) holds.

(ii) Let v be any vertex of V_2. Since $\gamma_I(G) \ge 3$, there exists some vertex $u \in V_1 \cup V_2$. Let $F = \{ux : x \in pn(v, V_2) \cap V_0\}\backslash E(G)$. Then the function $g_2 = (V_0, V_1 \cup \{v\}, V_2\backslash\{v\})$ is an IDF on $G + F$ with $\omega(g_2) = \omega(f) - 1$. Therefore F is an IR-set of G and so

$$\begin{aligned} r_I(G) &\le |F| \\ &= |\{ux : x \in pn(v, V_2) \cap V_0\}\backslash E(G)| \\ &\le |pn(v, V_2) \cap V_0|. \end{aligned}$$

The proof is completed. □

We remark that the upper bound of Theorem 2 is sharp. (i) Let $n \ge 8$ be an even number and $C_n = v_1 v_2 \cdots v_n v_1$. It is easy to check that the function f defined by $f(v_i) = 1$ for each odd i and $f(v_i) = 0$ for each even i, is an IDF on C_n with $\omega(f) = n/2$ and hence by Proposition A(ii) in Section III, f is a $\gamma_I(C_n)$-function. Observe that for each odd i, $|(N(v_i)\backslash pn(v_i, V_1)) \cap V_0| = |N(v_i) \cap V_0| = 2$ and so by Theorem 6 in Section III, we obtain $r_I(C_n) = 4 = |(N(v_i)\backslash pn(v_i, V_1)) \cap V_0| + 2$.

(ii) Let $X_1 = \{u\}$ and $Y_1 = \{u'_1, u'_2, \ldots, u'_s\}$ be the partite sets of $K_{1,s}$ and let $X_2 = \{v_1, v_2\}$ and $Y_2 = \{v'_1, v'_2, \ldots, v'_t\}$ be the partite sets of $K_{2,t}$ $(3 \le s \le t)$. We denote the graph G obtained from $K_{1,s}$ and $K_{2,t}$ by joining u and v_1. It is not difficult to verify that the function f defined by $f(u) = 2$, $f(v_1) = f(v_2) = 1$ and $f(x) = 0$ otherwise, is the unique $\gamma_I(G)$-function and so $\gamma_I(G) = 4$.

We now claim that $r_I(G) = s$. Let $F' = \{v_1 u'_i : 1 \le i \le s\}$. Then the function g_2 defined by $g_2(u) = g_2(v_1) = g_2(v_2) = 1$ and $g_2(x) = 0$ otherwise, is an IDF on $G + F'$ with $\omega(g_2) = 3 < \gamma_I(G)$. This implies that F' is an IR-set of G and so $r_I(G) \le |F'| = s$. Hence it suffices to show that $r_I(G) \ge s$. Let F be an $r_I(G)$-set and let h be a $\gamma_I(G + F)$-function. By Lemma 1(ii), we have $\omega(h) = \gamma_I(G+F) = \gamma_I(G)-1 = 3$.

If $h(V(K_{2,t})) \le 1$, then at least t vertices in $V(K_{2,t})\backslash\{v_1\}$ are incident with an edge in F; and if $h(V(K_{2,t})) = 3$, then $h(V(K_{1,s})) = 0$ and so each vertex in Y_1 is incident with an edge in F. In both cases, we obtain $|F| \ge s$. Suppose next that $h(V(K_{2,t})) = 2$. This forces $h(V(K_{1,s})) = 1$. If $h(u) = 1$, then each vertex in Y_1 is incident with an edge in F and so $|F| \ge s$. Hence we may assume that $h(u) = 0$. Then there exists some vertex, say u'_1, in Y_1 such that $h(u'_1) = 1$. If $h(v_1) = h(v_2) = 1$, then $|N_{G+F}(u'_i) \cap \{u'_1, v_1, v_2\}| \ge 2$ for $2 \le i \le s$ and so $|F| \ge 2(s-1) > s$. If exactly one of v_1 and v_2 is assigned 2 under h, then the other is assigned 0 and hence $v_1 v_2 \in F$ and $\{u'_2, u'_3, \ldots, u'_s\} \subseteq N_{G+F}(\{v_1, v_2\})$, implying that $|F| \ge s$. As a result, we obtain $r_I(G) = s$.

Recall that f is the unique $\gamma_I(G)$-function and u is the unique vertex assigned 2 under f. Thus $r_I(G) = s = |pn(u, V_2^f) \cap V_0^f|$.

Theorem 3: Let G be a graph of order n with $\gamma_I(G) \ge 3$. Then

$$r_I(G) \le \min\{\Delta + 2, n - \Delta - \gamma_I(G) + 2\}.$$

Proof: Using Theorem 2, we obtain $r_I(G) \le \Delta + 2$. Thus it suffices to show that $r_I(G) \le n - \Delta - \gamma_I(G) + 2$. Let v be a vertex of degree Δ. Since $\gamma_I(G) \ge 3$, we have $|V(G)\backslash N_G[v]| = n - \Delta - 1 \ge n - \Delta - \gamma_I(G) + 2$. Therefore, there exists a subset F of $\{uv \in E(\overline{G}) : u \in V(G)\backslash N[v]\}$ such that $|F| = n - \Delta - \gamma_I(G) + 2$. Then the function f defined by $f(v) = 2$, $f(x) = 0$ for each $x \in N_{G+F}(v)$ and $f(x) = 1$ otherwise, is an IDF on $G+F$ with $\omega(f) = n - |N_G[v]| - |F| + 2 = n - (\Delta + 1) - (n - \Delta - \gamma_I(G) + 2) + 2 = \gamma_I(G) - 1$. Thus F is an IR-set of G and so $r_I(G) \le |F| = n - \Delta - \gamma_I(G) + 2$, establishing the desired upper bound. □

We remark that the upper bound of Theorem 3 is sharp. For any integer $m \ge 2$, let G be the corona graph $K_m \circ K_1$. It is easy to verify that $|V(G)| = 2m$, $\Delta = m$, $\gamma_I(G) = m + 1$ and $r_I(G) = 1$, implying that $r_I(G) = \min\{\Delta + 2, |V(G)| - \Delta - \gamma_I(G) + 2\}$. Moreover, we conclude from Proposition A(ii) and Theorem 6 in Section III that for even $n \ge 8$, $r_I(C_n) = 4 = \min\{\Delta + 2, n - \Delta - \gamma_I(C_n) + 2\}$.

Next result is an immediate consequence of Theorem 3.

Corollary 1: For any graph G of order n with $\gamma_I(G) \ge 3$, $r_I(G) \le \lceil n/2 \rceil$.

Proof: If $\Delta \le \lceil n/2 \rceil - 2$, then Theorem 3 yields $r_I(G) \le \Delta + 2 \le \lceil n/2 \rceil$. If $\Delta \ge \lceil n/2 \rceil - 1$, then by Theorem 3, we obtain $r_I(G) \le n - \Delta - \gamma_I(G) + 2 \le n - (\lceil n/2 \rceil - 1) - 3 + 2 = \lfloor n/2 \rfloor$. □

As a special case, Theorem 3 implies that every graph G with $\delta = 1$ and $\gamma_I(G) \ge 3$ satisfies $r_I(G) \le \Delta + 2$. Next, we shall improve this upper bound. For this purpose, we first derive the following result.

Lemma 2: Let G be a graph with $\gamma_I(G) \ge 3$. If v is a support vertex of G, then $r_I(G) \le \max\{d(v), 3\}$.

Proof: Let f be a $\gamma_I(G)$-function and let u be a leaf adjacent to v. If $f(u) \ge 1$, then we deduce from Theorem 2 that $r_I(G) \le 3$. If $f(u) = 0$, then this forces $f(v) = 2$ and it follows from Theorem 2(ii) that $r_I(G) \le d(v)$. □

Theorem 4: Let G be a graph of order n with $\delta = 1$ and $\gamma_I(G) \ge 3$. Then $r_I(G) \le \Delta$.

Proof: If $\Delta \ge 3$, then the result follows from Lemma 2. Suppose that $\Delta \le 2$. Then G is a disjoint union of paths and cycles. Since $\delta = 1$, some connected components of G are paths. If some connected component of G is a path of order 2 or 3, then it is not different to verify that $r_I(G) \le \Delta$. If some connected component of G is a path of order at least 4, then by Theorem 5 in Section III, we have $r_I(G) \le 2 = \Delta$. □

It should be mentioned that the upper bound of Theorem 4 is sharp. Let G be a disjoint union of $k \ge 2$ copies of P_2. It can be easily checked that $r_I(G) = 1 = \Delta$. Moreover, it follows from Theorem 5 in Section III that any path of odd order $n \geq 5$ satisfies $r_I(P_n) = 2 = \Delta$. Let $\Delta \geq 3$. We now construct infinitely many trees T with $r_I(T) = \Delta(T) = \Delta$. Let H be a tree of order no less than 2 with $\Delta(H) \leq \Delta - 1$. For each $v \in V(H)$, let S_v be a star of order Δ and let c_v be the center of S_v. We let $T(\Delta, H)$ denote the tree obtained from $H \cup (\bigcup_{v\in V(H)} S_v)$ by joining v and c_v for each $v \in V(H)$.

Proposition 1: Let $\Delta \geq 3$ be an integer, H be a tree of order no less than 2 with $\Delta(H) \leq \Delta - 1$ and let $T = T(\Delta, H)$. Then $r_I(T) = \Delta(T) = \Delta$.

Proof: We first show that $\gamma_I(T) = 2|V(H)|$. Let g be a $\gamma_I(T)$-function. For each $v \in V(H)$, if a leaf u of S_v satisfies $g(u) = 0$, then $g(c_v) = 2$ and if all leaves u of S_v satisfies $g(u) \geq 1$, then $g(V(S_v)\backslash\{c_v\}) \geq \Delta - 1 \geq 2$. In either case, we obtain $g(V(S_v)) \geq 2$ for each $v \in V(H)$. Therefore,

$$\gamma_I(T) = \omega(g) \geq \sum_{v\in V(H)} g(V(S_v)) \geq 2|V(H)|.$$

Moreover, we observe that the function h defined by $h(c_v) = 2$ for each $v \in V(H)$ and $h(x) = 0$ otherwise, is an IDF on T and so $\gamma_I(T) \leq \omega(h) = 2|V(H)|$. As a result, we obtain $\gamma_I(T) = 2|V(H)|$.

We next claim that $r_I(T) = \Delta$. By Theorem 4, we have $r_I(T) \leq \Delta$. Hence it is sufficient to show that $r_I(T) \geq \Delta$. Let F be an $r_I(T)$-set and f be a $\gamma_I(T + F)$-function.

Claim 1: If $f(V(S_v)) \leq 1$ for some $v \in V(H)$, then there exists an edge in F incident with a vertex in $V(S_v)$ assigned 0 under f.

Proof of Claim 1: Since $f(V(S_v)) \leq 1$, we have $f(c_v) \leq 1$ and there exists a leaf u of S_v such that $f(u) = 0$. Thus u must be incident with an edge in F and so this claim is true.

Claim 2: Let $f(V(S_v) \cup \{v\}) \leq 1$ for some $v \in V(H)$. Then

(*i*) *There exist at least $\Delta - 1$ edges in F incident with a vertex in $V(S_v) \cup \{v\}$ assigned 0 under f.*

(*ii*) *If the number of edges in F incident with a vertex in $V(S_v) \cup \{v\}$ is $\Delta - 1$, then $f(V(S_v)) = 1$ and no edge in F is incident with v.*

Proof of Claim 2: Since $f(V(S_v) \cup \{v\}) \leq 1$, we have $f(V(S_v)) + f(v) \leq 1$. If $f(V(S_v)) = 0$, then each vertex of S_v is assigned 0 under f and hence is adjacent with an edge in F. Suppose that $f(V(S_v)) = 1$. Obviously, $f(v) = 0$. Thus exactly $\Delta - 1$ vertices of S_v are assigned 0 under f and hence they are adjacent with an edge in F. This implies that this claim holds.

By Lemma 1(*ii*), we obtain $\gamma_I(T + F) = \gamma_I(T) - 1 = 2|V(H)| - 1$ and so there exists some vertex $v \in V(H)$ such that $f(V(S_v) \cup \{v\}) \leq 1$. If there exists some vertex $v' \in V(H)\backslash\{v\}$ such that $f(V(S_{v'}) \cup \{v'\}) \leq 1$, then by Lemma 1(*i*) and Claim 2(*i*), $|F| \geq 2(\Delta - 1) > \Delta$. Therefore, v is the unique vertex such that $f(V(S_v) \cup \{v\}) \leq 1$. If $f(V(S_v)) = 0$ or v is incident with some edge in F, then it follows from Claim 2(*i*) and (*ii*) that $|F| \geq \Delta$. Hence we may assume that $f(V(S_v)) = 1$ and no edge in F is incident with v. This implies that $f(v) = 0$ and $f(V(S_v) \cup \{v\}) = 1$. Moreover, since $\omega(f) = \gamma_I(T + F) = 2|V(H)| - 1$ and v is the unique vertex such that $f(V(S_v) \cup \{v\}) \leq 1$, this forces $f(V(S_{v'}) \cup \{v'\}) = 2$ for each $v' \in V(H)\backslash\{v\}$. Noting that $f(V(S_v)) = 1, f(v) = 0$ and no edge in F is incident with v, we have that there exists some vertex $u \in N_H(v)$ such that $f(u) \geq 1$, implying that $f(V(S_u)) \leq 1$ since $f(V(S_u) \cup \{u\}) = 2$. Using Claim 1, there exists an edge in F incident with a vertex in $V(S_u)$ assigned 0 under f. Moreover, since $f(V(S_v) \cup \{v\}) = 1$, we conclude from Claim 2(*i*) that there exists at least $\Delta - 1$ edges in F incident with a vertex in $V(S_v) \cup \{v\}$ assigned 0 under f. Recall that each edge in F is incident with exactly one vertex assigned 0 under f by Lemma 1(*i*). As a result, we have $r_I(T) = |F| \geq \Delta$, which completes our proof. □

III. SPECIAL CLASSES OF GRAPHS

In this section, we mainly obtain the exact value of $r_I(G)$ for some specific families of graphs, such as paths, cycles, complete multipartite graphs and ladders.

A. PATHS AND CYCLES

In order to determine the Italian reinforcement number of paths and cycles, we need the following well-known result due to Chellali *et al.* [10].

***Proposition** A: ([10]).*

(*i*) *For any integer $n \geq 1$, $\gamma_I(P_n) = \lceil (n + 1)/2 \rceil$.*

(*ii*) *For any integer $n \geq 3$, $\gamma_I(C_n) = \lceil n/2 \rceil$.*

Theorem 5: For any integer $n \geq 4$,

$$r_I(P_n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let $P_n = v_1v_2 \cdots v_n$. If n is even, then by Proposition A, we have $\gamma_I(P_n + v_1v_n) < \gamma_I(P_n)$ and hence $r_I(P_n) = 1$. Suppose next that n is odd. By Proposition A(*i*), it is easy to verify that the function g defined by $g(v_i) = 1$ for each odd i and $f(v_i) = 0$ for each even i, is the unique $\gamma_I(P_n)$-function. Then g and each vertex v_i do not satisfy one of the conditions (*i*) and (*ii*) of Theorem 1 and hence $r_I(P_n) \geq 2$. On the other hand, the function h defined by $h(v_i) = 0$ for each odd i and $h(v_i) = 1$ for each even i, is an IDF on $P_n + \{v_1v_{n-1}, v_2v_n\}$ with $\omega(h) = (n-1)/2 = \gamma_I(P_n) - 1$. Thus the set $\{v_1v_{n-1}, v_2v_n\}$ is an IR-set of P_n and so $r_I(P_n) \leq 2$. Consequently, we have $r_I(P_n) = 2$.

The proof is completed. □

Theorem 6: For any integer $n \geq 5$,

$$r_I(C_n) = \begin{cases} 2, & \text{if } n \text{ is odd,} \\ 3, & \text{if } n = 6, \\ 4, & \text{if } n \geq 8 \text{ is even.} \end{cases}$$

Proof: Let $C_n = v_0v_1 \cdots v_{n-1}v_0$. Suppose first that n is odd. Observe that the function g defined by $g(v_i) = 1$ for each even $i \leq n - 3$ and $g(v_i) = 0$ otherwise, is an IDF on $C_n + \{v_0v_{n-2}, v_2v_{n-1}\}$ and so by Proposition A(*ii*), $\omega(g) = (n - 1)/2 = \gamma_I(C_n) - 1$. Thus the set $\{v_0v_{n-2}, v_2v_{n-1}\}$ is an IR-set of C_n and so $r_I(C_n) \leq 2$. Hence it suffices to show that $r_I(C_n) \geq 2$. Suppose, to the contrary, that $r_I(C_n) = 1$. Using Theorem 1, we have that there exist a $\gamma_I(C_n)$-function $f = (V_0, V_1, V_2)$ and a vertex $v \in V_1$ satisfying one of the conditions (*i*) and (*ii*) given in Theorem 1. If (*i*) holds, then we may assume, without loss of generality, that $v_1, v_2 \in V_1, v_3 \in V_0$ and $v_4 \in V_2$. If (ii) holds, then we may assume, without loss of generality, that $v_2 \in V_1, v_1, v_3 \in V_0$ and $v_0, v_4 \in V_2$. In either case, the restriction f^* of f on $V(C_n)\backslash\{v_2\}$ is an IDF on $C_n - v_2$ ($\cong P_{n-1}$). Using Proposition A,

$$\frac{n+1}{2} = \gamma_I(C_n) = \omega(f) = \omega(f^*) + 1 \geq \gamma_I(P_{n-1}) + 1 = \frac{n+1}{2} + 1,$$

a contradiction. Therefore, we obtain $r_I(C_n) \geq 2$.

Suppose next that n is even. It is easy to see that $r_I(C_6) = 3$. Let $n \geq 8$. Using Theorem 3, we have $r_I(C_n) \leq \Delta + 2 = 4$. Hence it suffices to show that $r_I(C_n) \geq 4$. In the remainder of the proof, we emphasize that the index of each vertex of C_n is taken modulo n.

Let F be an $r_I(C_n)$-set and f be a $\gamma_I(C_n+F)$-function such that $V_2^f = \emptyset$. We first assume that C_n has three consecutive vertices $v_i, v_{i+1}, v_{i+2} \in V_0^f$. Then the following hold:

(a) For each $j \in \{i, i+2\}$, F has an edge joining v_j to a vertex assigned 1 under f.

(b) F has two edges joining v_{i+1} to two vertices assigned 1 under f.

As a result, we obtain $|F| \geq 4$. Hence we may assume that $\sum_{j=i}^{i+2} f(v_j) \geq 1$ for each $0 \leq i \leq n-1$. It follows from Lemma 1(ii) and Proposition A(ii) that $\gamma_I(C_n+F) = \gamma_I(C_n) - 1 = n/2 - 1$ and hence there exist two indices i_1 and i_2 such that $|i_1 - i_2| \geq 3$ and $\{v_{i_1}, v_{i_1+1}, v_{i_2}, v_{i_2+1}\} \subseteq V_0^f$. Moreover, since $V_2^f = \emptyset$, we have that F has an edge joining v_j to a vertex assigned 1 under f for each $j \in \{i_1, i_1+1, i_2, i_2+1\}$. As a result, we also obtain $|F| \geq 4$. So in the following we may assume that any $r_I(C_n)$-set F and any $\gamma_I(C_n+F)$-function f satisfy $V_2^f \neq \emptyset$.

Claim 3: There exists an $r_I(C_n)$-set F and a $\gamma_I(C_n+F)$-function f such that $v_0 \in V_2^f$ and every edge in F is incident with v_0.

Proof of Claim 3: Let F' be an $r_I(C_n)$-set and f be a $\gamma_I(C_n+F')$-function. From our earlier assumptions, we note that $V_2^f \neq \emptyset$. Without loss of generality, assume that $v_0 \in V_2^f$. Using Lemma 1(i), each edge in F' is incident with exactly one vertex assigned 0 under f. Let $X = \{v \in V_0^f :$ v is incident with an edge in $F'\}$ and let $F = \{vv_0 : v \in X\}\backslash E(C_n)$. It is easy to see that f is an IDF on C_n+F and hence by Lemma 1(ii), $\gamma_I(C_n+F) \leq \omega(f) = \gamma_I(C_n+F') = \gamma_I(C_n)-1$, implying that F is an IR-set of C_n. Moreover, since $|F| \leq |F'|$, we have that F is also an $r_I(C_n)$-set. Again by Lemma 1(ii), $\omega(f) = \gamma_I(C_n+F') = \gamma_I(C_n)-1 = \gamma_I(C_n+F)$ and hence f is also a $\gamma_I(C_n+F)$-function. As a result, F and f is a desired pair of an $r_I(C_n)$-set and a $\gamma_I(C_n+F)$-function. So, this claim is true.

Let F and f be defined as in Claim 3. We may choose f so that $|N_{C_n+F}(v_0) \cap V_0^f|$ is as large as possible. Suppose that $N_{C_n+F}(v_0) \nsubseteq V_0^f$. Let $v_s \in N_{C_n+F}(v_0)\backslash V_0^f$ for some $0 \leq s \leq n-1$. Moreover, since each edge in F is incident with exactly one vertex assigned 0 under f by Lemma 1(i), we have $s \in \{1, n-1\}$. Note that $d_{C_n+F}(v_s) = 2$. Let $N_{C_n+F}(v_s) = \{v_0, v_t\}$.

If $v_0v_t \in F$, or $v_0v_t \notin F$ and $f(v_t) \geq 1$, then the function f_1 defined by $f_1(v_s) = 0$ and $f_1(v_i) = f(v_i)$ otherwise, is an IDF on C_n+F with $\omega(f_1) \leq \omega(f) - 1 < \gamma_I(C_n+F)$, a contradiction. Assume that $v_0v_t \notin F$ and $f(v_t) = 0$. Observe that the function f_2 defined by $f_2(v_s) = 0, f_2(v_t) = f(v_s)$ and $f_2(v_i) = f(v_i)$ otherwise, is an IDF on C_n+F with $\omega(f_2) = \omega(f) = \gamma_I(C_n+F)$, and so f_2 is also a $\gamma_I(C_n+F)$-function. However, F and f_2 satisfy the properties of Claim 3 with $|N_{C_n+F}(v_0) \cap V_0^{f_2}| = |N_{C_n+F}(v_0) \cap V_0^f| + 1$, contradicting to the choice of f. As a result, we get $N_{C_n+F}(v_0) \subseteq V_0^f$.

Let $H = (C_n+F) - N_{C_n+F}[v_0]$ and let $H_1, H_2, \ldots, H_k$ be the connected components of H. Clearly $|V(H)| = n - |N_{C_n+F}[v_0]| = n - |F| - 3$ and H_i is a path for each $1 \leq i \leq k$. Observe that the restriction f^* of f on $V(H)$ is an IDF on H. Using Proposition A(i), we have

$$\omega(f^*) \geq \sum_{1 \leq i \leq k} \gamma_I(H_i) = \sum_{1 \leq i \leq k} \left\lceil \frac{|V(H_i)|+1}{2} \right\rceil. \quad (1)$$

Moreover, by Lemma 1(ii) and Proposition A(ii),

$$\omega(f^*) = \omega(f) - f(N_{C_n+F}[v_0]) = \gamma_I(C_n+F) - 2 = \gamma_I(C_n) - 3 = \frac{n}{2} - 3. \quad (2)$$

Combining (1) and (2), we obtain

$$\frac{n}{2} - 3 = \omega(f^*) \geq \sum_{1 \leq i \leq k} \left\lceil \frac{|V(H_i)|+1}{2} \right\rceil > \sum_{1 \leq i \leq k} \frac{|V(H_i)|}{2} = \frac{n - |F| - 3}{2},$$

implying that $r_I(C_n) = |F| \geq 4$, which completes our proof. □

B. COMPLETE MULTIPARTITE GRAPHS

According to the following results presented in [15], we derive the exact value of Italian domination number of a complete multipartite graph, based on which we shall determine its Italian reinforcement number.

Proposition B: *([15]) Let G be a graph of order $n \geq 3$. Then $\gamma_I(G) = 3$ if and only if one of the following holds*:

(i) $\Delta < n-2$ and $\gamma_2(G) = 3$, where $\gamma_2(G)$ is 2-domination number of G.

(ii) $\Delta = n-2$ and $\{v \in V(G) : d(v) = n-2\}$ is a clique.

Proposition C: *([15]) Let $G \vee H$ denote the join of two graphs G and H. Then $\gamma_I(G \vee H) \leq 4$. Moreover, if $k = \gamma_I(G) \leq \gamma_I(H)$, then*

(i) $k \leq 2$ if and only if $\gamma_I(G \vee H) = 2$.

(ii) $k = 3$ or $k = 4$ and $\gamma(H) = 2$ if and only if $\gamma_I(G \vee H) = 3$.

Using Propositions B and C, we can derive the following result.

Proposition 2: For any positive integers $n_1 \le n_2 \le \cdots \le n_t$ with $t \ge 2$,

$$\gamma_I(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} 2, & \text{if } 1 \le n_1 \le 2, \\ 3, & \text{if } n_1 = 3, \text{ or } n_1 \ge 4 \text{ and } t \ge 3, \\ 4, & \text{if } n_1 \ge 4 \text{ and } t = 2. \end{cases}$$

Theorem 7: For any positive integers $3 \le n_1 \le n_2 \le \cdots \le n_t$ with $t \ge 2$,

$$r_I(K_{n_1,n_2,\ldots,n_t}) = \begin{cases} n_1 - 1, & \text{if } n_1 = 3, \text{ or } n_1 \ge 4 \text{ and } t \ge 3, \\ n_1 - 2, & \text{if } n_1 \ge 4 \text{ and } t = 2. \end{cases}$$

Proof: Since $\Delta = |V(K_{n_1,n_2,\ldots,n_t})| - n_1$, we deduce from Theorem 3 and Proposition 2 that

$$\begin{aligned} r_I(K_{n_1,n_2,\ldots,n_t}) &\le |V(K_{n_1,n_2,\ldots,n_t})| - \Delta - \gamma_I(K_{n_1,n_2,\ldots,n_t}) + 2 \\ &= \begin{cases} n_1 - 1, & \text{if } n_1 = 3, \text{ or } n_1 \ge 4 \text{ and } t \ge 3, \\ n_1 - 2, & \text{if } n_1 \ge 4 \text{ and } t = 2. \end{cases} \end{aligned}$$

To prove the inverse inequality, let $X_1, X_2, \ldots, X_t$ be the partite sets of $K_{n_1,n_2,\cdots,n_t}$ with $|X_i| = n_i$ $(1 \le i \le t)$ and let $X_i = \{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$. We let F and $f = (V_0, V_1, V_2)$ be an $r_I(K_{n_1,n_2,\ldots,n_t})$-set and a $\gamma_I(K_{n_1,n_2,\ldots,n_t} + F)$-function, respectively.

Suppose first that $n_1 = 3$, or $n_1 \ge 4$ and $t \ge 3$. By Lemma 1(*ii*) and Proposition 2, $\omega(f) = \gamma_I(K_{n_1,n_2,\ldots,n_t} + F) = \gamma_I(K_{n_1,n_2,\ldots,n_t}) - 1 = 2$. Thus we have $|V_1| = 2$ and $|V_2| = 0$, or $|V_1| = 0$ and $|V_2| = 1$.

Assume now that $|V_1| = 2$ and $|V_2| = 0$. Without loss of generality, assume that $v_1^k, v_2^l \in V_1$ $(1 \le k, l \le t)$. If $k = l$, then $\{v_1^k, v_2^k\} \subseteq N_{K_{n_1,n_2,\ldots,n_t}+F}(v_i^k)$ for each $3 \le i \le n_k$ and so $|F| \ge 2(n_k - 2) \ge n_1 - 1$; and if $k \ne l$, then $\{v_2^k, v_3^k, \ldots, v_{n_k}^k\} \subseteq N_{K_{n_1,n_2,\ldots,n_t}+F}(v_1^k)$ and so $|F| \ge n_k - 1 \ge n_1 - 1$. Assume next that $|V_1| = 0$ and $|V_2| = 1$. Without loss of generality, assume that $v_1^k \in V_2$ $(1 \le k \le t)$. Obviously, $\{v_2^k, v_3^k, \ldots, v_{n_k}^k\} \subseteq N_{K_{n_1,n_2,\ldots,n_t}+F}(v_1^k)$ and so $|F| \ge n_k - 1 \ge n_1 - 1$.

Suppose second that $n_1 \ge 4$ and $t = 2$. By Lemma 1(*ii*) and Proposition 2, $\omega(f) = \gamma_I(K_{n_1,n_2,\ldots,n_t} + F) = \gamma_I(K_{n_1,n_2,\ldots,n_t}) - 1 = 3$. Thus we have $|V_1| = 3$ and $|V_2| = 0$, or $|V_1| = |V_2| = 1$.

Assume now that $|V_1| = 3$ and $|V_2| = 0$. Without loss of generality, assume that $v_1^k, v_2^k, v_3^l \in V_1$ $(1 \le k, l \le 2)$. If $k = l$, then $|N_{K_{n_1,n_2,\ldots,n_t}+F}(v_i^k) \cap \{v_1^k, v_2^k, v_3^k\}| \ge 2$ for each $4 \le i \le n_k$ and so $|F| \ge 2(n_k - 3) \ge n_1 - 2$; and if $k \ne l$, then $|N_{K_{n_1,n_2,\ldots,n_t}+F}(v_i^k) \cap \{v_1^k, v_2^k\}| \ge 1$ for each $3 \le i \le n_k$ and so $|F| \ge n_k - 2 \ge n_1 - 2$. Assume next that $|V_1| = |V_2| = 1$. Without loss of generality, assume that $v_1^k \in V_1$ and $v_2^l \in V_2$ $(1 \le k, l \le 2)$. If $k = l$, then $\{v_i^k : 3 \le i \le n_k\} \subseteq N_{K_{n_1,n_2,\ldots,n_t}+F}(v_2^k)$ and so $|F| \ge n_k - 2 \ge n_1 - 2$; and if $k \ne l$, then $\{v_i^l : 1 \le i \le n_l \text{ and } i \ne 2\} \subseteq N_{K_{n_1,n_2,\ldots,n_t}+F}(v_2^l)$ and so $|F| \ge n_l - 1 > n_1 - 2$.

The proof is completed. □

C. LADDERS

In this subsection, we restrict our attention to the ladder $P_2 \square P_n$, where $G \square H$ is the Cartesian product of two graphs G and H.

We emphasize that $V(P_2 \square P_n) = \{v_j^i : 1 \le i \le 2, 1 \le j \le n\}$ and $E(P_2 \square P_n) = \{v_j^1 v_j^2 : 1 \le j \le n\} \cup \{v_j^i v_{j+1}^i : 1 \le i \le 2, 1 \le j \le n-1\}$, throughout our argument. Let f be an IDF on $P_2 \square P_n$. Then for each $1 \le j \le n$, we denote $a_j = f(v_j^1) + f(v_j^2)$.

In order to determine the Italian reinforcement number of a ladder, we need the following result and some lemmas.

Proposition D: *([15]) For any integer $n \ge 2$, $\gamma_I(P_2 \square P_n) = n$.*

Next, we shall determine the Italian reinforcement number of $P_2 \square P_n$. Recall that if f is an IDF on $P_2 \square P_n$, then we denote $a_j = f(v_j^1) + f(v_j^2)$ for each $1 \le j \le n$.

Lemma 3: Let $n \ge 1$ be an integer and let f be an IDF on $P_2 \square P_n$. If $a_j \ge 2$ for some $j \in \{1, n\}$, then $\omega(f) \ge n + 1$.

Proof: By symmetry, it suffices to show that if $a_1 \ge 2$, then $\omega(f) \ge n + 1$. We proceed by induction on n. The basis step of the induction is obvious for $n = 1$. Assume that the result holds for any integer $1 \le n' < n$.

If $a_1 \ge 3$, then the function g defined by $g(v_1^1) = 0$, $g(v_1^2) = 1$, $g(v_2^1) = 2$ and $g(x) = f(x)$ otherwise, is an IDF on $P_2 \square P_n$ and the restriction g^* of g on $V(P_2 \square P_n) \backslash \{v_1^1, v_1^2\}$ is an IDF on $P_2 \square P_{n-1}$ with $g^*(v_2^1) + g^*(v_2^2) \ge 2$ and hence by the induction hypothesis, $\omega(f) \ge \omega(g) = \omega(g^*) + 1 \ge n + 1$. So in the following we may assume that $a_1 = 2$.

Case 1: $f(v_1^1) = 2$ and $f(v_1^2) = 0$ (the case $f(v_1^1) = 0$ and $f(v_1^2) = 2$ is similar).

If $a_2 \ge 2$, or $f(v_2^1) = 1$ and $f(v_2^2) = 0$, then it is easy to verify that the restriction f_1^* of f on $V(P_2 \square P_n) \backslash \{v_1^1, v_1^2\}$ is an IDF on $P_2 \square P_{n-1}$ and hence by Proposition D, $\omega(f) = \omega(f_1^*) + 2 \ge (n-1) + 2 = n + 1$. If $f(v_2^1) = 0$ and $f(v_2^2) = 1$, then the function g_1 defined by $g_1(v_1^1) = 1$ and $g_1(x) = f(x)$ otherwise, is also an IDF on $P_2 \square P_n$ and hence by Proposition D, $\omega(f) = \omega(g_1) + 1 \ge n + 1$. If $f(v_2^1) = f(v_2^2) = 0$, then $f(v_3^2) = 2$ and so the restriction f_2^* of f on $V(P_2 \square P_n) \backslash \{v_1^1, v_1^2, v_2^1, v_2^2\}$ is an IDF on $P_2 \square P_{n-2}$ and hence by the induction hypothesis, $\omega(f) = \omega(f_2^*) + 2 \ge (n-1) + 2 = n + 1$.

Case 2: $f(v_1^1) = f(v_1^2) = 1$.

If $a_2 = 0$, then $a_3 \ge 2$ and so the restriction f_3^* of f on $V(P_2 \square P_n) \backslash \{v_1^1, v_1^2, v_2^1, v_2^2\}$ is an IDF on $P_2 \square P_{n-2}$ and hence by the induction hypothesis, $\omega(f) = \omega(f_3^*) + 2 \ge (n-1) + 2 = n + 1$. If $a_2 \ge 2$, then the restriction f_4^* of f on $V(P_2 \square P_n) \backslash \{v_1^1, v_1^2\}$ is an IDF on $P_2 \square P_{n-1}$ and so by the induction hypothesis, $\omega(f) = \omega(f_4^*) + 2 \ge n + 2$. Suppose now that $a_2 = 1$. Without loss of generality, assume that $f(v_2^1) = 1$ and $f(v_2^2) = 0$. Then the function g_2 defined by $g_2(v_1^1) = 0$ and $g_2(x) = f(x)$ otherwise, is also an IDF on $P_2 \square P_n$ and hence by Proposition D, $\omega(f) = \omega(g_2) + 1 \ge n + 1$.

The proof is completed. □

Lemma 4: Let $n \geq 2$ be an integer and let f be an IDF on $P_2 \Box P_n$. If there exists some $k \in \{1, 2, \dots, n-1\}$ such that $a_k \geq 2$ and $a_{k+1} \geq 2$, then $\omega(f) \geq n+2$.

Proof: Observe that the restriction f_1^* of f on $\{v_j^i : 1 \leq i \leq 2, 1 \leq j \leq k\}$ is an IDF on $P_2 \Box P_k$ with $f_1^*(v_k^1) + f_1^*(v_k^2) = a_k \geq 2$ and the restriction f_2^* of f on $\{v_j^i : 1 \leq i \leq 2, k+1 \leq j \leq n\}$ is an IDF on $P_2 \Box P_{n-k}$ with $f_2^*(v_{k+1}^1) + f_2^*(v_{k+1}^2) = a_{k+1} \geq 2$. Using Lemma 3, we have $\omega(f) = \omega(f_1^*) + \omega(f_2^*) \geq (k+1) + (n-k+1) = n+2$, as desired. □

The proof of the next result is similar to the proof of Lemma 4 and therefore omitted.

Lemma 5: Let $n \geq 3$ be an integer and let f be an IDF on $P_2 \Box P_n$. If there exist integers $k, l \geq 1$ such that $a_k \geq 2$, $a_{k+l+1} \geq 2$ and $\sum_{i=1}^{l} a_{k+i} \geq l-1$, then $\omega(f) \geq n+1$.

Lemma 6: Let $n \geq 3$ be an integer and let f be an IDF on $P_2 \Box P_n$. If there exists some $k \in \{1, 2, \dots, n-2\}$ such that $a_k \geq 2$, $a_{k+1} \geq 1$ and $a_{k+2} \geq 1$, then $\omega(f) \geq n+1$.

Proof: Suppose that there exists some $k \in \{1, 2, \dots, n-2\}$ such that $a_k \geq 2$, $a_{k+1} \geq 1$ and $a_{k+2} \geq 1$. If $a_{k+1} \geq 2$ or $a_{k+2} \geq 2$, then by Lemma 4 or 5, we have $\omega(f) \geq n+1$. So in the following we may assume that $a_{k+1} = a_{k+2} = 1$. By symmetry, we may assume that one of the following holds:

(a) $f(v_{k+1}^1) = f(v_{k+2}^2) = 1$ and $f(v_{k+1}^2) = f(v_{k+2}^1) = 0$.

(b) $f(v_{k+1}^1) = f(v_{k+2}^1) = 1$ and $f(v_{k+1}^2) = f(v_{k+2}^2) = 0$.

Noting that $a_k \geq 2$, it is easy to check that the restriction f_1^* of f on $\{v_j^i : 1 \leq i \leq 2, 1 \leq j \leq k\}$ is an IDF on $P_2 \Box P_k$ with $f_1^*(v_k^1) + f_1^*(v_k^2) = a_k \geq 2$ and hence by Lemma 3, $\omega(f_1^*) \geq k+1$.

Suppose that (a) holds. Observe that the restriction f_2^* of f on $\{v_j^i : 1 \leq i \leq 2, k+1 \leq j \leq n\}$ is an IDF on $P_2 \Box P_{n-k}$ and hence by Proposition D, $\omega(f_2^*) \geq n-k$, implying that $\omega(f) = \omega(f_1^*) + \omega(f_2^*) \geq (k+1) + (n-k) = n+1$.

Suppose that (b) holds. Obviously, $f(v_{k+3}^2) \geq 1$. Then the restriction f_3^* of f on $\{v_j^i : 1 \leq i \leq 2, k+2 \leq j \leq n\}$ is an IDF on $P_2 \Box P_{n-k-1}$ and hence by Proposition D, $\omega(f_3^*) \geq n-k-1$, implying that $\omega(f) = \omega(f_1^*) + \omega(f_3^*) + a_{k+1} \geq (k+1) + (n-k-1) + 1 = n+1$, which completes the proof. □

Lemma 7: Let $n \geq 4$ be an integer and let f be an IDF on $P_2 \Box P_n$. If there exists some k such that $a_k = 1$, $a_{k+1} = 0$ and $a_{k+2} = 4$, then $\omega(f) \geq n+1$.

Proof: Since $a_k = 1$, we may assume that $f(v_k^1) = 1$ and $f(v_k^2) = 0$ by symmetry. Noting that $a_{k+2} = 4$, we have $f(v_{k+2}^1) = f(v_{k+2}^2) = 2$. Observe that the function g defined by $g(v_{k+1}^2) = 1$, $g(v_{k+2}^2) = 0$, $g(v_{k+3}^2) = \max\{1, f(v_{k+3}^2)\}$ and $g(x) = f(x)$ otherwise, is an IDF on $P_2 \Box P_n$ with $\omega(g) \leq \omega(f)$. Furthermore, we have $g(v_k^1) + g(v_k^2) = g(v_{k+1}^1) + g(v_{k+1}^2) = 1$ and $g(v_{k+2}^1) + g(v_{k+2}^2) = 2$, and so by symmetry and Lemma 6, $\omega(f) \geq \omega(g) \geq n+1$, as desired. □

Lemma 8: Let $n \geq 3$ be an integer and let f be a $\gamma_I(P_2 \Box P_n)$-function. If there exists some $k \geq 2$ such that $a_k = 1$ and $a_{k+1} = 0$, then $n \geq k+5$.

Proof: By symmetry, it suffices to show that if there exists some $k \geq 2$ such that $f(v_k^1) = f(v_{k+1}^1) = f(v_{k+1}^2) = 0$ and $f(v_k^2) = 1$, then $n \geq k+5$. To Italian dominate the vertices v_{k+1}^1, v_{k+1}^2, we must have $f(v_{k+2}^1) = 2$ and $f(v_{k+2}^2) \geq 1$. If $f(v_{k+2}^2) = 2$, then the function g_1 defined by $g_1(v_{k+1}^1) = g_1(v_{k+2}^2) = 1$, $g_1(v_{k+2}^1) = 0$, $g_1(v_{k+3}^1) = \max\{1, f(v_{k+3}^1)\}$ and $g_1(x) = f(x)$ otherwise, is an IDF on $P_2 \Box P_n$ with $\omega(g_1) \leq \omega(f) - 1$, a contradiction. Thus $f(v_{k+2}^2) = 1$.

Claim 4: $f(v_{k+3}^1) = f(v_{k+3}^2) = 0$.

Proof of Claim 4: If $n = k+2$, then Lemma 3 implies that $\omega(f) \geq n+1$, a contradiction to Proposition D. Therefore, $n \geq k+3$. Note that $a_{k+2} = 3$. If $a_{k+3} \geq 2$, then by Lemma 4, we have $\omega(f) \geq n+2$, a contradiction to Proposition D. If $a_{k+3} = 1$, then the function g_2 defined by $g_2(v_{k+1}^1) = 1$, $g_2(v_{k+2}^1) = 0$, $g_2(v_{k+3}^1) = f(v_{k+3}^1) + 1$ and $g_2(x) = f(x)$ otherwise, is also a $\gamma_I(P_2 \Box P_n)$-function with $g_2(v_{k+1}^1) + g_2(v_{k+1}^2) = g_2(v_{k+2}^1) + g_2(v_{k+2}^2) = 1$ and $g_2(v_{k+3}^1) + g_2(v_{k+3}^2) = 2$, and hence by symmetry and Lemma 6, $\omega(f) = \omega(g_2) \geq n+1$, a contradiction to Proposition D. Thus we have $a_{k+3} = 0$, implying that Claim 4 is ture.

Claim 5: $f(v_{k+4}^1) = 0$ and $f(v_{k+4}^2) = 1$.

Proof of Claim 5: Recall that $a_{k+2} = 3$. If $a_{k+4} \geq 2$, then by Lemma 5, we have $\omega(f) \geq n+1$, a contradiction to Proposition D. Therefore, $a_{k+4} \leq 1$. Moreover, since $f(v_{k+2}^2) = 1$ and $f(v_{k+3}^1) = f(v_{k+3}^2) = 0$, $f(v_{k+4}^2) \geq 1$. As a result, we have $f(v_{k+4}^2) = 1$ and $f(v_{k+4}^1) = 0$. Claim 5 follows.

Since $f(v_{k+3}^1) = f(v_{k+4}^1) = 0$ and $f(v_{k+4}^2) = 1$ by Claims 4 and 5, this forces $f(v_{k+5}^1) \geq 1$, implying that $n \geq k+5$, establishing the desired lower bound. The proof is completed. □

Now we are ready to state the main result of this subsection.

Theorem 8: For any integer $n \geq 3$,

$$r_I(P_2 \Box P_n) = \begin{cases} 3, & \text{if } n = 4, \\ 2, & \text{if } 3 \leq n \leq 9 \text{ and } n \neq 4, \\ 1, & \text{if } n \geq 10. \end{cases}$$

Proof: By a tedious check, we can verify that $r_I(P_2 \Box P_4) = 3$. If $n \geq 10$, then the function h_1 defined by

$$h_1(v_j^i) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j = 1, \text{ or } i = 1 \text{ and } j \geq 7 \text{ is odd}, \\ & \text{or } i = 2 \text{ and } j \neq 8 \text{ is even}, \\ 2, & \text{if } i = 1 \text{ and } j = 4, \\ 0, & \text{otherwise}, \end{cases}$$

is an IDF on $P_2 \Box P_n + \{v_4^1 v_8^2\}$ and so $\omega(h_1) = n-1 < \gamma_I(P_2 \Box P_n)$ by Proposition D, implying that the set $\{v_4^1 v_8^2\}$ is an IR-set of $P_2 \Box P_n$ and so $r_I(P_2 \Box P_n) = 1$. If $3 \leq n \leq 9$ and $n \neq 4$, then the function h_2 defined by

$$h_2(v_j^i) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \geq 4 \text{ is even}, \\ & \text{or } i = 2 \text{ and } j \geq 5 \text{ is odd}, \\ 2, & \text{if } i = 1 \text{ and } j = 2, \\ 0, & \text{otherwise}, \end{cases}$$

is an IDF on $P_2\Box P_n + \{v_2^1v_1^2, v_2^1v_3^2\}$ and so $\omega(h_2) = n - 1 < \gamma_I(P_2\Box P_n)$ by Proposition D, implying that the set $\{v_2^1v_1^2, v_2^1v_3^2\}$ is an IR-set of $P_2\Box P_n$ and hence $r_I(P_2\Box P_n) \le 2$.

It remains to show that if $3 \le n \le 9$ and $n \ne 4$, then $r_I(P_2\Box P_n) \ge 2$. Suppose, to the contrary, that $r_I(P_2\Box P_n) = 1$. By Theorem 1, there must exist a $\gamma_I(P_2\Box P_n)$-function $f = (V_0, V_1, V_2)$ and a vertex $v \in V_1$ satisfying one of the conditions (i) and (ii) in Theorem 1. For each $1 \le j \le n$, we let $a_j = f(v_j^1) + f(v_j^2)$.

Suppose now that (i) holds. Without loss of generality, assume that there exists some k such that one of the following holds: (a) $f(v_k^1) = f(v_k^2) = 1, f(v_{k-1}^1) = f(v_{k+1}^1) = 0$, $f(v_{k-2}^1)+f(v_{k-1}^2) \ge 2$ and $f(v_{k+1}^2)+f(v_{k+2}^1) \ge 2$. (b) $f(v_k^1) = f(v_{k+1}^1) = 1, f(v_{k+1}^2) = f(v_{k+2}^1) = 0, f(v_k^2) + f(v_{k+2}^2) \ge 2$ and $f(v_{k+2}^2) + f(v_{k+3}^1) \ge 2$.

Assume first that (a) is true. If $n = k$, then $a_n = 2$ and hence by Lemma 3, we have $\omega(f) \ge n + 1$, a contradiction to Proposition D. Thus $n > k$. Noting that $a_k = 2$, if $f(v_{k+1}^2) = 0$, then $a_{k+2} \ge 2$ and hence by Lemma 5, $\omega(f) \ge n + 1$; if $f(v_{k+1}^2) = 1$, then $f(v_{k+2}^1) \ge 1$ since $f(v_{k+1}^2)+f(v_{k+2}^1) \ge 2$, and hence by Lemma 6, $\omega(f) \ge n+1$; and if $f(v_{k+1}^2) = 2$, then by Lemma 4, $\omega(f) \ge n + 2$. In each case, we have a contradiction to Proposition D.

Assume second that (b) is true. If $n = k + 1$, then $f(v_{n-1}^2) = 2$ since $f(v_k^2) + f(v_{k+2}^2) \ge 2$, implying that the restriction f_1^* of f on $V(P_2\Box P_n)\backslash\{v_n^1, v_n^2\}$ is an IDF on $P_2\Box P_{n-1}$ and so by Lemma 3, $\omega(f) = \omega(f_1^*) + 1 \ge n + 1$, a contradiction to Proposition D. Thus $n > k + 1$. Note that $a_k \ge 1$ and $a_{k+1} = 1$. If $f(v_{k+2}^2) = 2$, then by symmetry and Lemma 6, we have $\omega(f) \ge n + 1$, a contradiction to Proposition D. Suppose that $f(v_{k+2}^2) = 1$. Since $f(v_k^2) + f(v_{k+2}^2) \ge 2$, $f(v_k^2) \ge 1$ and so $a_k \ge 2$. Moreover, since $a_{k+1} = a_{k+2} = 1$, we have $\omega(f) \ge n + 1$ by Lemma 6, a contradiction to Proposition D. Hence we may assume that $f(v_{k+2}^2) = 0$. Moreover, since $f(v_{k+2}^2) + f(v_{k+3}^1) \ge 2$ (resp., $f(v_{k+1}^2) = f(v_{k+2}^1) = 0$), we have $f(v_{k+3}^1) = 2$ (resp., $f(v_{k+3}^2) = 2$). Noting that $a_{k+1} = 1, a_{k+2} = 0$ and $a_{k+3} = 4$, it follows from Lemma 7 that $\omega(f) \ge n + 1$, a contradiction to Proposition D.

Suppose next that (ii) holds. Then $V_2 \ne \emptyset$. Without loss of generality, assume that there exists some k such that $f(v_k^1) = 1$, $f(v_{k-1}^1) = f(v_k^2) = f(v_{k+1}^1) = 0$, $f(v_{k-2}^1) + f(v_{k-1}^2) \ge 2$, $f(v_{k-1}^2) + f(v_{k+1}^2) \ge 2$ and $f(v_{k+1}^2)+f(v_{k+2}^1) \ge 2$.

Assume that $k = 1$ (the case $k = n$ is similar). Then clearly $f(v_2^2) = 2$. Observe that the restriction f_2^* of f on $V(P_2\Box P_n)\backslash\{v_1^1, v_1^2\}$ is an IDF on $P_2\Box P_{n-1}$ and hence by Lemma 3, $\omega(f) = \omega(f_2^*) + 1 \ge n + 1$, a contradiction to Proposition D. Consequently, we have $k \in \{2, 3, \ldots, n-1\}$.

Assume first that $f(v_{k+1}^2) = 0$ (the case $f(v_{k-1}^2) = 0$ is similar). Moreover, since $f(v_k^2) = f(v_{k+1}^1) = 0$ (resp., $f(v_{k+1}^2) + f(v_{k+2}^1) \ge 2$), this forces $f(v_{k+2}^2) = 2$ (resp., $f(v_{k+2}^1) = 2$). Noting that $a_k = 1, a_{k+1} = 0$ and $a_{k+2} = 4$, it follows from Lemma 7 that $\omega(f) \ge n + 1$, a contradiction to Proposition D.

Assume second that $f(v_{k+1}^2) = 2$ (the case $f(v_{k-1}^2) = 2$ is similar). Recall that $a_k = 1$ and $a_{k+1} = 2$. If $f(v_{k-1}^2) \ge 1$, then $a_{k-1} \ge 1$ and so by symmetry and Lemma 6, $\omega(f) \ge n+1$, a contradiction to Proposition D. Hence we may assume that $f(v_{k-1}^2) = 0$. Moreover, since $f(v_{k-2}^1)+f(v_{k-1}^2) \ge 2$, this forces $f(v_{k-2}^1) = 2$. Noting that $a_{k-2} \ge 2$, $a_{k-1}+a_k = 1$ and $a_{k+1} = 2$, we conclude from Lemma 5 that $\omega(f) \ge n + 1$, a contradiction to Proposition D.

Now we consider the last case that $f(v_{k-1}^2) = f(v_{k+1}^2) = 1$. Moreover, since $f(v_{k+1}^2) + f(v_{k+2}^1) \ge 2$, this forces $f(v_{k+2}^1) \ge 1$. Note that $a_k = a_{k+1} = 1$. If $a_{k+2} \ge 2$, then by symmetry and Lemma 6, we have $\omega(f) \ge n + 1$, a contradiction to Proposition D. Thus we have $a_{k+2} = f(v_{k+2}^1) + f(v_{k+2}^2) \le 1$, implying that $f(v_{k+2}^1) = 1$ and $f(v_{k+2}^2) = 0$. By symmetry, we obtain $f(v_{k-2}^1) = 1$ and $f(v_{k-2}^2) = 0$. This implies that $k \ge 3$.

We proceed to show that $a_j = 1$ for each $j \le k - 3$ and $j \ge k + 3$ by induction on j. By symmetry, it suffices to show that $a_j = 1$ for each $j \ge k + 3$.

Assume that $j = k + 3$. Recall that $a_{k+1} = a_{k+2} = 1$. If $a_{k+3} \ge 2$, then by symmetry and Lemma 6, we have $\omega(f) \ge n+1$, a contradiction to Proposition D. Noting that $a_{k+2} = 1$, if $a_{k+3} = 0$, then by Lemma 8, $n \ge (k + 2) + 5 \ge 10$ since $k \ge 3$, a contradiction to the assumption that $n \le 9$. Therefore, we obtain $a_{k+3} = 1$. Assume that the result holds for all $k + 3 \le j' < j$.

Note that $a_{k+2} = 1$ and if $k + 3 \le j' < j$, then by the induction hypothesis, $a_{j'} = 1$. If $a_j \ge 2$, then by symmetry and Lemma 6, $\omega(f) \ge n+1$, a contradiction to Proposition D; and if $a_j = 0$, then by Lemma 8, we have $n \ge (j - 1) + 5 = j + 4 > (k + 3) + 4 \ge 10$ since $k \ge 3$, a contradiction to the assumption that $n \le 9$. As a result, $a_j = 1$. Therefore, we have $a_j = 1$ for each $j \le k - 3$ and $j \ge k + 3$. Recall that $a_j = 1$ for each $k - 2 \le j \le k + 2$. This implies that $V_2 = \emptyset$, a contradiction to (ii). Therefore, we have that if $3 \le n \le 9$ and $n \ne 4$, then $r_I(P_2\Box P_n) \ge 2$, which completes our proof. □

IV. CONCLUSION

As a variation of domination, the Italian domination was introduced by Chellali *et al.* [10], where it was called Roman {2}-domination. This paper initiate the study of Italian reinforcement number in graphs. We give some sharp bounds on the Italian reinforcement number and we also determine exact values of Italian reinforcement number of several special graph classes including paths, cycles, complete multipartite graph and ladders.

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