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Low-Hit-Zone Frequency/Time Hopping Sequence Sets With Large Family Size

XING LIU [1,2], **QIN SHU** [1], **AND QI ZENG** [1]
[1]College of Electrical Engineering, Sichuan University, Chengdu 610065, China
[2]College of Mathematics, Sichuan University, Chengdu 610064, China

Corresponding author: Xing Liu (liuxing4@126.com)

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ABSTRACT For a quasi-synchronous frequency/time hopping (FH/TH) code-division multiple-access (CDMA) system which involves many users, the low-hit-zone (LHZ) FH/TH sequence set with large family size is necessary. In this paper, an upper bound on the family size of LHZ FH/TH sequence sets is derived. The new bound includes the Singleton bound on FH sequence sets as a special case. Further, three constructions of LHZ FH/TH sequence sets are presented, which have optimal/near optimal family size and also optimal maximum Hamming correlation (MHC) with respect to the new bound. They have very large family size compared with the known LHZ FH/TH sequence sets in the literature.

INDEX TERMS Frequency hopping sequences, time hopping sequences, low hit zone, Hamming correlation, theoretical bound.

I. INTRODUCTION

In quasi-synchronous frequency/time hopping (FH/TH) code-division multiple-access (CDMA) systems, the time delay can be limited in low hit zone (LHZ). In LHZ, FH/TH sequences can be designed with low Hamming correlation value [18], [30]. If the time delay is controlled within LHZ, then interferences can be reduced effectively. For a quasi-synchronous FH/TH CDMA system which involves many users, the design of LHZ FH/TH sequence sets with large family size is necessary. Although there are many FH/TH sequence sets studied in the literature [1]–[3], [5], [6], [8], [11], [12], [14], [19], [23], [25], [27], [31], [32], most of them have small family size. In [7], [9], [10], [16], [17], [26], FH/TH sequence sets with large family size are obtained, but they are not LHZ FH/TH sequence sets. These FH/TH sequence sets are not suitable for quasi-synchronous FH/TH CDMA systems.

In recent years, some LHZ FH sequence sets are reported in the literature. In 2006, Peng *et al.* [24] established lower bounds on the maximum Hamming correlation (MHC) of LHZ FH sequence sets (Peng-Fan-Lee bounds). In 2011, Ma and Sun [20] obtained some LHZ FH sequence sets which have almost optimal or optimal MHC with respect to the Peng-Fan-Lee bounds. The next year Niu *et al.* [21] constructed LHZ FH sequence sets with optimal MHC by interleaving techniques. Chung and Yang [4] utilized Cartesian products to construct LHZ FH sequence sets with optimal MHC in 2013. In the same year, Niu *et al.* [22] also used interleaving techniques to construct new LHZ FH sequence sets with optimal MHC. In 2017, Han *et al.* [13] obtained LHZ FH sequence sets with optimal MHC based upon m sequences. In November of the same year, Zhou *et al.* [29] also utilized m sequences to get two classes of LHZ FH sequence sets with optimal MHC. The same month, Zhou *et al.* [28] presented several classes of LHZ FH sequence sets with optimal MHC also by Cartesian products. Note that all the LHZ FH sequence sets in the literature have optimal or almost optimal MHC according to the Peng-Fan-Lee bounds. However, they have small family size which are not suitable for quasi-synchronous FH/TH CDMA systems involving many users.

In this paper, we pay our attention to LHZ FH/TH sequence sets with large family size. We first derive an upper bound on the family size of LHZ FH/TH sequence sets and then construct three classes of LHZ FH/TH sequence sets which have large family size. The LHZ FH/TH sequence sets have optimal/near optimal family size and also optimal MHC with

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respect to the new bound. They are suitable for many users to share limited bandwidth in quasi-synchronous FH/TH CDMA systems.

The rest of this paper is organized as follows. In Section II, we give some definitions and notations. In Section III, we establish an upper bound the family size of LHZ FH/TH sequence sets. In Section IV, we construct two classes of LHZ FH/TH sequence sets by utilizing cyclic codes, which have optimal family size and also optimal MHC with respect to the new bound. In Section V, we construct a class of LHZ FH/TH sequence sets, which have near optimal family size and also optimal MHC with respect to the new bound. In Section VI, we summarize the work of this paper.

II. PRELIMINARIES

Let $H = \{H^0, H^1, \cdots, H^{F-1}\}$ be an FH/TH sequence set consisting of F sequences of length L over a frequency/time slot set $\nu = \{\nu_0, \nu_1, \cdots, \nu_{\lambda-1}\}$. Let $c(u, v) = 1$ for $u = v$ and $c(u, v) = 0$ otherwise. The least nonnegative residue of a modulo b is denoted by $\langle a\rangle_b$. For $H^i = (h^i_0, h^i_1, \cdots, h^i_{L-1})$, $H^j = (h^j_0, h^j_1, \cdots, h^j_{L-1})$, $0 \le i, j \le F-1$, the Hamming correlation function between them is defined by

$$C_{ij}(\tau) = \sum_{k=0}^{L-1} c(h^i_k, h^j_{\langle k+\tau\rangle_L}), \tag{1}$$

where τ is the time delay.

Let H defined above be an LHZ FH/TH sequence set which has LHZ W. The maximum Hamming auto-correlation C_{auto}, the maximum Hamming cross-correlation C_{cross}, and the MHC C_{max} of H are defined as follows, respectively:

$$\begin{aligned} C_{auto} &= \max\{C_{ii}(\tau) : 0 \le i \le F-1,\ \tau = 1, 2, \cdots, W\}, \\ C_{cross} &= \max\{C_{ij}(\tau) : 0 \le i, j \le F-1,\ i \ne j,\ \tau = 0, 1, \\ &\qquad \cdots, W\}, \\ C_{max} &= \max\{C_{auto}, C_{cross}\}. \end{aligned}$$

In the remainder of this paper, denote an LHZ FH/TH sequence set consisting of F sequences of length L over a frequency/time slot set of size λ, which has LHZ W, by $[L, F, \lambda, W]$.

In 2006, Peng *et al.* [24] derived the lower bounds on the MHC of LHZ FH sequence sets.

Lemma 1 (Peng-Fan-Lee Bounds): For an $[L, F, \lambda, W]$ LHZ FH sequence set with MHC C_{max}, we have

$$C_{max} \ge \frac{(FW + F - \lambda)L}{(FW + F - 1)\lambda} \tag{2}$$

and

$$C_{max} \ge \frac{(W+1)(2ILF + LF - I\lambda - I^2\lambda) - FL^2}{(FW + F - 1)LF} \tag{3}$$

where I is the integer part of $\frac{LF}{\lambda}$.

Definition 1: Let H be an $[L, F, \lambda, W]$ LHZ FH/TH sequence set with MHC C_{max}. If C_{max} is the minimum integer solution of (2) or (3), then the LHZ FH/TH sequence set H is said to have optimal MHC with respect to the Peng-Fan-Lee bounds.

Several constructions of LHZ FH sequence sets with optimal MHC with respect to the Peng-Fan-Lee bounds can be found in the literature [4], [13], [20]–[22], [28], [29].

III. AN UPPER BOUND ON THE FAMILY SIZE OF LHZ FH/TH SEQUENCE SETS

In this section, we establish an upper bound on the family size of LHZ FH/TH sequence sets.

Theorem 1: For an $[L, F, \lambda, W]$ LHZ FH/TH sequence set with MHC C_{max}, $C_{max} \ne L$, we have

$$F \le \frac{\lambda^{C_{max}+1}}{W+1}. \tag{4}$$

Proof: Let $H = \{H^0, H^1, \cdots, H^{F-1}\}$ be an $[L, F, \lambda, W]$ LHZ FH/TH sequence set with MHC C_{max} over a frequency/time slot set $\nu = \{\nu_0, \nu_1, \cdots, \nu_{\lambda-1}\}$ where $H^i = (h^i_0, h^i_1, \cdots, h^i_{L-1})$, $i = 0, 1, \cdots, F-1$. Since the MHC is C_{max}, we have

$$\sum_{k=0}^{L-1} c(h^i_k, h^j_{\langle k+\tau\rangle_L}) \le C_{max} \tag{5}$$

for $i, j = 0, 1, \cdots, F-1$, $\tau = 0, 1, \cdots, W$, and $(i-j)^2 + \tau^2 \ne 0$.

For two vectors $(h^i_{k_1}, h^i_{\langle k_1+1\rangle_L}, \cdots, h^i_{\langle k_1+C_{max}\rangle_L})$ and $(h^j_{k_2}, h^j_{\langle k_2+1\rangle_L}, \cdots, h^j_{\langle k_2+C_{max}\rangle_L})$ where $0 \le k_1, k_2 \le W$, $0 \le i, j \le F-1$, $(k_1-k_2)^2 + (i-j)^2 \ne 0$, we have

$$\begin{aligned} \sum_{t=0}^{C_{max}} c(h^i_{\langle k_1+t\rangle_L}, h^j_{\langle k_2+t\rangle_L}) &\le \sum_{t=0}^{L-1} c(h^i_{\langle k_1+t\rangle_L}, h^j_{\langle k_2+t\rangle_L}) \\ &= \sum_{t=0}^{L-1} c(h^i_t, h^j_{\langle t+k_2-k_1\rangle_L}). \end{aligned} \tag{6}$$

By (5) and (6), we can obtain that

$$\sum_{t=0}^{C_{max}} c(h^i_{\langle k_1+t\rangle_L}, h^j_{\langle k_2+t\rangle_L}) \le C_{max}. \tag{7}$$

This indicates that $(h^i_{k_1}, h^i_{\langle k_1+1\rangle_L}, \cdots, h^i_{\langle k_1+C_{max}\rangle_L}) \ne (h^j_{k_2}, h^j_{\langle k_2+1\rangle_L}, \cdots, h^j_{\langle k_2+C_{max}\rangle_L})$. Then in the vector set

$$\{(h^i_k, h^i_{\langle k+1\rangle_L}, \cdots, h^i_{\langle k+C_{max}\rangle_L}) : k = 0, 1, \cdots, W, \\ i = 0, 1, \cdots, F-1\}$$

the elements are distinct. It is known that there are $\lambda^{C_{max}+1}$ different vectors in $C_{max}+1$ dimensional vector space over an alphabet of size λ. Hence $(W+1)F \le \lambda^{C_{max}+1}$ which leads to

$$F \le \frac{\lambda^{C_{max}+1}}{W+1}.$$

□

Let $W = L-1$ in (4). Then a bound for conventional FH/TH sequence sets can be obtained, which is just the Singleton bound on FH sequence sets [7], [26].

Corollary 1 (Singleton Bound on FH Sequence Sets): For a conventional FH/TH sequence set consisting of F sequences of length L over a frequency/time slot set of size λ, whose MHC is C_{max}, we have

$$F \leq \frac{\lambda^{C_{max}+1}}{L}. \tag{8}$$

Definition 2: Let H be an $[L, F, \lambda, W]$ LHZ FH/TH sequence set with MHC C_{max}. If F is the maximum integer solution of (4), then the LHZ FH/TH sequence set H is said to have optimal family size. If $F+1$ is the maximum integer solution of (4), then the LHZ FH/TH sequence set H is said to have near optimal family size.

Example 1: Let $\nu = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a frequency/time slot set. A $[9, 21, 8, 2]$ LHZ FH/TH sequence set H is given as follows:

$$\begin{aligned} H = \{&(0,1,6,7,4,4,7,6,1),\ (7,0,7,1,2,5,5,2,1),\\ &(7,2,0,2,7,3,6,6,3),\ (4,2,3,0,3,2,4,1,1),\\ &(7,5,3,4,0,4,3,5,7),\ (2,2,6,4,5,0,5,4,6),\\ &(1,3,3,1,5,6,0,6,5),\ (7,6,1,0,1,6,7,4,4),\\ &(5,2,1,7,0,7,1,2,5),\ (6,6,3,7,2,0,2,7,3),\\ &(4,1,1,4,2,3,0,3,2),\ (3,5,7,7,5,3,4,0,4),\\ &(5,4,6,2,2,6,4,5,0),\ (0,6,5,1,3,3,1,5,6),\\ &(7,4,4,7,6,1,0,1,6),\ (1,2,5,5,2,1,7,0,7),\\ &(2,7,3,6,6,3,7,2,0),\ (0,3,2,4,1,1,4,2,3),\\ &(4,0,4,3,5,7,7,5,3),\ (4,5,0,5,4,6,2,2,6),\\ &(1,5,6,0,6,5,1,3,3)\}. \end{aligned}$$

We can easily check that the MHC of LHZ FH/TH sequence set H is 1. By the new bound (4), we have

$$F \leq \left\lfloor \frac{64}{3} \right\rfloor = 21$$

which indicates that the LHZ FH/TH sequence set H has optimal family size.

IV. TWO CLASSES OF LHZ FH/TH SEQUENCE SETS WITH OPTIMAL FAMILY SIZE

In this section, we construct two classes of LHZ FH/TH sequence sets by utilizing cyclic codes, which have optimal family size with respect to the new bound. First, we give the following notations which will be used in this section:

- q —— a prime power;
- $GF(q^n)$ —— Galois field with q^n elements;
- $GF(q^n)^*$ —— multiplicative group of $GF(q^n)$;
- $\Lambda(x)$ —— $\Lambda(x) = \min\{i : i|x,\ i > 1\}$;
- $\mu(x)$ —— Möbius function defined as

$$\mu(x) = \begin{cases} 1, & x = 1 \\ (-1)^s, & x \text{ is a product of } s \text{ different primes} \\ 0, & \text{otherwise.} \end{cases}$$

A. THE FIRST CLASS

Let l be an integer such that $l|q-1$. Let α be a generator of $GF(q)^*$ and β a primitive lth root of $GF(q)$. For an integer r, $1 \leq r \leq l$, we define

$$\mathbb{R} = \left\{ \sum_{i=1}^{r} f_i x^i : f_i \in GF(q), i = 1, 2, \cdots, r \right\}$$

and

$$\mathbb{H} = \{(f(\alpha^j), f(\alpha^j\beta), \cdots, f(\alpha^j\beta^{l-1})) : f(x) \in \mathbb{R},\ j = 0, 1, \cdots, \frac{q-1}{l} - 1\}.$$

Definition 3: For a sequence $h = (h_0, h_1, \cdots, h_{L-1})$, define a cyclic shift operator Γ as $\Gamma^i(h) = (h_i, h_{\langle i+1\rangle_L}, \cdots, h_{\langle i+L-1\rangle_L})$, $0 \leq i \leq L-1$.

Definition 4: For any $a, b \in \mathbb{H}$, $a \neq b$, if there exists an integer τ, $1 \leq \tau \leq l-1$, such that $a = \Gamma^\tau(b)$, then a and b are said to be cyclic equivalent.

The cyclic equivalence divides $\mathbb{H}$ into disjoint subsets (cyclic equivalence classes [17]). In each cyclic equivalence class, any two elements are cyclic equivalent. The number of elements in the cyclic equivalence class is said to be its cycle length. By picking up one element from each equivalence class with cycle length l, it forms a subset of $\mathbb{H}$, denoted by $\mathbb{H}^*$.

Let w be an integer such that $w|l$, $w \neq 1$. Construct an LHZ FH/TH sequence set H as follows:

$$H = \{\Gamma^{tw}(h) : h \in \mathbb{H}^*,\ t = 0, 1, \cdots, \frac{l}{w} - 1\}. \tag{9}$$

For the LHZ FH/TH sequence set H, we have the following theorem.

Theorem 2: The family size of the LHZ FH/TH sequence set H is

$$|H| = \frac{1}{w} \sum_{k|l} \mu(k) q^{\lfloor \frac{r}{k} \rfloor}. \tag{10}$$

Proof: By Lemma 19 in [26], the family size of $\mathbb{H}^*$ is

$$|\mathbb{H}^*| = \frac{1}{l} \sum_{k|l} \mu(k) q^{\lfloor \frac{r}{k} \rfloor}. \tag{11}$$

Together with (9), we have

$$|H| = \frac{l}{w} \times |\mathbb{H}^*| = \frac{l}{w} \times \frac{1}{l} \sum_{k|l} \mu(k) q^{\lfloor \frac{r}{k} \rfloor} = \frac{1}{w} \sum_{k|l} \mu(k) q^{\lfloor \frac{r}{k} \rfloor}.$$

□

Theorem 3: H is an $[l, |H|, q, w-1]$ LHZ FH/TH sequence set with MHC $r-1$, where $|H|$ is given by (10).

Proof: Note that the Hamming distance between any two elements in $\mathbb{H}$ is greater than or equal to $l-r+1$. For $h^i, h^j \in \mathbb{H}^*$, $0 \leq \tau \leq l-1$, and $(i-j)^2 + \tau^2 \neq 0$, we then have

$$C_{ij}(\tau) \leq l - (l - r + 1) = r - 1. \tag{12}$$

For $0 \leq t_1, t_2 \leq \frac{l}{w} - 1$, the Hamming correlation between $\Gamma^{t_1 w}(h^i)$ and $\Gamma^{t_2 w}(h^j)$ at time delay τ, is given by

$$C_{\Gamma^{t_1 w}(h^i)\Gamma^{t_2 w}(h^j)}(\tau) = C_{ij}(\tau + t_2 w - t_1 w). \tag{13}$$

Case 1) $i \neq j$. In this case, $C_{ij}(\tau + t_2 w - t_1 w) \leq r-1$. Thus, $C_{\Gamma^{t_1 w}(h^i)\Gamma^{t_2 w}(h^j)}(\tau) \leq r-1$.

Case 2) $i = j$, $t_1 \neq t_2$, $0 \leq \tau \leq w-1$. Since $\langle \tau + t_2 w - t_1 w\rangle_l \neq 0$, we have $C_{ij}(\tau + t_2 w - t_1 w) \leq r-1$. This leads to $C_{\Gamma^{t_1 w}(h^i)\Gamma^{t_2 w}(h^j)}(\tau) \leq r-1$.

Case 3) $i = j$, $t_1 = t_2$, $1 \leq \tau \leq l-1$. Since $\tau + t_2 w - t_1 w = \tau$, we have $C_{\Gamma^{t_1 w}(h^i)\Gamma^{t_2 w}(h^j)}(\tau) = C_{ij}(\tau) \leq r-1$.

That is to say, the MHC of H is $r-1$ in LHZ $w-1$. Therefore, H is an $[l, |H|, q, w-1]$ LHZ FH/TH sequence set with MHC $r-1$. □

Theorem 4: H has optimal family size by the new bound (4) if $r < \Lambda(l)$.

Proof: Since $r < \Lambda(l)$, (10) becomes

$$\begin{aligned} |H| &= \frac{1}{w}\left(\mu(1)q^r + \sum_{k|l,k\neq 1} \mu(k) q^{\lfloor \frac{r}{k} \rfloor}\right) \\ &= \frac{1}{w}\left(q^r + \sum_{k|l,k\neq 1}\mu(k)\right) \\ &= \frac{1}{w}\left(q^r + \sum_{k|l}\mu(k) - 1\right). \end{aligned} \tag{14}$$

Note that $\sum_{k|l}\mu(k) = 0$. Then

$$|H| = \frac{q^r - 1}{w}. \tag{15}$$

For the $[l, F, q, w-1]$ LHZ FH/TH sequence set with MHC $r-1$, by bound (4) we have

$$F \leq \left\lfloor \frac{q^r}{w} \right\rfloor = \left\lfloor \frac{q^r - 1}{w} + \frac{1}{w} \right\rfloor = \frac{q^r - 1}{w} + \left\lfloor \frac{1}{w} \right\rfloor = \frac{q^r-1}{w}. \tag{16}$$

Thus, H has optimal family size. □

Example 2: Let $q = 512$, $l = 511$, $r = 3$, $w = 73$. A $[511, 1838599, 512, 72]$ LHZ FH/TH sequence set with MHC 2 can be obtained. By bound (4), we have

$$F \leq \left\lfloor \frac{512^3}{73} \right\rfloor = 1838599.$$

Hence, it has optimal family size.

B. THE SECOND CLASS

Let l' be an odd integer such that $l' | q+1$. Let γ be a primitive l'th root of unity in GF(q^2). Let B_0, B_i be cyclotomic cosets defined as

$$B_0 = \{0\},\ B_i = \{i, l' - i\}, i = 1, 2, \cdots, \frac{l'-1}{2}, \tag{17}$$

respectively.

Let r' be an integer such that $1 \leq r' < \frac{\Lambda(l')+1}{2}$. Construct a polynomial

$$R(x) = R_{\frac{l'-1}{2}-r'+1}(x) R_{\frac{l'-1}{2}-r'+2}(x) \cdots R_{\frac{l'-1}{2}}(x) \tag{18}$$

where

$$R_j(x) = \prod_{k\in B_j}(x - \gamma^k),\ j = \frac{l'-1}{2} - r' + 1, \frac{l'-1}{2} - r' + 2, \dots, \frac{l'-1}{2}.$$

Let $R(x)$ be the parity-check polynomial of a cyclic code $\mathbb{E}$ of length l'. Then the generator polynomial of $\mathbb{E}$ is given by $T(x) = \frac{x^{l'}-1}{R(x)}$. The cyclic equivalence divides $\mathbb{E}$ into cyclic equivalence classes. By picking up one element from each equivalence class with cycle length l', it forms a subset of $\mathbb{E}$, denoted by $\mathbb{E}^*$.

Let w' be an integer such that $w' | l'$, $w' \neq 1$. We construct an LHZ FH/TH sequence set H' as follows:

$$H' = \{\Gamma^{tw'}(h) : h \in \mathbb{E}^*,\ t = 0, 1, \cdots, \frac{l'}{w'} - 1\}. \tag{19}$$

First, we give the BCH bound by the following lemma.

Lemma 2 (BCH Bound [15]): Let E be a cyclic code of length l' over GF(q). $T(x)$ is the generator polynomial of E. If $\gamma^i, \gamma^{i+1}, \cdots, \gamma^{i+z-2}$ are roots of $T(x)$, where γ is a primitive l'th root of unity in GF(q^n), then the minimum Hamming distance of E is greater than or equal to z. Moreover, the minimum Hamming distance of E is z if $\gamma^i, \gamma^{i+1}, \cdots, \gamma^{i+z-2}$ are all the roots of $T(x)$.

The following lemma is derived by Liu and Zeng [16] in 2019.

Lemma 3: Let $\mathbb{E}$ be a cyclic code of length l' whose parity-check polynomial is defined in (18). The number of codewords with cycle length l' is $q^{2r'} - 1$.

Then we give the size of the LHZ FH/TH sequence set H'.

Theorem 5: The size of H' is

$$|H'| = \frac{q^{2r'} - 1}{w'}. \tag{20}$$

Proof: Since each equivalence class with cycle length l' consists of l' codewords, by Lemma 3 we have $|\mathbb{E}^*| = \frac{q^{2r'}-1}{l'}$. Hence, the size of H' is

$$|H'| = \frac{l'}{w'} \times |\mathbb{E}^*| = \frac{q^{2r'}-1}{w'}.$$

□

Theorem 6: H' is an $[l', |H'|, q, w'-1]$ LHZ FH/TH sequence set with MHC $2r'-1$, where $|H'|$ is given by (20).

Proof: Note that

$$T(x) = \frac{x^{l'} - 1}{R(x)} = T_0(x) T_1(x) \cdots T_{\frac{l'-1}{2}-r'}(x) \tag{21}$$

is the generator polynomial of $\mathbb{E}$, where

$$T_j(x) = \prod_{k\in B_j}(x - \gamma^k),\ j = 0, 1, \cdots, \frac{l'-1}{2} - r'.$$

$\gamma^{\frac{l'+1}{2}+r'}, \gamma^{\frac{l'+1}{2}+r'+1}, \cdots, \gamma^0, \cdots, \gamma^{\frac{l'-1}{2}-r'-1}, \gamma^{\frac{l'-1}{2}-r'}$ are all the roots of $T(x)$. By the BCH bound in Lemma 2, the minimum Hamming distance of $\mathbb{E}$ is $l' - 2r' + 1$.

For $0 \leq t_1, t_2 \leq \frac{l'}{w'} - 1$ and $h^i, h^j \in \mathbb{E}^*$, the Hamming correlation between $\Gamma^{t_1 w'}(h^i)$ and $\Gamma^{t_2 w'}(h^j)$ at time delay τ, is given by

$$C_{\Gamma^{t_1 w'}(h^i)\Gamma^{t_2 w'}(h^j)}(\tau) = C_{ij}(\tau + t_2 w' - t_1 w'). \quad (22)$$

Case 1) $i \neq j$. Since $i \neq j$, we have $C_{ij}(\tau + t_2 w' - t_1 w') \leq l' - (l' - 2r' + 1) = 2r' - 1$. Therefore, $C_{\Gamma^{t_1 w'}(h^i)\Gamma^{t_2 w'}(h^j)}(\tau) \leq 2r' - 1$.

Case 2) $i = j$, $t_1 \neq t_2$, $0 \leq \tau \leq w' - 1$. In this case, $\langle \tau + t_2 w' - t_1 w' \rangle_{l'} \neq 0$. This implies that $C_{ij}(\tau + t_2 w' - t_1 w') \leq l' - (l' - 2r' + 1) = 2r' - 1$. Then we have $C_{\Gamma^{t_1 w'}(h^i)\Gamma^{t_2 w'}(h^j)}(\tau) \leq 2r' - 1$.

Case 3) $i = j$, $t_1 = t_2$, $1 \leq \tau \leq l' - 1$. In this case, $\tau + t_2 w' - t_1 w' = \tau$. Then $C_{\Gamma^{t_1 w'}(h^i)\Gamma^{t_2 w'}(h^j)}(\tau) = C_{ij}(\tau) \leq l' - (l' - 2r' + 1) = 2r' - 1$.

Hence, the MHC of H' is $2r' - 1$ in LHZ $w' - 1$. Then H' is an $[l', |H'|, q, w' - 1]$ LHZ FH/TH sequence set with MHC $2r' - 1$. □

Theorem 7: H' has optimal family size by the new bound (4).

Proof: For the $[l', F, q, w' - 1]$ LHZ FH/TH sequence set with MHC $2r' - 1$, by bound (4) we have

$$F \leq \left\lfloor \frac{q^{2r'}}{w'} \right\rfloor = \left\lfloor \frac{q^{2r'} - 1}{w'} + \frac{1}{w'} \right\rfloor = \frac{q^{2r'} - 1}{w'} + \left\lfloor \frac{1}{w'} \right\rfloor = \frac{q^{2r'} - 1}{w'}. \quad (23)$$

Then H' has optimal family size. □

Example 1: Let $q = 64$, $l' = 65$, $r' = 2$, $w' = 13$. We can obtain a $[65, 1290555, 64, 12]$ LHZ FH/TH sequence set with MHC 3. By bound (4), we have

$$F \leq \left\lfloor \frac{64^4}{13} \right\rfloor = 1290555.$$

Thus, it has optimal family size.

V. A CLASS OF LHZ FH/TH SEQUENCE SETS WITH NEAR OPTIMAL FAMILY SIZE

In this section, we give a construction of LHZ FH/TH sequence sets which have near optimal family size with respect to the new bound.

Step 1: For a prime power q, we suppose that α is a primitive element in GF(q^2). Let $\beta = \alpha^{q-1}$ be a primitive $(q+1)$th root in GF(q^2). Define a trace function from GF(q^2) to GF(q) by $\mathrm{Tr}_{q^2/q}(x)$. Construct A as follows:

$$A = \{(\mathrm{Tr}_{q^2/q}(\alpha^i), \mathrm{Tr}_{q^2/q}(\alpha^i \beta), \cdots, \mathrm{Tr}_{q^2/q}(\alpha^i \beta^q)) : i = 0, 1, \cdots, q-2\}.$$

Step 2: Let $f(x)$ be a one-to-one function from $\{0, 1, \cdots, q - 1\}$ to GF(q). Define $A' = \{h^k = (h_0^k, h_1^k, \cdots, h_q^k) : k = 0, 1, \cdots, q(q-1) - 1\}$ where

$$h_i^{a(q-1)+b} = \mathrm{Tr}_{q^2/q}(\alpha^b \beta^i) + f(a)$$

for $a = 0, 1, \cdots, q-1, b = 0, 1, \cdots, q-2, i = 0, 1, \cdots, q$.

Step 3: Construct an LHZ FH/TH sequence set H'' as follows:

$$H'' = \{\Gamma^{tw''}(h^k) : t = 0, 1, \cdots, \frac{q+1}{w''} - 1, \; k = 0, 1, \cdots, q(q-1) - 1\} \quad (24)$$

where w'' is an integer such that $w''|q+1$ and $w'' > \frac{q}{2}$.

Theorem 8: H'' is a $[q+1, \frac{q(q^2-1)}{w''}, q, w'' - 1]$ LHZ FH/TH sequence set with MHC 2.

Proof: By Theorem 7 in [17], the MHC of A' is 2. For $h^{k_1}, h^{k_2} \in A'$, $0 \leq \tau \leq q$, and $(k_1 - k_2)^2 + \tau^2 \neq 0$, we have

$$C_{k_1 k_2}(\tau) \leq 2. \quad (25)$$

For $0 \leq t_1, t_2 \leq \frac{q+1}{w''} - 1$, the Hamming correlation between $\Gamma^{t_1 w''}(h^{k_1})$ and $\Gamma^{t_2 w''}(h^{k_2})$ at time delay τ, is given by

$$C_{\Gamma^{t_1 w''}(h^{k_1})\Gamma^{t_2 w''}(h^{k_2})}(\tau) = C_{k_1 k_2}(\tau + t_2 w'' - t_1 w''). \quad (26)$$

Case 1) $k_1 \neq k_2$. In this case, $C_{k_1 k_2}(\tau + t_2 w'' - t_1 w'') \leq 2$. Thus, $C_{\Gamma^{t_1 w''}(h^{k_1})\Gamma^{t_2 w''}(h^{k_2})}(\tau) \leq 2$.

Case 2) $k_1 = k_2$, $t_1 \neq t_2$, $0 \leq \tau \leq w'' - 1$. Since $\langle \tau + t_2 w'' - t_1 w'' \rangle_{q+1} \neq 0$, we have $C_{k_1 k_2}(\tau + t_2 w'' - t_1 w'') \leq 2$. This leads to $C_{\Gamma^{t_1 w''}(h^{k_1})\Gamma^{t_2 w''}(h^{k_2})}(\tau) \leq 2$.

Case 3) $k_1 = k_2$, $t_1 = t_2$, $1 \leq \tau \leq q$. Since $\tau + t_2 w'' - t_1 w'' = \tau$, we have $C_{\Gamma^{t_1 w''}(h^{k_1})\Gamma^{t_2 w''}(h^{k_2})}(\tau) = C_{k_1 k_2}(\tau) \leq 2$.

Then the MHC of H'' is 2 in LHZ $w'' - 1$. Hence, H'' is a $[q + 1, \frac{q(q^2-1)}{w''}, q, w'' - 1]$ LHZ FH/TH sequence set with MHC 2. □

Theorem 9: H'' has near optimal family size by the new bound (4).

Proof: For the $[q + 1, F, q, w'' - 1]$ LHZ FH/TH sequence set with MHC 2, by bound (4) we have

$$F \leq \left\lfloor \frac{q^3}{w''} \right\rfloor = \left\lfloor \frac{q^3 - q}{w''} + \frac{q}{w''} \right\rfloor = \frac{q^3 - q}{w''} + \left\lfloor \frac{q}{w''} \right\rfloor = \frac{q^3 - q}{w''} + 1. \quad (27)$$

Therefore, H'' has near optimal family size. □

Example 4: Let $q = 11$ and $w'' = 6$. Then we can get a $[12, 220, 11, 5]$ LHZ FH/TH sequence set with MHC 2. By bound (4), we have

$$F \leq \left\lfloor \frac{11^3}{6} \right\rfloor = 221.$$

Thus, it has near optimal family size.

Remark 1: For an $[L, F, \lambda, W]$ LHZ FH/TH sequence set with MHC C_{max}, if C_{max} is the minimum integer solution of (4), then the LHZ FH/TH sequence set is said to have optimal MHC with respect to the new bound (4). It is easy to verify that the LHZ FH/TH sequence sets in this paper also have optimal MHC with respect to the new bound (4).

Table 1 lists the parameters of LHZ FH/TH sequence sets in the literature and this paper. It can easily be seen that the LHZ FH/TH sequence sets in this paper have very large family size compared with the LHZ FH/TH sequence sets in the literature. Moreover, the LHZ FH/TH sequence

TABLE 1. The parameters of LHZ FH/TH sequence sets in the literature and this paper.

Parameters $[L, F, \lambda, W], C_{max}$	Restriction	According to the Peng-Fan-Lee bounds (2) and (3)	According to the new bound (4)	Reference
$[m(q^n-1), F', q, W']$, $m(q^{n-1}-1)$	$q^n-1 = F'(W'+1)$, $\gcd(m, q^n-1)=1$, $m < F'$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[20]
$[m(q^n-1), F'q^{n_1}, q^{n_1}, W']$, $m(q^{n-n_1}-1)$	$0<n_1\leq n$, $q^n-1 = F'(W'+1)$, $m\equiv 1 \bmod W'+1$, $m<\frac{q^{n_1}(q^n-2)}{q^{n_1}-1}$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[21]
$[p^2(q_1-1)(q_2-1), pq_1q_2, pq_1q_2, \min\{p^2-1, q_1-2, q_2-2\}], p$	$\gcd(p, q_1-1, q_2-1)=1$, $3p < \min\{q_1-1, q_2-1\}$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[4]
$[m(q^n-1), F', q, W']$, $m(q^{n-1}-1)$	$q^n-1 = F'(W'+1)$, $\gcd(m^*, q^n-1)=1$, $m\equiv m^* \bmod W'+1$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[22]
$[\frac{q^n-1}{d}, q-1, q^{n_1}, \frac{q^n-1}{q-1}-1]$, $\frac{q^{n-n_1}-1}{d}$	$0<n_1\leq n$, $d\mid q-1$, $\gcd(d,n)=1$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[13]
$[\frac{q^n-1}{d}, F', q^{n_1}, W']$, $\frac{q^{n-n_1}-1}{d}$	$0<n_1\leq n$, $q^n-1 = F'(W'+1)$, $d\mid q-1$, $\gcd(d,n)=1$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[29]
$[(q_1-1)(q_2^n-1), q_1, q_1q_2, q_2^n-2]$, $q_2^{n-1}-1$	$\gcd(q_1-1, q_2^n-1)=1$, $q_1 > q_2^n$	Optimal MHC	Not optimal/near optimal family size Not optimal MHC	[28]
$[l, \frac{q^r-1}{w}, q, w-1], r-1$	$l\mid q-1$, $w\mid l$, $w\neq 1$, $1\leq r<\Lambda(l)$	Not optimal MHC	Optimal family size Optimal MHC	This paper
$[l', \frac{q^{2r'}-1}{w'}, q, w'-1], 2r'-1$	l' is odd integer, $l'\mid q+1$, $w'\mid l'$, $w'\neq 1$, $1\leq r'<\frac{\Lambda(l')+1}{2}$	Not optimal MHC	Optimal family size Optimal MHC	This paper
$[q+1, \frac{q(q^2-1)}{w''}, q, w''-1], 2$	$w''\mid q+1$, $w''>\frac{q}{2}$	Optimal MHC	Near optimal family size Optimal MHC	This paper

- q, q_1, q_2 are prime powers.
- p is an odd prime.

sets in this paper have optimal/near optimal family size and also optimal MHC with respect to the new bound while the LHZ FH/TH sequence sets in the literature do not. The LHZ FH/TH sequence sets in this paper are suitable for many users to share limited bandwidth in quasi-synchronous FH/TH CDMA systems.

VI. CONCLUSION

In this paper, we first established an upper bound on the family size of LHZ FH/TH sequence sets which includes the Singleton bound on FH sequence sets as a special case. Then we presented three constructions of LHZ FH/TH sequence sets with large family size. They have optimal/near optimal family size and also optimal MHC according to the new bound while the known LHZ FH/TH sequence sets in the literature do not. Compared with the LHZ FH/TH sequence sets in the literature, our new LHZ FH/TH sequence sets have very large family size. They are suitable for many users to share limited bandwidth in quasi-synchronous FH/TH CDMA systems.

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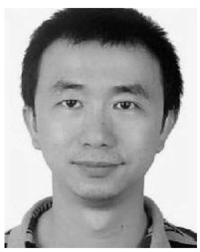
XING LIU was born in Sichuan, China, in July 1986. He received the B.S. and Ph.D. degrees in communication engineering from Southwest Jiaotong University, Chengdu, China, in 2009 and 2014, respectively. From January 2015 to May 2016, he was an Engineer with the No.38 Research Institute, China Electronics Technology Group Corporation, Hefei, China. From July 2016 to July 2018, he was a Postdoctoral Member with the National Key Laboratory of Science and Technology on Communications, University of Electronic Science and Technology of China, Chengdu. He is currently an Assistant Research Fellow with the College of Electrical Engineering and a Postdoctoral Member with the College of Mathematics, Sichuan University, Chengdu. His research interests include sequence design and coding theory.

QIN SHU received the Ph.D. degree in information and signal processing from the University of Electronic Science and Technology of China, in 1991. He has more than 20 years' experience in the area of signal processing. He is currently a Professor with the College of Electrical Engineering, Sichuan University. He has expertise in signal theory, blind signal processing, and mathematical foundations in signal processing.



QI ZENG received the Ph.D. degree in communication and information system from Southwest Jiaotong University, Chengdu, China, in 2013. From 2010 to 2012, he was a Visiting Ph.D. Student with the University of Tennessee, Knoxville, TN, USA. From 2017 to 2018, he was a Visiting Scholar with the Department of Electrical and Electronic Engineering, University of Nottingham, U.K. In 2016, he joined the College of Electrical Engineering and Information Technology, Sichuan University, China, as a Lecturer. His research interests include coding theory, interference mitigation, and wireless communications. Dr. Zeng is a Reviewer of the IEEE COMMUNICATIONS LETTERS, the IEEE TRANSACTIONS ON INDUSTRIAL ELECTRONICS, *IET Networks*, *IET Communications*, and the *Journal of Wireless Communications and Mobile Computing*.