IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received October 30, 2019, accepted December 2, 2019, date of publication December 5, 2019, date of current version December 23, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2957942

A Construction of Optimal (r, δ)-Locally Recoverable Codes

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The work of B. Chen was supported by the National Natural Science Foundation of China under Grant 11871025, Grant 11971175, and Grant 11601158, and in part by the Science and Technology Program of Guangzhou under Grant 201804010102. The work of J. Huang supported by China Postdoctoral Science Foundation under Grant 2019M662883.

ABSTRACT Locally recoverable codes (LRCs) play a significant role in distributed and cloud storage systems. The key ingredient for constructing such optimal LRCs is to characterize the parity-check matrix for LRCs. In this letter based on the parity-check matrix for generalized Reed-Solomon codes we mainly present new constructions of optimal (r, δ)-locally recoverable codes with unbounded lengths in terms of the properties of the Vandermonde matrices, of which the parameters contain the known ones.

INDEX TERMS Distributed storage systems, locally recoverable codes, singleton-type bound.

I. INTRODUCTION

In the application of distributed storage, locally recoverable codes (LRCs for convenience) with locality r are used to design specifically for the failed storage nodes, which was put forward by Gopalan *et al.* [3]. Soon afterwards, for increasing the chances of successful recover, Prakash *et al.* [4] proposed the concept of locally recoverable codes with locality (r, δ), which generalizes the notion of locally recoverable codes with locality r (in this case $\delta = 2$).

More precisely, throughout this letter suppose that F_q is the finite field with q elements, where q is a prime power. Let $n \geq 1$ be an integer. For a vector $u = (u_1, u_2, \cdots, u_n) \in F_q^n$ and a subset $I \subseteq \{1, 2, \cdots, n\}$, let u_I be the projection of u on I, i.e., $u_I = (u_i)_{i \in I}$. Let C be an $[n, k, d]$ linear code over F_q. Write $C_I = \{c_I \mid c \in C\}$. If for any $i \in \{1, 2, \cdots, n\}$ there is a subset I_i such that $i \in I_i$, $|I_i| \leq r + \delta - 1$ and $d(C_{I_i}) \geq \delta$, then C is called an $[n, k, d]$ *locally recoverable code with locality* (r, δ). An $[n, k, d]$ locally recoverable code with locality (r, δ) is abbreviated to an (r, δ)-LRC with parameters $[n, k, d]$. When $\delta = 2$, an $(r, 2)$-LRC is usually called an LRC with locality r.

Like the classical Singleton bound of a linear code, the parameters $[n, k, d]$ of any (r, δ)-LRC over F_q satisfy

$$d \leq n - k - (\lceil \frac{k}{r} \rceil - 1)(\delta - 1) + 1. \qquad (1)$$

The upper bound (1) for (r, δ)-LRCs with parameters $[n, k, d]$ is called *Singleton-type bound* due to Prakash et al. [4]. An (r, δ)-LRC with parameters $[n, k, d]$ is called optimal if its parameters meet the singleton-type bound (1) with equality.

Recently, LRCs have become more and more important and have received much attention. It is particularly appealing and great challenging to determine and construct optimal (r, δ)-LRCs. The studies on the constructions of optimal (r, δ)-LRCs have many related works. The constructions of optimal $(r, 2)$-LRCs can be found in [3]–[22] and the references therein. Very recently, Chen et al. in [5] and [6] constructed some new classes of optimal (r, δ)-LRCs ($\delta \geq 2$) with lengths $n \leq q + 1$ via constacyclic MDS codes (include cyclic MDS codes). In [1], Fang and Fu constructed four families of optimal (r, δ)-LRCs with unbounded lengths through cyclic codes. In [2], Sun et al. utilized constacyclic MDS codes to construct several families of optimal (r, δ)-LRCs with unbounded lengths.

In this letter, motivated by the above works, we construct a class of new optimal (r, δ)-LRCs with unbounded lengths via generalized Reed-Solomon codes (GRS codes for short), of which the parameters cover the known ones in [1] and [2].

The rest of this letter is arranged in the following manner. Preliminary facts about GRS codes are introduced in the following section. We present a new and explicit construction of optimal (r, δ)-LRCs via GRS codes in Section III. We then give several corollaries of our main results in order to compare the parameters of our results with the previously known ones in the literature in Section IV. Lastly, Section V concludes this letter.

II. PRELIMINARIES

In this letter we apply GRS codes to obtain new constructions of optimal (r, δ)-LRCs with unbounded lengths. For this

The associate editor coordinating the review of this manuscript and approving it for publication was Xueqin Jiang.

purpose, we state some known results about GRS codes in this section. For detailed information about GRS codes the reader may refer to [23] or [24].

As before, let F_q be the finite field with q elements, where q is a prime power. Let n be an integer satisfying $1 < n \le q$. Suppose that $a = (a_1, a_2, \cdots, a_n) \in F_q^n$ and $v = (v_1, v_2, \cdots, v_n) \in F_q^n$, where $a_1, a_2, \cdots, a_n$ are distinct in F_q and $v_i \ne 0$ for all $1 \le i \le n$. Let k be an integer with $1 \le k \le n$, and $P_k[x]$ is described by the following set: $\{f(x) \in F_q[x] | \deg(f(x)) \le k-1\}$. Then for $k \le n$, the GRS code is defined to be

$$GRS_k(a, v) = \{(v_1 f(a_1), v_2 f(a_2), \cdots, v_n f(a_n)) \,|\, f \in P_k[x]\}.$$

We summarize some results on GRS codes in the following Propositon 1.

Proposition 1: Let notation be as above. Then

(1) [24, Theorem 9.1.4] $GRS_k(a, v)$ is an MDS code with parameters $[n, k, n-k+1]$.

(2) [24, Theorem 9.1.6] There exists an n-tuple $w = (w_1, w_2, \cdots, w_n)$ of nonzero elements of F_q such that $GRS_k(a, v)^\perp = GRS_{n-k}(a, w)$ for $1 \le k \le n-1$. In addition, the vector w belongs to the 1-dimensional code $GRS_{n-1}(a, v)^\perp$ and has no zero components.

(3) [24, Corollary 9.1.7] A parity-check matrix for $GRS_k(a, v)$ has the following form

$$H = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ w_1 a_1 & w_2 a_2 & \cdots & w_n a_n \\ \vdots & \vdots & \ddots & \vdots \\ w_1 a_1^{n-k-1} & w_2 a_2^{n-k-1} & \cdots & w_n a_n^{n-k-1} \end{pmatrix}.$$

III. CONSTRUCTIONS

In this letter we rely principally on the parity-check matrix for a q-ary GRS code with length $r+\delta-1$ in order to give the constructions of optimal (r, δ)-LRCs with unbounded lengths, where both r and δ are two positive integers. Let D be a GRS code and let the following matrix A be the parity-check matrix for D:

$$A = \begin{pmatrix} w_1 & w_2 & \cdots & w_{r+\delta-1} \\ w_1 a_1 & w_2 a_2 & \cdots & w_{r+\delta-1} a_{r+\delta-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 a_1^{\delta-2} & w_2 a_2^{\delta-2} & \cdots & w_{r+\delta-1} a_{r+\delta-1}^{\delta-2} \end{pmatrix}.$$

Thus D is an $[r+\delta-1, r, \delta]$ GRS code. Our first goal is to construct an optimal (r, δ)-LRC of which the length n is unbounded and the minimum distance d is equal to δ.

Lemma 1: Let $r \ge 2$ be a positive integer. Define H_1 as the following block diagonal matrix over F_q:

$$H_1 = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix},$$

where there are m blocks A lying along the diagonal of H_1. Let C_1 be a linear code over F_q with parity-check matrix H_1 as follows. Then C_1 is an optimal (r, δ)-LRC and the parameters of C_1 is given by

$$n = m(r+\delta-1), \quad k = n - m(\delta-1), \quad d = \delta,$$

where $r+\delta-1 \le q$.

Proof: Obviously, the length of C_1 is $n = m(r+\delta-1)$. It is easy to see that the rank $R(H_1)$ of the matrix H_1 is $mR(A) = m(\delta-1)$, so $k = n - m(\delta-1)$. By the Singleton-type bound routine computation shows that

$$\begin{aligned} d &\le n - k - \left(\lceil \frac{k}{r} \rceil - 1\right)(\delta - 1) + 1 \\ &= m(\delta-1) - (m-1)(\delta-1) + 1 \\ &= \delta. \end{aligned}$$

To show that C_1 is optimal, it suffices to prove that $d \ge \delta$. To this end, we claim that any $\delta-1$ column vectors of H_1 are linearly independent.

Let $h_{i_1}, h_{i_2}, \cdots, h_{i_{\delta-1}}$ be any $\delta-1$ column vectors of H_1, where $i_1, i_2, \cdots, i_{\delta-1}$ are $\delta-1$ positive integers. Suppose that

$$x_1 h_{i_1} + x_2 h_{i_2} + \cdots + x_{\delta-1} h_{i_{\delta-1}} = 0$$

with $x_1, x_2, \cdots, x_{\delta-1}$ being elements of F_q.

Let v be a positive integer with $1 \le v \le m$. In general, we suppose that there exist v positive integers $t_1, t_2, \cdots, t_v$ such that

$$\begin{aligned} 1 &\le i_1, i_2, \cdots, i_{t_1} \le r+\delta-1; \\ r+\delta &\le i_{t_1+1}, i_{t_1+2}, \cdots, i_{t_1+t_2} \\ &\le 2(r+\delta-1); \cdots; \\ (v-1)(r+\delta-1)+1 &\le i_{t_1+\cdots+t_{v-1}+1}, \cdots, i_{t_1+\cdots+t_v} \\ &\le v(r+\delta-1), \end{aligned}$$

where $t_1 + t_2 + \cdots + t_v = \delta - 1$.

Denote $\boldsymbol{\alpha_i}$ by the i-th column vector of A for $i = 1, 2, \cdots, r+\delta-1$. Then

$$A = (\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \cdots, \boldsymbol{\alpha_{r\mathcal{C}\delta-1}}).$$

Set

$$\begin{aligned} A_1 &= (\boldsymbol{\alpha_{i_1}}, \boldsymbol{\alpha_{i_2}}, \cdots, \boldsymbol{\alpha_{i_{t_1}}}), \\ A_j &= (\boldsymbol{\alpha_{i_{t_1 \mathcal{C} \cdots \mathcal{C} t_{j-1} \mathcal{C} 1} - j(r\mathcal{C}\delta-1)}}, \cdots, \boldsymbol{\alpha_{i_{t_1 \mathcal{C} \cdots \mathcal{C} t_j} - j(r\mathcal{C}\delta-1)}}), \end{aligned}$$

for all $j = 2, 3, \cdots, v$;

$$\begin{aligned} x_1 &= (x_1, x_2, \cdots, x_{t_1}) \in F_q^{t_1}, \\ x_j &= (x_{t_1+\cdots+t_{j-1}+1}, \cdots, x_{t_1+\cdots+t_{j-1}+t_j}) \in F_q^{t_j}, j = 2, \cdots, v. \end{aligned}$$

It follows that

$$A_j x_j^T = 0, \quad j = 1, 2, \cdots, v.$$

Since A is the parity-check matrix for the above GRS code D, any $t_j (1 \le t_j \le \delta-1, j = 1, 2, \cdots, v)$ column vectors of A are linearly independent, then the column vectors of A_j are linearly independent. Hence

$$x_j^T = 0, \quad j = 1, 2, \cdots, v.$$

This forces all of the $x_1, x_2, \cdots, x_{\delta-1}$ are zeros. Therefore the columns $h_{i_1}, h_{i_2}, \cdots, h_{i_{\delta-1}}$ are linearly independent. It follows that $d = \delta$. In conclusion, C_1 is an optimal $[n = m(r+\delta-1), k = n - m(\delta-1), d = \delta]$-LRC with locality (r, δ) over F_q. ■

Lemma 1 is helpful in finding optimal (r, δ)-LRCs of unbounded lengths with $d > \delta$. Indeed, our strategy is to add suitable rows to the bottom of the matrix H_1; if one suitable row (say β) has been added in the bottom of H_1, we have a new code C_2 whose parity-check matrix can be described by $\binom{H_1}{\beta}$ and the dimension of C_1 is equal to the dimension of C_2 plus one. This says that the right-hand side of the Singleton-type bound (1) has increased by one. However, we want to ensure that C_2 is optimal, which requires that the Hamming distance of C_2 must also increase by one. Continuing this strategy λ times, we get the result as follows. We assume that λ is a positive integer with $1 \le \lambda < r$ and let B be the matrix

$$B = \begin{pmatrix} w_1 a_1^{\delta-1} & w_2 a_2^{\delta-1} & \cdots & w_{r+\delta-1} a_{r+\delta-1}^{\delta-1} \\ w_1 a_1^{\delta} & w_2 a_2^{\delta} & \cdots & w_{r+\delta-1} a_{r+\delta-1}^{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 a_1^{\delta-2+\lambda} & w_2 a_2^{\delta-2+\lambda} & \cdots & w_{r+\delta-1} a_{r+\delta-1}^{\delta-2+\lambda} \end{pmatrix}.$$

Theorem 1: Let $r \ge 2$ be a positive integer. Define the following matrix H_2 over F_q:

$$H_2 = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \\ -- & -- & -- & -- \\ B & B & \cdots & B \end{pmatrix},$$

where there are m blocks A in H_2. Let C_2 be the linear code over F_q with parity-check matrix H_2. Then C_2 is an optimal-LRC with locality (r, δ) over F_q, and has parameters

$$n = m(r+\delta-1), \quad k = n - m(\delta-1) - \lambda, \quad d = \lambda + \delta,$$

where r, δ, λ satisfy $r + \delta - 1 \le q$, $1 \le \lambda < r$ and $\lambda \le \delta$.

Proof: Clearly, C_2 has code length $n = m(r+\delta-1)$. We proceed the proof by showing the following two claims.

Claim 1: The rank $R(H_2)$ of the matrix H_2 is $m(\delta-1)+\lambda$.

Suppose that there exists a row vector

$$x_0 = (x_1, x_2, \cdots, x_{m(\delta-1)+\lambda})$$

such that $x_0 H_2 = 0$. Let

$$x_i = (x_{(i-1)(\delta-1)+1}, x_{(i-1)(\delta-1)+2}, \cdots, x_{i(\delta-1)})$$

for $i = 1, 2, \cdots, m$ and

$$x_{m+1} = (x_{m(\delta-1)+1}, x_{m(\delta-1)+2}, \cdots, x_{m(\delta-1)+\lambda}).$$

Clearly, $x_0 = (x_1, x_2, \cdots, x_m, x_{m+1})$. Thus from $x_0 H_2 = 0$ we have that $x_i A + x_{m+1} B = 0$, $i = 1, 2, \cdots, m$, i.e.,

$$\big(x_i,\ x_{m+1}\big) \begin{pmatrix} A \\ B \end{pmatrix} = 0, \quad i = 1, 2, \cdots, m.$$

Note that the first $\delta - 1 + \lambda$ column vectors of $\begin{pmatrix} A \\ B \end{pmatrix}$ are the following matrix

$$\begin{pmatrix} w_1 & w_2 & \cdots & w_{\delta-1+\lambda} \\ w_1 a_1 & w_2 a_2 & \cdots & w_{\delta-1+\lambda} a_{\delta-1+\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 a_1^{\delta-2+\lambda} & w_2 a_2^{\delta-2+\lambda} & \cdots & w_{\delta-1+\lambda} a_{\delta-1+\lambda}^{\delta-2+\lambda} \end{pmatrix},$$

whose determinant is

$$w_1 w_2 \cdots w_{\delta-1+\lambda} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{\delta-1+\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{\delta-2+\lambda} & a_2^{\delta-2+\lambda} & \cdots & a_{\delta-1+\lambda}^{\delta-2+\lambda} \end{pmatrix}$$
$$= w_1 w_2 \cdots w_{\delta-1+\lambda} \prod_{1 \le j < i \le \delta-1+\lambda} (a_i - a_j) \ne 0.$$

It follows that $\big(x_i,\ x_{m+1}\big) = 0$, $i = 1, 2, \cdots, m$, which yields $x_0 = 0$. Hence $R(H_2) = m(\delta-1)+\lambda$. This shows that **Claim** 1 is true. Therefore by **Claim 1** we get that

$$k = n - r(H_2) = n - m(\delta-1) - \lambda = mr - \lambda.$$

Now by the Singleton-type bound we obtain that

$$\begin{aligned} d &\le n - k - \big(\lceil \frac{k}{r} \rceil - 1\big)\big(\delta - 1\big) + 1 \\ &= m(\delta-1) + \lambda - (\lceil m - \frac{\lambda}{r} \rceil - 1)(\delta-1) + 1 \\ &= m(\delta-1) + \lambda - (m-1)(\delta-1) + 1 \\ &= \delta + \lambda. \end{aligned}$$

To get that C_2 is optimal, it suffices to prove that $d \ge \delta + \lambda$. To this end, we need to prove that

Claim 2: Any $\delta + \lambda - 1$ column vectors of H_2 are linearly independent.

Let $h_{i_1}, h_{i_2}, \cdots, h_{i_{\delta+\lambda-1}}$ be any $\delta + \lambda - 1$ column vectors of H_2, and there exists a row vector

$$x = \big(x_1, x_2, \cdots, x_{\delta+\lambda-1}\big)$$

such that

$$x_1 h_{i_1} + x_2 h_{i_2} + \cdots + x_{\delta+\lambda-1} h_{i_{\delta+\lambda-1}} = 0.$$

Our goal is to show $x = 0$. Let u be a positive integer and there exist two groups of positive integers: $t_1, t_2, \cdots, t_u$ and $s_1, s_2, \cdots, s_u$ such that

$$\begin{aligned} (t_1 - 1)(r+\delta-1) &\le i_1, i_2, \cdots, i_{s_1} \le t_1(r+\delta-1), \\ (t_2 - 1)(r+\delta-1) &\le i_{s_1+1}, i_{s_1+2}, \cdots, i_{s_1+s_2} \\ &\le t_2(r+\delta-1), \cdots, \\ (t_u - 1)(r+\delta-1) &\le i_{s_1+\cdots+s_{u-1}+1}, \cdots, i_{s_1+\cdots+s_u} \\ &\le t_u(r+\delta-1), \end{aligned}$$

where $1 \le t_1 < t_2 < \cdots < t_u \le m$ and

$$s_1 + s_2 + \cdots + s_u = \lambda + \delta - 1.$$

Now we proceed by analyzing the following three cases separately.

Case 1: $u = 1$. In general, we assume that $\triangle_{t_1} = \triangle_1$. By $x_1 h_{i_1} + x_2 h_{i_2} + \cdots + x_{\delta+\lambda-1} h_{i_{\delta+\lambda-1}} = 0$, we see that

$$\begin{pmatrix} w_{i_1} & \cdots & w_{i_{\lambda+\delta-1}} \\ w_{i_1} a_{i_1} & \cdots & w_{i_{\lambda+\delta-1}} a_{i_{\lambda+\delta-1}} \\ \vdots & \ddots & \vdots \\ w_{i_1} a_{i_1}^{\lambda+\delta-2} & \cdots & w_{i_{\lambda+\delta-1}} a_{i_{\lambda+\delta-1}}^{\lambda+\delta-2} \end{pmatrix} x^T = 0. \quad (2)$$

Observe that the coefficient matrix of the above homogeneous system of linear equations becomes a Vandermonde matrix when extracting $w_{i_1}, w_{i_2}, \cdots, w_{i_{\lambda+\delta-1}}$ from each of columns. By the property of the Vandermonde determinant we know the coefficient matrix is invertible, and therefore we have that $x = 0$.

Case 2: $u = 2$. Without loss of generality, let $\triangle_{t_1} = \triangle_1$ and $\triangle_{t_2} = \triangle_2$. Set $x_1 = (x_1, x_2, \cdots, x_{s_1})$ and $x_2 = (x_{s_1+1}, x_{s_1+2}, \cdots, x_{\lambda+\delta-1})$. In this case, assume that $j_1 = i_{s_1+1} - (r+\delta-1), j_2 = i_{s_1+2} - (r+\delta-1), \cdots, j_{s_2} = i_{s_1+s_2} - (r+\delta-1)$. Note that $s_1 + s_2 = \lambda+\delta-1 \leq \delta + (\delta-1) = 2\delta - 1$. Then we need to consider the following two subcases.

Subcase 2.1: $s_1 \leq \delta - 1$ and $s_2 \leq \delta - 1$. Since

$$x_1 h_{i_1} + x_2 h_{i_2} + \cdots + x_{\delta+\lambda-1} h_{i_{\delta+\lambda-1}} = 0,$$

we deduce that

$$\begin{pmatrix} w_{i_1} & w_{i_2} & \cdots & w_{i_{s_1}} \\ w_{i_1} a_{i_1} & w_{i_2} a_{i_2} & \cdots & w_{i_{s_1}} a_{i_{s_1}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i_1} a_{i_1}^{s_1-1} & w_{i_2} a_{i_2}^{s_1-1} & \cdots & w_{i_{s_1}} a_{i_{s_1}}^{s_1-1} \end{pmatrix} x_1^T = 0 \quad (3)$$

and

$$\begin{pmatrix} w_{j_1} & w_{j_2} & \cdots & w_{j_{s_2}} \\ w_{j_1} a_{j_1} & w_{j_2} a_{j_2} & \cdots & w_{j_{s_2}} a_{j_{s_2}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{j_1} a_{j_1}^{s_2-1} & w_{j_2} a_{j_2}^{s_2-1} & \cdots & w_{j_{s_2}} a_{j_{s_2}}^{s_2-1} \end{pmatrix} x_2^T = 0. \quad (4)$$

Observe that A is the parity-check matrix for the GRS code D with minimum distance δ, which follows that any $s_i (s_i \leq \delta - 1, i = 1, 2)$ column vectors of A are linearly independent. So we can obtain that $x_1^T = x_2^T = 0$, which yields $x = 0$.

Subcase 2.2: $s_1 > \delta - 1, s_2 \leq \delta - 1$ or $s_1 \leq \delta - 1, s_2 > \delta - 1$. Since the proofs of two subcases are similar, we shall only work with the latter. According to $x_1 h_{i_1} + x_2 h_{i_2} + \cdots + x_{\delta+\lambda-1} h_{i_{\delta+\lambda-1}} = 0$ again, we have that

$$\begin{pmatrix} w_{i_1} & w_{i_2} & \cdots & w_{i_{s_1}} \\ w_{i_1} a_{i_1} & w_{i_2} a_{i_2} & \cdots & w_{i_{s_1}} a_{i_{s_1}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i_1} a_{i_1}^{s_1-1} & w_{i_2} a_{i_2}^{s_1-1} & \cdots & w_{i_{s_1}} a_{i_{s_1}}^{s_1-1} \end{pmatrix} x_1^T = 0, \quad (5)$$

which follows that $x_1^T = 0$. Note that $s_2 \geq \delta$. Then we construct the Vandermonde matrix by taking the first $s_2 - \delta + 1$ rows from the matrix B, where $1 \leq s_2 - \delta + 1 \leq \lambda + \delta - 2 - \delta + 1 = \lambda - 1 < \lambda$. Thus

$$\begin{pmatrix} w_{j_1} & w_{j_2} & \cdots & w_{j_{s_2}} \\ w_{j_1} a_{j_1} & w_{j_2} a_{j_2} & \cdots & w_{j_{s_2}} a_{j_{s_2}} \\ \vdots & \vdots & \ddots & \vdots \\ w_{j_1} a_{j_1}^{s_2-1} & w_{j_2} a_{j_2}^{s_2-1} & \cdots & w_{j_{s_2}} a_{j_{s_2}}^{s_2-1} \end{pmatrix} x_2^T = 0. \quad (6)$$

By the property of the Vandermonde matrix we immediately obtain that $x_2^T = 0$. Hence $x = 0$.

Case 3: $u \geq 3$. In this case, note that $s_1 + s_2 + \cdots + s_u = \lambda + \delta - 1 \leq 2\delta - 1$, which implies that there exists at most one s_i $(1 \leq i \leq u)$ such that $s_i \geq \delta$. If there is only one s_i $(1 \leq i \leq u)$ such that $s_i \geq \delta$, then this case is boiled down to the Subcase 2.2. Otherwise, $s_i \leq \delta - 1$ for all $i = 1, 2, \cdots, u$. In this case the proof is similar to that of Subcase 2.1.

This finishes the proof of **Claim 2**.

According to **Claims 1** and **2**, C_2 is an optimal-LRC with locality (r, δ) over F_q, and has parameters

$$n = m(r + \delta - 1), \quad k = n - m(\delta - 1) - \lambda, \quad d = \lambda + \delta,$$

where r, δ, λ satisfy $r + \delta - 1 \leq q$, $1 \leq \lambda < r$ and $\lambda \leq \delta$. ∎

IV. COROLLARIES AND CODE COMPARISONS

According to Theorem 1, we construct a family of new optimal (r, δ)-LRC codes whose parameters are

$$n = m(r + \delta - 1), \quad k = n - m(\delta - 1) - \lambda, \quad d = \lambda + \delta,$$

where $r + \delta - 1 \leq q$, $0 \leq \lambda < r$ and $\lambda \leq \delta$.

In the following we give some corollaries in order to compare the parameters of our results with the previously known ones in the literature.

- Take $\lambda = 1, 2, \delta$ in Theorem 1, then we have

  Corollary 1:

  (i) Let $r, \delta \geq 2$. Then there is an optimal (r, δ)-LRC over F_q with parameters $[n = m(r + \delta - 1), k = n - m(\delta - 1) - 1, d = \delta + 1]$.

  (ii) Let $r \geq 3, \delta \geq 2$. Then there is an optimal (r, δ)-LRC over F_q with parameters $[n = m(r + \delta - 1), k = n - m(\delta - 1) - 2, d = \delta + 2]$.

  (iii) Let $r \geq \delta + 1$. Then there is an optimal (r, δ)-LRC over F_q with parameters $[n = m(r + \delta - 1), k = n - m(\delta - 1) - \delta, d = 2\delta]$.

  It is easy to see that the parameters of the codes listed in Corollary 1 contain those given in [1, Theorem 1-3] and [2, Theorem 2,4], respectively. We do not need the constraints imposed in [1, Theorem 1-3] and [2, Theorem 2,4], which are presented as follows.

  (i) For $d = \delta + 1$, it is under the conditions that $\gcd(q, n) = 1, \gcd(n, q - 1) \equiv 0 \pmod{r + \delta - 1}$ ([1, Theorem 1]).

  (ii) For $d = \delta + 2$, it needs the conditions that $\gcd(q, n) = 1, \gcd(n, q - 1) \equiv 0 \pmod{r + \delta - 1}$, and $\gcd(\frac{n}{r+\delta-1}, r + \delta - 1)$ divides δ ([1, Theorem 2]), or $\gcd(n, q) = 1, \gcd(n, q + 1) \equiv 0 \pmod{r + \delta - 1}$, $2 \mid \delta$ and $\gcd(\frac{n}{r+\delta-1}, r + \delta - 1)$

divides $\frac{\delta}{2}$ ([2, Theorem 2]); $\gcd(n, q) = 1$, $\gcd(n, q+1) \equiv 0 \pmod{r+\delta-1}$, $2 \mid \delta$ and $\gcd(\frac{n}{r+\delta-1}, r+\delta-1)$ divides δ ([2, Theorem 4]).

(iii) For $d = 2\delta$, it is suffice under the conditions that $\gcd(q, n) = 1$, $\gcd(n, q-1) \equiv 0 \pmod{r+\delta-1}$, and $\gcd(\frac{n}{r+\delta-1}, r+\delta-1) = 1$ ([1, Theorem 3]).

- Take $\lambda = \delta = 3$ in Theorem 1, then we get

  Corollary 2: Let $5 < r+2 \leq q$. Then there is a q-ary optimal $(r, 3)$-LRC with $d = 6$.

  In fact, these parameters of the code given in Corollary 2 contain those in [1, Theorem 4]. We do not need the constrains assumed in [1, Theorem 4]: n is odd, $\gcd(n, q+1) \equiv 0 \pmod{r+2}$ and $\gcd(\frac{n}{r+2}, r+2) = 1$.
- When $\lambda = 2\epsilon, 1 \leq \epsilon \leq \frac{\delta}{2}$ or $\lambda = 2\epsilon, 1 \leq \epsilon \leq \frac{\delta-1}{2}$ in Theorem 1, we get the result as follows, which extends [2, Theorem 1,3] in the sense that the condition that $\gcd(r+\delta-1, \frac{n}{r+\delta-1}) = 1$ is removed here.

  Corollary 3: Suppose that $r+\delta-1 \leq q$ and $0 \leq \lambda < r$. Then there is an optimal (r, δ)-LRC with $d = \delta + 2\epsilon$.
- When $\delta = 2$, $\lambda = 1$ and 2 in Theorem 1, we deduce Corollary 4 as follows.

  Corollary 4: Suppose that q is a prime power with $q \geq r+1$. Then

  (i) there exists a q-ary optimal $[m(r+1), n-m-1, 3]$-LRC with locality $r \geq 2$.

  (ii) there exists a q-ary optimal $[m(r+1), n-m-2, 4]$-LRC with locality $r \geq 3$.

From Corollary 4, it is easy to see that the constraints about the paraments of the optimal LRC is much less than those given in the main result of [17, Theorem 1]. More precisely, we have removed the conditions $(r+1) | \gcd(n, q-1)$ and $\gcd(\frac{n}{r+1}, r+1) | 2$. Note that Corollary 4 has appeared in [7].

V. CONCLUSION

In this work, we present a construction of optimal (r, δ)-LRCs via GRS codes, which is different from the known ones. Our parameters are new, mainly because we have removed the constraints required in the known results. In addition, it remains an open problem to study whether there are families of optimal LRCs in the case when $\lambda \geq \delta + 1$.

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