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N-Systems Function Projective Combination Synchronization–A Review of Real and Complex Continuous Time Chaos Synchronization

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ABSTRACT As an important branch in the field of chaos, chaos synchronization (CHS) has attracted great attentions since it was firstly proposed. In this review article, numerous types of continuous time CHS are presented for the past 29 years. Previous CHS are divided into two major parts: one is CHS in real number field, and the other is CHS in complex number field. Some classic types of CHS, controllers and corresponding stability theories are preliminarily reviewed. Besides, a new type of chaos synchronization is proposed, which is called N-systems combination function projective synchronization (NCOFPS). Interestingly and importantly, most of these proposed types of CHS belong to NCOFPS. Then, time delay function projective synchronization (TDFPS) is also proposed, which is one of the types of NCOFPS and never studied in existed papers. And TDFPS controller, corresponding mathematical proof are obtained and simulation experiments illustrate the effectiveness of investigated controller. This paper provides a generalized view to catch the development of CHS in different fields. Finally, we give the conclusions and prospects of CHS.

INDEX TERMS Controller, synchronization, complex chaotic systems, N-systems combination function projective synchronization.

I. INTRODUCTION

Since the first specific chaotic system was proposed by Lorenz in 1963 [1], chaos, as the third great physical revolution after the generation of relativity and quantum mechanics, have made significant historical progress in recent years. It is the interdisciplinary and multidisciplinary field of science [2], which promotes connections between different disciplines and demonstrates the complexity of human society and nature.

The phenomena of synchronization widely exist in real world. For instance, the frogs croak together in the pond, people's applause gradually tend to be consistence, fireflies glow together and go out together and so on. Two or more identical chaotic systems that begin from small different initial or parametric conditions can separate exponentially with time, because the evolutions of the chaotic systems sensitively depend on original states. Due to the special dynamic characters, scholars found it was difficult to control chaos until Pecora and Carroll [3] achieved **CHS** in real electronic circuit. Although Aranson [4], Volkovskii [5] and Fujisaka [6] previously explored so-call synchronization in pure mathematics, people always regarded the synchronization in actual application as the significant milestone. Since then, **CHS** becomes a hotspot in the fields of chaos. It is deserving for researchers to study **CHS** and explore more meanings of **CHS**.

In 1982, Fowler proposed complex Lorenz chaotic system [7]. In the same year, the real physical background of complex Lorenz equations were also discovered in describing

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rotating fluids, analyzing the characters of detuned lasers and modeling thermal convection process [8]–[11]. It was another major breakthrough in chaos. As complex chaotic systems does not limit real number and it can double the number of variables without increasing the complexity of actual implementation, the applications of complex **CHS** have natural advantages over the traditional **CHS**, especially in increasing the contents and security of the transmitted information for secure communication. Although most of proposed review papers [12]–[16] gave a deep and comprehensive description of **CHS**, most of them only focused on the real **CHS**, while complex **CHS** is rarely reviewed.

By means of the unpredictability and complexity of chaotic systems [17], [18], secure communication based on **CHS** was also widely studied in [19]–[33]. The diversity of **CHS** can improve the security performance. Thus, it is really meaningful for researchers to make a profound study on **CHS**. This paper review numerous types of real and complex **CHS** and summarize the corresponding research methods. From an interesting perspective, we reveal the intrinsic laws that how does complex chaotic systems develop from real chaotic systems and beyond them.

The report is organized as follows. In Section 2 we describe the types of synchronization for real chaotic system. Complex **CHS** forms are reviewed in Section.3. In Section 4, N-systems combination function projective synchronization (**NCOFPS**) is proposed and some remarks are also obtained. Most of the above **CHS** in real and complex chaos are special cases of **NCOFPS**. Especially, time delay function projective synchronization (**TDFPS**) is investigated. It is a special case of **NCOFPS** and is never reported in existed works. Finally, we draw the conclusions and discuss several prospective topics in Section 5.

II. REAL CHAOS SYNCHRONIZATION

In 1990, Pecora and Carroll [3] accomplished complete synchronization (**CS**) by means of drive-response method, then a variety of synchronization and a great number of control means were developed. **CHS** is still a hot research area nowadays. According to chronological order, generalized synchronization (**GS**) in 1995, phase synchronization (**PHS**) in 1996, lag synchronization (**LS**) in 1997, projective synchronization(**PS**) in 1999, were presented and became indispensable parts of chaos's history. At the same time, some stability methods and robustness theorems, such as active control [34], sliding mode control [35]–[37], $H\infty$ synchronization [38], fuzzy control [39], adaptive control [40], backstepping control [41] and so on were also applied to (**CHS**). By rigorous theoretical proof with the help of Barbalat lemma and Lyapunov stability, satisfied conditions were derived to guarantee **CHS**. We introduce most types of synchronization in chronological order, with a view to restore a true development process of chaos synchronization science.

A. COMPLETE SYNCHRONIZATION

CS is the earliest and simplest type of **CHS**, which means two real chaotic systems with different initial conditions tend to remain the same trajectories. The synchronization appears to be structurally stable [3]. This is a milestone in the history of chaos. People begin to realize that chaos is controllable. The process of synchronization can be viewed as a response system that ''knows'' what state (attractor) to go to when driven (stimulated) by a particular signal. Therefore, it is easy to understand the following definition.

The drive (master) system is

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \quad \mathbf{y} \in \mathbf{R}^n, \tag{1}$$

and the response (slave) system is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{u}, \quad \mathbf{x} \in \mathbf{R}^n, \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$ are state vectors of the two systems (T denotes transposition), $\mathbf{u}$ is the synchronization controller. $\mathbf{f}(\mathbf{x}) \in \mathbf{R}^n$, $\mathbf{g}(\mathbf{y}) \in \mathbf{R}^n$ and $\mathbf{u} \in \mathbf{R}^n$ are n dimensional nonlinear functions in this part. If a proper controller is designed to satisfy

$$\lim_{t \to +\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0, \tag{3}$$

it means that the trajectories of $\mathbf{x}(t)$ will converge to the same values as $\mathbf{y}(t)$ and they will remain in step with each other, which is called complete synchronization (**CS**). People start to observe that there are many **CS** phenomena such as some neural processes in human brain, fireflies glow synchronously and extinguish synchronously, crickets sing in unison, the frequency of applause at the end of performances, and so on.

Furthermore, the proposed **CS** feedback controller and corresponding stability theorem are introduced in [42].

Lemma 1: For any $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \varsigma$ (ς are the sets of numbers), there is a constant $l > 0$ and it satisfies

$$|f(x) - f(y)| \leq l|x - y|. \tag{4}$$

One call the condition as the uniform Lipschitz condition and the l stands for the uniform Lipschitz constant.

Consider the drive system

$$\dot{y} = f(y), \tag{5}$$

and the response system is

$$\dot{x} = f(x) + \varepsilon(x - y), \tag{6}$$

where $\varepsilon_i(x - y) = (\varepsilon_1 e_1, \varepsilon_2 e_2, \cdots)$ are the feedback coupling controllers and error state variables $e_i = x_i - y_i$. One choose the following update law of ε,

$$\dot{\varepsilon}_i = -\eta_i e_i^2, \tag{7}$$

where $\eta_i > 0$, $i = 1, 2, \cdots, n$. To illustrate the effectiveness of the proposed feedback coupling controllers, the Lyapunov function is introduced,

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\eta_i}(\varepsilon_i + L)^2, \tag{8}$$

where L is a constant bigger than nl.

Then we can get the derivative of V,

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n} e_i(\dot{x}_i - \dot{y}_i) + \sum_{i=1}^{n}(L+\varepsilon_i)\dot{\varepsilon} \\ &= \sum_{i=1}^{n} e_i(f_i(x) - f_i(y) + \varepsilon_i e_i) - \sum_{i=1}^{n}(\varepsilon_i + L)e_i^2 \\ &\leq (-L+nl)\sum_{i=1}^{n} e_i^2 \leq 0. \end{aligned} \tag{9}$$

We assume $x, y \in \varsigma$ and ς is globally attractive. By means of Lipschitz condition (4), one can easily get $\dot{V} = 0$ if and only if $e_i = 0$. In other word, the set $E = [(e,\varepsilon) \in R^{2n} : e = 0, \varepsilon = \varepsilon_0 \in R^n]$ as the largest invariant set is contained in $\dot{V} = 0$ for the augment system. Following the Lasalle invariance principle [43], one can know that starting with different initial conditions of the augment system, the system orbit converges asymptotically to the largest invariant set E and $x \to y$ with $t \to \infty$. To get a perspicuous result of **CS**, one introduce two Lorenz systems with different initial conditions, which is shown as follows. The response system is

$$\begin{cases}\dot{x}_1 = 10(x_2 - x_1) + u_1, \\ \dot{x}_2 = 28x_1 - x_1x_3 - x_2 + u_2, \\ \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3 + u_3,\end{cases} \tag{10}$$

and the drive system is

$$\begin{cases}\dot{y}_1 = 10(y_2 - y_1), \\ \dot{y}_2 = 28y_1 - y_1y_3 - y_2, \\ \dot{y}_3 = y_1y_2 - \frac{8}{3}y_3.\end{cases} \tag{11}$$

Following the proposed method of error feedback controller, one can get

$$\begin{cases}u_1 = \varepsilon_1(x_1 - y_1), \dot{\varepsilon}_1 = -\eta_1(x_1 - y_1)^2, \\ u_2 = \varepsilon_2(x_2 - y_2), \dot{\varepsilon}_2 = -\eta_2(x_2 - y_2)^2, \\ u_3 = \varepsilon_3(x_3 - y_3), \dot{\varepsilon}_3 = -\eta_3(x_3 - y_3)^2.\end{cases} \tag{12}$$

where $\eta_1 = \eta_2 = \eta_3 = 10$ and the initial condition is $[x_1(0), x_2(0), x_3(0)] = [1, 2, 3]$, $[y_1(0), y_2(0), y_3(0)] = [8, 9, 5]$. Fig.1 and Fig.2 are the diagrams of error states and phase states, respectively. The black line ($e_1 = x_1 - y_1$), the red line ($e_2 = x_2 - y_2$) and the blue line ($e_3 = x_3 - y_3$) fleetly tend to zero with the feedback controller (12) in Fig.1. Fig.2 also vertify the effectiveness of the control method.

Remark 1: **CS** is the most classic and simplest type of real **CHS**. Numerous proposed new **CHS** were obtained based on **CS**. And it also lays a foundation for the chaotic synchronization communications.

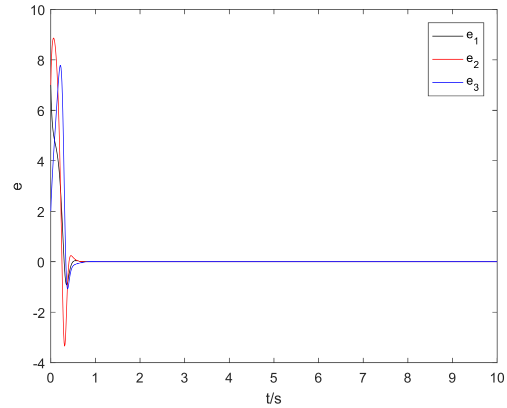

FIGURE 1. Diagram of error states of CS.

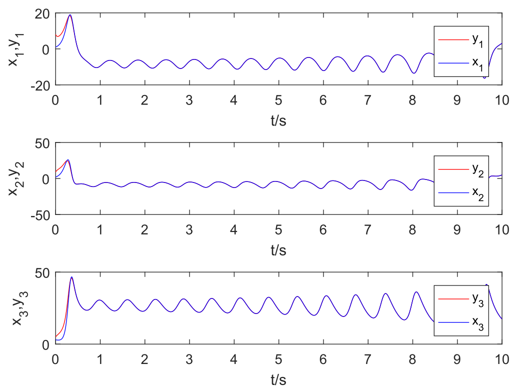

FIGURE 2. Diagram of phase states of CS.

Remark 2: As for the feedback controller, the structure of the controller is simple and easy, it has great control ability in accomplishing **CS**, which is shown from the formula deduction and simulation results. There also exists some limitations of the feedback controller. It mainly focuses on the same chaotic systems with different initial conditions, while the synchronization of different chaotic systems is hard to be achieved.

When one solve the chaotic differential equation, there were many numerical methods, such as the classical runge-kutta method and predictor-corrector method. Note that computer-generated chaotic results given by traditional algorithms in double precision are a kind of mixture of "true" (convergent) solution and numerical noises at the same level [44] and systematic distortions are uncovered in the statistical properties of chaotic dynamical systems when represented and simulated on digital computers using standard IEEE floating-point numbers [45]. Some significant work [44]–[46], for instance clean numerical simulation (CNS), focused on this qusetion and it is another issue that might need attention.

B. GENERALIZED SYNCHRONIZATION

In 1995, based on actual equality of the variables of **CHS** as they evolve in time, Rulkov [47] studied a generalization of this condition, which can equate state variables from one system with a function of the variables of another system. This means that synchronization implies a collapse of the overall evolution onto a subspace of the system attractor in full space. As for the drive system (1) and the response system (2), the generalized synchronization (**GS**) can be expressed as

$$\lim_{t\to+\infty} \|\mathbf{x}(t) - \varphi(\mathbf{y}(t))\| = 0. \tag{13}$$

Remark 3: **GS** shows a general relation between two different chaotic systems. It demonstrates the randomness and universality of relation in synchronization process. Obviously, **CS** belongs to a simple form of **GS**. However, as the special forms of φ are flexible and it is difficult to design corresponding controller $\mathbf{u}$, we do not find a general controller on **GS**.

C. PHASE SYNCHRONIZATION

In 1996, Rosenblum etal. found an interesting phenomenon that the phases were locked in the **CS** of coupled Rossler attractors, while the amplitudes varied chaotically and were practically uncorrelated [48]. When he tried to couple a hyperchaotic oscillator with the other, the phase difference was unbounded and the frequencies were entrained. Therefore, the phase synchronization (**PHS**) was obtained. Assuming ϕ_1, ϕ_2 are the phases of two chaotic oscillators, respectively. The definition of **PHS** is as follows,

$$\|a_1\phi_1 - a_2\phi_2\| < a_3, \tag{14}$$

where a_1, a_2 are two positive constants and a_3 is a small positive constant.

Remark 4: Phase synchronization is characterized by that the phase difference between two chaotic oscillators which are locked within 2π, while their amplitudes are still chaotic and uncorrelated.

However, Rosenblum etal. also mentioned that the phenomenon of phase synchronization was a characteristic feature of autonomous continuous-time systems, and cannot be observed in discrete-time or periodically forced models. Four years later, Liu *et al.* [49] observed biased anti-phase synchronization (**APHS**) in coupled map lattices, and reduced the biased error of anti-phase synchronization efficiently by adjusting the coupling constant. Almost at the same time, Hu *et al.* [50] reported an antiphase phase-synchronized state in a system of diffusively coupled Rossler oscillators, then H. L. Yang [51] explored this antiphase state in detail. Later, Ho *et al.* [52] adopted active control techniques and achieved phase and anti-phase synchronization for two coupled chaotic systems. **PHS** and **APHS** are classical synchronization forms. They are discovered in an early time, however, they are not as popular as projective synchronization which is covered in the following paragraphs.

D. LAG AND ANTICIPATING SYNCHRONIZATION

In 1997, Rosenblum etal. focused on increasing the coupling strength of two coupled self-sustained chaotic oscillators [53]. The states of two chaotic systems firstly reached phase synchronization. As further increase of coupling strength, a new synchronous regime was observed, where the states of one system always lag behind to the other. It was called lag synchronization (**LS**) as follows,

$$\lim_{t\to+\infty} \|\mathbf{x}(t) - \mathbf{y}(t-\tau)\| = 0, \tag{15}$$

where τ is time-delay or time lag. In practice, time delay is unavoidable in the process of transmitting information. Although many literatures regarded the synchronization as a no-delay status, one can not neglect the minor time delay, technically. Thus, **LS** can be more frequently encountered in experiments with coupled systems than **CS**. Actually, **LS** is similar to the anticipating synchronization [54], which means the response systems can anticipate the future states of drive systems or the past states of drive systems can anticipate current states of response systems.

Remark 5: When $\tau < 0$, the response system will synchronize the future state of drive system. In other word, the response system anticipates the state of drive system and the lag synchronization changes into anticipating synchronization.

Remark 6: If we set $\tau = 0$, there is no time lag factor in response system and drive system. The **LS** will predigest into **CS**.

E. PROJECTIVE SYNCHRONIZATION AND ITS EXTENSIONS

In 1999, Mainieri and Rehacek [55] found a new way of synchronization which was up to a scaling factor and explained its mechanism for a class of three-dimensional systems. It described a constant projective relation between different chaotic systems. Projective synchronization (**PS**) was firstly proposed and greatly attracted widespread attentions from later generations. The form of **PS** is shown as follows,

$$\lim_{t\to+\infty} \|\mathbf{x}(t) - h\mathbf{y}(t)\| = 0, \tag{16}$$

where h is the scaling factor. And when h is a genernal scaling factor, the **PS** also was called general projective synchronization (**GPS**) [56]. Consider hyperchaotic Lorenz system as the response system in [57], which is shown as follows,

$$\begin{cases} \dot{x}_1 = x_4 - a_1(x_1 - x_2) + u_1, \\ \dot{x}_2 = -x_2 + a_2x_1 - x_1x_3 + u_2, \\ \dot{x}_3 = -a_3x_3 + x_1x_2 + u_3, \\ \dot{x}_4 = -x_1x_3 + a_4x_4 + u_4, \end{cases} \tag{17}$$

and the same drive system is

$$\begin{cases} \dot{y}_1 = y_4 - a_1(y_1 - y_2), \\ \dot{y}_2 = -y_2 + a_2y_1 - y_1y_3, \\ \dot{y}_3 = -a_3y_3 + y_1y_2, \\ \dot{y}_4 = -y_1y_3 + a_4y_4, \end{cases} \tag{18}$$

where u_1, u_2, u_3, u_4 are controllers and a_1, a_2, a_3, a_4 are system parameters.

Following the definition of **PS**, we can set error state vector $e_1 = x_1 - hy_1, e_2 = x_2 - hy_2, e_3 = x_3 - hy_3, e_4 = x_4 - hy_4$. Then the error systems is

$$\begin{cases} \dot{e}_1 = -a_1(x_1 - x_2) + x_4 + ha_1(y_1 - y_2) - hy_4 - hu_1, \\ \dot{e}_2 = -x_1x_3 + a_2x_1 - x_2 + hy_1y_3 - ha_2y_1 + hy_2 + hu_2, \\ \dot{e}_3 = x_1x_2 - a_3x_3 - hy_1y_2 + ha_3y_3 - hu_3, \\ \dot{e}_4 = -x_1x_3 + a_4x_4 + hy_1y_3 - ha_4y_4 - hu_4. \end{cases} \tag{19}$$

One can choose the controller according to active control, which is shown as follows,

$$\begin{cases} u_1 = \frac{1}{h}[-a_1(x_1 - x_2) + x_4 + ha_1(y_1 - y_2) - hy_4 + e_1], \\ u_2 = \frac{1}{h}[-x_1x_3 + a_2x_1 - x_2 + hy_1y_3 - ha_2y_1 + hy_1 + e_2], \\ u_3 = \frac{1}{h}[x_1x_2 - a_3x_3hy_1y_2 + ha_3y_3 + e_3], \\ u_4 = \frac{1}{h}[-x_1x_3 + a_4x_4 + hy_1y_3 - ha_4y_4 + e_4], \end{cases} \tag{20}$$

Proof: We can choose the Lyapunov function in **PS**

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2),$$

then

$$\begin{aligned} \dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \\ &= e_1(-a_1(x_1 - x_2) + x_4 + ha_1(y_1 - y_2) - hy_4 - hu_1) \\ &\quad + e_2(-x_1x_3 + a_2x_1 - x_2 + hy_1y_3 - ha_2y_1 + hy_2 - hu_2) \\ &\quad + e_3(x_1x_2 - a_3x_3 - hy_1y_2 + ha_3y_3 - hu_3) \\ &\quad + e_4(-x_1x_3 + a_4x_4 + hy_1y_3 - ha_4y_4 - hu_4). \end{aligned} \tag{21}$$

Put the controller (20) of **PS** into error system (19). Thus,

$$\dot{V} = -(e_1^2 + e_2^2 + e_3^2 + e_4^2) < 0$$

The proof is finished. Following the Lyapunov stability theory, the error system is asymptotically stable and the drive system (17) and response system (18) accomplish **PS** with the controller (20). We also inspect the proposed control technique in **PS** by reappearing simulations. Set the initial condition is $[x_1(0), x_2(0), x_3(0), x_4(0)] = [1, 2, 3, 4]$, $[y_1(0), y_2(0), y_3(0), y_4(0)] = [-1, -2, -3, -6]$, $h = 2$. Fig.3 is diagram of error state variables. It demonstrates that each error variables tend to zero at $t = 5s$ with the **PS** controller. Fig.4 clearly illustrates the proportional relationship between response system and drive system.

Remark 7: We can choose various values of h. When $h = 1$, **PS** will become **CS**. Thus, **CS** is a special form of **PS**.

Remark 8: Consider an extreme case, one set $h = 0$ and the question of **PS** becomes a problem of chaos control.

Remark 9: Active control theory is widely used in obtaining chaos synchronization, because it has a simple theory and mechanism in constructing synchronization controllers. Although active control has many natural advantages in this field, active controllers (such as (20)) eliminate the nonlinear parts of response systems in fact. As a result, some characters of response chaotic system may vanish by the effect of active controller.

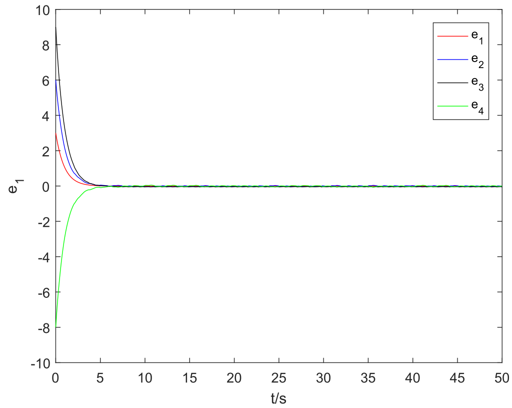

FIGURE 3. Diagram of error states of PS.

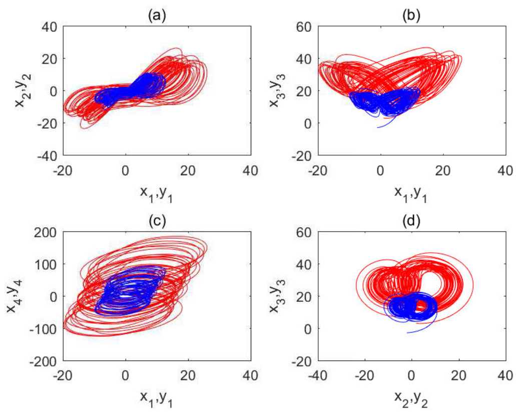

FIGURE 4. Diagram of phase states of PS.

Remark 10: As for error feedback controllers (such as (6)), it changes performances of response system as little as possible. It has the simple structure and is easy to be implemented, it is usually applicable to the case that the drive system and response system have the same chaotic model with different initial conditions.

In 2001, Sivaprakasam *et al.* [58] found a transition from a positive gradient to a negative gradient when plotting the output power of the slave laser against that of the master at each instant in time. This new phenomenon was called inverse synchronization. In fact, it is anti-synchronization (**AS**). Two years later, Kim [59] proposed the anti-synchronization, which can be characterized by vanishing of the sum of relevant state variables. It has the similar structure to **CS**. **AS** is

$$\lim_{t \to +\infty} \|x(t) + y(t)\| = 0. \tag{22}$$

Remark 11: When one consider $h = -1$ in (16), the **PS** will change into **AS**. In other words, **AS** is the special case of **PS**.

In 2007, Li [60] extensively investigated **PS**, and extended it to modified projective synchronization (**MPS**). The major difference between **PS** and **MPS** is that the scaling factor h in **MPS** is a diagonal scaling matrix $h = diag(h_1, h_2, \ldots, h_n)$. In other words, the **CS**, **AS** and **PS** are the special cases of **MPS** when $h = 1, h = -1$ and $h_1 = h_2 = \cdots = h_n$, respectively. There also was another definition of MPS, which was depicted as full state hybrid projective synchronization (**FSHPS**) [61]. As a matter of fact, **FSHPS** is same as **MPS**.

Later, Rui [62] studied hybrid projective synchronization (**HPS**), which means two different chaotic systems with different dimensions can synchronize up to an arbitrary scaling matrix.

Hybrid projective dislocated synchronization (**HPDS**) with unknown parameters was proposed in [63], which means the state variables in response system dislocatedly synchronize the state variables in drive system. Consider x_1, x_2, x_3 are three state variables in response chaotic system and y_1, y_2, y_3 are three state variables in drive chaotic system. The scaling factors are h_1, h_2, h_3. It is said one accomplish **HPDS** such that $e_1 = x_1 - h_1y_2, e_2 = x_2 - h_2y_3, e_3 = x_3 - h_3y_1$.

F. FUNCTION PROJECTIVE SYNCHRONIZATION

Enlightened by **PS** and **MPS**, Chen *et al.* [64] introduced a more general synchronization form in 2007, which was called function projective synchronization (**FPS**). It demonstrates that the drive and response chaotic systems can synchronize to a scaling function $h(t)$. **FPS** is shown as follows,

$$\lim_{t\to+\infty} \|x(t) - h(t)y(t)\| = 0. \tag{23}$$

Xu *et al.* [65] considered a special case that the scaling function was a bounded scaling function. Thus, the **FPS** can be bounded scaling function projective synchronization, which had the same structure with **FPS**. When it comes to the unknown parameters, Xu [66] studied adaptive synchronization of uncertain chaotic systems with adaptive scaling function by proposed adaptive controllers, which was significant in accomplishing **FPS** with unknown parameters.

Then Du *et al.* [67] made an extension of **FPS** and realized the application of modified function projective synchronization (**MFPS**), which was similar to the extension from **PS** to **MPS**. **MFPS** can also synchronize up to a desired diagonal scaling function matrix $h(t) = diag(h_1(t), h_2(t), \ldots, h_n(t))$.

Remark 12: **FPS** and **PS** are the special cases of **MFPS** when $h_1(t) = h_2(t) = \cdots = h_n(t)$ and $h_1(t) = h_2(t) = \cdots = h_n(t) = constant$, respectively. The summary about relations of various synchronization types is shown in Tab 1.

Here we give a simple simulation to illustrate **MFPS**. Consider the drive system (11) and response system (10) with different initial conditions. The scaling function is $h(t) = [sin(t), cos(t), 1]^T$. Then we have the following error states,

$$\begin{cases} e_1 = x_1 - sin(t)y_1, \\ e_2 = x_2 - cos(t)y_2, \\ e_3 = x_3 - y_3, \end{cases} \tag{24}$$

TABLE 1. Relations of various synchronization types.

Scaling matrix $\mathbf{h}$	The type of synchronization
$\mathbf{h}(t) = \mathrm{diag}\{h_1(t), h_2(t), \cdots, h_n(t)\} \in \mathbb{R}^{n\times n}$	MFPS
$\mathbf{h} \in \mathbb{R}^{m\times n}$	HPS
$\mathbf{h} = \mathrm{diag}\{h_1, h_2, \cdots, h_n\} \in \mathbb{R}^{n\times n}$	MPS
$\mathbf{h}(t) = \mathrm{diag}\{h(t), h(t), \cdots, h(t)\} \in \mathbb{R}^{n\times n}$	FPS
$\mathbf{h} = \mathrm{diag}\{h, h, \cdots, h\} \in \mathbb{R}^{n\times n}$	PS
$\mathbf{h} = \mathrm{diag}\{1, 1, \cdots, 1\}$	CS
$\mathbf{h} = \mathrm{diag}\{-1, -1, \cdots, -1\}$	AS

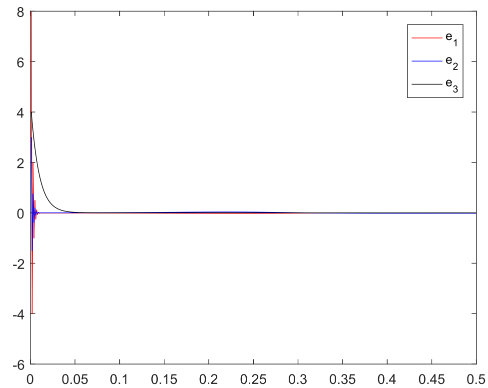

FIGURE 5. Diagram of error states of MFPS.

and the error states system is

$$\begin{cases} \dot{e}_1 = 10(x_2 - x_1) - cos(t)y_1 - 10sin(t)(y_2 \\ \qquad -y_1) + u_1, \\ \dot{e}_2 = 28x_1 - x_1x_3 - x_2 + y_2sin(t) \\ \qquad -cos(t)(28y_1 - y_1y_3 - y_2) + u_2, \\ \dot{e}_3 = x_1x_2 - 8/3x_3 - y_1y_2 + 8/3e_3 + u_3. \end{cases} \tag{25}$$

According to active control theory, the **MFPS** controller is designed as follows,

$$\begin{cases} u_1 = -10e_2 + y_1cos(t) - k_1e_1, \\ u_2 = -28e_1 + x_1x_3 - y_1y_3cos(t) \\ \qquad -y_2sin(t) - k_2e_2, \\ u_3 = -x_1x_2 + y_1y_2 - k_3e_3, \end{cases} \tag{26}$$

where k_1, k_2, k_3 are control strength parameters and they are real positive constants.

Related simulations are made to clarify the **MFPS**. Set $[x_1(0), x_2(0), x_3(0)] = [8, 4, 6], [y_1(0), y_2(0), y_3(0)] = [5, 1, 2], k_1 = 1500, k_2 = 1500, k_3 = 100$. Fig.5 is the diagram of error states. It demonstrates that the error system (25) tends to zero and two different Lorenz systems obtain **MPFS** with the controller (26). Note that the parameters of k_1, k_2 are larger than k_3 due to the complexity of scaling functions $h_1(t), h_2(t)$. Generally, larger control strengths ensure better stability of error systems.

In 2009, Li *et al.* [68] combined the characteristics of **LS** and **PS** and proposed projective lag synchronization (**PLS**). It meant that the drive chaotic system can synchronize the past states of the response system up to a scaling factor h, which is shown as follows,

$$\lim_{t\to+\infty} \|x(t-\tau) - hy(t)\| = 0. \tag{27}$$

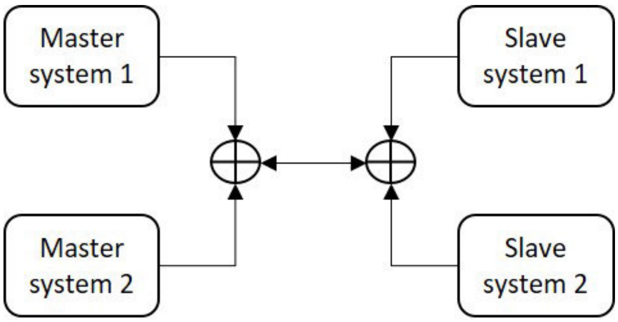

FIGURE 6. **Diagram of principle structure of DS.**

In the same year, Tae *et al.* [69] extended **PLS** to function projective lag synchronization (**FPLS**) and achieved **FPLS** by means of adaptive controller. The difference between **PLS** and **FPLS** is that the scaling factor in **FPLS** is a function $h(t)$, which is as follows,

$$\lim_{t\to+\infty} \|x(t-\tau) - h(t)y(t)\| = 0. \tag{28}$$

Remark 13: Obviously, **FPLS** will change into **PLS** when scaling function $h(t)$ is a constant matrix.

Remark 14: If one set $\tau = 0$ in (28), the **FPLS** completely becomes **FPS** and the various extensions of **FPS** also belong to **FPLS**.

G. DUAL SYNCHRONIZATION

The above synchronization forms are limited in one to one chaotic system. Can the **CHS** happen in two drive systems and two response systems? Early in 2000, Liu and Davis [70] treated the problem of synchronizing two different pairs of chaotic system. Then the dual synchronization (**DS**) of chaos was presented, and the diagram of principle structure of **DS** is shown in Fig.6, where Master system 1 and Slave system 1 are a pair of chaotic systems, and so are Master system 2 and Slave system 2.

Remark 15: It is said that **DS** means two different pairs of chaotic systems simultaneously and respectively synchronize with a single scalar signal. It is the characteristic that is different from previous synchronization.

H. COMBINATION SYNCHRONIZATION AND ITS EXTENSIONS

In 2011, Luo [71] attempted to realize **CHS** in three chaotic systems and proposed the definition of combination synchronization (**COS**) as follows,

$$\lim_{t\to+\infty} \|A_1x(t) + A_2y(t) - A_3z(t)\| = 0, \tag{29}$$

where $x(t), y(t)$ are both drive systems and $z(t)$ is one response system, A_1, A_2, A_3 are three constant matrixes and $A_3 \neq 0$ in **COS**.

We inspect the controller and corresponding stability theorem in [71]. Consider the first drive system

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = -b_1x_1 - x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - c_1x_3. \end{cases} \tag{30}$$

The second drive system is Chen system, which is shown as follows,

$$\begin{cases} \dot{y}_1 = a_2(y_2 - y_1), \\ \dot{y}_2 = (c_2 - a_2)y_1 - c_2y_2 - y_1y_3, \\ \dot{y}_3 = y_1y_2 - b_2y_3. \end{cases} \tag{31}$$

The response system is

$$\begin{cases} \dot{z}_1 = a_3(z_2 - z_1) + u_1, \\ \dot{z}_2 = c_3z_2 - z_1z_3 + u_2, \\ \dot{z}_3 = z_1z_2 - b_3z_3 + u_3, \end{cases} \tag{32}$$

where a, b, c are systems parameters. We set $e_1 = \gamma_1 z_1 - \alpha_1 x_1 - \beta_1 y_1, e_2 = \gamma_2 z_2 - \alpha_2 x_2 - \beta_2 y_2, e_3 = \gamma_3 z_3 - \alpha_3 x_3 - \beta_3 y_3$. Thus, the error systems is shown as follows,

$$\begin{cases} \dot{e}_1 = \frac{\gamma_1 a_3}{\gamma_2}e_2 - a_3e_1 + f + \gamma_1u_1, \\ \dot{e}_2 = e_2c_3 - \frac{\gamma_2}{\gamma_3\gamma_1}e_1e_3 - \frac{\gamma_2}{\gamma_3\gamma_1}(\alpha_3x_3 + y_3\beta_3)e_1 \\ -\frac{\gamma_2}{\gamma_3\gamma_1}(\alpha_1x_1 + y_1\beta_1)e_3 + g + \gamma_2u_2, \\ \dot{e}_3 = \frac{\gamma_3}{\gamma_2\gamma_1}e_2e_1 + \frac{\gamma_3}{\gamma_2\gamma_1}e_1(x_2\alpha_2 + y_2\beta_2) \\ +\frac{\gamma_3}{\gamma_2\gamma_1}e_2(\alpha_1x_1 + \beta_1y_1) - b_3e_3 + h + \gamma_3u_3 \end{cases} \tag{33}$$

where

$$\begin{cases} f = \frac{\gamma_1\alpha_2a_3}{\gamma_2}x_2 + \frac{\gamma_1\beta_2a_3}{\gamma_2}y_2 - a_3\alpha_1x_1 \\ \quad -a_3\beta_1y_1 - a_1x_2\alpha_1 + a_1x_1\alpha_1 - \beta_1y_2a_2 + \beta_1y_1a_2, \\ g = c_3(\alpha_2x_2 + \beta_2y_2) - \frac{\gamma_2}{\gamma_3\gamma_1}(\alpha_1x_1 + y_1\beta_1)(\beta_3y_3 + x_3\alpha_3) \\ \quad +b_1x_1\alpha_2 + x_2\alpha_2 + \alpha_2x_1x_3 - \beta_2y_1(c_2 - a_2) \\ \quad -\beta_2y_2c_2 + \beta_2y_3y_1, \\ h = \frac{\gamma_3}{\gamma_2\gamma_1}e_2(\alpha_1x_1 + \beta_1y_1) + \frac{\gamma_3}{\gamma_2\gamma_1}(x_1\alpha_1 + y_1\beta_1) \\ \quad *(x_2\alpha_2 + y_2\beta_2) - b_3(x_3\alpha_3 + y_3\beta_3) - x_1x_2\alpha_3 \\ \quad +\alpha_3c_1x_3 - y_1y_3\beta_3 + b_3y_3\beta_3 + \gamma_3u_3. \end{cases} \tag{34}$$

The controller is

$$\begin{cases} u_1 = -\frac{1}{\gamma_1}f, \\ u_2 = -\frac{1}{\gamma_2}((\frac{\gamma_2(a_3-1)}{\gamma_1a_3} + \frac{\gamma_2(a_3-1)c_3}{\gamma_1a_3} \\ \quad -\frac{\gamma_2}{\gamma_1\gamma_3}(\alpha_3x_3 + y_3\beta_3))v_1 + \frac{\gamma_1a_3}{\gamma_2}v_1 \\ \quad +v_2(c_3 - a_3 + 2) + g), \\ u_3 = -\frac{1}{\gamma_3}(v_3(1 - b_3) + \frac{(a_3-1)}{a_3\gamma_1^2}v_1^2\gamma_3 \\ \quad +\frac{\gamma_3}{\gamma_1\gamma_2}v_1(x_2\alpha_2 + \beta_2y_2) + \frac{\gamma_3(-1+a_3)}{\gamma_1^2a_3} \\ \quad \times(\beta_1y_1 + x_1\alpha_1)v_1 \\ \quad +(\frac{\gamma_3}{\gamma_1\gamma_2} - \frac{\gamma_2}{\gamma_1\gamma_3}v_2(\alpha_1x_1 + v_1 + \beta_1y_1)) + h), \end{cases} \tag{35}$$

where $v_1 = e_1, v_2 = e_2 - v_1\frac{\gamma_2(a_3-1)}{\gamma_1a_3}, v_3 = e_3$.

Proof: Firstly, we consider the $v_1 = e_1$, so

$$\dot{v}_1 = \frac{a_3\gamma_1}{\gamma_2}e_2 - e_1 a_3. \tag{36}$$

Set $e_2 = \alpha_1(v_1)$ and regard it as a virtual controller. To stabilize v_1 system, we can get the Lyapunov function:

$$V_1 = \frac{1}{2}v_1^2,$$

and

$$\dot{V}_1 = v_1\frac{\gamma_1 a_3}{\gamma_2}\alpha_1(v_1) - v_1^2 a_3.$$

Assume $\alpha_1(v_1) = \frac{\gamma_2 a_3 - \gamma_2}{\gamma_1 a_3}$, then we can get $\dot{V}_1 = -v_1^2 < 0$. Thus, $\dot{V}_1$ is negative definite, the v_1 systems (36) is asymptotically stable. Considering that the function $\alpha_1(v_1)$ is a hypothetical estimative function, we denote $v_2 = e_2 - \alpha_1(v_1)$. So we can get

$$\begin{cases} \dot{v}_1 = \frac{\gamma_1 a_3}{\gamma_2}v_2 - v_1, \\ \dot{v}_2 = -\frac{\gamma_2}{\gamma_1\gamma_3}((\alpha_1 x_1 + \beta_1 y_1) + v_1)e_3 - \frac{\gamma_1 a_3}{\gamma_2}v_1 - v_2, \end{cases} \tag{37}$$

where $e_3 = \alpha_2(v_1, v_2)$. We regard it as an another virtual controller.

Secondly, to stabilize the $v_1 - v_2$ systems (37), we give the Lyapunov function of (37), which is shown as follows,

$$V_2 = V_1 + \frac{1}{2}v_2^2.$$

Then,

$$\dot{V}_2 = v_1(\frac{\gamma_1 a_3}{\gamma_2}v_2 - v_1) - \frac{\gamma_2 v_2}{\gamma_1\gamma_3}[(\alpha_1 x_1 + y_1\beta_1) + v_1]\alpha_2 \times (v_1, v_2) - \frac{\gamma_1 a_3}{\gamma_2}v_1 v_2 - v_2^2. \tag{38}$$

Choose $\alpha_2(v_1, v_2) = 0$, we can know $\dot{V}_2 = -v_1^2 - v_2^2 \leq 0$. It demonstrates that the $v_1 - v_2$ system (37) is asymptotically stable. Following the method of selecting v_2, one can get $v_3 = e_3 - \alpha_2(v_1, v_2)$. The derivative of v_3 is

$$\dot{v}_3 = -v_3 + \frac{\gamma_2}{\gamma_1\gamma_3}[v_1 + \alpha_1 x_1 + \beta_1 y_1]v_2.$$

Then, we get the following $v_1 - v_2 - v_3$ system,

$$\begin{cases} \dot{v}_1 = \frac{\gamma_1 a_3}{\gamma_2}v_2 - v_1, \\ \dot{v}_2 = -\frac{\gamma_2}{\gamma_1\gamma_3}(\alpha_1 x_1 + \beta_1 y_1 + v_1)v_3 - \frac{\gamma_1 a_3}{\gamma_2}v_1 - v_2, \\ \dot{v}_3 = -v_3 + \frac{\gamma_2}{\gamma_1\gamma_3}[v_1 + \alpha_1 x_1 + \beta_1 y_1]v_2, \end{cases} \tag{39}$$

We can choose the following Lyapunov function,

$$V_3 = \frac{1}{2}v_3^2 + V_2.$$

and

$$\dot{V}_3 = v_3\dot{v}_3 + \dot{V}_2 = -v_1^2 - v_2^2 - v_3^2 \leq 0.$$

According to Lyapunov stability theorem, one can get that the $v_1 - v_2 - v_3$ system is asymptotically stable. In other words, if we choose suitable $v_1 = e_1, v_2 = e_2 - \alpha_1(v_1), v_3 = e_3 - \alpha_2(v_1, v_2)$, then

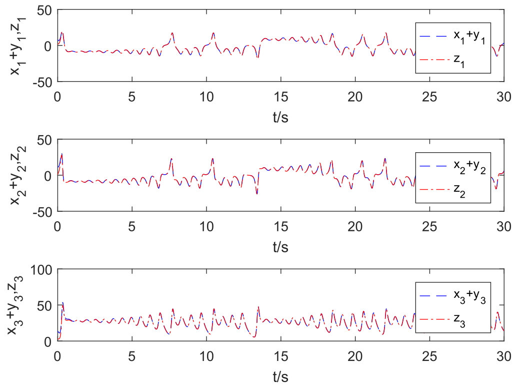

FIGURE 7. Diagram of phase states of COS.

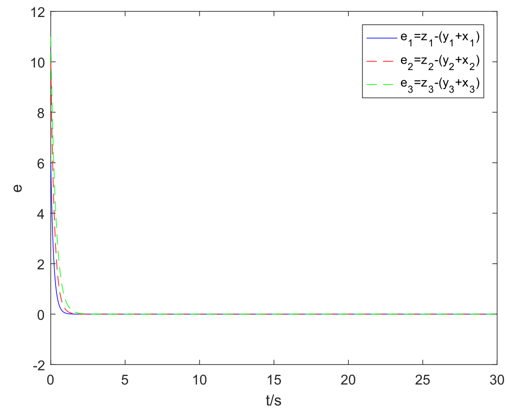

FIGURE 8. Diagram of error phase states of COS.

$\lim_{t\to+\infty} e_1(t) = 0$, $\lim_{t\to+\infty} e_2(t) = 0$, $\lim_{t\to+\infty} e_3(t) = 0$. It illustrates that the two drive systems (30),(31) and the response system (32) can reach combination synchronization with the controller (35). The proof is finished. The corresponding simulations are shown in Fig.7 and Fig.8, where the initial conditions are $[x_1(0), x_2(0), x_3(0)] = [1, 3, 5]$, $[y_1(0), y_2(0), y_3(0)] = [7, 8, 9]$, $[z_1(0), z_2(0), z_3(0)] = [2, 1, 3]$. One can see that the response system and two drive systems accomplish **COS** with the controller and the error system gradually tends to zero.

Remark 16. Traditional **CHS** mainly focus on different relations between two chaotic systems, while **COS** proposed a totally different **CHS** form. Three chaotic systems are the necessary condition to obtained **COS**. In the practical sense, **COS** can modulate the information signal into two parts, which increase the difficulty of anti-attacking in the secure communication of **CHS**.

Remark 17: If $A_1 = 1, A_2 = 0$ and A_3 is a scaling constant matrix, then **COS** will become **PS**. If $A_1 = A_3 = 1, A_2 = 0$, **COS** will become into **CS**, accordingly.

In the sequel, many scholars made a host of innovations of **COS**. Based on **COS**, Luo [72] extensively studied the equal combination synchronization (**ECOS**). The form of **ECOS** is similar to **COS**, which is as follows,

$$\lim_{t\to+\infty} \|A_1 x(t) + A_2 y(t) + A_3 z(t)\| = 0. \tag{40}$$

In 2013, Sun *et al.* [73] achieved **CHS** in four identical chaotic systems, which consists of two drive systems and two response systems. It is called combination-combination synchronization (**COCOS**) and is shown as follows,

$$\lim_{t\to+\infty} \|A_1x(t) + A_2y(t) - A_3z(t) - A_4w(t)\| = 0, \quad (41)$$

where $x(t), y(t)$ are two drive systems and $z(t), w(t)$ are two response systems, A_1, A_2, A_3, A_4 are four constant matrixes in **COCOS**.

Obviously, **COCOS** is different from **DS**. **DS** mainly emphasizes that two pairs of chaotic systems simultaneously accomplish synchronization.

Moreover, Sun [74] continued to investigate the **CHS** forms of four systems. Previous types of **CHS** only focused on the forms of adding and subtracting from different chaotic systems. Sun firstly proposed a significant **CHS**, which was called Compound Synchronization. Compound Synchronization has two categories of drive systems, one parts are base drive systems, and the other is scaling drive system. It has the natural advantages in secure communication because of the complex and unpredictable dynamic characteristics. Compound Synchronization is shown as follows,

$$\lim_{t\to+\infty} \|By(t)(A_1x_1(t) + A_2x_2(t)) - Cz(t)\| = 0, \quad (42)$$

where $x_1(t), x_2(t)$ are the basic drive systems, $y(t)$ is the scaling drive system and $z(t)$ is response system.

In 2016, Sun [75] made a small extension of Compound Synchronization. He added a response system and derived compound-combination synchronization of five chaotic systems, which is as follows,

$$\lim_{t\to+\infty} \|By(t)(A_1x(t)+A_2x_2(t)) - C_1z_1(t)-C_2z_2(t)\|=0, \quad (43)$$

where $x_1(t), x_2(t)$ are the basic drive systems, $y(t)$ is the scaling drive system, $z_1(t), z_2(t)$ are two response systems and A_1, A_2, B, C_1, C_2 are five constant matrixes.

At the same time, based on dual synchronization and combination synchronization of three chaotic systems, Junwei Sun [76] discussed dual combination synchronization (**DCOS**) for six chaotic systems as follows,

$$\lim_{t\to+\infty} (||A_1x_1(t) + B_1y_1(t) - C_1z_1(t)|| + ||A_2x_2(t) + B_2y_2(t) - C_2z_2(t)||) = 0, \quad (44)$$

where $x_1(t), x_2(t), y_1(t), y_2(t)$ are drive systems and $z_1(t), z_2(t)$ are response systems, $A_1, A_2, B_1, B_2, C_1, C_2$ are six constant matrixes. Therefore, if $A_1 = B_1 = C_1 = 0$ (or $A_2 = B_2 = C_2 = 0$), the **DCOS** problem will be changed into **COS** problem.

In 2018, Mahmoud *et al.* [77] introduced double compound combination synchronization of eight chaotic systems according to double compound synchronization and compound combination synchronization as follows,

$$\lim_{t\to+\infty} ||(B_1y_1(t) + B_2y_2(t))(A_1x_1(t) + A_2x_2(t)) - C_1z_1(t) - C_2z_2(t) - C_3z_3(t) - C_4z_4(t)|| = 0. \quad (45)$$

The compound combination synchronization is a special case of one of $A_i = 0$ (or $B_i = 0$, $i = 1, 2$) and two of $C_i = 0$ $(i = 1, 2, 3, 4)$.

There also exists function combination synchronization (**FCOS**), function combination-combination synchronization (**FCOCOS**) and so on. Due to the small innovations, we do not make detailed reviews of them.

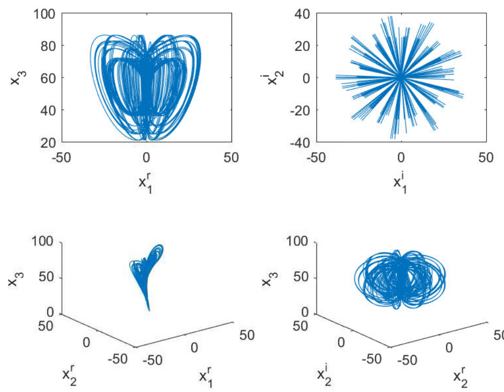

FIGURE 9. Diagram of projection spaces of complex Lorenz system without separating.

III. COMPLEX CHAOS SYNCHRONIZATION

In 1982, Fowler *et al.* [78] firstly presented the complex Lorenz system. It is applied to rotating fluids [79], [80], detuned laser [9], [79], [81], thermal convection process and so on. Since Mahmoud *et al.* [82] gave the detailed properties and the method of synchronization of complex lorenz system in 2007, complex-variable chaotic systems (**CVCSs**) have been getting more and more attentions. The relations between complex chaos and real chaos are established by means of separating the real and imaginary parts, thus complex variables can carry more messages and additionally enhance the security of communication. Nowadays, complex chaos are widely used in numerous applications such as plasma physics [11], electromagnetic fields [83] and secure communication [84].

The classic complex Lorenz system model is shown as follows,

$$\begin{cases} \dot{x}_1 = 35(x_2 - x_1), \\ \dot{x}_2 = 55x_1 - x_2 - x_1x_3, \\ \dot{x}_3 = \frac{1}{2}(\bar{x}_1x_2 + \bar{x}_2x_1) - \frac{8}{3}x_3, \end{cases} \quad (46)$$

where x_1, x_2 are complex variables, x_3 is real variable.

Without separating the real and imaginary parts, we compute complex variables by MatLAB directly and get Fig.9 which shows the diagrams of different projection spaces. For comparison, we obtain Fig.10 by separating the real

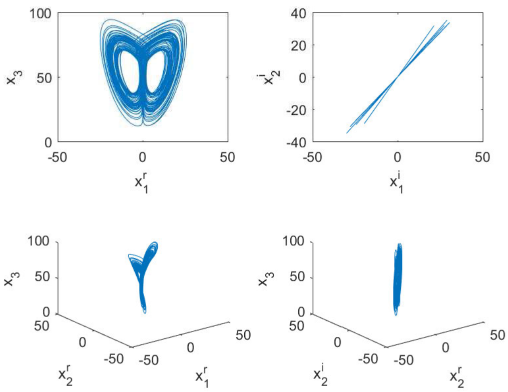

FIGURE 10. Diagram of projection spaces of complex Lorenz system with separating.

and imaginary parts. Although the same initial conditions, system parameters and iterations are used, there are some differences between Fig.9 and Fig.10. We may not give an actual answer about which one is completely right. Different solutions may correspond to different results. Then, we will describe the types of synchronization in **CVCSs** in the following paragraphs.

Synchronization of **CVCSs** has gradually been understood in recently ten years and many scholars applied existing synchronization types to **CVCSs**. For example, Hu *et al.* [85] observed the **HPS** in a coupled **CVCS** in 2008; Mahmoud *et al.* [86] investigated **CS** between complex Chen and Lü systems via active controller in 2009 and went on to discuss **CS** of **CVCSs** with uncertain parameters in 2010 [87]. To illustrate the basic process of synchronization of **CVCSs**, here we introduce the achieved complex synchronization in [86].

Consider the complex Chen systems as the drive system and the complex Lü system as response systems. The complex Chen is

$$\begin{cases}\dot{x}_1 = p_1(y_1 - x_1),\\ \dot{y}_1 = (p_3 - p_1)x_1 - x_1 z_1 + p_3 y_1,\\ \dot{z}_1 = \frac{1}{2}(\bar{x}_1 y_1 + \bar{y}_1 x_1) - p_2,\end{cases} \tag{47}$$

where $x_1 = m_{11} + jm_{21}, y_1 = m_{31} + jm_{41}, z_1 = m_{51}, x_1, y_1$ are complex variables, z_1 is real variables and p_1, p_2, p_3 are real system parameters. The form of complex Lü system is

$$\begin{cases}\dot{x}_2 = p_4(y_2 - x_2) + (u_1 + ju_2),\\ \dot{y}_2 = -x_2 z_2 + p_6 y_2 + (u_3 + ju_4),\\ \dot{z}_2 = \frac{1}{2}(\bar{x}_2 y_2 + \bar{y}_2 x_2) - p_5 z_2 - u_5,\end{cases} \tag{48}$$

where $x_2 = m_{12} + jm_{22}, y_2 = m_{32} + jm_{42}, z_2 = m_{52}, x_2, y_2$ are complex variables, p_4, p_5, p_6 are real system parameters. u_1, u_2, u_3, u_4, u_5 are undetermined control functions in this part, which will accomplish complete synchronization.

Then, separating imaginary and real parts, one can get the new form of complex chaotic systems. The complex Chen (47) is transformed into

$$\begin{cases}\dot{m}_{11} = p_1(m_{43} - m_{11}),\\ \dot{m}_{21} = p_1(m_{41} - m_{21}),\\ \dot{m}_{31} = (p_3 - p_1)m_{11} - m_{11}m_{51} + p_3 m_{31},\\ \dot{m}_{41} = (p_3 - p_1)m_{21} - m_{21}m_{51} + p_3 m_{31},\\ \dot{m}_{51} = m_{11}m_{31} + m_{21}m_{41} - p_2 m_{51},\end{cases} \tag{49}$$

and the new form of complex Lü system (48) is

$$\begin{cases}\dot{m}_{12} = p_4(m_{32} - m_{12}) + u_1,\\ \dot{m}_{22} = p_4(m_{42} - m_{22}) + u_2,\\ \dot{m}_{32} = -m_{12}m_{52} + p_6 m_{32} + u_3,\\ \dot{m}_{42} = -m_{22}m_{52} + p_6 m_{42} + u_4,\\ \dot{m}_{52} = m_{12}m_{32} + m_{22}m_{42} - p_5 m_{52} + u_5.\end{cases} \tag{50}$$

Then, the error states between drive variable and response variable are shown as follows,

$$\begin{cases}e_{m_1} + je_{m_2} = x_2 - x_1,\\ e_{m_3} + je_{m_4} = y_2 - y_1,\\ e_{m_5} = z_2 - z_1,\end{cases} \tag{51}$$

The derivative of error state variables is

$$\begin{cases}\dot{e}_{m_1} = p_4(e_{m_3} - e_{m_1}) + (p_4 - p_1)(m_{31} - m_{11}) + u_1,\\ \dot{e}_{m_2} = p_4(e_{m_4} - e_{m_2}) + (p_4 - p_1)(m_{41} - m_{21}) + u_2,\\ \dot{e}_{m_3} = -m_{12}e_{m_5} - m_{51}e_{m_1} + p_6 e_{m_3} + (p_6 - p_3)m_{31}\\ -(p_3 - p_1)m_{11} + u_3,\\ \dot{e}_{m_4} = -m_{22}e_{m_5} - m_{51}e_{m_2} + p_6 e_{m_4} + (p_6 - p_3)m_{41}\\ -(p_3 - p_1)m_{21} + u_4,\\ \dot{e}_{m_5} = -p_5 e_{m_5} + m_{11}e_{m_3} + m_{32}e_{m_1} + m_{21}e_{m_4}\\ +m_{42}e_{m_2} + (p_2 - p_5)m_{51} + u_5,\end{cases} \tag{52}$$

One can choose the Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{5} e_{m_i}^2.$$

and the derivative of Lyapunov function is

$$\begin{aligned}\dot{V} = &-(p_4 e_{m_1}^2 + p_4 e_{m_2}^2 + p_5 e_{m_5}^2) + e_{m_1}(p_4 e_{m_3}\\ &+ (p_4 - p_1)(m_{31} - m_{11})) + e_{m_2}(p_4 e_{m_4} + (p_4\\ &- p_1)(m_{41} - m_{21})) + e_{m_3}(-m_{12}e_{m_5} - m_{51}e_{m_1}\\ &+ p_6 e_{m_3} + (p_6 - p_3)m_{31} - (p_6 - p_3)m_{31}\\ &- (p_3 - p_1)m_{11}) + e_{m_4}(-m_{22}e_{m_5} - m_{51}e_{m_2} + ve_{m_4}\\ &+ (p_6 - p_3)m_{41} - (p_3 - p_1)m_{21}) + e_{m_5}(m_{11}e_{m_3}\\ &+ m_{32}e_{m_1}c + m_{21}e_{m_4} + m_{42}e_{m_2} + (p_2 - p_5)m_{51})\\ &+ \sum_{i=1}^{5} u_i e_{m_i}.\end{aligned} \tag{53}$$

According to active controller, one can get

$$u_1 = -p_4 e_{m_3} - (p_4 - p_1)(m_{31} - m_{11}),$$

$$u_2 = -p_4 e_{m_4} - (p_4 - p_1)(m_{41} - m_{21}),$$

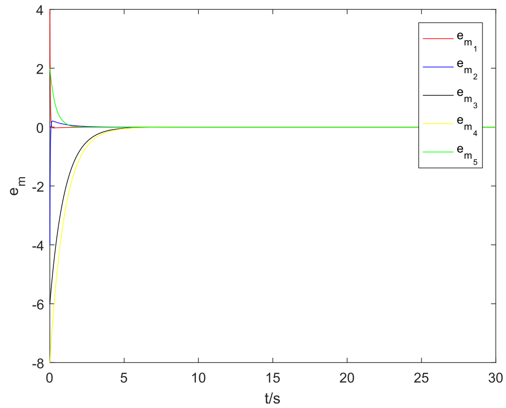

FIGURE 11. Diagram of error variable phases in complex chaotic synchronization.

$$\begin{aligned} u_3 &= -(p_6 e_{m_3} - m_{12} e_{m_5} - m_{51} e_{m_1} + (p_6 - p_3) m_{31} \\ &\quad -(p_3 - p_1) m_{11}), \\ u_4 &= -(p_6 e_{m_4} - m_{22} e_{m_5} - m_{51} e_{m_2} + (p_6 - p_3) m_{31} \\ &\quad -(p_3 - p_1) m_{21}), \\ u_5 &= -(m_{11} e_{m_3} - m_{32} e_{m_1} + m_{21} e_{m_4} + m_{42} e_{m_2} \\ &\quad +(p_2 - p_5) m_{51}). \end{aligned} \tag{54}$$

Putting (54) into (53), one can get

$$\dot{V} = -(p_4 e_{m_1}^2 + p_4 e_{m_2}^2 + p_5 e_{m_5}^2) < 0$$

Since $\dot{V}$ is negative definite and V is positive definite, the error system is asymptotically stable and the complex Chen and complex Lü can accomplish synchronization with active control function (54). Fig.11 is the diagram of error system (52), where the initial conditions are $[x_1, y_1, z_1] = [5 - 2i, -3 - 4i, 1]$, $[x_2, y_2, z_2] = [1 + 2i, 3 + 4i, -1]$. The simulations are in accord with the proof.

Remark 18: Note that CVCSs synchronization requests the real part of state variable and imaginary part simultaneously accomplish accordance.

Mahmoud *et al.* [88] designed a scheme to achieve phase synchronization (**PHS**) and antiphase synchronization (**APS**) in 2010; Mahmoud and Ahmed [89] presented a modified projective synchronization of complex Chen and Lü systems; Mahmoud *et al.* [90] introduced the modified projective lag synchronization (**MPLS**) of two nonidentical hyperchaotic complex nonlinear systems in 2011; Liu and Liu [91] provided anti-synchronization scheme between a new complex chaotic system and complex Lorenz system; Wang and Wei [92] investigated modified function projective lag synchronization of hyperchaotic complex systems in 2014. However, the above synchronization types are the extensions of previous synchronization types of real chaos in essence, and these scholars applied them to **CVCSs**.

In 2013, when some scholars tried to apply the **PS** to **CVCSs**, they found an interesting phenomenon, scaling factors can be complex numbers. **PS** with complex scaling factors was respectively and coincidentally obtained by Zhang *et al.* [93], Wu *et al.* [94] and Mahmoud and Mahmoud [95] at almost the same time. It is called as complex modified projective synchronization (**CMPS**) by most scholars. The complex scaling factors establish a link between real chaotic systems and complex chaotic systems [93]. We can adopt a complex system to synchronize the real drive system with complex scaling factors. A real system to synchronize the real (imaginary) part of the product of complex drive system and complex scaling factors can be realized. In the sequel, complex hybrid projective synchronization (**CHPS**) for **CVCSs** with different dimensions was also studied in [96].

Remark 19: CMPS is totally different from PS in real number, because the scaling factors are complex numbers. When we multiply the chaotic system with the complex scaling factors, the real parts and imaginary parts of CVCSs can be changed, while PS just multiply two real parts. It means linear combination using complex scaling factors while proportion for real scaling factors.

The similar situation appeared in the extension from real **FPS** to complex **FPS**. In 2013, Wu and Wang [97] studied adaptive generalized function projective synchronization for different dimensional **CVCSs**, where the scaling functions were still real. One year later, Liu and Zhang [98] considered the scaling functions can be complex. She firstly proposed complex function projective synchronization (**CFPS**) and studied its application in secure communications. We inspect the detailed synchronization method and corresponding controllers in [98]. Consider a general form of coupled complex n-dimensional chaotic system, which is shown as follows,

$$\begin{cases} \dot{x} = p(x, z) + u, \\ \dot{y} = f(y, z), \\ \dot{z} = g(y, z), \end{cases} \tag{55}$$

where x, y, z are state variables and u is the controller.

For the complex state variables in (55), it is said to accomplish **CFPS** between $x(t)$ and $y(t)$ if it exists a complex vector function such as

$$\begin{aligned} &\lim_{t \to +\infty} ||x(t) - H(y, z, t) y(t)|| \\ &\quad = ||x(t)^r - H(y, z, t)^r y(t)^r + H(y, z, t)^i y(t)^i|| \\ &\qquad + ||x(t)^i - H(y, z, t)^r y(t)^i + H(y, z, t)^i y(t)^r|| \\ &\quad = 0, \end{aligned} \tag{56}$$

where $H(y, z, t)$ is the bounded complex vector function. Superscripts r, i are the real and imaginary parts of complex function, respectively. The error state of **CFPS** is

$$e(t) = x(t) - H(y, z, t) y(t),$$

where $e(t) = e^r + je^i$. To be specific,

$$\begin{aligned} e^r &= x(t)^r - H(y, z, t)^r y(t)^r + H(y, z, t)^i y(t)^i \\ e^i &= x(t)^i - H(y, z, t)^r y(t)^i + H(y, z, t)^i y(t)^r. \end{aligned} \tag{57}$$

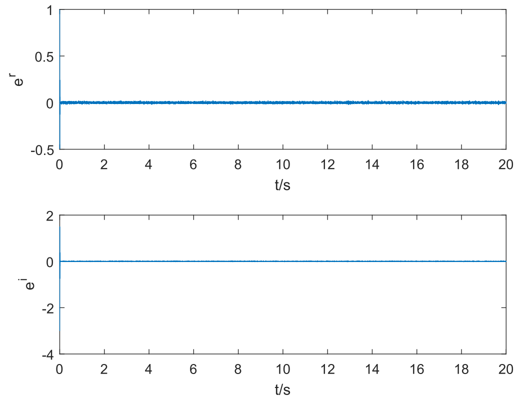

FIGURE 12. Diagram of error variable in CFPS.

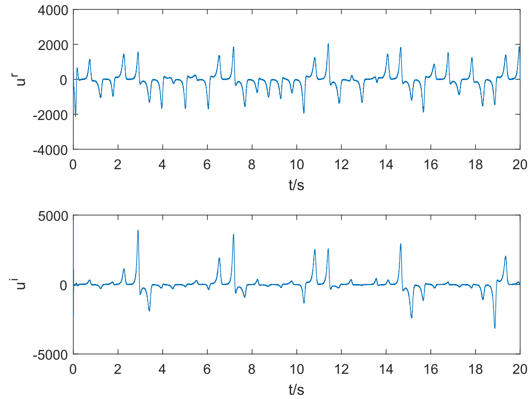

FIGURE 13. Diagram of the evolutions of controllers in CFPS.

According to the definition of **CFPS**, one can get the derivative of error states, which is shown as follows,

$$\dot{e} = \dot{x}(t) - \frac{d(H(y,z,t))y(t)}{dt} = \dot{x}(t) - \dot{H}(y,z,t)y(t) + \dot{y}(t)H(y,z,t). \quad (58)$$

Set $\frac{d(H(y,z,t))y(t)}{dt} = M$, then $\dot{e}(t) = p - M + u$. The proposed controller is shown as follows,

$$u = -p + M - ke, \quad (59)$$

where k is the control strength matrix and it is usually a positive value.

Proof: Put the controller (59) into the error system (58), one get

$$\dot{e} = \dot{e}^r + j\dot{e}^i = p + M - p + M - ke = -ke = -(ke^r + ke^i). \quad (60)$$

Consider the following Lyapunov function,

$$V(e,t) = \frac{1}{2}[(e^r)^T e^r + [(e^i)^T e^i], \quad (61)$$

and the derivative of (61) is shown,

$$\dot{V} = (\dot{e}^r)^T e^r + (\dot{e}^i)^T e^i = -(k(e^r)^2 + k(e^i)^2) = -ke^2 < 0. \quad (62)$$

Therefore, the Lyapunov function is negative definite in sense of that the general coupled complex chaotic system achieve **CFPS** with the controller (59). Because just one group of state variable was applied to transmit information in [98], we plot one error state and controller state in Fig.12 and Fig.13, respectively. It demonstrates that the drive system and response system fastly obtain **CFPS** with the controller (59).

Remark 20: **CFPS** is a more general type of **CHS**, which accounts for the diverse relation and complex links between two chaotic systems.

Remark 21: When we set that the complex function matrix is a complex constant factor, the **CFPS** change into **CMPS**. In a similar way, one consider that the function matrix is a real number function matrix, the **CFPS** can become into **FPS**.

Complex hybrid function projective synchronization (**CHFPS**) for **CVCSs** with different dimensions followed in [99]. As there are real parts and imaginary parts of complex variables, Mahmoud [87] introduced a synchronization only for complex nonlinear systems in 2013. That is complex complete synchronization (**CCS**) as follows,

$$\lim_{t\to+\infty} \|\mathbf{x}(t) - j\mathbf{y}(t)\| = 0. \quad (63)$$

Remark 22. **CCS** is an interesting type of **CHS** and it only exists in **CVCSs**. The distinction of **CCS** is that it shows the relation between real part of drive system and imaginary part of response system, so are imaginary part of drive system and real part of response system.

Later, considering time lag in **CVCSs**, he and K. M. Abualnaja [100] presented the definition of complex **LS** (**CLS**) as follows,

$$\lim_{t\to+\infty} ||\mathbf{x}(t) - j\mathbf{y}(t-\tau)|| = ||[\mathbf{x}^r(t) + \mathbf{y}^i(t-\tau)] + j[\mathbf{x}^i(t) - \mathbf{y}^r(t-\tau)]|| = 0, \quad (64)$$

It can be seen that **CLS** is greatly different from **LS**. The **CLS** includes **ALS** of real parts and **LS** of imaginary parts. **ALS** occurs between the real part of the response system and the imaginary part of a drive system, while **LS** occurs between an imaginary part of the response system and a real part of the drive system.

We recall the specific process of **CLS**. Consider the coupled complex nonlinear systems,

$$\begin{cases} \dot{x} = \dot{x}^r + j\dot{x}^i = \phi x + F(x,z) + u, \\ \dot{y} = \dot{y}^r + j\dot{y}^i = \phi y + F(y,z), \\ \dot{z} = g(x,y,z), \end{cases} \quad (65)$$

where $\dot{x}^r$, $\dot{x}^i$ are the real and imaginary parts and ϕ, F are the linear and nonlinear part functions, respectively. Following the definition of **CLS**, we can get

$$e = e^r + je^i = \lim_{t\to+\infty} ||x(t) - jy(t-\tau)||, \quad (66)$$

then, the derivative of e is

$$\dot{e} = \dot{e}^r + j\dot{e}^i = (\dot{x}^r(t) + \dot{y}^i(t-\tau)) + j(\dot{x}^i(t) - \dot{y}^r(t-\tau)) \quad (67)$$

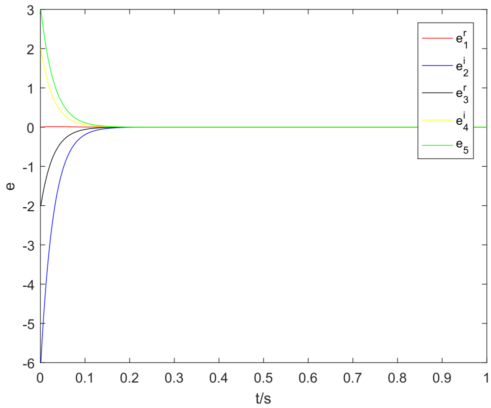

FIGURE 14. Diagram of error variable in CLS.

The proposed controller is shown as follows,

$$\begin{aligned} u &= u^r + ju^i \\ &= -\phi x(t) - F(x(t), z(t)) \\ &\quad + j[\phi y(t-\tau) + F(y(t-\tau), z(t-\tau))] - ke, \end{aligned} \tag{68}$$

where k is a positive constant.

To illustrate the proposed the feasibility of proposed controller, that is to say, the coupled complex chaotic systems achieve **CLS** with the controller (68), one define the Lyapunov function as follows,

$$V(t) = \frac{1}{2}((e^r)^T e^r + (e^i)^T e^i). \tag{69}$$

We can get the derivative of (69),

$$\begin{aligned} \dot{V}(t) &= (\dot{e}^r)^T e^r + (\dot{e}^i)^T e^i \\ &= (\phi x^r(t) + F^r(x(t), z(t)) + \phi y^i(t-\tau) \\ &\quad + F^i(y(t-\tau), z(t-\tau)) + u^r)^T e^r \\ &\quad + (\phi x^i(t) + F^i(x(t), z(t)) + \phi y^r(t-\tau) \\ &\quad + F^r(y(t-\tau), z(t-\tau)) + u^i)^T e^i. \end{aligned} \tag{70}$$

Put the controller (68) into (69), then

$$\begin{aligned} \dot{V}(t) &= k((e^r)^T e^r + (e^i)^T e^i) \\ &= -k((e^r)^2 + (e^i)^2) < 0. \end{aligned} \tag{71}$$

It demonstrates that $V(t)$ is positive definite and the derivative of $V(t)$ is negative definite, the error system (67) is asymptotically stable and the coupled complex chaotic system (65) accomplish **CLS** with the controller (68). To get a clear result, two complex Lorenz systems with different initial conditions are used to obtain **CLS**. Fig.14 is the diagram error system, where $e_1^r = x_1^r - y_1^i, e_1^i = x_1^i - y_1^r, e_2^r = x_2^r - y_2^i, e_2^i = x_2^i - y_2^r, e_3 = x_3 - y_3, \tau = 0$ and initial conditions are $[x_1(0), x_2(0), x_3(0)] = [1+2j, 3+4j, 5], [y_1(0), y_2(0), y_3(0)] = [8+j, 2+5j, 2]$. The simulations are as expected and in accord with the proof.

Then Emad E. Mahmoud and Fatimah S. Abood [101] developed the following complex antilag synchronization (**CALS**) in 2017,

$$\begin{aligned} \lim_{t\to+\infty} ||\mathbf{x}(t) + j\mathbf{y}(t-\tau)|| &= ||[\mathbf{x}^r(t) - \mathbf{y}^i(t-\tau)] \\ &\quad + j[\mathbf{x}^i(t) + \mathbf{y}^r(t-\tau)]|| = 0, \end{aligned} \tag{72}$$

The term **CALS** can be seen as synchronization among **ALS** and **LS**, similar to **CLS**.

For the **CHS** of multiple complex chaotic systems, considering combination synchronization with regard to the complex scaling matrices, Sun *et al.* [102] firstly researched complex combination synchronization(**CCOS**) among three complex chaotic systems in 2014. Since then, Zhang *et al.* [103] achieved complex combination-combination synchronization (**CCOCOS**) for four complex memristor oscillator systems. In 2015, Jiang [104] proposed a complex generalized combination synchronization, where two drive systems and one response system can be synchronized to two complex scaling matrices which are non-square matrices. Sun [105] finished complex compound synchronization for four complex chaotic systems and provided relevant controllers. Although aforehand papers made some innovations in **CHS** and color the diversity of chaos synchronization, most of them put previous type of **CHS** into **CVCSs** in essence.

So far, the above types of **CHS** are projective relation and produced by the proportion of drive and response systems. What is their difference? In 2018, Chen *et al.* [106] studied modified difference function synchronization (**MDFS**), which aimed to synchronize the error system up to a desired scaling function matrice. **MDFS** is as follows,

$$\lim_{t\to+\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = \mathbf{h}(t), \tag{73}$$

where $\mathbf{h}(t)$ is the expected complex scaling function matrice. In fact, we can design the controller $\mathbf{u}$ to realize any desired $\mathbf{h}(t)$, and it is the problem of tracking control for **CVCSs**. **MDFS** further increases the complexity and diversity of chaos synchronization, thus increasing the diversity and security of communications.

IV. N-SYSTEMS COMBINATION FUNCTION PROJECTIVE SYNCHRONIZATION

A. THE GENERAL FORM OF CHAOTIC SYNCHRONIZATIONC

Enlightended by all reviewed types of **CHS**, in this part, we propose N-systems combination function projective synchronization (**NCOFPS**) which is as follows.

The N drive (master) systems are

$$\begin{cases} \dot{\mathbf{y}}_1 = \mathbf{g_1}(\mathbf{y_1}), \ \mathbf{y_1} \in \mathbf{C}^n, \\ \dot{\mathbf{y}}_2 = \mathbf{g_2}(\mathbf{y_2}), \ \mathbf{y_2} \in \mathbf{C}^n, \\ \cdots \\ \dot{\mathbf{y}}_N = \mathbf{g_N}(\mathbf{y_N}), \ \mathbf{y_N} \in \mathbf{C}^n \end{cases} \tag{74}$$

and the N' response (slave) systems are

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f_1}(\mathbf{x_1}) + \mathbf{u_1},\ \mathbf{x_1} \in \mathbf{C}^n, \\ \dot{\mathbf{x}}_2 = \mathbf{f_2}(\mathbf{x_2}) + \mathbf{u_2},\ \mathbf{x_2} \in \mathbf{C}^n, \\ \cdots \\ \dot{\mathbf{x}}_{N'} = \mathbf{f_{N'}}(\mathbf{x_{N'}}) + \mathbf{u_{N'}},\ \mathbf{x'_N} \in \mathbf{C}^n \end{cases} \tag{75}$$

where $\mathbf{x_1},\ \mathbf{x_2}, \cdots, \mathbf{x_{N'}}$ are complex-variable state vectors of N' response systems while $\mathbf{y_1},\ \mathbf{y_2},\ \cdots,\ \mathbf{y_N}$ are those of N drive (master) systems. $\mathbf{u_1},\ \mathbf{u_2}, \cdots, \mathbf{u_{N'}}$ are the synchronization controllers. $\mathbf{g_1}, \mathbf{g_2}, \cdots,\ \mathbf{g_N}, \mathbf{f_1},\ \mathbf{f_2}, \cdots, \mathbf{f_{N'}}$ are complex-variable nonlinear function vectors.

For the drive systems (74) and response systems (75), they are said to be **NCOFPS** if exists

$$\lim_{t\to+\infty} ||J_1x_1(t-\tau_1) + J_2x_2(t-\tau_2) + \cdots + J_{N'}x_{N'}(t-\tau_{N'}) \\ -]\,O_1y_1(t-\tau_1) - O_2y_2(t-\tau_2) - \cdots - O_Ny_N(t-\tau_N)|| \\ = 0, \tag{76}$$

where $J_1, J_2, \cdots, J_{N'}$ are the scaling function matrixes of response systems, while $O_1, O_2, \cdots, O_N$ are the scaling function matrixes of drive systems and τ is the potential time lag.

If the systems in **NCOFPS** are integer chaos, we have

Remark 23: when $J_1 = I_n$, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be the convergence of chaotic systems to zero.

Remark 24: when $J_1 = I_n, O_1 = I_n$, $\tau = 0$ and others $J, O = 0$, **NCOFPS** become **CS**.

Remark 25: when $J_1 = I_n, O_1 = -I_n$, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be **AS**.

Remark 26: when $J_1 = I_n$, the elements of scaling matrixes O_1 are the same constant, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be **PS**.

Remark 27: when $J_1 = I_n$, the elements of scaling matrixes O_1 are arbitrary constants, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can become **MPS**.

Remark 28: when $J_1 = I_n$, the elements of scaling matrixes O_1 are the same function, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be **FPS**.

Remark 29: when $J_1 = I_n$, the elements of scaling matrixes O_1 are arbitrary functions, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be turn into **MFPS**.

Remark 30: when J_1, J_2, O_1 are scaling constant matrixes, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be **COS**.

Remark 31: when J_1, J_2, O_1 are scaling function matrixes, $\tau = 0$ and others $J, O = 0$, **NCOFPS** can be **FCOS**.

Remark 32: when J_1, J_2, O_1, O_2 are scaling constant matrixes, $\tau = 0$ and others $J, O = 0$, **NCOFPS** turn into **CCOS**.

Remark 33: when J_1, J_2 are same scaling state variables, O_1 is scaling constant matrix, $\tau = 0$ and others $J, O = 0$, **NCOFPS** turn into Compound Synchronization.

Remark 34: when J_1, J_2 are the same scaling state variables, O_1, O_2 are scaling constant matrixes, $\tau = 0$ and others $J, O = 0$, **NCOFPS** turn into Compound Combination Synchronization.

Remark 35: when $J_1 = I_n$, O_1 is a complex scaling factor, $\tau = 0$ and others $J, O = 0$, **NCOFPS** turn into **CMPS**.

Remark 36: when $J_1 = I_n, O_1$ is a complex scaling function factor, $\tau = 0$ and others $J, O = 0$, **NCOFPS** become **CFPS**, etc.

Above remarks illustrate that **NCOFPS** is the generalized form in most proposed types of **CHS** in real and complex numbers. Can **NCOFPS** product more new forms of **CHS**? When the scaling functions are time delay function, few papers investigated this circumstance. For instance,

Remark 37: when J_1, or O_1 is the time delay scaling function, and others $J, O = 0$, **NCOFPS** turn into time delay function projective synchronization (**TDFPS**).

Remark 38: when J_1, or O_1 is the complex time delay scaling function, and others $J, O = 0$, **NCOFPS** become complex time delay function projective synchronization.

Remark 39: when J_1, J_2, and O_1 are the time delay scaling function, and others $J, O = 0$, **NCOFPS** turn into combination time delay function projective synchronization.

Remark 40: when J_1, J_2, and O_1 are the time delay scaling function, and others $J, O = 0$, **NCOFPS** turns into combination complex time delay function projective synchronization.

NCOFPS provides a macro and directional summary for the forms of chaos synchronization. It is really a benefit for the advance of chaos control. We can recognize that **NCOFPS** consists of most mentioned **CHS** forms in this paper. It is fully demonstrated the generalization of **NCOFPS**. Furthermore, the types of **NCOFPS** do not limit in mentioned forms absolutely, and studying new different **CHS** forms of **NCOFPS** are our future work.

B. TIME DELAY FUNCTION PROJECTIVE SYNCHRONIZATION

1) QUESTION FORMULATION

In this part, time delay function projective synchronization (**TDFPS**) is illustrated, which belongs to one of the types of **NCOFPS** and never studied in existed work. There is always one question in secure communications. Sometimes, we need not transmit all of the information signals. For example, the recording of meetings usually is too long. Only part of the recording should we transmit and encrypted. When we send some short confidential information, the recording editing must be done, especially in some long recording information. Therefore, how can we transmit useful and short messages instead of the whole long information? To address this problem, **TDFPS** is presented, which means the scaling function lag behind drive and response system.

Consider the response system

$$\dot{x}(t) = f(x(t)) + u(t), \tag{77}$$

the drive system

$$\dot{y}(t) = g(y(t)) \tag{78}$$

and the time delay function matrix is

$$h(t-\tau) = h_\tau. \tag{79}$$

Definition 1: If there exists suitable controller $u(t)$ satisfying

$$\lim_{t\to+\infty} ||x(t) - h_\tau y(t)|| = 0, \tag{80}$$

we said the response system (77) and drive system (78) have accomplished time delay function projective synchronization and the error system $\dot{e} = \dot{x} - B, (B = \frac{d(h_\tau y)}{dt})$ tends to zero.

Remark 40: h_τ is a time delay function transition matrix. From this point, many kinds of **CHS** are special cases of **TDFPS**, such as **FPS**, **PS**, **CS**.

Theorem 1: If the controller is designed as follows,

$$u(t) = -f(x) + B + ke, \tag{81}$$

where $e = x - h_\tau y$ and k is a negative definite matrix, the **TDFPS** will be obtained by means of controller (81).

Proof: Introduce the Lyapunov function

$$V(t) = \frac{1}{2}e^T e, \tag{82}$$

the derivative of V is

$$\dot{V} = \frac{1}{2} \times (2e^T \times \dot{e}) = e\dot{e}. \tag{83}$$

Putting the (77), (78) and (81) into the derivative Lyapunov function, then

$$\begin{aligned}\dot{V} &= e^T\dot{e}\\ &= e^T(\dot{x} - \dot{h}_\tau y - h_\tau \dot{y})\\ &= e^T(f(x) + u - \dot{h}_\tau y - h_\tau \dot{y})\\ &= e^T(f(x) - \dot{h}_\tau y - h_\tau \dot{y} - f(x) + B + ke)\\ &= ke^T e.\end{aligned} \tag{84}$$

Because the scaling matrix k is a negative definite matrix, the $\dot{V} < 0$. It demonstrates that the error system $\dot{e}$ is asymptotically stable and the error state variables tend to zeros with the controller (81). Thus, the proof is finished.

2) NUMERICAL SIMULATIONS

To effectively and intuitively illustrate the **TDFPS** and the proposed controller, corresponding simulations are obtained in this part. We use the fourth order runge-kutta method to solve the differential equations.

The response system is

$$\begin{cases}\dot{x}_1 = 10(x_2 - x_1) + u_1,\\ \dot{x}_2 = 28x_1 - x_2 - x_1x_3 + u_2,\\ \dot{x}_3 = x_1x_2 - \frac{8}{3}x_3 + u_3,\end{cases} \tag{85}$$

and the drive system is shown as follows,

$$\begin{cases}\dot{y}_1 = 10(y_2 - y_1),\\ \dot{y}_2 = 28y_1 - y_2 - y_1y_3,\\ \dot{y}_3 = y_1y_2 - \frac{8}{3}y_3.\end{cases} \tag{86}$$

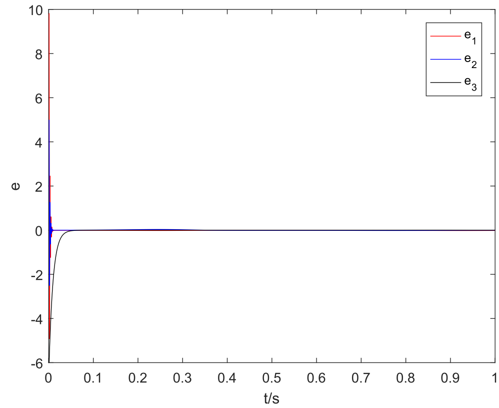

FIGURE 15. Diagram of error variable phases in TDFPS.

The scaling time delay function matrix is

$$h_\tau = \begin{bmatrix} sin[\pi(t-\tau)] & 0 & 0\\ 0 & cos[\pi(t-\tau)] & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Therefore, the error state variables are $e_1 = x_1 - sin[\pi(t-\tau)]y_1$, $e_2 = x_2 - cos[\pi(t-\tau)]y_2$, $e_3 = x_3 - y_3$ and the error system is

$$\begin{cases}\dot{e}_1 = 10(x_2 - x_1) - y_1cos[\pi(t-\tau)]\\ \quad -10sin[\pi(t-\tau)](y_2 - y_1) + u_1,\\ \dot{e}_2 = 28x_1 - x_2 - x_1x_3 + y_2sin[\pi(t-\tau)]\\ \quad -cos[\pi(t-\tau)](28y_1 - y_1y_3 - y_2) + u_2c,\\ \dot{e}_3 = x_1x_2 - \frac{8}{3}x_3 - y_1y_2 + \frac{8}{3}y_3 + u_3.\end{cases} \tag{87}$$

Following the contol law in (81), we can construct controller as,

$$\begin{cases}u_1 = 10e_2 - y_1cos[\pi(t-\tau)] + k_1e_1,\\ u_2 = -28e_1 + x_1x_3 - y_1y_3cos[\pi(t-\tau)]\\ \quad -y_2sin[\pi(t-\tau)] + k_2e_2,\\ u_3 = -x_1x_2 + y_1y_3 + k_3e_3.\end{cases} \tag{88}$$

The initial conditions are $[x_1(0), x_2(0), x_3(0)] = [10, 6, 2]$, $[y_1(0), y_2(0), y_3(0)] = [5, 1, 8]$, the time delay $\tau = 10$ and $k_1 = k_2 = 1500, k_3 = 100$. Fig.15 is the diagram of error variable phases. One can see that the proposed controller fastly make the evolutions of error states tend to zero, which fully illustrates that the feasibility of **TDFPS** and the effectiveness of controller (81).

V. CONCLUSIONS AND PROSPECTS

In this paper, we review the proposed mainstream **CHS** forms. The types of **CHS** are changed into two major categories, real **CHS** and complex **CHS**. Different types of chaos synchronization are generated by means of various physical or mathematics background. Complete synchronization means that two chaotic systems accomplish accordance with time. Projective synchronization and function projective synchronization request that certain proportion relations surrounding them. When it comes to complex number field, the proportion relations may be more complicated because of the existence of real and imaginary parts. Combination and dual synchronization mainly focus on multiple chaotic systems synchronization. etc. Based on these **CHS**, we present a new type of **CHS**, which is called **NCOFPS**. It is a directional summary of proposed **CHS** forms, which has great benefits in understanding diversity of **CHS**. **NCOFPS** is the most generalized form of **CHS** in real and complex chaotic systems. According to Remark 23 to Remark 36, we can see that **NCOFPS** will contain most types of the synchronization. Most types of the synchronization are special types of **NCOFPS**. After summarizing all forms of synchronization, it is easy to get **NCOFPS**. It is an induction of all the previous forms of synchronization. It is a scientific development of great inevitability and objectivity. Therefore, it is interesting and valuable to summarize **NCOFPS**.

We also discuss time delay function projective synchronization **TDFPS**, which is one of **NCOFPS** types and has never been studied in existed papers. We design **TDFPS** controller and give corresponding mathematical proof and simulation experiments. However, there are still many unsolved problems in **CHS** and its applications. Many worthwhile and meaningful works deserve to do in the future. The authors put forward the following research directions for readers' references.

A. RESEARCH ON THE CHARACTERISTIC OF COMPLEX CHAOTIC SYSTEM AND COMPLEX CHAOS SYNCHRONIZATION

Many practical phenomena can be described by complex chaotic systems, such as thermal convection of liquids, generators and so on. Furthermore, complex chaotic systems can even be employed to represent complex dielectric constant, electromagnetic wave etc. Therefore, it is of great value in theory and practical applications to study the characteristics and synchronization of complex chaotic systems with certain physical background. It is still a research hotspot in recent years.

B. TIME-DELAY COMPLEX CHAOTIC SYSTEMS AND SYNCHRONIZATION

Time-delay widely exists in real systems, which affects the dynamic characteristics of the systems. Time-delay chaotic systems usually have more complicated and unpredictable dynamic behaviors, but most of chaotic characteristics have not been discovered. Especially, the relationship between the time-delay factor and the characteristics of time-delay complex chaotic systems is still ambiguous, which should be deeply discussed in the level of mechanism and complete mathematical proof.

C. CHAOTIC COMMUNICATION

At present, chaotic communication is still a hotspot. However, many literatures only put forward the communication scheme of real chaotic systems and complex-variable chaotic systems in theory. More performance indexes of actual communication schemes and circuit implementation will be considered. Moreover, we also can adopt chaotic modulation, chaotic keying and chaotic cryptosystem, especially using **CVCSs** and time-delay complex chaotic systems to improve the secrecy performance of communication.

D. THE COMPLEXITY OF CONTROLLERS

As the advance of **CHS** diversity, more complicated **CHS** forms were proposed. Complicated **CHS** forms need more complex controllers simultaneously and absolutely. Although many feasible controllers can be presented in mathematical field, such as active controllers, adaptive controllers, sliding model controllers and so on, they are hardly realized in reality. Thus, implementable controllers for complicated **CHS** are also deserved to be extensively studied in practical application.

E. MORE FORMS OF NCOFPS

Although many proposed types of **CHS** belong to **NCOFPS**, due to its variety that consists of integer chaotic systems, complex chaotic systems and time delay chaotic systems, there are still some forms that have not studied in previous literatures. On condition that we have appropriate controllers, the increasing diversities of **CHS** absolutely enhance the potential applications of **CHS**. Therefore, more types of **NCOFPS** are also meaningful.

F. SYNCHRONIZATION OF CHAOTIC MAP

Most of chaotic maps have simple structures and they can be easily implemented in in electronic circuits. Chaotic maps have wide applications in image encryption and pseudo-random number generator. However, our paper mainly focus on continuous time chaos synchronizations. Therefore, how to build and synchronize enhanced chaotic map are interesting things.

G. PARAMETERS ESTIMATION IN CHAOS SYNCHRONIZATION

In real synchronization implementation or physical systems, parameters may be changed with the time or probably unknown. Although numerous types of chaos synchronization and related controllers are obtained, it is ineffective for the proposed controllers to accomplish chaos synchronization with unknown parameters. Thus, studying chaos synchronization with partly or completely unknown parameters is also important.

Finally, we really hope this paper will make new learners understand this topic quickly and easily. We do hope our review paper may motivate many significant work in the future.

DISCLOSURE OF POTENTIAL CONFLICTS OF INTEREST ETHICAL AND FINANCIAL

The authors declare that they have no conflict of interest.

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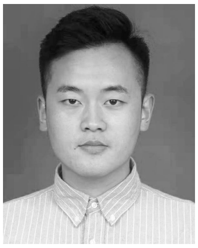

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