IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received November 18, 2019, accepted December 12, 2019, date of publication December 17, 2019, date of current version December 27, 2019.

*Digital Object Identifier 10.1109/ACCESS.2019.2960378*

# Testing Deterministic Chaos: Incorrect Results of the 0-1 Test and How to Avoid Them

**WIESLAW MARSZALEK [1], MACIEJ WALCZAK [2], AND JAN SADECKI [1]**
[1]Institute of Computer Science, Opole University of Technology, 45-758 Opole, Poland
[2]SP ZOZ, ZOZ Głuchołazy (Hospital), 48-340 Głuchołazy, Poland

Corresponding author: Wieslaw Marszalek (w.marszalek@po.edu.pl)

**ABSTRACT** The 'false-negative' and 'false-positive' outcomes of the 0-1 test for chaos in continuous dynamical systems are described and analyzed in this paper. First, typical false outcomes of the 0-1 test for chaos are illustrated through several numerical examples of the solutions of chaotic continuous systems. Those examples are based on computation of the $K$ values in the 0-1 test ($0 \leq K \leq 1$) for a selection of two parameters, namely the $dt$, output step in the numerical solver, and the $T$ value (integer denoting the step of the output sample selection). The central role in the 'false-negative' outcome is played by the oversampling phenomenon in the 0-1 test, while the 'false-positive' results are possible for a complicated periodic signal having a spectrum with multiple frequencies. Analyzing the spectra of the signals is the key method to avoid the false outcomes and also an important tool in the process of reconstructing of chaotic attractors from the time series signals. The correct computing process for continuous dynamical systems and selection of the parameters $dt$ and $T$ depend on the analyzed system (dynamical model) and should always be preceded (or combined with) the frequency analysis of the examined signals. The computation of special multi-parameter ($n-$parameter; $n \geq 2$) bifurcation diagrams for the 0-1 test should, in most cases, be done by parallel computing, since, obtaining one such multi-parameter bifurcation diagram in practice requires solving of the underlying mathematical model (system of ODEs) millions of times.

**INDEX TERMS** Oscillatory chaotic and periodic circuits and systems, the 0-1 test for chaos, bifurcation diagrams, oversampling, reconstruction of chaotic attractors.

## I. INTRODUCTION: THE 0-1 TEST FOR CHAOS

The relatively new 0-1 test for chaos continues to interest more and more researchers dealing with various continuous and discrete oscillatory dynamical systems. The area of possible application of the test is quite wide, from engineering systems through financial markets, industry manufacturing processes, transportation systems, atmospheric signal analysis and weather prediction, biology, chemistry, medicine to energy harvesting. Typical continuous systems being tested with the 0-1 test are the well-known Lorenz, Rössler, Lü and Chua systems or circuits, oscillatory memristor and arc circuits, mechanical systems, stock market, chaotic plasma, epilepsy and traffic models, laser systems, interactions in industrial production models, experimental data and others [1]–[14]. The test can be applied to a time series obtained, for example, from an experiment, even when a mathematical model is unknown [15]. The reliability of the 0-1 test has been questioned in [16] for the cases of edge of chaos, weak chaos and $1/f^{\alpha}$ noise in which the 0-1 test for chaos may misclassify the signals. Apart from these border-like cases, in certain scenarios, the 0-1 test for chaos may give a 'false-negative' outcome and a chaotic system or signal is classified as a periodic one [17], [18]. On the other hand, it is also possible to obtain a 'false-positive' outcome, when a periodic signal is classified as a chaotic one. Both of such cases are, obviously, undesirable and lead to wrong conclusions.

In this paper we illustrate the false outcomes, focus on the 'false-negative' ones, and answer the question: what should be done to avoid those undesirable outcomes? The answer is surprisingly simple and an appropriate way of handling the 0-1 test is to implement the well-known discrete Fourier transform to identify the maximum frequency of the chaotic signal or time series. Our analysis applies to the following scenarios. First, we assume that a mathematical model of a continuous dynamical process is available in the form of a system of ordinary differential equations (ODEs). The system

The associate editor coordinating the review of this manuscript and approving it for publication was Chao-Yang Chen.

**TABLE 1. Results of 0-1 test for Lorenz system: $\sigma = 10$, $\rho = 28$, $\beta = 8/3$ [19].**

| dt | | | | | T |
|---|---|---|---|---|---|
| 0.005 | 0.01 | 0.02 | 0.04 | 0.08 | |
| -0.0202 | -0.0131 | -0.0250 | 0.1190 | **0.9963** | 1 |
| -0.0094 | -0.0144 | 0.0738 | **0.9954** | **0.9986** | 2 |
| -0.0158 | 0.1340 | **0.9948** | **0.9975** | | 4 |
| 0.1950 | **0.9767** | **0.9960** | | | 8 |
| **0.9830** | **0.9980** | | | | 12 |
| **0.9953** | **0.9981** | | | | 16 |
| **0.9981** | | | | | 20 |
| **0.9975** | | | | | 24 |
| **0.9981** | | | | | 28 |
| **0.9979** | | | | | 32 |

**TABLE 2. Results of 0-1 test for Rössler system: $a = b = 0.1$, $c = 14$.**

| dt | | | | | T |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.4 | 0.8 | 1.6 | |
| -0.0021 | 0.0721 | 0.3267 | 0.6847 | **0.9237** | 1 |
| 0.1069 | 0.1367 | 0.5454 | 0.8782 | **0.9830** | 2 |
| 0.2858 | 0.6805 | 0.8558 | **0.9863** | **0.9911** | 4 |
| 0.6957 | 0.8867 | **0.9579** | **0.9956** | | 8 |
| 0.8608 | **0.9459** | **0.9499** | | | 12 |
| 0.8880 | **0.9794** | **0.9871** | | | 16 |
| **0.9401** | **0.9467** | | | | 20 |
| **0.9391** | **0.9408** | | | | 24 |
| **0.9538** | **0.9830** | | | | 28 |
| **0.9867** | **0.9880** | | | | 32 |
| **0.9654** | | | | | 36 |
| **0.9524** | | | | | 40 |
| **0.9211** | | | | | 44 |
| **0.9321** | | | | | 48 |
| **0.9441** | | | | | 52 |
| **0.9814** | | | | | 56 |
| **0.9892** | | | | | 60 |
| **0.9858** | | | | | 64 |

is solved be a numerical solver with constant or variable step size and an output is created at uniform spacing, say $0, dt, 2dt, 3dt, \ldots$. Besides the $dt$ value, another important parameter, $T$, a positive integer, plays a crucial role when using the 0-1 test: from all possible output values we select every $T$ sample and the new sequence of discrete values is tested by the 0-1 test for chaos. An example of such a case obtained for electric arc circuits via numerical solutions in Matlab with the *options* feature is given in [19].

## II. PROBLEMS WITH USING THE 0-1 TEST FOR CHAOS

Problems with using the 0-1 test in the cases of edge of chaos, weak chaos and stochastic noise are well-reported in the literature, see, for example [16], and those with the oversampling phenomenon in [18], [19]. The later issue is related to the $dt$ step size and another parameter $T$, which is a specially chosen integer, preventing the oversampling phenomenon to occur. This paper presents an extension of some results from [19], in which the data shown in Table 1 has been described in details. The numbers are the $K$ values obtained by using the 0-1 test for chaos for the chaotic Lorenz system with parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The numbers written in red indicate erroneously a periodic system ($K$ is close to 0), while those in black (and bold) correctly identify the Lorenz system as a chaotic one ($K$ close to 1). Notice the peculiar combination of the $dt$ and $T$ values for the incorrect and correct outcomes of the 0-1 test for chaos. Also, paper [18] includes examples when a periodic signal with multiple frequencies in its spectrum can be erroneously classified as a chaotic with the $K$ parameter from the 0-1 test being close to 1. Thus, when the oversampling phenomenon occurs, the well-known chaotic systems of Lorenz, Rössler and Chua, may be classified as periodic by the 0-1 test, even when other tools (i.e. Lyapunov exponents, power spectrum) indicate otherwise. The values of $K$ in the empty spaces in Table 1 do not exist because of the assumed value of $t_{span}$ in $0 \leq t \leq t_{span}$ and the fact that $5,000$ discrete values were used for the 0-1 test [19].

*Example 1:* A similar result to that presented in Table 1 but obtained for Rössler system is illustrated in Table 2. It is well-known that Rössler system is chaotic for the coefficients $a = 0.1$, $b = 0.1$ and $c = 14$. The $K$ values shown in Table 2 were obtained from the 0-1 test by using *tspan=[0:dt:32000]*, initial condition *[-10, 0, 1]*, the *options* feature (in the *ode45* Matlab's solver) and various $dt$ and $T$ values. In addition to the 'false-negative' cases represented by the $K$ values written in red, we also marked the cases with $0.5 < K < 0.9$ with regular black fonts. As in Table 1, all values $K \geq 0.9$ are written in bold (black) font.

As the next case in this example consider the chaotic arc circuit described in Appendix A. The result of using the 0-1 test with *tspan=[0:dt:10000]*, initial conditions *[0.5, 4.0, 1.0]*, the *options* feature and various $dt$ and $T$ values is shown in Table 3. The parameters of the model were $L = 0.147$, $C = 4.437$, $R = 15$ and $m = -2/3$ for which the system is a chaotic one (see [20]). Notice the smaller values of $dt$ and larger values of $T$ used in this case compared with the previous two cases of Lorenz and Rössler systems. The cases with values of $K$ highlighted in yellow and green in the three tables are analyzed in more detail below. ♦

The three cases of the Lorenz, Rössler and electric arc circuits discussed above indicate that the correct outcomes of the 0-1 test are obtained with the proper value of the product $dt \cdot T$. It turns out that the product $dt \cdot T$ is related to the chaotic signal's maximum frequency. We now illustrate this fact by analyzing the graphs in Figs. 1 and 2. The first row in Fig. 1 shows the time series, its DFT and the outcome of the 0-1 test for the Lorenz system with $dt = 0.01$ and $T = 2$. Notice that the maximum frequency of the signal is $f_{max} \approx 13$ Hz and the 0-1 test yields $K \approx 0$. The actual value of $K$ is highlighted in yellow in Table 1. The outcome is 'false-negative'. Since $dt = 0.01$, therefore $f_s/2 = 1/(2 \cdot dt) = 50$ Hz in this

**TABLE 3.** Results of 0-1 test for electric arc circuits described by (1) in Appendix A with $L = 0.147$, $C = 4.437$, $R = 15$, $m = -2/3$.

| dt | | | | | T |
|---|---|---|---|---|---|
| 0.001 | 0.002 | 0.004 | 0.008 | 0.016 | |
| -0.0488 | -0.0221 | -0.1066 | -0.1318 | 0.1325 | 1 |
| 0.0958 | -0.0217 | 0.0225 | -0.0028 | 0.0066 | 25 |
| 0.0429 | -0.0559 | -0.0031 | 0.0096 | **0.9138** | 50 |
| -0.0239 | -0.0099 | 0.0067 | 0.6179 | **0.9942** | 100 |
| -0.0164 | -0.0062 | 0.3964 | **0.9929** | | 200 |
| 0.0154 | 0.1909 | **0.9838** | | | 300 |
| 0.0003 | 0.8574 | **0.9934** | | | 400 |
| 0.0201 | **0.9579** | **0.9938** | | | 500 |
| 0.0461 | **0.9696** | | | | 600 |
| 0.8616 | **0.9921** | | | | 700 |
| **0.9294** | **0.9922** | | | | 800 |
| **0.9338** | **0.9931** | | | | 900 |
| **0.9747** | **0.9924** | | | | 1000 |
| **0.9823** | | | | | 1100 |
| **0.9797** | | | | | 1200 |
| **0.9900** | | | | | 1300 |
| **0.9938** | | | | | 1400 |
| **0.9858** | | | | | 1500 |
| **0.9942** | | | | | 1600 |
| **0.9944** | | | | | 1700 |
| **0.9941** | | | | | 1800 |
| **0.9946** | | | | | 1900 |
| **0.9945** | | | | | 2000 |

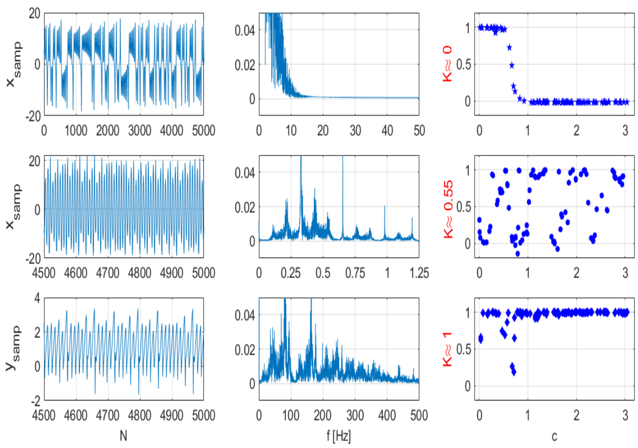



**FIGURE 1.** Results of the 0-1 test for Lorenz (first row) Rössler (middle row) and electric arc circuits (bottom row) for selected values of $dt$ and $T$. The time series are shown in the first column (only the last 500 samples are shown for the Rössler and arc circuit cases). The DFTs are shown in the second column, while the graphs of $K_c$'s and the final $K$ values from the 0-1 test (see Appendix B) are shown in the third column.

case. Also, the values of $c$ in the 0-1 test are chosen randomly from the interval $(0, \pi)$ [1]. The ratio $13/50$ translates into the corresponding graph in the third column as $(13/50)\pi = 0.82$. Notice from the third graph in the first row in Fig. 1 that the $K_c$ values ($K = \text{median}[K_1, \ldots, K_{100}]$, $c = 1, \ldots, 100$, [1]) are close to 1 exactly in the interval $c \in (0, 0.82)$ and they are close to 0 in the interval $c \in (0.82, \pi)$. Since the later interval contain more $c$ values than the former one, therefore the median of all $K_c$ values is close to 0. Thus the 0-1 test results in the incorrect 'false-negative' outcome. Table 1 indicates that for $dt = 0.01$ one obtains the correct outcome (that is the values of $K$ close to 1) for $T$ equal 8, 12 and 16 (see the green highlighted values in Table 1). Of course, other values of $T$ greater than 8, say 9, 10, 11, 13, etc. will also yield the correct outcomes. Such cases were not analyzed in this paper. It may also be expected that the minimum value of $T$ guaranteeing a correct outcome is probably equal 7.

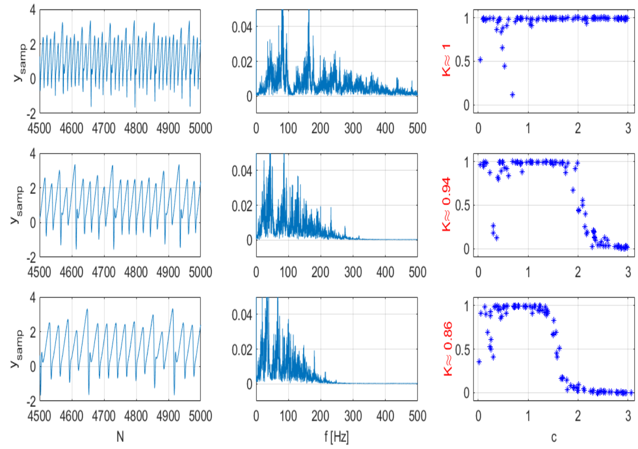



**FIGURE 2.** Results of the 0-1 test for the electric arc circuit and selected values of $T$ =1700 (top), 900 (middle) and 700 (bottom). The $dt = 0.001$ in all three cases.

The third row in Fig. 1 is for the electric arc chaotic system with $dt = 0.001$ and $T = 1700$, thus $f_s/2 = 500$ Hz. From the second graph in the third row in Fig. 1 we obtain $f_{max} \approx 500$ Hz, and $\pi f_{max}/(f_s/2) = \pi$. Thus, for almost all values $c \in (0, \pi)$ we have $K_c \approx 1$, and the median value of all those 100 values of $K_c$ is close to 1. The 0-1 test outcome is correct. See the yellow highlighted value of $K$ in Table 3. Notice that much smaller values of $T$ in this case will also yield a correct outcome. It follows from the first column in Table 3 that the values of $T$ greater or equal 700 will all yield the correct outcome - the 0-1 test will result in $K$ values close to 1 (for $dt = 0.001$).

Finally, the second row in Fig. 1 illustrates the *borderline* outcome of the 0-1 test for the Rössler system. The $dt = 0.4$, yielding $f_s/2 = 1.25$ Hz. The second graph in the middle row in Fig. 1 shows that while the maximum frequency of the analyzed signal is close to 1.25 Hz, in the interval $(0, 1.25)$ Hz we have certain gaps, for example the intervals $(0, 0.1)$ Hz, $(0.83, 0.93)$ Hz and others, without any frequency components in the spectrum of that signal. The corresponding $c$ intervals result in $K_c \approx 0$. On the other hand, for the respective intervals of $c$ in the third graph of the second row for which the spectrum is non-zero we have $K_c \approx 1$. Overall, we get that median value of all the one hundred values of $K_c$ is approximately in the middle between 0 and 1, as the yellow highlighted value of 0.5454 indicates in Table 2.

Increasing the value of $T$ for the fixed value $dt = 0.4$ will result in the correct values of $K$ from the 0-1 test, indicating chaotic Rössler system. Those cases have their $K$ values highlighted in green in Table 2.

Notice the one-to-one correspondence shown in Fig. 2 between the spectrum of the analyzed signal and the intervals where the $K_c$ values from the 0-1 test are close to 0 or 1. The first row in Fig. 2 is equivalent to the last row in Fig. 1. However, different seed values of the *rand* generator in Matlab were used in the two runs of the 0-1 test, resulting in a slightly different sequences of the $K_c$ values in both runs. The outcomes of both runs are correct though, as we get $K \approx 1$ in both cases. The second row in Fig. 2 is for $T = 900$. Notice, that the second graph in the second row in Fig. 2 indicates the maximum frequency $f_{max} = 300$ Hz. This translates into the interval $(0, 0.60)\pi$ in the third graph in the second row, where the $K_c$ values are close to 1. The value of 0.60 comes from $f_{max}/(f_s/2) = 300/500 = 0.60$. There are no spectral components in the frequency range $(300, 500)$ Hz, so the values of $K_c$ in the corresponding interval for $c$, that is in the interval $(0.60, 1)\pi$ are close to 0. Overall, there are more values of $K_c$ close to 1 than those close to 0, so the median value is $K = 0.94$, as indicated by the corresponding value of 0.9338 in the first column in Table 3. The third row in Fig. 2 illustrates the case with $T = 700$, which is practically the border line for the 0-1 test for chaos. Again, the one-to-one correspondence between the lengths of the non-zero spectrum interval in the second graph of the third row and the length of the interval where the values of $K_c$ are close to 1 in the third graph in the third row are clearly visible. The median of those $K_c$ values is $K = 0.86$. Reducing the $T$ value slightly below 700 results in an incorrect outcome of the 0-1 test. For example, for $T = 600$ we get $K = 0.0461$ (see column one in Table 3).

*Example 2:* The 0-1 test has also been applied to create two-parameter bifurcation diagrams for the electric arc circuit as shown in Fig. 3. Those diagrams are of size $600 \times 600$, that is 600 discrete values of each of the two parameters, $L$ and $C$, have been used. The ODE model of the circuit has been solved $360,000$ times. For each pair of discrete values of the two parameters the ODE system was solved in the interval $0 \leq t \leq 10000$ and the solutions were output with the constant time step of $dt = 0.001$ to form various sequences being analyzed with the 0-1 test. Notice that the pair of parameters $(L, C) = (0.147, 4.437)$ used in the third case in Example 1, is inside the range of the parameters $L$ and $C$ considered in this example (see Fig. 3). As in the previous example (with the values of $K$ close to 1 obtained for $T \geq 700$), the results shown in Fig. 3 confirm that the excellent quality picture obtained for $T = 1700$ deteriorates with the decreased $T$ values. The outcomes of the 0-1 test for $T$ less than 700 are problematic, as the 0-1 test fails to identify chaotic signals represented by the white color and $K \approx 1$. Practically, no values of $K$ greater than 0.9 are obtained in the diagrams shown in Figs. 3(d)-3(f). ♦

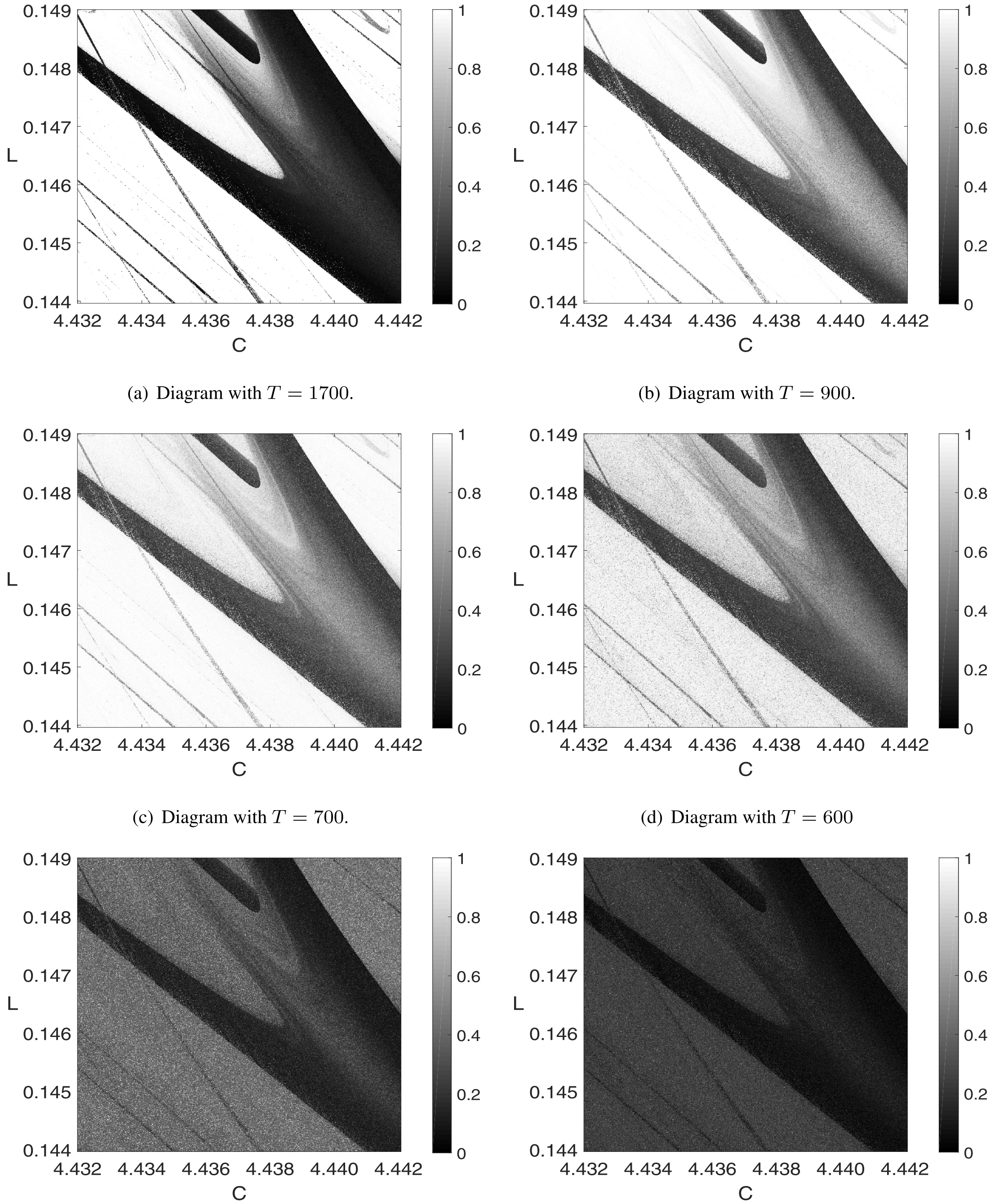
## III. PRACTICAL HINTS FOR THE USE OF 0-1 TEST FOR CHAOS

The examples presented in the previous section, the fact that the relatively new test 0-1 for chaos relies on the randomly chosen parameters $c \in (0, \pi)$ [2], and the fact that $c = 2\pi f/f_s$ in the 0-1 test [18], all show that there is a one-to-one relation between the outcome of the 0-1 test and the frequency analysis of the tested signal or solution of a system of ODEs. The values of $c$, spread out uniformly in the interval $c \in (0, \pi)$ in the 0-1 test and work as *identifiers* of the frequency components in the analyzed signal. If a **chaotic** signal with spectrum $f \in (0, f_{max})$ is analyzed by the 0-1 test and the output sampling rate is $f_s = 1/dt$, then

- if $2\pi f_{max}/f_s < \pi/2$ (or equivalently if $f_s > 4f_{max}$), then such a chaotic signal will always result in $K \approx 0$, since the number of $K_c$ values close to 1 is smaller than the number of $K_c$ values close to 0. This yields the median $K = median\{K_c\} \approx 0$. This is the 'false-negative' outcome. Those are all the cases with $dt$ and $T$ values that have their corresponding $K$ values written in the red color ($K < 0.5$ in Tables 1-3). This is also the graphing case illustrated in the first row in Fig. 1 (solution of the Lorenz system with $dt = 0.01$ and $T = 2$). All such cases are commonly referred to as the *oversampling* cases of the 0-1 test.
- if $f_s < 4f_{max}$, then the 0-1 test for chaos will yield a correct result, that is $K \approx 1$ for a chaotic signal. This case is illustrated by all the cases with $dt$ and $T$ yielding $K$ close to 1 in Tables 1 through 3 and by the cases in Figs. 1 and 2 with $K \approx 1$.
- if $f_s \approx 4f_{max}$, then the outcome of the 0-1 test is problematic (or a borderline case for using the 0-1 test). This is illustrated by those cases with $dt$ and $T$ yielding $K$ values greater than approximately 0.25 and smaller than 0.75. See the cases illustrated by the second row in Fig. 1 and also the third row in Fig. 2. The given value of 0.25 and 0.75 are somewhat blurry, may be different for different individuals, as they, in fact, depend of subjective perception.

The above analysis gives the following practical hints about the 0-1 test for chaos in continuous systems. When one expect to deal with a **chaotic** signal, then it is advisable to check the signal's spectrum.

1) Continuous nature of that spectrum in a certain frequency interval is the first indication of a possibility of having a chaotic signal.
2) Second, if the continuous spectrum is, say within the lowest frequency $f_0$ and maximum frequency $f_{max}$, then, for the chosen $f_s = 1/dt$, the length of the corresponding $c$ interval should be such that $\Delta c = 2\pi(f_{max} - f_0)/f_s > \pi/2$ (or equivalently $f_s < 4(f_{max} - f_0)$). This will guarantee that the number of discrete values of $c$ with their corresponding $K_c$



(a) Diagram with $T = 1700$.

(b) Diagram with $T = 900$.

(c) Diagram with $T = 700$.

(d) Diagram with $T = 600$

(e) Diagram with $T = 500$

(f) Diagram with $T = 450$

**FIGURE 3.** **Two-parameter diagrams with varying *L* and *C*, various values of *T* and constant *dt* = 0.001, *R* = 15. Obtaining the above and similar two-parameter diagrams of larger than 600 × 600 sizes requires using parallel computation. Details of such computation are provided in [19], [21].**

values close to 1 will be more than 50% of all values of $c$ in the interval $(0, \pi)$. This will further yield the median to be close to 1, indicating correctly the chaotic nature of the analyzed signal.

3) If the condition $f_s < 4(f_{max} - f_0)$ is not satisfied, then increasing the $T$ value, that is creating a signal with increased value of $f_{max} - f_0$ will eventually make the condition $f_s < 4(f_{max} - f_0)$ to be satisfied. This concept is clearly illustrated and supported by the data in Tables 1 through 3, and also by the graphs in Fig. 2.

Finally, when a complex **periodic** signal with multiple individual frequencies (the number must be more than 50 when 100 values of $c$ are used in the 0-1 test) is misclassified by the 0-1 test as a chaotic one (see [18]), then the spectrum of such a periodic signal has a different nature - is not continuous as in the chaotic case - therefore a spectrum analysis should be sufficient to identify the false outcome of the 0-1 test.

## IV. CONCLUSION

We have analyzed possibilities of false outcomes of the 0-1 test for chaos and presented practical hints to avoid such outcomes. The recommended steps are based on using DFTs of the analyzed signal to establish the range of its spectrum. Such a range is further used to identify a proper combination of two parameters, $dt$ and $T$, that are crucial in a creation of time series sequences tested by the 0-1 test. This confirms, once again, that while the 0-1 test is simple, it can rarely be used alone in testing oscillatory circuits, systems and signals.

Another conclusion one can draw, based on a visual inspection of the diagrams in Fig. 3, is that the oversampling phenomenon in the 0-1 test for chaos results in blurred diagrams of the two-parameter bifurcation diagrams. While the spread of the gray levels in Fig. 3a is full (from 0 to 255, as described in [19]), such a spread of gray levels decreases and becomes narrower with the decreased $T$ value. This indicates the increased oversampling problem in the 0-1 test for chaos.

Finally, it is worth mentioning that the method of analyzing oscillatory signals presented in this paper (based on both DFT and appropriate choice of the parameters $dt$ and $T$) is important when one wants to reconstruct chaotic attractors based on the time series $x(t)$ and its values $x(n \cdot dt \cdot T)$. As shown in [18] for the Lorenz system and Chua's modified circuit, for an assumed $dt$ value, an excellent reconstruction result is obtained with the pair $\{x(t), x(t+T)\}$, where $T$ is the parameter used in this paper. Further work in this direction is under way.

## APPENDIXES

## APPENDIXA

## ELECTRIC CHAOTIC ARC CIRCUITS

The electric arc circuits from [19] are used in Examples 1 and 2 in this paper. The circuits are described by the three equations on the left side of (1) below and their equivalent versions are given on the right side [20]

$$\begin{aligned} \frac{di}{d\tau} &= \frac{1}{L}\left(u_C - \frac{U(i_\theta)}{i_\theta} i\right) & \frac{dx}{dt} &= \frac{1}{L}(y - xz^m) \\ \frac{du_C}{d\tau} &= \frac{1}{RC}(E - u_C - Ri) \rightarrow & \frac{dy}{dt} &= \frac{1}{RC}(R+1-y-Rx) \\ \frac{di_\theta^2}{d\tau} &= \frac{1}{\theta}(i^2 - i_\theta^2) & \frac{dz}{dt} &= x^2 - z \end{aligned} \quad (1)$$

where $i$, $u_C$, $i_\theta$ are the arc current, current through $L$ (inductance) and voltage across $C$ (capacitance), respectively. The three variables correspond to the variables $x$, $y$ and $z$ on the right side of (1). The $R$, $L$, $C$ and $m$ are the parameters used in this paper in the bifurcation analysis of the arc circuits through the 0-1 test for chaos. Color bifurcation diagrams obtained for (1) by using a different method are presented in [20].

## APPENDIX B

## THE 0-1 TEST FOR CHAOS

The test results have two forms: a single real number $0 \le K \le 1$, and a two-dimensional graph of variables $(p_c(n), q_c(n))$ [1], [18]. When a chaotic sequence is fed into the test, the number $K$ should be close to 1, while for regular sequences the number $K$ is close to 0. There are two methods to compute $K$: regression or correlation. For a time-series $\{N_k\}$, $k = 0, \ldots, \overline{N} - 1$, with the recommended value $\overline{N} = 5000$, the $p_c$ and $q_c$ are computed by

$$p_c(n) = \sum_{j=0}^{n} N_j cos[(j+1)c], \quad q_c(n) = \sum_{j=0}^{n} N_j sin[(j+1)c] \quad (2)$$

with $n = 0, \ldots, \overline{N} - 1$ and a randomly chosen real number $c \in (0, \pi)$. Then, the quantity $M_c(n)$, $n = 0, 1, \ldots, n_{cut}$, called the mean square displacement of $p_c(n)$ and $q_c(n)$, is computed as follows

$$M_c(n) = \lim_{\overline{N}\to\infty} \frac{1}{\overline{N} - 1} \sum_{j=0}^{\overline{N}-1} [p_c(j+n) - p_c(j)]^2 + [q_c(j+n) - q_c(j)]^2 \quad (3)$$

with the recommended value $n_{cut} \approx (\overline{N} - 1)/10$. If the regression method is applied, then the $K_c$ value, the asymptotic growth rate of the mean square displacement, is computed as follows

$$K_c = \lim_{n\to\infty} \frac{log\, M_c(n)}{log\, n}. \quad (4)$$

For the correlation method, two vectors $\xi = (0, 1, 2, \ldots, n_{cut})$ and $\Delta = (M_c(0), M_c(1), M_c(2), \ldots, M_c(n_{cut}))$ are created and the correlation coefficient $K_c$ is obtained as follows

$$K_c = corr(\xi, \Delta) \equiv \frac{cov(\xi, \Delta)}{\sqrt{var(\xi) var(\Delta)}} \quad (5)$$

with the $cov$ and $var$ denoting the covariance and variance, respectively [1].

In both the regression and correlation methods the above steps are repeated for $N_c$ values of $c$ chosen randomly from the interval $(0, \pi)$. For the random selection of $N_c$ values of $c \in (0, \pi)$ one can use the $rand$ function in Matlab. It is recommended that $N_c = 100$. Computing the median of the $N_c$ values of $K_c$ gives the number $K$. The $K \approx 1$ indicates a chaotic sequence, while $K \approx 0$ indicates regular (non-chaotic) dynamics. The third columns of Figs. 1 and 2 show the graphs of $N_c = 100$ values of $K_c$, as well as the final value of $K$. All sequences tested in this paper have length of 5000 real values. They were the solutions of the Lorenz, Rössler and arc circuits in Example 1 (Figs. 1 and 2), and exclusively the arc circuits in Example 2 (Fig. 3).

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**WIESLAW MARSZALEK** received the D.Sc. and Ph.D. degrees in electrical engineering from the Warsaw University of Technology, Poland, and the Ph.D. degree in applied mathematics from North Carolina State University, Raleigh, NC, USA. In the past, he held various research and teaching positions at the state and private universities in North Carolina, Ohio, and New Jersey (USA). He was a Humboldt Research Fellow in Bochum, from 1990 to 1992, Hamburg, in 1996, and Halle, in 2011, Germany and a Fulbright Research Fellow, Warsaw, from 2005 to 2006 and Opole, in 2018, Poland. He currently teaches computer science with the Opole University of Technology, Poland, and mathematics with Rutgers University, New Jersey, USA. He has published over 100 journal and conference papers in the areas of 2D discrete equations, differential-algebraic equations, nonlinear circuits and electric arcs, memristors, singularity induced bifurcations, and chaos.

**MACIEJ WALCZAK** received the M.Sc. degree in computer networks and database management. He is currently pursuing the Ph.D. degree with the Institute of Computer Science, Opole University of Technology, Opole, Poland, in 2011. He is currently a Computer Network Administrator with SP ZOZ, ZOZ Głuchołazy (Hospital), Głuchołazy, Poland. His past research interests were in the areas overlapping phenomenon and static optimization using parallel computing. He is currently interested in using parallel computing in the analysis of dynamical systems. In his spare time he enjoys traveling and reading about astronomy.

**JAN SADECKI** received the D.Sc. and Ph.D. degrees in automatic control and robotics from the Warsaw University of Technology, Poland, in 1988 and 2004, respectively. He is currently with the Institute of Computer Science, Department of Electrical Engineering, Opole University of Technology, Opole, Poland. From 1992 to 1993, he spent a sabbatical year with the Centre for Mathematical Software Research, University of Liverpool, U.K. From 2012 to 2016, he was an Associate Chair of the Department of Electrical Engineering. Since 2017, he has been the Director of the Institute of Computer Science, Opole University of Technology. His research and teaching interests are in the area of parallel computing, distributed computing networks, and optimization.