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Multiple Criteria Group Decision Making Using a Parametric Linear Programming Technique for Multidimensional Analysis of Preference Under Uncertainty of Pythagorean Fuzziness

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ABSTRACT This paper aims to utilize the core structure of linear programming technique for multidimensional analysis of preference (LINMAP) to propose a parametric LINMAP methodology for addressing multiple criteria group decision-making problems based on Pythagorean fuzzy sets. To compare Pythagorean membership grades, this paper presents a Hamming distance-based approach for identifying closeness-based order relations based on Pythagorean fuzzy closeness indices. The concept of comprehensive closeness measures is introduced to measure individual order consistency and inconsistency between subjective preference relations and objective order relations. In the spirit of LINMAP, this paper determines individual goodness of fit and poorness of fit and further constructs a novel parametric LINMAP model. The applicability of the developed approach is explored by a practical application of railway project investment. Some comparative analyses are conducted to demonstrate the usefulness and advantages of the proposed methodology.

INDEX TERMS Multiple criteria group decision making, Pythagorean fuzzy set, Pythagorean membership grade, closeness-based order relation.

I. INTRODUCTION

The linear programming technique for multidimensional analysis of preference (LINMAP), initiated by Srinivasan and Shocker [1], is a well-known compromising model in the decision-making field [2], [3]. LINMAP is capable of handling preference information of alternatives, determining objective weights of criteria, and making decisions through identifying the best compromise alternative [4]–[7]. Based on the decision matrix and a set of preference relations of decision makers on pairwise comparisons of alternatives, LINMAP defines the consistency and inconsistency indices concerning each paired comparison for acquiring the poorness of fit and the goodness of fit [3], [8]. LINMAP constructs a linear programming model to obtain an imaginary ideal solution against each criterion and the optimal weights of criteria [5], [6], [9]. Each alternative is compared with the others based on its Euclidean distance from the ideal solution [9], [10], which is defined as the point on the Pareto front that each criterion is optimized regardless of counting the other criteria [2], [11]. The alternative that has the shortest distance from the ideal solution is selected as the best compromise solution [2], [8].

In multiple criteria group decision-making (GDM) problems, LINMAP can be used to reflect decision makers' preferences over alternatives more effectively. In this regard, numerous LINMAP methods and techniques have been proposed to solve GDM problems. For example, Zhang *et al.* [12] employed the LINMAP to propose an interval-valued intuitionistic fuzzy programming technique for multiple criteria GDM based on Shapley values and incomplete preference information. Similarly, Liu *et al.* [8] investigated a double-hierarchy hesitant fuzzy linguistic mathematical programming method to solve GDM problems with Shapley values and incomplete preference information. Quan *et al.* [6] employed the LINMAP to determine the

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optimal weights of criteria and presented a hybrid GDM approach for large group green supplier selection with interval-valued intuitionistic uncertain linguistic information. Based on interval type-2 fuzzy sets, Haghighi *et al.* [4] proposed a new GDM approach with the linear assignment method, in which the weights of evaluation factors are determined based on a new developed version of LINMAP. Zuo *et al.* [13] developed a large GDM method of generalized multi-attribute and multi-scale based on LINMAP. It can be observed that fuzzy sets are frequently utilized in the extended LINMAP methods and techniques because decision makers tend to express ambiguous evaluations due to the vagueness of human thinking [6]. Decision information contained in GDM problems is incomplete and ambiguous in most real-world cases [14], [15]. Thus, more general and high-order fuzzy extensions of LINMAP should be capable of providing an effective way to tackle GDM problems within complex and uncertain environments. This is the first motivation of this paper.

The concept of Pythagorean fuzzy (PF) sets, originally developed by Yager [16]–[18] and Yager and Abbasov [19], is useful to represent ambiguous and uncertain decision information [20], [21]. As a valuable extension of intuitionistic fuzzy sets, Pythagorean membership grades involved in a PF set relax the condition that the sum of membership and non-membership degrees is less than or equal to one with the square sum is less than or equal to one [15], [22]–[24]. Accordingly, PF sets have been widely popular in handling complex uncertainty involved in practical decision-making problems, and they have attracted numerous scholars' research interests in recent years. A series of methodologies have been developed to handle a variety of decision-making problems, such as PF techniques for order preference by similarity to ideal solutions (TOPSIS) [23], [25], [26], PF preference ranking organization methods for enrichment evaluations (PROMETHEE) [27], [28], cubic PF weighted averaging and weighted geometric operations [20], weighted distance based approximation with new score functions [29], and GDM approaches based on PF preference relations [14] and based on order relations for PF numbers [15]. Because PF sets provide a powerful and flexible tool in modeling real-world uncertainty, the extension of the LINMAP structure can create a more promising research subject and capture vagueness and incompleteness in human evaluations. This is the second motivation of this paper.

Wan *et al.* [30] and Xue *et al.* [31] employed the core concept of LINMAP to propose novel multiple criteria GDM approaches within the PF environment. Wan *et al.* [30] developed a new PF mathematical programming method to solve GDM problems with PF truth degrees. The main feature of their approach is the utilization of the cross-entropy theory for determining the weights of decision makers and the collective relative closeness degrees of alternatives. Xue et al. [31] based on the consistency index and the PF entropy to establish a novel PF LINMAP model that could realize high consistency index and acquire an amount of knowledge. The prominent feature of their approach is the construction of PF entropy and interval-valued PF entropy for measuring the fuzziness and uncertainty of PF sets and interval-valued PF sets. The aforesaid methods have solid theoretical bases and possess comprehensive computation procedures. However, the mathematical works and notations in Wan *et al.* [30] and Xue *et al.* [31] are rather difficult for decision makers or relevant practitioners. Furthermore, the technical aspects and characters as well as their influences on the solution results are not always well understood. To make sure the usefulness and flexibility of PF sets in characterizing uncertainty and fuzziness, the LINMAP-based methodology should be extended to the PF environment using a straightforward and uncomplicated manner. This is the third motivation of this paper.

According to the above discussions, the motivations of this paper are summarized as follows:

(1) The core structure of LINMAP can be improved or enriched to more general and high-order fuzzy environments for handling GDM problems in intricate circumstances.

(2) Developing the extended LINMAP methods in PF contexts can describe vagueness and incompleteness in subjective assessments and accommodate more complex uncertainties.

(3) A simple and effective LINMAP-based approach is needed to enhance an understanding of the value and merits of the PF extension in enriching the LINMAP-based methodology.

To address the three motivational issues, this paper intends to develop a novel PF LINMAP approach which is very simple and easily understood by decision makers. The purpose of this paper is to utilize PF closeness-based order relations via a recently developed Hamming distance measure and construct a novel parametric PF LINMAP model to address multiple criteria GDM problems. Instead of the popular Euclidean distance-based approach in the classical LINMAP procedure, this paper presents the PF closeness indices via a Hamming distance-based approach to conduct criterion-wise comparisons between PF evaluative ratings for the sake of PF closeness-based order relations. The proposed Hamming distance-based approach is based on the essential characteristics of PF sets, i.e., membership, non-membership, strength, and direction, which makes it a very useful technique for general decision making in PF contexts. Next, this paper introduces the concept of comprehensive closeness measures to acquire synthetic effects over all evaluative criteria and specify objective order relations. These comprehensive measures and relations can be used to identify individual indices of order consistency and order inconsistency between the preorders of the alternatives in the preference set for each decision maker. The measurements of individual goodness of fit and poorness of fit can then be acquired for each decision maker. Based on a bi-objective optimization model that aims to maximize the total collective comprehensive closeness measure and minimize the collective poorness of fit, this paper establishes a parametric PF LINMAP model for determining the optimal weight vector and individual degrees of violation. An effective algorithmic procedure is proposed to applying the PF LINMAP methodology to handle a GDM problem in PF contexts. Finally, this paper conducts an illustrative application and some comparative analyses concerning a GDM problem of railway project investment to examine the usefulness and advantages of the proposed PF LINMAP approach in the real world.

The main innovation and advantages of this paper are highlighted as three aspects:

(1) Empowerment of decision makers to make group decisions based on PF uncertainty

Complicated and volatile decision environments pose severe challenges for decision makers in knowing how to cope with vague or imprecise information in the GDM process. PF sets empower decision makers to describe uncertain evaluation information more flexibly than popular intuitionistic fuzzy sets. This paper makes use of PF sets to model inherent fuzziness and subjectivity within uncertain environments and resolve situations where decision makers hesitate in assessing alternatives under complex uncertainty.

(2) Construction of an effective procedure for manipulating PF information

In the light of the flexibility and complexity of PF sets, the PF specification in GDM problems has evident difficulty in handling sophisticated PF information, which would reduce the quality of being practical. The natural quasi-ordering between Pythagorean membership grades does not often appear in the PF context. Thus, the question then arises about the comparison for PF information. To differentiate the dominance relationships among PF information, this paper advances a new order relation, named PF closeness-based order relation, based on the concept of PF closeness indices. In contrast to natural quasi-ordering of PF sets, the proposed approach can differentiate the dominance relationships among PF information more accurately and convincingly. Although there are major doubts about the practicality of the theory of Pythagorean fuzziness, this paper puts forward a simple and noncomplex procedure to handle PF data for enhancing the feasibility and practicability of the PF set theory.

(3) Development of an easy-to-use PF LINMAP model containing valuable concepts

This paper proposes a parametric PF LINMAP model that is straightforward and simple to use. In contrast to the current PF LINMAP methods [30], [31], this paper adopts a straightforward way to deal with complex PF decision information in uncertain GDM problems. The parametric PF LINMAP model is built upon solid theoretical bases, including several helpful concepts of the PF closeness-based order relation via PF closeness indices, the objective order relation via comprehensive closeness measures, individual order consistency/inconsistency indices, and new fitness measurements. Unlike complicated and troublesome computation procedures, this paper develops an effective PF LINMAP methodology that is easy to implement and understand for the sake of managing GDM problems within PF environments.

The remainder of this paper is organized as follows. Section II briefly reviews some basic concepts of Pythagorean membership grades and PF sets. Section III formulates a multiple criteria GDM problem involving decision makers' force-choice ordered paired comparison judgments over alternatives within the PF environment. Section IV presents useful PF closeness indices based on a recently developed Hamming distance measure for acquiring PF closeness-based order relations. Section V constructs a novel parametric PF LINMAP model to solve GDM problems under uncertainty of Pythagorean fuzziness. Section VI applies the proposed methodology to railway project investment decision making to validate its feasibility and practicality. Moreover, four comparative studies and discussions are conducted to show the usefulness and advantages of the proposed methodology. Finally, Section VII presents the conclusions.

II. PRELIMINARIES

This section reviews basic definitions of Pythagorean membership grades and PF sets for facilitating the subsequent study. The concept of a generalized PF distance measures is introduced as well.

Definition 1 ([17], [21], [27], [32]): Let P be a PF set in a finite universe of discourse X. Let p denote a Pythagorean membership grade of the element $x \in X$ belonging to P; it is characterized by the degree of membership $\mu_P(x)$, the degree of non-membership $\nu_P(x)$, the strength of commitment $r_P(x)$, and the direction of commitment $d_P(x)$, as follows:

$$p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x)), \tag{1}$$

where $\mu_P(x), \nu_P(x), r_P(x), d_P(x) \in [0, 1]$ such that $(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. The PF set P is expressed as the collection of Pythagorean membership grades for all $x \in X$ as follows:

$$P = \{\langle x, (\mu_P(x), \nu_P(x); r_P(x), d_P(x))\rangle | x \in X\}. \tag{2}$$

Definition 2 ([17], [19]): Let $\theta_P(x)$ be expressed in radians in the range $[0, \pi/2]$. For a Pythagorean membership grade p, the parameters $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$ are defined as follows:

$$\mu_P(x) = r_P(x) \cdot \cos(\theta_P(x)), \tag{3}$$

$$\nu_P(x) = r_P(x) \cdot \sin(\theta_P(x)), \tag{4}$$

$$r_P(x) = \sqrt{(\mu_P(x))^2 + (\nu_P(x))^2}, \tag{5}$$

$$d_P(x) = 1 - \frac{2 \cdot \theta_P(x)}{\pi}. \tag{6}$$

Definition 3 ([21], [27], [32]): For a PF set P in a finite universe of discourse X, the degree of indeterminacy $\tau_P(x)$ of the element $x \in X$ to P is defined as follows:

$$\tau_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}. \tag{7}$$

where $\tau_P(x) \in [0, 1]$. The duality property exists between $\tau_P(x)$ and $r_P(x)$ because $(\tau_P(x))^2 + (r_P(x))^2 = 1$ for each x.

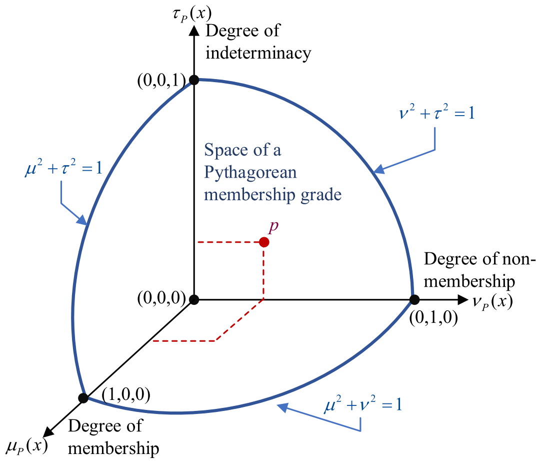


FIGURE 1. **Three-dimensional space represented by $\mu_P(x)$, $\nu_P(x)$, and $\tau_P(x)$ for a Pythagorean membership grade p.**

The Pythagorean membership grade p in a PF set P can be geometrically described as a point in a three-dimensional space by means of the three coordinates labeled by $\mu_P(x)$, $\nu_P(x)$, and $\tau_P(x)$, as shown in Figure 1. It can be observed that the relaxed constraint conditions (i.e., $(\mu_P(x))^2+(\nu_P(x))^2 \leq 1$ and $(\tau_P(x))^2 = 1-(\mu_P(x))^2-(\nu_P(x))^2)$ make PF sets possess a significant advantage for a wider coverage of information span [22].

Certain useful generalized distance measures have been developed in PF contexts, such as the Minkowski distance measures [30], [33], the generalized distance measures based on four characteristics [34] and five characteristics [35], and the generalized PF distance measure [27], [32]. In particular, Chen [27], [32] utilized the essential characteristics of Pythagorean membership grades (i.e., membership degree, non-membership degree, strength of commitment, and direction of commitment) to propose a novel generalized PF distance measure. Because Chen's proposed measure has the advantages of furnishing a suitable normalization approach, addressing the double weighting issue, and utilizing the square terms in Pythagorean membership degrees, this paper attempts to utilize the special case of the generalized PF distance measure, i.e., the Hamming distance model, to measure the separation between Pythagorean membership grades in PF contexts.

Definition 4 ([27], [32]): Let p_1 and p_2 be two Pythagorean membership grades in a PF set P on the universe of discourse X, where $p_1 = (\mu_{P_1}(x), \nu_{P_1}(x); r_{P_1}(x), d_{P_1}(x))$ and $p_2 = (\mu_{P_2}(x), \nu_{P_2}(x); r_{P_2}(x), d_{P_2}(x))$. Let β denote a distance parameter, where $\beta \geq 1$. The generalized PF distance measure D^β between p_1 and p_2 is defined as follows:

$$D^\beta(p_1,p_2) = \left[\frac{1}{3}\left(\left|(\mu_{P_1}(x))^2-(\mu_{P_2}(x))^2\right|^\beta + \left|(\nu_{P_1}(x))^2-(\nu_{P_2}(x))^2\right|^\beta + \left|(r_{P_1}(x))^2 - (r_{P_2}(x))^2\right|^\beta + \left|d_{P_1}(x)-d_{P_2}(x)\right|^\beta\right)\right]^{\frac{1}{\beta}}. \quad (8)$$

When $\beta = 1$, the generalized PF distance reduces to the Hamming distance measure D (i.e., D^1):

$$D(p_1,p_2) = \frac{1}{3}\left(\left|(\mu_{P_1}(x))^2-(\mu_{P_2}(x))^2\right| + \left|(\nu_{P_1}(x))^2 - (\nu_{P_2}(x))^2\right| + \left|(r_{P_1}(x))^2-(r_{P_2}(x))^2\right| + \left|d_{P_1}(x)-d_{P_2}(x)\right|\right). \quad (9)$$

Theorem 1: Let $p_i = (\mu_{P_i}(x), \nu_{P_i}(x); r_{P_i}(x), d_{P_i}(x))$ $(i = 1, 2, 3)$ be three Pythagorean membership grades in a PF set P on the universe of discourse X. The Hamming distance measure D satisfies the following properties:

(T1.1) $D(p_1, p_1) = 0$ (reflexivity);
(T1.2) $D(p_1, p_2) = 0$ if and only if $p_1 = p_2$ (separability);
(T1.3) $D(p_1, p_2) = D(p_2, p_1)$ (symmetry);
(T1.4) $0 \leq D(p_1, p_2) \leq 1$ (boundedness);
(T1.5) $D(p_1, p_3) \leq D(p_1, p_2) + D(p_2, p_3)$ (triangle inequality).

Proof: Refer to the proof process in Chen [27], [32].

III. PROBLEM FORMULATION

This section attempts to describe a multiple criteria GDM problem involving PF evaluation information and decision makers' force-choice ordered paired comparison judgments over candidate alternatives.

Consider a multiple criteria GDM problem within the PF environment. Let $E = \{e_1, e_2, \cdots, e_K\}$ denote the set of decision makers involved in the group decision-making process. Let $A = \{a_1, a_2, \cdots, a_m\}$ denote a discrete set of m candidate alternatives, where $m \geq 2$. Let $C = \{c_1, c_2, \cdots, c_n\}$ denote a finite set of n evaluative criteria, where $n \geq 2$. In general, the set C is divided into two disjoint sets, namely, the set of benefit criteria C_{I} and the set of cost criteria C_{II}, where $C_{\mathrm{I}} \cap C_{\mathrm{II}} = \emptyset$ and $C_{\mathrm{I}} \cup C_{\mathrm{II}} = C$. For each decision maker $e_k \in E$, the PF evaluative rating of an alternative $a_i \in A$ with respect to a criterion $c_j \in C$ is represented as a Pythagorean membership grade $p_{ij}^k = (\mu_{ij}^k, \nu_{ij}^k; r_{ij}^k, d_{ij}^k)$, in which $\mu_{ij}^k = r_{ij}^k \cdot \cos(\theta_{ij}^k)$, $\nu_{ij}^k = r_{ij}^k \cdot \sin(\theta_{ij}^k)$, $r_{ij}^k = ((\mu_{ij}^k)^2 + (\nu_{ij}^k)^2)^{0.5}$, and $d_{ij}^k = 1 - (2 \cdot \theta_{ij}^k \big/ \pi)$ for $\theta_{ij}^k \in [0, \pi/2]$. For the decision maker e_k, the PF decision matrix P^k that involves PF evaluative ratings is represented as follows:

$$\mathrm{P}^k = \left[p_{ij}^k\right]_{m\times n} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{cccc} \quad c_1 & \quad c_2 & \cdots & \quad c_n \end{array} \\ \left[\begin{array}{cccc} (\mu_{11}^k, \nu_{11}^k; r_{11}^k, d_{11}^k) & (\mu_{12}^k, \nu_{12}^k; r_{12}^k, d_{12}^k) & \cdots & (\mu_{1n}^k, \nu_{1n}^k; r_{1n}^k, d_{1n}^k) \\ (\mu_{21}^k, \nu_{21}^k; r_{21}^k, d_{21}^k) & (\mu_{22}^k, \nu_{22}^k; r_{22}^k, d_{22}^k) & \cdots & (\mu_{2n}^k, \nu_{2n}^k; r_{2n}^k, d_{2n}^k) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}^k, \nu_{m1}^k; r_{m1}^k, d_{m1}^k) & (\mu_{m2}^k, \nu_{m2}^k; r_{m2}^k, d_{m2}^k) & \cdots & (\mu_{mn}^k, \nu_{mn}^k; r_{mn}^k, d_{mn}^k) \end{array}\right] \end{array}. \quad (10)$$

According to the decision maker's knowledge, expertise, and decision-making experience, the preference relations between alternatives can be conveniently expressed using force-choice ordered paired comparison judgments. Let the symbol "$\succsim$" denote a preference relation provided by the decision maker. More specifically, the preference relation $a_{i'} \succsim a_{i''}$ indicates that either the decision maker prefers the alternative $a_{i'}$ to the alternative $a_{i''}$ or the decision maker feels indifferent between $a_{i'}$ and $a_{i''}$. Let Ω^k denote a preference set provided by the decision maker e_k. The set Ω^k contains the ordered pairs (i', i'') in line with the decision maker's paired preference relations for the alternatives in Z and is defined as follows:

$$\Omega^k = \left\{ (i', i'') \middle| a_{i'} \succsim a_{i''}, i', i'' \in \{1, 2, \cdots, m\} \right\}. \tag{11}$$

It is worthy to note that incomplete and inconsistent information exists among some paired preference relations in the sets Ω^k for $k \in \{1, 2, \cdots, K\}$. There are at most $m(m-1)/2$ paired preference relations in the preference set Ω^k. However, the decision makers often provide incomplete preference information about alternatives in practice. Furthermore, inconsistent preference relations may be found among some ordered pairs provided by different decision makers.

Let $\mathrm{w} = (w_1, w_2, \cdots, w_n)$ denote the weight vector of n criteria which is unknown a priori and must be determined. The weight w_j satisfies the normalization conditions, namely, $w_j \in [0, 1]$ for all $j \in \{1, 2, \cdots, n\}$ and $\sum_{j=1}^{n} w_j = 1$. However, it is anticipated that non-zero weights can be derived through the LINMAP procedure because the n evaluative criteria are salient attributes in the group decision-making process. In a similar way to Li [36], this paper considers a non-zero boundary condition and assumes that $w_j \geq \varepsilon$ for all $c_j \in C$, where ε is a sufficiently small positive number and $\varepsilon \in (0, 1]$.

IV. PF CLOSENESS-BASED ORDER RELATIONS

This section attempts to employ a Hamming distance-based approach to define PF closeness indices and identify PF closeness-based order relations that can furnish a basis for measuring order consistency and inconsistency.

For a benefit criterion $c_j \in C_{\mathrm{I}}$, a higher PF evaluative rating p_{ij}^k indicates a stronger preference. Conversely, for a cost criterion $c_j \in C_{\mathrm{II}}$, a lower p_{ij}^k indicates a stronger preference. Following the rationale, this paper specifies the positive-ideal PF evaluative rating $p_{*j}^k = (\mu_{*j}^k, \nu_{*j}^k; r_{*j}^k, d_{*j}^k)$ and the negative-ideal PF evaluative rating $p_{\#j}^k = (\mu_{\#j}^k, \nu_{\#j}^k; r_{\#j}^k, d_{\#j}^k)$ for anchored judgments, in which $\theta_{*j}^k = \cos^{-1}(\mu_{*j}^k / r_{*j}^k) = \sin^{-1}(\nu_{*j}^k / r_{*j}^k)$ and $\theta_{\#j}^k = \cos^{-1}(\mu_{\#j}^k / r_{\#j}^k) = \sin^{-1}(\nu_{\#j}^k / r_{\#j}^k)$. Let the symbols "$\wedge$" and "$\vee$" denote the minimum and maximum operators, respectively.

Definition 5: For a PF decision matrix P^k, the positive- and negative-ideal PF evaluative ratings p_{*j}^k and $p_{\#j}^k$, respectively, with respect to each benefit criterion $c_j \in C_{\mathrm{I}}$ are defined as follows:

$$p_{*j}^k = \left(\bigvee_{i=1}^{m} \mu_{ij}^k, \bigwedge_{i=1}^{m} \nu_{ij}^k; \sqrt{\left(\bigvee_{i=1}^{m} \mu_{ij}^k \right)^2 + \left(\bigwedge_{i=1}^{m} \nu_{ij}^k \right)^2}, 1 - \frac{2 \cdot \theta_{*j}^k}{\pi} \right), \tag{12}$$

$$p_{\#j}^k = \left(\bigwedge_{i=1}^{m} \mu_{ij}^k, \bigvee_{i=1}^{m} \nu_{ij}^k; \sqrt{\left(\bigwedge_{i=1}^{m} \mu_{ij}^k \right)^2 + \left(\bigvee_{i=1}^{m} \nu_{ij}^k \right)^2}, 1 - \frac{2 \cdot \theta_{\#j}^k}{\pi} \right). \tag{13}$$

Definition 6: For a PF decision matrix P^k, the positive- and negative-ideal PF evaluative ratings p_{*j}^k and $p_{\#j}^k$, respectively, with respect to each cost criterion $c_j \in C_{\mathrm{II}}$ are defined as follows:

$$p_{*j}^k = \left(\bigwedge_{i=1}^{m} \mu_{ij}^k, \bigvee_{i=1}^{m} \nu_{ij}^k; \sqrt{\left(\bigwedge_{i=1}^{m} \mu_{ij}^k \right)^2 + \left(\bigvee_{i=1}^{m} \nu_{ij}^k \right)^2}, 1 - \frac{2 \cdot \theta_{*j}^k}{\pi} \right), \tag{14}$$

$$p_{\#j}^k = \left(\bigvee_{i=1}^{m} \mu_{ij}^k, \bigwedge_{i=1}^{m} \nu_{ij}^k; \sqrt{\left(\bigvee_{i=1}^{m} \mu_{ij}^k \right)^2 + \left(\bigwedge_{i=1}^{m} \nu_{ij}^k \right)^2}, 1 - \frac{2 \cdot \theta_{\#j}^k}{\pi} \right). \tag{15}$$

Theorem 2: For the PF decision matrix P^k, there is a natural quasi-ordering between $p_{i'j}^k$ and $p_{i''j}^k$, namely, $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ if and only if $\mu_{i'j}^k \leq \mu_{i''j}^k$ and $\nu_{i'j}^k \geq \nu_{i''j}^k$. The PF evaluative ratings p_{ij}^k, p_{*j}^k, and $p_{\#j}^k$ satisfy the following quasi-orderings:

(T2.1) $p_{\#j}^k \precsim_{\mathbb{Q}} p_{ij}^k \precsim_{\mathbb{Q}} p_{*j}^k$ for $c_j \in C_{\mathrm{I}}$;

(T2.2) $p_{*j}^k \precsim_{\mathbb{Q}} p_{ij}^k \precsim_{\mathbb{Q}} p_{\#j}^k$ for $c_j \in C_{\mathrm{II}}$.

Proof: (T2.1) and (T2.2) can be easily proven using Definitions 5 and 6, respectively. This completes the proof.

The important point to note is the natural quasi-ordering between PF evaluative ratings does not often appear in the PF context. For example, assume that $p_{i'j}^k = (0.70, 0.60; 0.9220, 0.5489)$ and $p_{i''j}^k = (0.50, 0.40; 0.6403, 0.5704)$. One can observe that the quasi-ordering does not exist between $p_{i'j}^k$ and $p_{i''j}^k$ because $\mu_{i'j}^k (= 0.70) > \mu_{i''j}^k (= 0.50)$ and $\nu_{i'j}^k (= 0.60) > \nu_{i''j}^k (= 0.40)$. That is, neither $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ nor $p_{i''j}^k \precsim_{\mathbb{Q}} p_{i'j}^k$ holds in this example. The question then arises about the comparison for PF information, because the use of natural quasi-ordering cannot effectively differentiate the dominance relationships among PF evaluative ratings. To overcome this difficulty, this paper advances a new order relation based on the concept of PF closeness indices to facilitate effective comparisons for PF evaluative ratings.

In classical LINMAP, the square of the weighted Euclidean distance between alternatives has been commonly used to determine the consistency and inconsistency measurements between the subjective and objective ranking orders. However, the Euclidean distance measure is costly as there involve expensive square and square root operations. Moreover, the squared Euclidean distance is not a metric, as it does not satisfy the triangle inequality. In contrast, the Hamming distance measure D is a metric, because it satisfies the properties of reflexivity, separability, symmetry, and triangle inequality, as demonstrated in Theorem 1. Compared to the Euclidean distance model, the Hamming distance model is uncomplicated and easy to implement. Instead of the Euclidean distance-based approach, this paper utilizes the Hamming distance measure D to define the useful PF closeness indices and studies some important properties.

Definition 7: The PF closeness index of a PF evaluative rating p_{ij}^k in the PF decision matrix P^k is defined as follows:

$$CI(p_{ij}^k) = \frac{D(p_{ij}^k, p_{\#j}^k)}{D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k)}. \tag{16}$$

Theorem 3: For each PF evaluative rating p_{ij}^k in the PF decision matrix P^k, the PF closeness index $CI(p_{ij}^k)$ satisfies the following properties:

(T3.1) $0 \le CI(p_{ij}^k) \le 1$;

(T3.2) $CI(p_{ij}^k) = 0$ if and only if $p_{ij}^k = p_{\#j}^k$;

(T3.3) $CI(p_{ij}^k) = 1$ if and only if $p_{ij}^k = p_{*j}^k$;

(T3.4) $CI(p_{ij}^k) \le D(p_{ij}^k, p_{\#j}^k) \big/ D(p_{*j}^k, p_{\#j}^k)$.

Proof: First, (T3.1) is inferred directly because of the boundedness property in (T1.4) (i.e., $0 \le D(p_{ij}^k, p_{\#j}^k) \le 1$ and $0 \le D(p_{ij}^k, p_{*j}^k) \le 1$). Next, for the necessity in (T3.2), the condition of $CI(p_{ij}^k) = 0$ implies that $D(p_{ij}^k, p_{\#j}^k) = 0$. Thus, one has $p_{ij}^k = p_{\#j}^k$ according to the separability property in (T1.2). For the sufficiency in (T3.2), if $p_{ij}^k = p_{\#j}^k$, then $D(p_{ij}^k, p_{\#j}^k) = 0$ using the reflexivity property in (T1.1), which follows that $CI(p_{ij}^k) = 0$. Similarly, for the necessity in (T3.3), the condition of $CI(p_{ij}^k) = 1$ indicates that $D(p_{ij}^k, p_{\#j}^k) = D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k)$, which follows that $D(p_{ij}^k, p_{*j}^k) = 0$. Thus, the equality $p_{ij}^k = p_{*j}^k$ is fulfilled based on (T1.2). For the sufficiency in (T3.3), if $p_{ij}^k = p_{*j}^k$, then $D(p_{ij}^k, p_{*j}^k) = 0$ based on (T1.1), which leads to $CI(p_{ij}^k) = 1$. For (T3.4), it is known that $D(p_{*j}^k, p_{\#j}^k) \le D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k)$ based on the symmetry property in (T1.3) and the triangle inequality in (T1.5). It is easy to see that $1\big/(D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k)) \le 1\big/D(p_{*j}^k, p_{\#j}^k)$, which implies that $CI(p_{ij}^k) \le D(p_{ij}^k, p_{\#j}^k)\big/D(p_{*j}^k, p_{\#j}^k)$. Therefore, (T3.1)–(T3.4) are valid, which completes the proof.

Theorem 4: Let p_{ij}^k, $p_{i'j}^k$, and $p_{i''j}^k$ be three PF evaluative ratings in the PF decision matrix P^k, in which p_{*j}^k and $p_{\#j}^k$ denote the ideal PF evaluative ratings. If $(r_{\#j}^k \wedge r_{*j}^k) \le r_{ij}^k, r_{i'j}^k, r_{i''j}^k \le (r_{\#j}^k \vee r_{*j}^k)$ for $c_j \in C$, then the following properties are satisfied:

(T4.1) $D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k) = D(p_{*j}^k, p_{\#j}^k)$;

(T4.2) $CI(p_{ij}^k) = D(p_{ij}^k, p_{\#j}^k) \big/ D(p_{*j}^k, p_{\#j}^k)$;

(T4.3) $CI(p_{i'j}^k) \le CI(p_{i''j}^k)$ in case of $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ for $c_j \in C_{\mathrm{I}}$;

(T4.4) $CI(p_{i'j}^k) \ge CI(p_{i''j}^k)$ in case of $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ for $c_j \in C_{\mathrm{II}}$.

Proof: For (T4.1), either $r_{\#j}^k \le r_{ij}^k \le r_{*j}^k$ or $r_{*j}^k \le r_{ij}^k \le r_{\#j}^k$ holds according to the assumption $r_{\#j}^k \wedge r_{*j}^k \le r_{ij}^k \le r_{\#j}^k \vee r_{*j}^k$. When $r_{\#j}^k \le r_{ij}^k \le r_{*j}^k$, $|(r_{ij}^k)^2 - (r_{*j}^k)^2| + |(r_{ij}^k)^2 - (r_{\#j}^k)^2| = (r_{*j})^2 - (r_{\#j}^k)^2$. When $r_{*j}^k \le r_{ij}^k \le r_{\#j}^k$, $|(r_{ij}^k)^2 - (r_{*j}^k)^2| + |(r_{ij}^k)^2 - (r_{\#j}^k)^2| = (r_{\#j}^k)^2 - (r_{*j}^k)^2$. Taking a cost criterion $c_j \in C_{\mathrm{II}}$ for example, the following results are correct:

$$\begin{aligned}
&D(p_{ij}^k, p_{*j}^k)\\
&= \frac{1}{3}\Big((\mu_{ij}^k)^2 - (\mu_{*j}^k)^2 + (\nu_{*j}^k)^2 - (\nu_{ij}^k)^2\\
&\quad + \left|(r_{ij}^k)^2 - (r_{*j}^k)^2\right| + d_{ij}^k - d_{*j}^k\Big),\\
&D(p_{ij}^k, p_{\#j}^k)\\
&= \frac{1}{3}\Big((\mu_{\#j}^k)^2 - (\mu_{ij}^k)^2 + (\nu_{ij}^k)^2 - (\nu_{\#j}^k)^2\\
&\quad + \left|(r_{ij}^k)^2 - (r_{\#j}^k)^2\right| + d_{\#j}^k - d_{ij}^k\Big),\\
&D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k)\\
&= \frac{1}{3}\Big((\mu_{\#j}^k)^2 - (\mu_{*j}^k)^2 + (\nu_{*j}^k)^2 - (\nu_{\#j}^k)^2 + \left|(r_{ij}^k)^2 - (r_{*j}^k)^2\right|\\
&\quad + \left|(r_{ij}^k)^2 - (r_{\#j}^k)^2\right| + d_{\#j}^k - d_{*j}^k\Big)\\
&= \frac{1}{3}\Big(\left|(\mu_{\#j}^k)^2 - (\mu_{*j}^k)^2\right| + \left|(\nu_{\#j}^k)^2 - (\nu_{*j}^k)^2\right| + \left|(r_{\#j}^k)^2 - (r_{*j}^k)^2\right|\\
&\quad + \left|d_{\#j}^k - d_{*j}^k\right|\Big) = D(p_{\#j}^k, p_{*j}^k).
\end{aligned}$$

The above result can be analogously obtained in case of $c_j \in C_{\mathrm{I}}$. Therefore, $D(p_{ij}^k, p_{*j}^k) + D(p_{ij}^k, p_{\#j}^k) = D(p_{*j}^k, p_{\#j}^k)$ holds for all $c_j \in C$, i.e., (T4.1) is valid. Accordingly, (T4.2) is correct based on (T4.1) and Definition 7. Next, according to the premise assumption in (T4.3) and the property in (T2.1), it is known that $p_{\#j}^k \precsim_{\mathbb{Q}} p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ for $c_j \in C_{\mathrm{I}}$. This implies that $D(p_{i'j}^k, p_{\#j}^k) \le D(p_{i''j}^k, p_{\#j}^k)$. By applying (T4.2), it is obtained that $CI(p_{i'j}^k) = D(p_{i'j}^k, p_{\#j}^k)\big/D(p_{*j}^k, p_{\#j}^k)$ and $CI(p_{i''j}^k) = D(p_{i''j}^k, p_{\#j}^k)\big/D(p_{*j}^k, p_{\#j}^k)$. Therefore, $CI(p_{i'j}^k) \le CI(p_{i''j}^k)$ for $c_j \in C_{\mathrm{I}}$, i.e., (T4.3) is valid. Based on the assumption in (T4.4) and (T2.1), one has $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k \precsim_{\mathbb{Q}} p_{\#j}^k$ for $c_j \in C_{\mathrm{II}}$, which follows that $D(p_{i'j}^k, p_{\#j}^k) \ge D(p_{i''j}^k, p_{\#j}^k)$. Thus, it can be inferred that $CI(p_{i'j}^k) \ge CI(p_{i''j}^k)$. This completes the proof.

Definition 8: For two PF evaluative ratings $p_{i'j}^k$ and $p_{i''j}^k$ in the PF decision matrix P^k, the PF closeness-based order relation between $p_{i'j}^k$ and $p_{i''j}^k$ is specified via PF closeness indices as follows:

(D8.1) If $CI(p_{i'j}^k) > CI(p_{i''j}^k)$, then $p_{i'j}^k$ is superior to $p_{i''j}^k$, denoted by $p_{i'j}^k \succ_{\mathbb{C}} p_{i''j}^k$;

(D8.2) If $CI(p_{i'j}^k) = CI(p_{i''j}^k)$, then $p_{i'j}^k$ is indifferent to $p_{i''j}^k$, denoted by $p_{i'j}^k \sim_{\mathbb{C}} p_{i''j}^k$;

(D8.3) If $CI(p_{i'j}^k) < CI(p_{i''j}^k)$, then $p_{i'j}^k$ is inferior to $p_{i''j}^k$, denoted by $p_{i'j}^k \prec_{\mathbb{C}} p_{i''j}^k$.

As demonstrated in the aforementioned theorems and discussions, the PF closeness indices possess several important and desirable properties and can facilitate conducting criterion-wise comparisons between PF evaluative ratings. Accordingly, the proposed PF closeness-based order relations, i.e., $\succ_{\mathbb{C}}$, $\sim_{\mathbb{C}}$, and $\prec_{\mathbb{C}}$, can be fully utilized to determine the measurements of order consistency and inconsistency between the subjective preference relations and objective order relations in the PF LINMAP methodology.

V. PROPOSED PARAMETRIC PF LINMAP MODELS

This section employs the PF closeness indices and PF closeness-based order relations to measure the amount of order consistency and order inconsistency and to establish a parametric PF LINMAP methodology for solving multiple criteria GDM problems based on PF sets.

To determine a synthetic effect of PF closeness-based order relations across all evaluative criteria, this paper combines the weights of criteria and PF closeness indices to define a comprehensive closeness measure.

Definition 9: Let $\mathrm{w} = (w_1, w_2, \cdots, w_n)$ be the weight vector of n criteria. The comprehensive closeness measure of the alternative $z_i \in Z$ for the decision maker $e_k \in E$ are defined as follows:

$$CM_i^k = \sum_{j=1}^{n} w_j \cdot CI(p_{ij}^k). \tag{17}$$

Theorem 5: The comprehensive closeness measure CM_i^k for each $z_i \in Z$ and $e_k \in E$ satisfies the following properties:

(T5.1) $0 \le CM_i^k \le 1$;

(T5.2) If $p_{ij}^k = p_{\#j}^k$ for all $c_j \in C$, then $CM_i^k = 0$;

(T5.3) If $p_{ij}^k = p_{*j}^k$ for all $c_j \in C$, then $CM_i^k = 1$.

Proof: (T5.1) is easily proven because $0 \le CI(p_{ij}^k) \le 1$ in (T3.1) and $0 \le w_j \le 1$ based on the normalization condition. (T5.2) is obvious according to the property in (T3.2). (T5.3) is valid based on (T3.3) and $\sum_{j=1}^{n} w_j = 1$. This completes the proof.

Theorem 6: For two PF evaluative ratings $p_{i'j}^k$ and $p_{i''j}^k$, assume that $(r_{\#j}^k \wedge r_{*j}^k) \le r_{i'j}^k, r_{i''j}^k \le (r_{\#j}^k \vee r_{*j}^k)$ holds for $c_j \in C$. If $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ and $p_{i''j}^k \precsim_{\mathbb{Q}} p_{i'j}^k$ for $c_j \in C_{\mathrm{I}}$ and $c_j \in C_{\mathrm{II}}$, respectively, then the comprehensive closeness measures $CM_{i'}^k \le CM_{i''}^k$.

Proof: For $c_j \in C_{\mathrm{I}}$, it is known that $CI(p_{i'j}^k) \le CI(p_{i''j}^k)$ from the premise condition $p_{i'j}^k \precsim_{\mathbb{Q}} p_{i''j}^k$ based on (T4.3). For $c_j \in C_{\mathrm{II}}$, the premise condition $p_{i''j}^k \precsim_{\mathbb{Q}} p_{i'j}^k$ implies that $CI(p_{i'j}^k) \le CI(p_{i''j}^k)$ by applying (T4.4). Thus, it can be inferred that $w_j \cdot CI(p_{i'j}^k) \le w_j \cdot CI(p_{i''j}^k)$ for $c_j \in C$, which follows that $CM_{i'}^k \le CM_{i''}^k$. This completes the proof.

Based on the desirable properties in Theorems 5 and 6, the concept of comprehensive closeness measures can be used to determine the objective order relations over the alternatives for each decision maker. Specifically, the comprehensive closeness measures $CM_{i'}^k$ and $CM_{i''}^k$ can be employed to rank the alternatives $a_{i'}$ and $a_{i''}$, which can be viewed as a kind of objective ranking order based on evaluation information in the PF decision matrix P^k.

Definition 10: For two alternatives $a_{i'}$ and $a_{i''}$ in a GDM problem involving the PF decision matrix P^k, the objective order relation between $a_{i'}$ and $a_{i''}$ is specified via comprehensive closeness measures as follows:

(D10.1) If $CM_{i'}^k > CM_{i''}^k$, then $a_{i'}$ is superior to $a_{i''}$, denoted by $a_{i'} \succ_{\mathbb{C}} a_{i''}$;

(D10.2) If $CM_{i'}^k = CM_{i''}^k$, then $a_{i'}$ is indifferent to $a_{i''}$, denoted by $a_{i'} \sim_{\mathbb{C}} a_{i''}$;

(D10.3) If $CM_{i'}^k < CM_{i''}^k$, then $a_{i'}$ is inferior to $a_{i''}$, denoted by $a_{i'} \prec_{\mathbb{C}} a_{i''}$.

In contrast, the ordered pair $(i', i'') \in \Omega^k$ given by the decision maker e_k belongs to subjective preference relations. In practical situations, there exists somewhat deviations between the subjective and objective order relations. To measure such deviations, the objective order relations would be contrasted with the subjective preference relations in the preference set Ω^k.

For each ordered pair $(i', i'') \in \Omega^k$, if $CM_{i'}^k \ge CM_{i''}^k$, the alternative $a_{i'}$ is closer to the positive-ideal point of reference and farther from the negative-ideal point of reference than the alternative $a_{i''}$. Based on Definition 10, the objective order relation $a_{i'} \succsim_{\mathbb{C}} a_{i''}$ is obtained, which is consistent with the subjective preference relation given by the decision maker e_k. On the contrary, if $CM_{i'}^k < CM_{i''}^k$, then the obtained objective order relation $a_{i'} \prec_{\mathbb{C}} a_{i''}$ is inconsistent with the ordered pair (i', i''). In other words, for each ordered pair $(i', i'') \in \Omega^k$, no error can be attributed to the paired preference relation between alternatives $a_{i'}$ and $a_{i''}$ if $CM_{i'}^k \ge CM_{i''}^k$, whereas errors exist if $CM_{i'}^k < CM_{i''}^k$. In this regard, this paper introduces the concepts of individual order consistency and inconsistency indices for each $(i', i'') \in \Omega^k$ to measure consistency and inconsistency, respectively, between the subjective preference relations and objective order relations.

The individual order consistency index $(CM_{i'}^k - CM_{i''}^k)^+$ and inconsistency index $(CM_{i'}^k - CM_{i''}^k)^-$ between the preorders of the alternatives $a_{i'}$ and $a_{i''}$ for each $(i', i'') \in \Omega^k$ are defined as follows:

$$\left(CM_{i'}^k - CM_{i''}^k\right)^+ = \begin{cases} CM_{i'}^k - CM_{i''}^k & \text{if } a_{i'} \succsim_{\mathbb{C}} a_{i''} \text{in } P^k, \\ 0 & \text{if } a_{i'} \prec_{\mathbb{C}} a_{i''} \text{in } P^k, \end{cases}$$
$$= \max\left\{0, CM_{i'}^k - CM_{i''}^k\right\}, \tag{18}$$

$$\left(CM_{i'}^k - CM_{i''}^k\right)^- = \begin{cases} CM_{i''}^k - CM_{i'}^k & \text{if } a_{i'} \prec_{\mathbb{C}} a_{i''} \text{in } P^k, \\ 0 & \text{if } a_{i'} \succsim_{\mathbb{C}} a_{i''} \text{in } P^k, \end{cases}$$
$$= \max\left\{0, CM_{i''}^k - CM_{i'}^k\right\}, \tag{19}$$

where $(CM_{i'}^k - CM_{i''}^k)^+ \ge 0$ and $(CM_{i'}^k - CM_{i''}^k)^- \ge 0$.

To determine the measurements of individual goodness of fit and poorness of fit, this paper combines individual order consistency and inconsistency indices, respectively, over all ordered pairs in each preference set. This paper sums the indices $(CM_{i'}^k - CM_{i''}^k)^+$ and $(CM_{i'}^k - CM_{i''}^k)^-$ for all $(i', i'') \in \Omega^k$ to determine the individual goodness of fit G^k

and individual poorness of fit B^k, respectively, as follows:

$$G^k = \sum_{(i',i'')\in\Omega^k} \left(CM_{i'}^k - CM_{i''}^k\right)^+, \tag{20}$$

$$B^k = \sum_{(i',i'')\in\Omega^k} \left(CM_{i'}^k - CM_{i''}^k\right)^-, \tag{21}$$

where $G^k \geq 0$ and $B^k \geq 0$.

In general, each decision maker e_k anticipates a solution of which the individual goodness of fit G^k is higher than the individual poorness of fit B^k to some degree. To this end, this paper designates a non-negative number h that represents the lowest acceptable level towards the difference between G^k and B^k. Then, the conditions of $G^k - B^k \geq h$ for all $k \in \{1, 2, \cdots, K\}$ are incorporated into the proposed PF LINMAP model. As mentioned earlier, motivated by Li [36], this paper modifies the normalization conditions of the weight vector $\mathrm{w} = (w_1, w_2, \cdots, w_n)$ to ensure that the obtained weights are not zero. Furthermore, this paper considers the number of evaluative criteria n to be an upper bound of the weight. Namely, it is suggested that the weights are not larger than $1/n$. Therefore, the constraints about the weight w_j contain $\sum_{j=1}^n w_j = 1$ and $w_j \geq \varepsilon$ for all $j \in \{1, 2, \cdots, n\}$ in the proposed PF LINMAP model, where ε is a sufficiently small number and $0 < \varepsilon \leq 1/n$. To minimize the collective poorness of fit $\sum_{k=1}^K B^k$, the following linear programming model is established:

$$\textbf{Model(I)} \ \min \left\{\sum_{k=1}^K B^k\right\}$$
$$\text{s.t.} \begin{cases} G^k - B^k \geq h(k = 1, 2, \cdots, K), \\ \sum_{j=1}^n w_j = 1, \ w_j \geq \varepsilon(j = 1, 2, \cdots, n). \end{cases} \tag{22}$$

On the other side, preference conflicts sometimes occur among some ordered pairs provided by different decision makers. For example, a decision maker prefers the alternative $a_{i'}$ to the alternative $a_{i''}$, whereas another decision maker prefers $a_{i''}$ to $a_{i'}$. Thus, more or less degrees of violation exist for the ordered pairs given by K decision makers. To acquire individual degrees of violation for each decision maker's paired preference relations, this paper denotes a non-negative variable $Z_{i'i''}^k$ that is defined as the maximum of 0 and $CM_{i''}^k - CM_{i'}^k$ for each (i', i''), as follows:

$$Z_{i'i''}^k = \max\left\{0, CM_{i''}^k - CM_{i'}^k\right\}, \tag{23}$$

where $Z_{i'i''}^k \geq 0$ and $Z_{i'i''}^k \geq CM_{i''}^k - CM_{i'}^k$. It is obvious that $Z_{i'i''}^k = (CM_{i'}^k - CM_{i''}^k)^-$. Therefore, the collective poorness of fit becomes:

$$\sum_{k=1}^K B^k = \sum_{k=1}^K \sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k. \tag{24}$$

Employing the individual degree of violation $Z_{i'i''}^k$, Model (I) can be transformed into the following linear programming model:

$$\textbf{Model(II)} \ \min \left\{\sum_{k=1}^K \sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k\right\}$$
$$\text{s.t.} \begin{cases} \sum_{(i',i'')\in\Omega^k} \left(CM_{i'}^k - CM_{i''}^k\right) \\ \geq h(k = 1, 2, \cdots, K), \\ CM_{i'}^k - CM_{i''}^k + Z_{i'i''}^k \geq 0 \text{ and } Z_{i'i''}^k \geq 0 \\ ((i', i'') \in \Omega^k \text{ and } k = 1, 2, \cdots, K), \\ \sum_{j=1}^n w_j = 1, \ w_j \geq \varepsilon(j = 1, 2, \cdots, n). \end{cases} \tag{25}$$

It is worthwhile to mention that Model (II) only considers the minimal objective of the collective poorness of fit $\sum_{k=1}^K B^k$ that represents the lowest extent of violation with respect to the conditions in the preference relationships over all preference sets Ω^k ($k \in \{1, 2, \cdots, K\}$). The collective comprehensive closeness measure of the alternative a_i is calculated by $\sum_{k=1}^K CM_i^k$. The decision makers are generally conceived to accept a GDM solution that enjoys the highest collective comprehensive closeness measure. The larger the sum of collective comprehensive closeness measures, the higher degree of satisfaction towards the solution result perceived by the decision makers. For these reasons, the total collective comprehensive closeness measure, namely, $\sum_{i=1}^m \sum_{k=1}^K CM_i^k$, should be designated as a maximal objective in the PF LINMAP model. Accordingly, the following bi-objective optimization model can be established to maximize the total collective comprehensive closeness measure and minimize the collective poorness of fit:

$$\textbf{Model(III)} \ \max \left\{\sum_{i=1}^m \sum_{k=1}^K CM_i^k\right\}, \min \left\{\sum_{k=1}^K \sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k\right\}$$
$$\text{s.t.} \begin{cases} \sum_{(i',i'')\in\Omega^k} \left(CM_{i'}^k - CM_{i''}^k\right) \\ \geq h(k = 1, 2, \cdots, K), \\ CM_{i'}^k - CM_{i''}^k + Z_{i'i''}^k \geq 0 \text{ and } Z_{i'i''}^k \geq 0 \\ ((i', i'') \in \Omega^k \text{ and } k = 1, 2, \cdots, K), \\ \sum_{j=1}^n w_j = 1, \ w_j \geq \varepsilon(j = 1, 2, \cdots, n). \end{cases} \tag{26}$$

To reduce the computation complexity of Model (III), this paper combines the two objectives by use of a weighting parameter. First, the minimal objective $\sum_{k=1}^K \sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k$ is equivalent to the maximal objective $-\sum_{k=1}^K \sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k$. Next, let a parameter η that denotes the weight of the "total collective comprehensive closeness measure" objective, while $1 - \eta$ denotes the weight of the "collective poorness of fit" objective, where $\eta \in [0, 1]$. Recall that $CM_i^k = \sum_{j=1}^n CI(p_{ij}^k) \cdot w_j$ from Definition 9. This follows that

$\sum_{i=1}^{m}\sum_{k=1}^{K} CM_i^k = \sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} CI(p_{ij}^k)\cdot w_j$ and $CM_{i'}^k - CM_{i''}^k = \sum_{j=1}^{n}\left(CI(p_{i'j}^k) - CI(p_{i''j}^k)\right)\cdot w_j$. This paper employs the parameter η to coordinate the two objectives in Model (III) and transforms the bi-objective model into a simple linear programming model. To determine the optimal weight vector and individual degrees of violation, the following parametric PF LINMAP model is constructed for solving the GDM problem within the PF environment:

Parametric PF LINMAP Model

$$\max\left\{\eta\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} CI(p_{ij}^k)w_j-(1-\eta)\sum_{k=1}^{K}\sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k\right\}$$

$$\text{s.t.}\begin{cases}\displaystyle\sum_{(i',i'')\in\Omega^k}\sum_{j=1}^{n}\left(CI(p_{i'j}^k)-CI(p_{i''j}^k)\right)\cdot w_j\geq h\\ (k=1,2,\cdots,K),\\ \displaystyle\sum_{j=1}^{n}\left(CI(p_{i'j}^k)-CI(p_{i''j}^k)\right)\cdot w_j+Z_{i'i''}^k\geq 0 \text{ and}\\ Z_{i'i''}^k\geq 0((i',i'')\in\Omega^k \text{ and } k=1,2,\cdots,K),\\ \displaystyle\sum_{j=1}^{n} w_j=1,\ w_j\geq\varepsilon(j=1,2,\cdots,n).\end{cases}\tag{27}$$

The optimal weight $\bar{w}_j$ of each criterion $c_j \in C$ and the optimal individual degree of violation $\bar{Z}_{i'i''}^k$ for each ordered pair $(i', i'') \in \Omega^k$ provided by the decision maker e_k can be obtained by solving the parametric PF LINMAP model using the Simplex method. Based on the optimal weight vector $\bar{w} = (\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_n)$, the optimal collective comprehensive closeness measure for each alternative $a_i \in A$ is determined as follows:

$$\sum_{k=1}^{K}\overline{CM}_i^k=\sum_{k=1}^{K}\sum_{j=1}^{n}\bar{w}_j\cdot CI(p_{ij}^k).\tag{28}$$

Finally, the ultimate priority ranking orders of the m alternatives can be obtained according to the decreasing order of the $\sum_{k=1}^{K}\overline{CM}_i^k$ values. The best compromise solution is ranked the best by the $\sum_{k=1}^{K}\overline{CM}_i^k$ values among all $a_i \in A$.

In a nutshell, the general framework and relevant core concepts of the proposed methodology are depicted in Figure 2. There are four phases in the parametric PF LINMAP methodology, consisting of ascertainment of PF closeness-based order relations, resolution of objective order relations, construction of the PF LINMAP models, and the final ranking phase.

The procedural steps of the proposed parametric PF LINMAP methodology for addressing a multiple criteria GDM problem within the PF uncertain environment can be summarized as the following algorithm:

Step 1 Problem Formulation: Construct a GDM problem with the set of candidate alternatives $A = \{a_1, a_2, \cdots, a_m\}$, the set of evaluative criteria $C = \{c_1, c_2, \cdots, c_n\}$, and the set of decision makers $E = \{e_1, e_2, \cdots, e_K\}$. The set C is

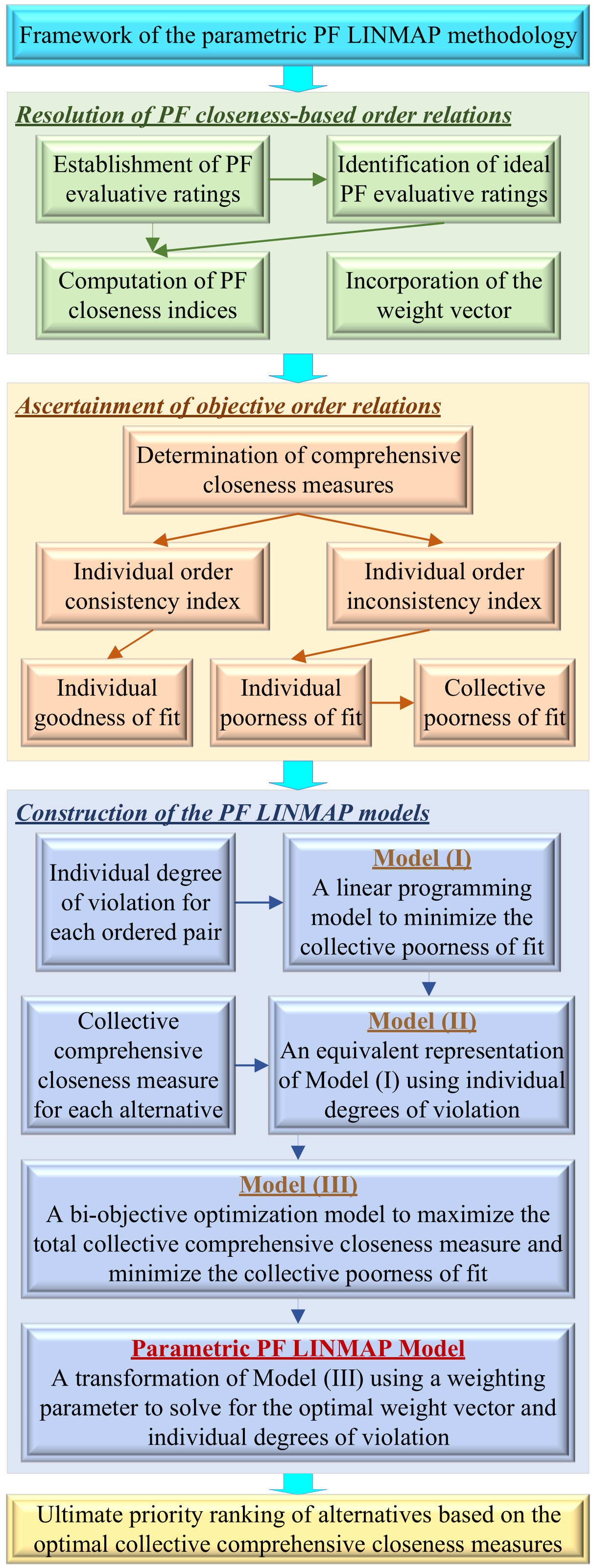


FIGURE 2. **The framework of the parametric PF LINMAP methodology.**

divided into C_{I} (set of benefit criteria) and C_{II} (set of cost criteria).

Step 2 Preference Judgment Over Alternatives: Inquire each decision maker e_k to express the subjective preference relations between alternatives. After that, specify the preference set $\Omega^k = \{(i', i'')|a_{i'} \succsim a_{i''}, i', i'' \in \{1, 2, \cdots, m\}\}$.

Step 3 Rating by Pythagorean Membership Grades: Build a PF decision matrix P^k, which is composed of the PF evaluative rating p_{ij}^k of each alternative $a_i \in A$ with respect to criterion $c_j \in C$ for the decision maker $e_k \in E$.

Step 4 Establishment of Ideal Ratings: Employ (12) and (13) to identify the positive-ideal PF evaluative rating p_{*j}^k and the negative-ideal PF evaluative rating $p_{\#j}^k$, respectively, with respect to $c_j \in C_{\mathrm{I}}$. Identify p_{*j}^k and $p_{\#j}^k$ in terms of $c_j \in C_{\mathrm{II}}$ using (14) and (15), respectively.

Step 5 Computation of PF Closeness Indices: Apply (16) to calculate the PF closeness index $CI(p_{ij}^k)$ for each PF evaluative rating p_{ij}^k in the PF decision matrix P^k.

Step 6 Setting of Parameter Values: Designate the lowest acceptable level h, the non-zero boundary condition ε, and the weighting parameter η, where $h \geq 0$, $0 < \varepsilon \leq 1/n$, and $0 \leq \eta \leq 1$.

Step 7 Construction of the PF LINMAP Model: Denote the weight vector $\mathrm{w} = (w_1, w_2, \cdots, w_n)$ such that $\sum_{j=1}^n w_j = 1$ and $w_j \geq \varepsilon$ for all j. Denote the individual degree of violation $Z_{i'i''}^k$ of the ordered pair $(i', i'') \in \Omega^k$. Establish the parametric PF LINMAP model using (27).

Step 8 Ranking of Alternatives: Solve for the optimal weight $\bar{w}_j$ and the optimal $\bar{Z}_{i'i''}^k$. Employ (28) to derive the optimal collective comprehensive closeness measure $\sum_{k=1}^K \overline{CM}_i^k$ of each a_i for acquiring the ultimate priority ranking of alternatives and the best compromise solution.

An essential issue on how the PF data are obtained should be further addressed in Step 3. Because decision makers often express subjective assessments or judgments by means of linguistic terms in practice, this paper suggests an approach via an appropriate linguistic rating system to effectively estimate the PF evaluative ratings. Table 1 presents some useful linguistic rating scales for evaluating candidate alternatives, consisting of commonly used five-point, seven-point, and nine-point scales. By applying these linguistic variables, decision makers can provide the performance evaluations of alternatives with respect to criteria in a simple and direct manner. Based on the linguistic scales in Table 1, decision makers' linguistic evaluations can be easily transformed into suitable Pythagorean membership grades for the sake of forming the PF decision matrix in Step 3. By means of the linguistic rating system in this table, decision makers can describe their opinions about the ratings of the alternatives with respect to each criterion more conveniently.

VI. APPLICATION AND COMPARISON ANALYSIS

This section applies the proposed methodology to investigate a real-world GDM problem concerning railway project investment decision making for validating the practicability and usefulness of the parametric PF LINMAP model. Furthermore, this section conducts four comparative analyses and discussions to demonstrate the advantages of the developed techniques in solving complicated GDM problems within PF uncertain environments.

TABLE 1. PF linguistic rating scales.

Linguistic term	Pythagorean membership grade	Source
Five-point linguistic rating scale		
Very high	(0.90, 0.10; 0.9055, 0.9296)	Mete [37]
High	(0.75, 0.40; 0.8500, 0.6881)	Pérez-Domínguez
Medium	(0.50, 0.45; 0.6727, 0.5335)	et al. [38]
Low	(0.35, 0.60; 0.6946, 0.3362)	
Very low	(0.10, 0.90; 0.9055, 0.0704)	
Seven-point linguistic rating scale		
Very high	(0.85, 0.15; 0.8631, 0.8888)	Chen [33]
High	(0.75, 0.25; 0.7906, 0.7952)	Cui et al. [39]
Moderately high	(0.65, 0.35; 0.7382, 0.6855)	Gul et al. [40]
Medium	(0.50, 0.45; 0.6727, 0.5335)	Yazdi et al. [41]
Moderately low	(0.35, 0.65; 0.7382, 0.3145)	
Low	(0.25, 0.75; 0.7906, 0.2048)	
Very low	(0.15, 0.85; 0.8631, 0.1112)	
Nine-point linguistic rating scale		
Absolutely high	(1.00, 0.10; 1.0050, 0.9365)	Mete [37]
Very high	(0.80, 0.44; 0.9130, 0.6799)	Pérez-Domínguez
High	(0.70, 0.60; 0.9220, 0.5489)	et al. [38]
Moderately high	(0.60, 0.71; 0.9296, 0.4467)	Oz et al. [26]
Medium	(0.50, 0.80; 0.9434, 0.3556)	
Moderately low	(0.40, 0.87; 0.9575, 0.2744)	
Low	(0.25, 0.92; 0.9534, 0.1689)	
Very low	(0.10, 0.97; 0.9751, 0.0654)	
Absolutely low	(0.10, 0.99; 0.9950, 0.0641)	

A. PRACTICAL APPLICATION

The GDM problem of railway project investment comes from Xue *et al.* [31]. To examine the feasibility of the PF LINMAP method based on PF entropy measures, Xue *et al.* [31] address a practical GDM problem about the railway project selection in China's "One Belt One Road" strategy that a global development strategy launched by the Chinese government. To enhance regional connectivity, the initial focus of the One Belt One Road is to improve the physical infrastructure along land corridors that roughly equate to the old silk road. In particular, the railway investment is an essential part of the infrastructure investment in this initiative. In the practical example provided by Xue *et al.* [31], four countries consisting of Germany, Russia, Singapore, and Malaysia were selected for further evaluation, because of their high cooperation intentions with China in the railway field. Three experts employed an indicator system of railway project selection from the perspectives of financial evaluation and noneconomic evaluation to provide each alternative's ratings in terms of criteria. Moreover, there are six evaluative criteria in the indicator system.

In Step 1, the set of candidate alternatives $A = \{a_1$ (Germany), a_2 (Russia), a_3 (Singapore), a_4 (Malaysia)$\}$. The set of evaluative criteria $C = \{c_1$ (financial internal rate of return), c_2 (net present value), c_3 (investment recovery period), c_4 (debt ratio and current ratio), c_5 (repayment period of loan), c_6 (public benefit and diplomatic influence)}, where $C_{\mathrm{I}} = C$ and $C_{\mathrm{II}} = \emptyset$. The set of decision makers $E = \{e_1, e_2, e_3\}$. Here, $m = 4$, $n = 6$, and $K = 3$.

In Step 2, based on the surveyed data in Xue *et al.* [31], the three experts expressed their subjective preference judgments between alternatives. The three preference sets were obtained as follows: $\Omega^1 = \{(3, 2), (4, 1), (3, 1)\}$ for e_1, $\Omega^2 = \{(2, 1), (4, 3), (2, 4), (3, 1)\}$ for e_2, and $\Omega^3 = \{(3, 1), (3, 4), (2, 3), (2, 4)\}$ for e_3. It can be observed that incomplete and inconsistent information exists in the three preference sets. Theoretically, there are at most 6 (i.e., (4×3)/2) paired preference comparisons over the four alternatives in the three preference sets. Nonetheless, the numbers of preference relations are 3, 4, and 4 in Ω^1, Ω^2, and Ω^3, respectively, which demonstrates incomplete information. Moreover, some preference information is inconsistent, such as (3,2) (i.e., $a_3 \succsim a_2$) in Ω^1 vs. (2,3) (i.e., $a_2 \succsim a_3$) in Ω^3 and (4,3) (i.e., $a_4 \succsim a_3$) in Ω^2 vs. (3,4) (i.e., $a_3 \succsim a_4$) in Ω^3.

In Step 3, the assessment information given by the three experts is expressed as the degrees of satisfaction and dissatisfaction of an alternative $a_i \in A$ with respect to a criterion $c_j \in C$. Such an assessment approach implies that all elements in C belong to benefit criteria. Based on the representation of Pythagorean membership grades, the PF evaluative rating p_{ij}^k in the PF decision matrices P^1 $(= [p_{ij}^1]_{4\times 6})$, P^2 $(= [p_{ij}^2]_{4\times 6})$, and P^3 $(= [p_{ij}^3]_{4\times 6})$ were indicated in Table 2.

In Step 4, because the six criteria belong to C_{I}, this paper employed (12) and (13) to identify the positive- and negative-ideal PF evaluative ratings, respectively, with respect to each criterion. The determination results of p_{*j}^k and $p_{\#j}^k$ are revealed in Table 3.

In Step 5, this paper employed (16) to derive the PF closeness index $CI(p_{ij}^k)$ for each PF evaluative rating p_{ij}^k. The computation results are indicated in Table 4.

In Step 6, this paper designated the lowest acceptable level $h = 0.4$, the non-zero boundary condition $\varepsilon = 0.025$, and the weighting parameter $\eta = 0.2$. In particular, the weight of the "total collective comprehensive closeness measure" objective was 0.2, while the weight of the "collective poorness of fit" objective was 0.8.

In Step 7, let $\mathrm{w} = (w_1, w_2, \cdots, w_6)$ denote the weight vector of criteria such that $\sum_{j=1}^{6} w_j = 1$ and $w_j \geq \varepsilon = 0.025$ for each $c_j \in C$. The individual degrees of violation were denoted as Z_{32}^1, Z_{41}^1, and Z_{31}^1 based on the preference set Ω^1, Z_{21}^2, Z_{43}^2, Z_{24}^2, and Z_{31}^2 based on Ω^2, and Z_{31}^3, Z_{34}^3, $Z_{23}^{\mathrm{I},3}$, and $Z_{24}^{\mathrm{I},3}$ based on Ω^3. Applying (27), the parametric PF LINMAP model was constructed as follows:

$$\max \Big\{0.2 \cdot (6.4773w_1 + 5.8649w_2 + 6.3193w_3 + 7.8502w_4 + 6.5943w_5 + 6.1464w_6) - 0.8 \cdot \Big(Z_{32}^1 + Z_{41}^1 + Z_{31}^1 + Z_{21}^2 + Z_{43}^2 + Z_{24}^2 + Z_{31}^2 + Z_{31}^3 + Z_{34}^3 + Z_{23}^3 + Z_{24}^3\Big)\Big\}$$

subject to:

$$0.6957w_1 - 0.1616w_2 + 1.4282w_3 + 1.5187w_4 - 1.4811w_5 + 0.4109w_6 \geq 0.4,$$

TABLE 2. PF evaluative ratings in the PF decision matrices.

c_j	a_i	p_{ij}^1 in P^1	a_i	p_{ij}^1 in P^1
c_1	a_1	(0.70, 0.60; 0.9220, 0.5489)	a_3	(0.70, 0.50; 0.8602, 0.6051)
c_2		(0.80, 0.60; 1.0000, 0.5903)		(0.40, 0.30; 0.5000, 0.5903)
c_3		(0.50, 0.50; 0.7071, 0.5000)		(0.90, 0.10; 0.9055, 0.9296)
c_4		(0.40, 0.70; 0.8062, 0.3305)		(0.80, 0.20; 0.8246, 0.8440)
c_5		(0.90, 0.40; 0.9849, 0.7338)		(0.70, 0.40; 0.8062, 0.6695)
c_6		(0.40, 0.90; 0.9849, 0.2662)		(0.60, 0.60; 0.8485, 0.5000)
c_1	a_2	(0.90, 0.40; 0.9849, 0.7338)	a_4	(0.90, 0.30; 0.9487, 0.7952)
c_2		(0.80, 0.60; 1.0000, 0.5903)		(0.70, 0.20; 0.7280, 0.8228)
c_3		(0.70, 0.70; 0.9899, 0.5000)		(0.40, 0.30; 0.5000, 0.5903)
c_4		(0.90, 0.30; 0.9487, 0.7952)		(0.90, 0.40; 0.9849, 0.7338)
c_5		(0.80, 0.20; 0.8246, 0.8440)		(0.50, 0.40; 0.6403, 0.5704)
c_6		(0.90, 0.40; 0.9849, 0.7338)		(0.60, 0.70; 0.9220, 0.4511)
		p_{ij}^2 in P^2		p_{ij}^2 in P^2
c_1	a_1	(0.50, 0.40; 0.6403, 0.5704)	a_3	(0.70, 0.60; 0.9220, 0.5489)
c_2		(0.30, 0.90; 0.9487, 0.2048)		(0.50, 0.30; 0.5831, 0.6560)
c_3		(0.40, 0.30; 0.5000, 0.5903)		(0.80, 0.10; 0.8062, 0.9208)
c_4		(0.90, 0.40; 0.9849, 0.7338)		(0.90, 0.10; 0.9055, 0.9296)
c_5		(0.30, 0.70; 0.7616, 0.2578)		(0.70, 0.30; 0.7616, 0.7422)
c_6		(0.40, 0.50; 0.6403, 0.4296)		(0.50, 0.50; 0.7071, 0.5000)
c_1	a_2	(0.90, 0.30; 0.9487, 0.7952)	a_4	(0.30, 0.90; 0.9487, 0.2048)
c_2		(0.80, 0.50; 0.9434, 0.6444)		(0.20, 0.30; 0.3606, 0.3743)
c_3		(0.70, 0.60; 0.9220, 0.5489)		(0.30, 0.70; 0.7616, 0.2578)
c_4		(0.90, 0.20; 0.9220, 0.8608)		(0.50, 0.80; 0.9434, 0.3556)
c_5		(0.90, 0.20; 0.9220, 0.8608)		(0.70, 0.60; 0.9220, 0.5489)
c_6		(0.80, 0.30; 0.8544, 0.7716)		(0.40, 0.70; 0.8062, 0.3305)
		p_{ij}^3 in P^3		p_{ij}^3 in P^3
c_1	a_1	(0.40, 0.30; 0.5000, 0.5903)	a_3	(0.20, 0.50; 0.5385, 0.2422)
c_2		(0.30, 0.40; 0.5000, 0.4097)		(0.30, 0.40; 0.5000, 0.4097)
c_3		(0.50, 0.40; 0.6403, 0.5704)		(0.80, 0.60; 1.0000, 0.5903)
c_4		(0.60, 0.60; 0.8485, 0.5000)		(0.90, 0.10; 0.9055, 0.9296)
c_5		(0.70, 0.70; 0.9899, 0.5000)		(0.30, 0.10; 0.3162, 0.7952)
c_6		(0.30, 0.70; 0.7616, 0.2578)		(0.60, 0.40; 0.7211, 0.6257)
c_1	a_2	(0.80, 0.30; 0.8544, 0.7716)	a_4	(0.60, 0.80; 1.0000, 0.4097)
c_2		(0.70, 0.10; 0.7071, 0.9097)		(0.40, 0.10; 0.4123, 0.8440)
c_3		(0.80, 0.20; 0.8246, 0.8440)		(0.20, 0.40; 0.4472, 0.2952)
c_4		(0.90, 0.10; 0.9055, 0.9296)		(0.70, 0.10; 0.7071, 0.9097)
c_5		(0.80, 0.30; 0.8544, 0.7716)		(0.60, 0.20; 0.6325, 0.7952)
c_6		(0.80, 0.10; 0.8062, 0.9208)		(0.80, 0.10; 0.8062, 0.9208)

TABLE 3. Positive- and negative-ideal PF evaluative ratings.

c_j	e_k	p_{*j}^k	e_k	$p_{\#j}^k$
c_1	e_1	(0.90, 0.30; 0.9487, 0.7952)	e_1	(0.70, 0.60; 0.9220, 0.5489)
c_2		(0.80, 0.20; 0.8246, 0.8440)		(0.40, 0.60; 0.7211, 0.3743)
c_3		(0.90, 0.10; 0.9055, 0.9296)		(0.40, 0.70; 0.8062, 0.3305)
c_4		(0.90, 0.20; 0.9220, 0.8608)		(0.40, 0.70; 0.8062, 0.3305)
c_5		(0.90, 0.20; 0.9220, 0.8608)		(0.50, 0.40; 0.6403, 0.5704)
c_6		(0.90, 0.40; 0.9849, 0.7338)		(0.40, 0.90; 0.9849, 0.2662)
c_1	e_2	(0.90, 0.30; 0.9487, 0.7952)	e_2	(0.30, 0.90; 0.9487, 0.2048)
c_2		(0.80, 0.30; 0.8544, 0.7716)		(0.20, 0.90; 0.9220, 0.1392)
c_3		(0.80, 0.10; 0.8062, 0.9208)		(0.30, 0.70; 0.7616, 0.2578)
c_4		(0.90, 0.10; 0.9055, 0.9296)		(0.50, 0.80; 0.9434, 0.3556)
c_5		(0.90, 0.20; 0.9220, 0.8608)		(0.30, 0.70; 0.7616, 0.2578)
c_6		(0.80, 0.30; 0.8544, 0.7716)		(0.40, 0.70; 0.8062, 0.3305)
c_1	e_3	(0.80, 0.30; 0.8544, 0.7716)	e_3	(0.20, 0.80; 0.8246, 0.1560)
c_2		(0.70, 0.10; 0.7071, 0.9097)		(0.30, 0.40; 0.5000, 0.4097)
c_3		(0.80, 0.20; 0.8246, 0.8440)		(0.20, 0.60; 0.6325, 0.2048)
c_4		(0.90, 0.10; 0.9055, 0.9296)		(0.60, 0.60; 0.8485, 0.5000)
c_5		(0.80, 0.10; 0.8062, 0.9208)		(0.30, 0.70; 0.7616, 0.2578)
c_6		(0.80, 0.10; 0.8062, 0.9208)		(0.30, 0.70; 0.7616, 0.2578)

TABLE 4. Results of PF closeness indices.

e_k	c_j	$CI(p^k_{1j})$	$CI(p^k_{2j})$	$CI(p^k_{3j})$	$CI(p^k_{4j})$
e_1	c_1	0.0000	0.8037	0.2497	1.0000
	c_2	0.5682	0.5682	0.3838	0.7753
	c_3	0.2954	0.3738	1.0000	0.3927
	c_4	0.0000	0.9142	0.8051	0.8227
	c_5	0.7776	0.7470	0.4106	0.0000
	c_6	0.0000	1.0000	0.5000	0.4109
e_2	c_1	0.5533	1.0000	0.5840	0.0000
	c_2	0.0762	0.7922	0.6860	0.5119
	c_3	0.4674	0.5044	1.0000	0.0000
	c_4	0.7513	0.9298	1.0000	0.0000
	c_5	0.0000	1.0000	0.6287	0.5341
	c_6	0.3078	1.0000	0.3818	0.0000
e_3	c_1	0.5735	1.0000	0.3337	0.3794
	c_2	0.0000	1.0000	0.0000	0.5030
	c_3	0.4271	1.0000	0.6395	0.2190
	c_4	0.0000	1.0000	1.0000	0.6271
	c_5	0.4301	0.8392	0.5499	0.6770
	c_6	0.0000	1.0000	0.5459	1.0000

$$
\begin{aligned}
&0.8934w_1+1.4320w_2+0.0740w_3+0.3569w_4+2.0000w_5\\
&\quad+1.3843w_6\geq 0.4,\\
&1.0015w_1+0.9940w_2+1.7744w_3+1.7458w_4+0.4441w_5\\
&\quad+0.5459w_6\geq 0.4,\\
&-0.5540w_1-0.1844w_2+0.6262w_3-0.1091w_4-0.3365w_5\\
&\quad-0.5000w_6+Z^1_{32}\geq 0,\\
&1.0000w_1+0.2071w_2+0.0973w_3+0.8227w_4-0.7776w_5\\
&\quad+0.4109w_6+Z^1_{41}\geq 0,\\
&0.2497w_1-0.1844w_2+0.7046w_3+0.8051w_4-0.3670w_5\\
&\quad+0.5000w_6+Z^1_{31}\geq 0,\\
&0.4467w_1+0.7160w_2+0.0370w_3+0.1784w_4+1.0000w_5\\
&\quad+0.6922w_6+Z^2_{21}\geq 0,\\
&-0.5840w_1-0.1741w_2-1.0000w_3-1.0000w_4\\
&\quad-0.0946w_5-0.3818w_6+Z^2_{43}\geq 0,\\
&1.0000w_1+0.2803w_2+0.5044w_3+0.9298w_4+0.4659w_5\\
&\quad+1.0000w_6+Z^2_{24}\geq 0,\\
&0.0307w_1+0.6098w_2+0.5326w_3+0.2487w_4+0.6287w_5\\
&\quad+0.0740w_6+Z^2_{31}\geq 0,\\
&-0.2397w_1+0.0000w_2+0.2124w_3+1.0000w_4\\
&\quad+0.1198w_5+0.5459w_6+Z^3_{31}\geq 0,\\
&-0.0456w_1-0.5030w_2+0.4205w_3+0.3729w_4-0.1271w_5\\
&\quad-0.4541w_6+Z^3_{34}\geq 0,\\
&0.6663w_1+1.0000w_2+0.3605w_3+0.0000w_4+0.2893w_5\\
&\quad+0.4541w_6+Z^3_{23}\geq 0,\\
&0.6206w_1+0.4970w_1+0.7810w_3+0.3729w_4+0.1622w_5\\
&\quad+0.0000w_6+Z^3_{24}\geq 0,\\
&Z^1_{32},Z^1_{41},Z^1_{31},Z^2_{21},Z^2_{43},Z^2_{24},Z^2_{31},Z^3_{31},Z^3_{34},Z^3_{23},Z^3_{24}\geq 0,\\
&\sum_{j=1}^{6}w_j=1,\quad w_j\geq 0.025 \text{ for } j=1,2,\cdots,6.
\end{aligned}
$$

In Step 8, this paper solved the above model and acquired the optimal weight vector $\bar{w} = (\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_6) = (0.0250, 0.3019, 0.2633, 0.2050, 0.1798, 0.0250)$ and the optimal individual degrees of violation $\bar{Z}^2_{43} = 0.5620$ and $\bar{Z}^1_{32} = \bar{Z}^1_{41} = \bar{Z}^1_{31} = \bar{Z}^2_{21} = \bar{Z}^2_{24} = \bar{Z}^2_{31} = \bar{Z}^3_{31} = \bar{Z}^3_{34} = \bar{Z}^3_{23} = \bar{Z}^3_{24} = 0$. Apply (28), the optimal collective comprehensive closeness measures were determined as follows: $\sum_{k=1}^{3}\overline{CM}^k_1 = 0.9149$, $\sum_{k=1}^{3}\overline{CM}^k_2 = 2.4003$, $\sum_{k=1}^{3}\overline{CM}^k_3 = 1.9437$, and $\sum_{k=1}^{3}\overline{CM}^k_4 = 1.2862$. The ultimate priority ranking of the four candidate alternatives was $a_2 \succ a_3 \succ a_4 \succ a_1$. Moreover, the best compromise solution was a_2. The obtained results are in conformity with those yielded by Xue *et al.*'s developed approach [31].

B. COMPARISON ANALYSIS

This subsection attempts to conducts four comparative studies and discussions to examine the usefulness and advantages of the proposed PF LINMAP methodology.

1) FIRST COMPARATIVE STUDY

The first comparative analysis focuses on the influences of distinct points of reference on the solution results yielded by the parametric PF LINMAP model. It is known that (1,0;1,1) and (0,1;1,0) are the largest and smallest Pythagorean membership grades, respectively. In this regard, this paper considers (1,0;1,1) and (0,1;1,0) as the benchmark points of reference and redefines the PF closeness index with respect to the two benchmark points.

For a PF evaluative rating p^k_{ij} in the PF decision matrix P^k, the benchmark-based PF closeness index $CI^0(p^k_{ij})$ is defined as follows:

$$
CI^0(p^k_{ij}) = \frac{D(p^k_{ij},(0,1;1,0))}{D(p^k_{ij},(1,0;1,1)) + D(p^k_{ij},(0,1;1,0))}. \tag{29}
$$

The parametric PF LINMAP model based on the benchmark-based PF closeness indices can be established as follows:

Benchmark−based PF LINMAP Model

$$
\max\left\{\eta\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n}CI^0(p^k_{ij})w_j-(1-\eta)\sum_{k=1}^{K}\sum_{(i',i'')\in\Omega^k}Z^k_{i'i''}\right\}
$$
$$
\text{s.t.}\begin{cases}\displaystyle\sum_{(i',i'')\in\Omega^k}\sum_{j=1}^{n}\left(CI^0(p^k_{i'j})-CI^0(p^k_{i''j})\right)\cdot w_j\geq h\\ (k=1,2,\cdots,K),\\ \displaystyle\sum_{j=1}^{n}\left(CI^0(p^k_{i'j})-CI^0(p^k_{i''j})\right)\cdot w_j+Z^k_{i'i''}\geq 0 \text{ and}\\ Z^k_{i'i''}\geq 0((i',i'')\in\Omega^k \text{ and } k=1,2,\cdots,K),\\ \displaystyle\sum_{j=1}^{n}w_j=1,\ w_j\geq\varepsilon(j=1,2,\cdots,n).\end{cases}
$$
(30)

Based on the optimal weight vector $\bar{w}$ and the $CI^0(p_{ij}^k)$ values, the benchmark-based collective comprehensive closeness measure $\sum_{k=1}^{K}\overline{CM}_i^{0,k}$ of each alternative $a_i \in A$ is determined as follows:

$$\sum_{k=1}^{K}\overline{CM}_i^{0,k} = \sum_{k=1}^{K}\sum_{j=1}^{n}\bar{w}_j \cdot CI^0(p_{ij}^k). \tag{31}$$

Consider the same GDM problem of railway project investment under the parameter settings of $h = 0.4$, $\varepsilon = 0.025$, and $\eta = 0.2$. Applying (30), the following linear programming model was constructed:

$$\max \Big\{0.2 \cdot (6.8893w_1 + 6.6206w_2 + 6.9587w_3 + 8.5161w_4 + 7.5751w_5 + 6.4875w_6) - 0.8 \cdot \Big(Z_{32}^1 + Z_{41}^1 + Z_{31}^1 + Z_{21}^2 + Z_{43}^2 + Z_{24}^2 + Z_{31}^2 + Z_{31}^3 + Z_{34}^3 + Z_{23}^3 + Z_{24}^3\Big)\Big\}$$

subject to:

$$0.1161w_1 - 0.1029w_2 + 0.7675w_3 + 0.7603w_4 - 0.5286w_5 + 0.2346w_6 \geq 0.4,$$
$$0.5577w_1 + 0.9664w_2 + 0.0371w_3 + 0.1077w_4 + 1.0198w_5 + 0.5409w_6 \geq 0.4,$$
$$0.5696w_1 + 0.2328w_2 + 0.7450w_3 + 0.6530w_4 + 0.2494w_5 + 0.2495w_6 \geq 0.4,$$
$$-0.1908w_1 - 0.0878w_2 + 0.3659w_3 - 0.0579w_4 - 0.1244w_5 - 0.2888w_6 + Z_{32}^1 \geq 0,$$
$$0.2630w_1 + 0.0727w_2 + 0.0356w_3 + 0.4238w_4 - 0.2504w_5 + 0.2346w_6 + Z_{41}^1 \geq 0,$$
$$0.0438w_1 - 0.0878w_2 + 0.3659w_3 + 0.3944w_4 - 0.1538w_5 + 0.2888w_6 + Z_{31}^1 \geq 0,$$
$$0.2789w_1 + 0.4832w_2 + 0.0186w_3 + 0.0539w_4 + 0.5099w_5 + 0.2705w_6 + Z_{21}^2 \geq 0,$$
$$-0.3714w_1 - 0.1102w_2 - 0.4513w_3 - 0.5319w_4 - 0.1130w_5 - 0.1350w_6 + Z_{43}^2 \geq 0,$$
$$0.6345w_1 + 0.2030w_2 + 0.2215w_3 + 0.5086w_4 + 0.2885w_5 + 0.3671w_6 + Z_{24}^2 \geq 0,$$
$$0.0158w_1 + 0.3904w_2 + 0.2484w_3 + 0.0771w_4 + 0.3345w_5 + 0.0384w_6 + Z_{31}^2 \geq 0,$$
$$-0.1415w_1 + 0.0000w_2 + 0.0851w_3 + 0.3659w_4 + 0.0782w_5 + 0.2495w_6 + Z_{31}^3 \geq 0,$$
$$0.0176w_1 - 0.1416w_2 + 0.1941w_3 + 0.1435w_4 - 0.0683w_5 - 0.2018w_6 + Z_{34}^3 \geq 0,$$
$$0.3379w_1 + 0.2580w_2 + 0.1359w_3 + 0.0000w_4 + 0.1539w_5 + 0.2018w_6 + Z_{23}^3 \geq 0,$$
$$0.3555w_1 + 0.1164w_1 + 0.3300w_3 + 0.1435w_4 + 0.0856w_5 + 0.0000w_6 + Z_{24}^3 \geq 0,$$
$$Z_{32}^1, Z_{41}^1, Z_{31}^1, Z_{21}^2, Z_{43}^2, Z_{24}^2, Z_{31}^2, Z_{31}^3, Z_{34}^3, Z_{23}^3, Z_{24}^3 \geq 0,$$
$$\sum_{j=1}^{6} w_j = 1, \quad w_j \geq 0.025 \text{ for } j = 1, 2, \cdots, 6.$$

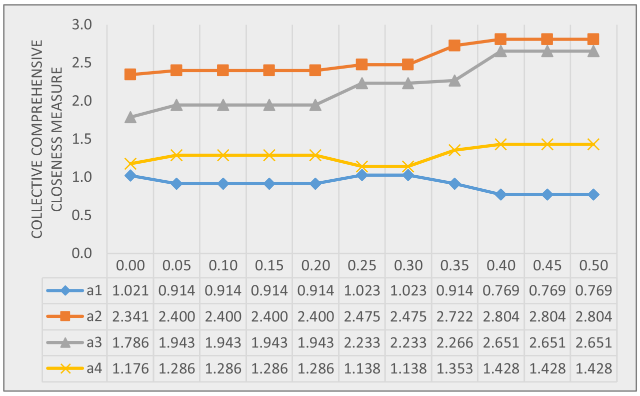
Solving the above benchmark-based PF LINMAP model, the following results were acquired: the optimal weight vector $\bar{w} = (\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_6) = (0.0250, 0.1957, 0.1918, 0.4357, 0.1268, 0.0250)$, the optimal individual degrees of violation $\bar{Z}_{43}^2 = 0.3668$ and $\bar{Z}_{32}^1 = \bar{Z}_{41}^1 = \bar{Z}_{31}^1 = \bar{Z}_{21}^2 = \bar{Z}_{24}^2 = \bar{Z}_{31}^2 = \bar{Z}_{31}^3 = \bar{Z}_{34}^3 = \bar{Z}_{23}^3 = \bar{Z}_{24}^3 = 0$, and the optimal collective comprehensive closeness measures $\sum_{k=1}^{3}\overline{CM}_1^{0,k} = 1.5426$, $\sum_{k=1}^{3}\overline{CM}_2^{0,k} = 2.2541$, $\sum_{k=1}^{3}\overline{CM}_3^{0,k} = 2.1459$, and $\sum_{k=1}^{3}\overline{CM}_4^{0,k} = 1.6931$. The ultimate priority ranking was $a_2 \succ a_3 \succ a_4 \succ a_1$, and the best compromise solution was a_2. It can be observed that the ranking results yielded by the proposed methodology and the benchmark-based version are identical. Additionally, these results are consistent with those rendered by Xue *et al.*'s PF LINMAP method [31] based on PF entropy measures.

2) SECOND COMPARATIVE STUDY

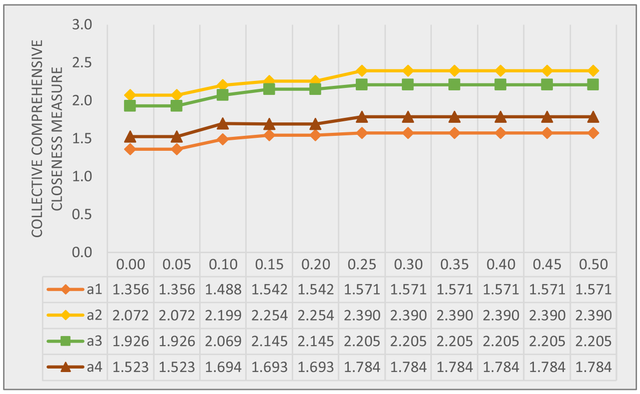
The second comparative analysis focuses on a comprehensive study of the solution outcomes under various settings of the weighting parameter η.

As mentioned before, the parameter η combines the two objectives in Model (III) for enhance computation efficiency; it can adjust the relative proportion of the two objectives. More specifically, the objective in the parametric PF LINMAP model reduces to $\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} CI(p_{ij}^k)\cdot w_j$ and $-\sum_{k=1}^{K}\sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k$ in the cases of $\eta = 1$ and $\eta = 0$, respectively. In particular, when $\eta = 0$, the maximal objective $-\sum_{k=1}^{K}\sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k$ is equivalent to the minimal objective $\sum_{k=1}^{K}\sum_{(i',i'')\in\Omega^k} Z_{i'i''}^k$ in Model (II), which inherits the merits of LINMAP-based methods. Because the proposed methodology originates from classical LINMAP, an appropriate range [0, 0.5] is suggested for the parameter η, i.e., $0 \leq \eta \leq 0.5$, to make sure that the weight of the "collective poorness of fit" objective is larger than or equal to the weight of the "total collective comprehensive closeness measure" objective.

This paper designates the η values varying from 0 to 0.5 under the settings of $h = 0.4$ and $\varepsilon = 0.025$ in this comparative study. The proposed parametric PF LINMAP model and its benchmark-based version were utilized to solve the GDM problem concerning railway project investment in the eleven scenarios of $\eta = 0.00, 0.05, \cdots, 0.50$. The comparison results of the optimal collective comprehensive closeness measures for four candidate alternatives are revealed in Figure 3. More precisely, Figure 3(a) and Figure 3(b) demonstrate the contrasts among the obtained $\sum_{k=1}^{3}\overline{CM}_i^k$ and $\sum_{k=1}^{3}\overline{CM}_i^{0,k}$, respectively, for each $a_i \in A$. Even though

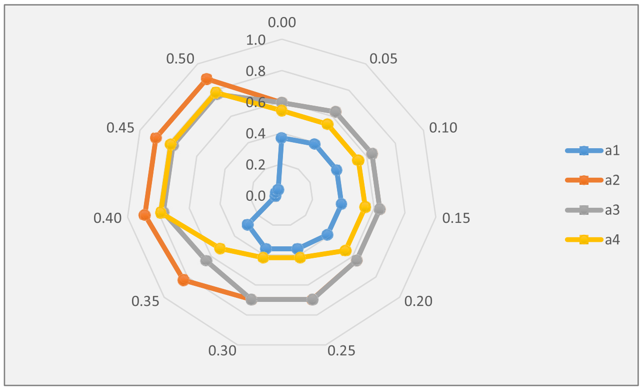


(a) **Results yielded by the parametric PF LINMAP model.**

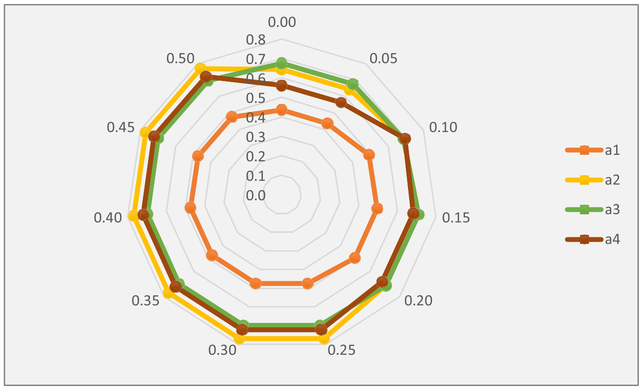


(b) **Results yielded by the benchmark-based PF LINMAP model.**

FIGURE 3. **Comparison results of the optimal collective comprehensive closeness measures under various settings of the weighting parameter.**



(a) **Contrasts yielded by the parametric PF LINMAP model.**



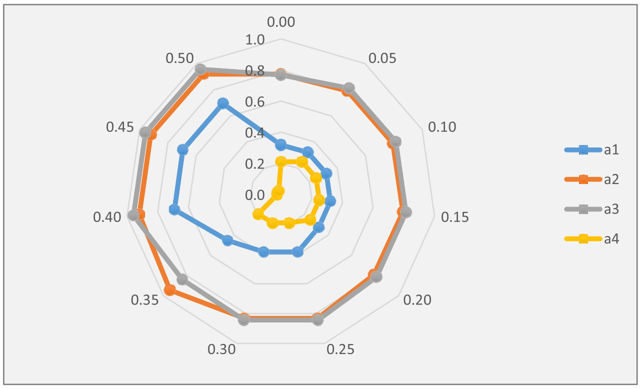
(b) **Contrasts yielded by the benchmark-based PF LINMAP model.**

FIGURE 4. **Optimal comprehensive closeness measures for the decision maker e_1 under various settings of the weighting parameter.**

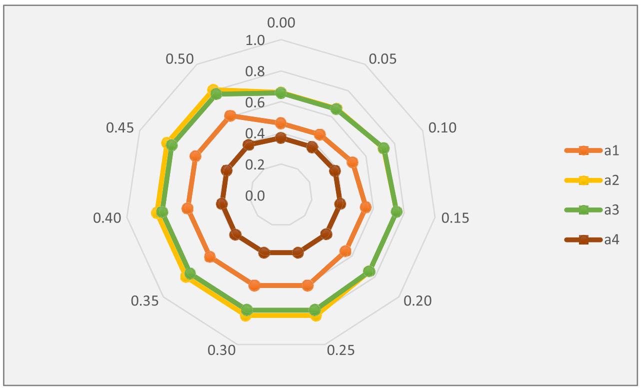
the same ultimate priority ranking of alternatives (i.e., $a_2 \succ a_3 \succ a_4 \succ a_1$) was determined using the two comparative models, the proposed methodology can provide a good way to distinguish between better choices and worse ones. As shown in Figure 3(a), the relative advantages of alternatives a_2 and a_3 are evidently higher than a_1 and a_4, especially in the cases of $\eta \geq 0.25$. However, the contrasts of the better choices (i.e., a_2 and a_3) and the worse choices (i.e., a_1 and a_4) are not significant based on Figure 3(b) regardless of the η values. As a result, the proposed methodology can effectively identify and differentiate different alternatives, which can facilitate decision aiding for GDM problems.

Furthermore, the comparison results of the optimal comprehensive closeness measures for the decision makers e_1, e_2, and e_3 are presented in Figures 4–6, respectively. Consider the contrast outcomes generated by the parametric PF LINMAP model. Based on Definition 9 and the optimal weight vector $\bar{w}$, the optimal comprehensive closeness measures for each decision maker $e_k \in E$ can be calculated as follows: $\overline{CM}_i^k = \sum_{j=1}^{6} \bar{w}_j \cdot CI(p_{ij}^k)$ for each alternative $a_i \in A$. In regard to each decision maker, the contrast patterns of $\overline{CM}_i^1$ (for e_1), $\overline{CM}_i^2$ (for e_2), and $\overline{CM}_i^3$ (for e_3) among the four alternatives are depicted in Figures 4(a)–6(a), respectively. In a similar manner, when the benchmark-based PF LINMAP model was employed, the optimal comprehensive closeness measures for each e_k can be derived as follows: $\overline{CM}_i^{0,k} = \sum_{j=1}^{6} \bar{w}_j \cdot CI^0(p_{ij}^k)$ for all a_i. The contrast patterns of $\overline{CM}_i^{0,1}$ (for e_1), $\overline{CM}_i^{0,2}$ (for e_2), and $\overline{CM}_i^{0,3}$ (for e_3) among the four alternatives are sketched in Figures 4(b)–6(b), respectively. Based on the results in these figures, the contrasts of the $\overline{CM}_i^k$ values among the four alternatives are more obvious than the $\overline{CM}_i^{0,k}$ values regardless of the decision makers e_1, e_2, and e_3. Moreover, the patterns of $\overline{CM}_i^{0,1}$, $\overline{CM}_i^{0,2}$, and $\overline{CM}_i^{0,3}$ are moderately steady under distinct settings of the weighting parameter η. In contrast, the patterns of $\overline{CM}_i^1$, $\overline{CM}_i^2$, and $\overline{CM}_i^3$ can appropriately reflect the influences of various η values on the optimal comprehensive closeness measures. The findings have also shown flexibility in adapting to the changeable proportion of the two objectives in the parametric PF LINMAP model. Moreover, the proposed methodology possesses greater capability than the comparative benchmark-based model in differentiating relatively better candidate alternatives and worse ones.

Consider the comparison results of the optimal comprehensive closeness measures for the decision maker e_1 in Figure 4. Based on Figure 4(a), the parametric PF LINMAP model rendered the priority rankings of the four alternatives as follows: $a_3 \succ a_2 \succ a_4 \succ a_1$, $a_2 \succ a_3 \succ a_4 \succ a_1$, and $a_2 \succ a_4 \succ a_3 \succ a_1$ in the cases of $\eta = 0.00, 0.05, \cdots, 0.30$, $\eta = 0.35$, and $\eta = 0.40, 0.45, 0.50$, respectively. In contrast, as revealed in Figure 4(b), the benchmark-based PF LINMAP model generated the following priority rankings of the alternatives: $a_3 \succ a_2 \succ a_4 \succ a_1$, $a_4 \succ a_3 \succ a_2 \succ a_1$, and $a_2 \succ a_4 \succ a_3 \succ a_1$ in the cases of $\eta = 0.00, 0.05, 0.15, 0.20$, $\eta = 0.10$, and $\eta = 0.25, 0.30, \cdots, 0.50$, respectively. It is noted that an abnormal ranking $a_4 \succ a_3 \succ a_2 \succ a_1$ was obtained via the benchmark-based model. When $\eta = 0.10$, the best compromise solution was a_4, which is unreasonable and unacceptable for the decision maker e_1 because a_4 is a worse choice in most situations.



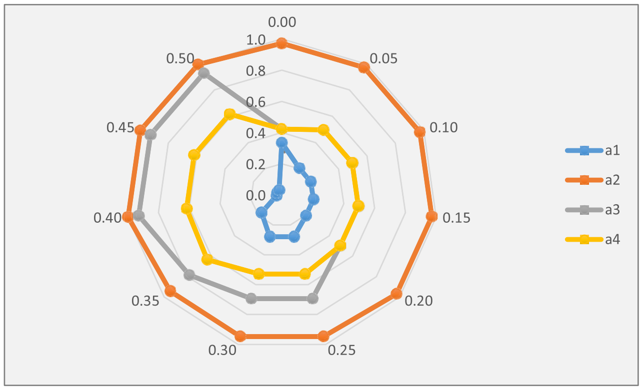
(a) Contrasts yielded by the parametric PF LINMAP model.



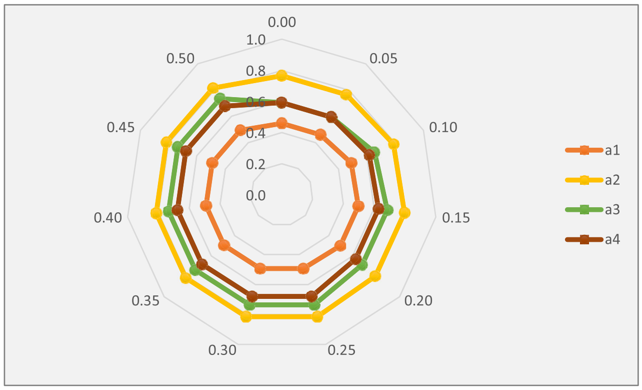
(b) Contrasts yielded by the benchmark-based PF LINMAP model.

FIGURE 5. Optimal comprehensive closeness measures for the decision maker e_2 under various settings of the weighting parameter.

Figure 5(a) demonstrates the comparison results of the optimal comprehensive closeness measures for the decision maker e_2. The parametric PF LINMAP model produced the priority ranking $a_2 \succ a_3 \succ a_1 \succ a_4$ in both cases of $\eta = 0.00$ and 0.35. Moreover, the ranking $a_3 \succ a_2 \succ a_1 \succ a_4$ was acquired in the remaining cases (i.e., $\eta = 0.05, 0.10, \cdots, 0.30, 0.40, 0.45, 0.50$). Based on the results in Figure 5(b), the benchmark-based PF LINMAP model yielded the two priority rankings $a_2 \succ a_3 \succ a_1 \succ a_4$ and $a_3 \succ a_2 \succ a_1 \succ a_4$ when $\eta = 0.00, 0.05, 0.25, 0.30, \cdots, 0.50$ and $\eta = 0.10, 0.15, 0.20$, respectively. As a whole, the two rankings of the alternatives rendered by the parametric PF LINMAP model and the benchmark-based version are the same for the decision maker e_2.



(a) Contrasts yielded by the parametric PF LINMAP model.



(b) Contrasts yielded by the benchmark-based PF LINMAP model.

FIGURE 6. Optimal comprehensive closeness measures for the decision maker e_3 under various settings of the weighting parameter.

According to the contrast results of the optimal comprehensive closeness measures for the decision maker e_3 in Figure 6 (a), the parametric PF LINMAP model rendered the two priority rankings $a_2 \succ a_4 \succ a_3 \succ a_1$ and $a_2 \succ a_3 \succ a_4 \succ a_1$ when $\eta = 0.00, 0.05, \cdots, 0.20$ and $\eta = 0.25, 0.30, \cdots, 0.50$, respectively. Based on the contrast results in Figure 6(b), the benchmark-based PF LINMAP model acquired the priority rankings $a_2 \succ a_4 \succ a_3 \succ a_1$ and $a_2 \succ a_3 \succ a_4 \succ a_1$ when $\eta = 0.00, 0.05$ and $\eta = 0.10, 0.15, \cdots, 0.50$, respectively. Obviously, the two PF LINMAP models yielded the identical ranking results of the four alternatives. Moreover, for the decision maker e_3, the best compromise solution was a_2 for all eleven cases of $\eta = 0.00, 0.05, \cdots, 0.50$.

3) THIRD COMPARATIVE STUDY

The third comparative analysis focuses on a discussion of the feasible ranges of relevant parameters. Moreover, some discussions based on the original preference data between alternatives are carried out by use of the optimal individual degrees of violation.

TABLE 5. Optimal solutions via the parametric PF LINMAP model.

η	The optimal weights of evaluative criteria					
	$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_4$	$\bar{w}_5$	$\bar{w}_6$
0.00	0.1931	0.2321	0.3548	0.0250	0.1700	0.0250
0.05	0.0250	0.3018	0.2633	0.2050	0.1798	0.0250
0.10	0.0250	0.3018	0.2633	0.2050	0.1798	0.0250
0.15	0.0250	0.3018	0.2633	0.2050	0.1798	0.0250
0.20	0.0250	0.3018	0.2633	0.2050	0.1798	0.0250
0.25	0.0250	0.0250	0.2827	0.3080	0.3343	0.0250
0.30	0.0250	0.0250	0.2827	0.3080	0.3343	0.0250
0.35	0.0250	0.0250	0.0250	0.5579	0.3421	0.0250
0.40	0.0250	0.0250	0.0250	0.8750	0.0250	0.0250
0.45	0.0250	0.0250	0.0250	0.8750	0.0250	0.0250
0.50	0.0250	0.0250	0.0250	0.8750	0.0250	0.0250

η	Feasible range	The optimal individual degrees of violation
0.00	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.5586 and the other $\bar{Z}^k_{i'i''} = 0$
0.05	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.5620 and the other $\bar{Z}^k_{i'i''} = 0$
0.10	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.5620 and the other $\bar{Z}^k_{i'i''} = 0$
0.15	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.5620 and the other $\bar{Z}^k_{i'i''} = 0$
0.20	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.5620 and the other $\bar{Z}^k_{i'i''} = 0$
0.25	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.6508 and the other $\bar{Z}^k_{i'i''} = 0$
0.30	$0 \le h < 0.9$	$\bar{Z}^2_{43}$ =0.6508 and the other $\bar{Z}^k_{i'i''} = 0$
0.35	$0 \le h < 0.9$	$\bar{Z}^1_{32}$ =0.1913, $\bar{Z}^2_{43}$ =0.6438, and the other $\bar{Z}^k_{i'i''} = 0$
0.40	$0 \le h < 0.9$	$\bar{Z}^1_{32}$ =0.1192, $\bar{Z}^2_{43}$ =0.9309, and the other $\bar{Z}^k_{i'i''} = 0$
0.45	$0 \le h < 0.9$	$\bar{Z}^1_{32}$ =0.1192, $\bar{Z}^2_{43}$ =0.9309, and the other $\bar{Z}^k_{i'i''} = 0$
0.50	$0 \le h < 0.9$	$\bar{Z}^1_{32}$ =0.1192, $\bar{Z}^2_{43}$ =0.9309, and the other $\bar{Z}^k_{i'i''} = 0$

TABLE 6. Optimal solutions via the benchmark-based PF LINMAP model.

η	The optimal weights of evaluative criteria					
	$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_4$	$\bar{w}_5$	$\bar{w}_6$
0.00	0.0250	0.1829	0.3988	0.0697	0.0250	0.2985
0.05	0.0250	0.1829	0.3988	0.0697	0.0250	0.2985
0.10	0.0250	0.3477	0.1865	0.3907	0.0250	0.0250
0.15	0.0250	0.1957	0.1918	0.4356	0.1268	0.0250
0.20	0.0250	0.1957	0.1918	0.4356	0.1268	0.0250
0.25	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250
0.30	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250
0.35	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250
0.40	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250
0.45	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250
0.50	0.0250	0.1516	0.0250	0.6179	0.1554	0.0250

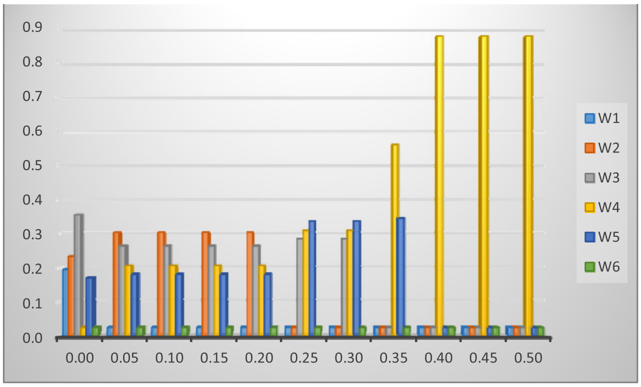
η	Feasible range	The optimal individual degrees of violation
0.00	$0 \le h < 0.5$	$\bar{Z}^2_{43}$ =0.2896 and the other $\bar{Z}^k_{i'i''} = 0$
0.05	$0 \le h < 0.5$	$\bar{Z}^2_{43}$ =0.2896 and the other $\bar{Z}^k_{i'i''} = 0$
0.10	$0 \le h < 0.5$	$\bar{Z}^2_{43}$ =0.3458 and the other $\bar{Z}^k_{i'i''} = 0$
0.15	$0 \le h < 0.5$	$\bar{Z}^2_{43}$ =0.3668 and the other $\bar{Z}^k_{i'i''} = 0$
0.20	$0 \le h < 0.5$	$\bar{Z}^2_{43}$ =0.3668 and the other $\bar{Z}^k_{i'i''} = 0$
0.25	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$
0.30	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$
0.35	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$
0.40	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$
0.45	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$
0.50	$0 \le h < 0.5$	$\bar{Z}^1_{32}$ =0.0713, $\bar{Z}^2_{43}$ =0.3869, and the other $\bar{Z}^k_{i'i''} = 0$

First, the designation of the lowest acceptable level h is essential for making sure of practical feasibility of the linear program in our proposed methodology. If the h value is too high, the parametric PF LINMAP model and the benchmark-based version would be infeasible because no solution satisfies all of the constraints. To address the issue that no feasible solution is constructed via the PF LINMAP model, this paper conducts computational experiments and investigates appropriate feasible ranges for the parameter h under various settings of the weighting parameter η. Consider the scenarios of $\varepsilon = 0.025$ and $\eta = 0.00, 0.05, \cdots, 0.50$. Table 5 presents the computational results yielded by the parametric PF LINMAP model, consisting of the optimal weight $\bar{w}_j$, the optimal individual degree of violation $\bar{Z}^k_{i'i''}$, and the feasible ranges of the h value for each η setting. The admissible filed of the lowest acceptable level is suggested as $0 \le h < 0.9$ because feasible solutions can be found in this range.

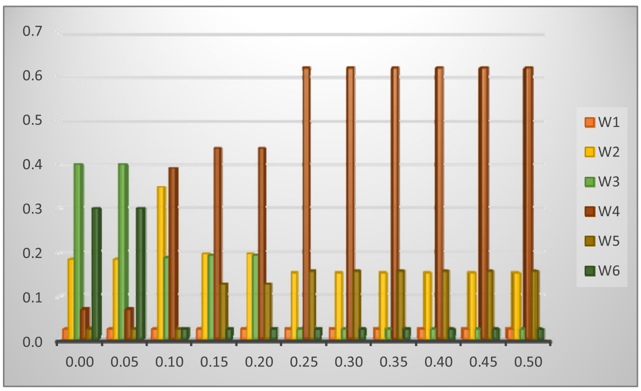
When employing the benchmark-based PF LINMAP model, the determination results of $\bar{w}_j$, $\bar{Z}^k_{i'i''}$, and the feasible ranges of h under various settings of η are summarized in Table 6. In particular, the admissible filed of the lowest acceptable level is suggested as $0 \le h < 0.5$ in the cases of $\varepsilon = 0.025$ and $\eta = 0.00, 0.05, \cdots, 0.50$. Comparing the obtained $0 \le h < 0.5$ with the feasible range $0 \le h < 0.9$ in Table 5, the parametric PF LINMAP model possesses greater feasibility and flexibility because of wider ranges in the admissible filed. Moreover, the proposed model can enhance the quality of solution results because a higher lowest acceptable level h can be designated within the range [0, 0.9) in this GDM problem, while the designation range is [0, 0.5) using the benchmark-based approach.

There are other things to note. The optimal individual degrees of violation can also demonstrate the better quality of solution results produced by the proposed model in compared to the comparative approach. As mentioned before, some preference conflicts exist for the ordered pairs given by three decision makers, i.e., (3,2) in Ω^1 vs. (2,3) in Ω^3 and (4,3) in Ω^2 vs. (3,4) in Ω^3. Accordingly, it is anticipated that some optimal individual degrees of violation cannot be equal to zero in the solution results. Consider that the collective poorness of fit is determined by the aggregation of individual degrees of violation. In this regard, the solution quality of application results can be examined with the use of the $\bar{Z}^k_{i'i''}$ values.

The results of the $\bar{Z}^k_{i'i''}$ values yielded by the proposed model and the benchmark-based approach are revealed in the below right parts of Tables 5 and 6, respectively. From Table 5, it is known that $\bar{Z}^2_{43} > 0$ when $\eta = 0.00, 0.05, \cdots, 0.30$ and $\bar{Z}^1_{32} > 0$ and $\bar{Z}^2_{43} > 0$ when $\eta = 0.35, 0.40, 0.45, 0.50$. Based on Table 6, one has $\bar{Z}^2_{43} > 0$ when $\eta = 0.00, 0.05, \cdots, 0.20$ and $\bar{Z}^1_{32} > 0$ and $\bar{Z}^2_{43} > 0$ when $\eta = 0.25, 0.30, \cdots, 0.50$. As noted, the decision makers generally anticipate the obtained solution that possesses the lowest extent of violation in regard to the preference relationships in the preference sets (i.e., to minimize the collective poorness of fit $\sum_{k=1}^{K} B^k$). Consider the two cases of $\eta = 0.25$ and 0.30. According to Table 5, only one conflict (i.e., $\bar{Z}_{43}^2 > 0$) exists between subjective preference relations and objective order relations. Namely, the obtained objective order relation $a_4 \prec_{\mathbb{C}} a_3$ is inconsistent with the preference relation $a_4 \succsim a_3$ in Ω^2 provided by the decision maker e_2. Nonetheless, from Table 6, two conflicts (i.e., $\bar{Z}_{32}^1 > 0$ and $\bar{Z}_{43}^2 > 0$) have been found when $\eta = 0.25$ and 0.30. By employing the benchmark-based approach, the obtained objective order relations $a_3 \prec_{\mathbb{C}} a_2$ and $a_4 \prec_{\mathbb{C}} a_3$ are inconsistent with the preference relations $a_3 \succsim a_2$ in Ω^1 and $a_4 \succsim a_3$ in Ω^2, respectively. It is easy to see that the proposed model performs better than the comparative approach in the quality of solution results.



(a) Results yielded by the parametric PF LINMAP model.



(b) Results yielded by the benchmark-based PF LINMAP model.

FIGURE 7. Comparison results of the optimal weights of criteria under various settings of the weighting parameter.

Furthermore, the comparison results of the optimal weights of criteria under various settings of the weighting parameter η are sketched in Figure 7. The six evaluative criteria in the set C belong to the salient factors in the GDM problem. Thus, it is reasonable and acceptable that the PF LINMAP model can render non-degenerate weights for the sake of reflecting the relative importance of each criterion. Figure 7(a) and Figure 7(b) reveal the patterns of the optimal weights yielded by the parametric PF LINMAP model and the benchmark-based version, respectively. As demonstrated in Figure 7(a), the issue of degenerate weights was not noticeable in the cases of $\eta = 0.00, 0.05, \cdots, 0.35$. Nevertheless, the specialized degenerate weighting issue has become obvious in the cases of $\eta = 0.40$, 0.45, and 0.50. In such cases, the highest importance has been assigned to the criterion c_4 (i.e., debt ratio and current ratio), in which $\bar{w}_4 = 0.8750$. On the contrary, based on Figure 7(b), the issue of degenerate weights was apparent in the obtained weighting patterns generated by the benchmark-based PF LINMAP model. As noted, the degenerate weighting problem has been found to a certain extent in the cases of $\eta = 0.15$ and 0.20. In particular, when $\eta = 0.25, 0.30, \cdots, 0.50$, the specialized degenerate weighting issue has become significant via the benchmark-based approach. Analogously, in the cases of $\eta = 0.25, 0.30, \cdots, 0.50$, the highest importance has been assigned to c_4, in which $\bar{w}_4 = 0.6179$. As a whole, the proposed parametric PF LINMAP model can yield more reasonable and acceptable results than the comparative approach because of non-degenerate weighting results in most cases.

4) FOURTH COMPARATIVE STUDY

The fourth comparative analysis focuses on the comparisons of the application results yielded by the PF TOPSIS method and the proposed model. TOPSIS is a renowned and widely used compromising model in the field of multiple criteria decision analysis, and it plays an important role in many practical applications. Because LINMAP belongs to the compromising model, this paper employs the core structure of TOPSIS to conduct the final comparative study.

TOPSIS is an individual decision-making procedure that specifies how multidimensional characterization information is to be processed in order to arrive at a choice that is closest to the ideal solution. Because TOPSIS cannot directly deal with multiple decision matrices, this comparative study first fuses decision information to form an aggregated decision matrix. By utilizing the ordered weighted averaging (OWA) operator [42], this paper aggregates evaluation information in the PF decision matrix P^k for all $k \in \{1, 2, \cdots, K\}$ to construct the collective PF decision matrix $\widehat{\mathrm{P}}$. This paper defines the following OWA operators that represent the mapping OWA: $R^K \to R$ that has an associated weight vector $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_K)$ such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{K} \varpi_k = 1$, as follows:

$$\widehat{\mu}_{ij} = \sum_{k=1}^{K} \varpi_k \dot{\mu}_{ij}^k, \tag{32}$$

$$\widehat{\nu}_{ij} = \sum_{k=1}^{K} \varpi_k \dot{\nu}_{ij}^k, \tag{33}$$

where $\dot{\mu}_{ij}^k$ and $\dot{\nu}_{ij}^k$ are the k-th largest of μ_{ij}^k and ν_{ij}^k, respectively ($k = 1, 2, \cdots, K$). The collective PF decision matrix $\widehat{\mathrm{P}}$ that involves the collective PF evaluative rating $\widehat{p}_{ij}$ is represented as follows:

$$\widehat{\mathrm{P}} = \left[\widehat{p}_{ij}\right]_{m\times n} = \left[(\widehat{\mu}_{ij}, \widehat{\nu}_{ij}; \widehat{r}_{ij}, \widehat{d}_{ij})\right]_{m\times n}. \tag{34}$$

Here, $\widehat{r}_{ij} = ((\widehat{\mu}_{ij})^2 + (\widehat{\nu}_{ij})^2)^{0.5}$ and $\widehat{d}_{ij} = 1 - (2 \cdot \widehat{\theta}_{ij} \big/ \pi)$, where $\widehat{\theta}_{ij} = \cos^{-1}(\widehat{\mu}_{ij}/\widehat{r}_{ij}) = \sin^{-1}(\widehat{\nu}_{ij}/\widehat{r}_{ij})$.

Analogous Definitions 5 and 6, the positive- and negative-ideal collective PF evaluative ratings $\widehat{p}_{*j}$ and $\widehat{p}_{\#j}$ with respect to each $c_j \in C$ can be identified as follows:

$$\widehat{p}_{*j} = \begin{cases} \left(\overset{m}{\underset{i=1}{\vee}} \widehat{\mu}_{ij}, \overset{m}{\underset{i=1}{\wedge}} \widehat{\nu}_{ij}; \sqrt{\left(\overset{m}{\underset{i=1}{\vee}} \widehat{\mu}_{ij}\right)^2 + \left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\nu}_{ij}\right)^2}, 1 - \dfrac{2 \cdot \widehat{\theta}_{*j}}{\pi}\right) \\ \text{if} c_j \in C_{\mathrm{I}}, \\ \left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\mu}_{ij}, \overset{m}{\underset{i=1}{\vee}} \widehat{\nu}_{ij}; \sqrt{\left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\mu}_{ij}\right)^2 + \left(\overset{m}{\underset{i=1}{\vee}} \widehat{\nu}_{ij}\right)^2}, 1 - \dfrac{2 \cdot \widehat{\theta}_{*j}}{\pi}\right) \\ \text{if} c_j \in C_{\mathrm{II}}; \end{cases} \tag{35}$$

$$\widehat{p}_{\#j} = \begin{cases} \left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\mu}_{ij}, \overset{m}{\underset{i=1}{\vee}} \widehat{\nu}_{ij}; \sqrt{\left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\mu}_{ij}\right)^2 + \left(\overset{m}{\underset{i=1}{\vee}} \widehat{\nu}_{ij}\right)^2}, 1 - \dfrac{2 \cdot \widehat{\theta}_{\#j}}{\pi}\right) \\ \text{if} c_j \in C_{\mathrm{I}}, \\ \left(\overset{m}{\underset{i=1}{\vee}} \widehat{\mu}_{ij}, \overset{m}{\underset{i=1}{\wedge}} \widehat{\nu}_{ij}; \sqrt{\left(\overset{m}{\underset{i=1}{\vee}} \widehat{\mu}_{ij}\right)^2 + \left(\overset{m}{\underset{i=1}{\wedge}} \widehat{\nu}_{ij}\right)^2}, 1 - \dfrac{2 \cdot \widehat{\theta}_{\#j}}{\pi}\right) \\ \text{if} c_j \in C_{\mathrm{II}}. \end{cases} \tag{36}$$

Combining the weight vector $\mathrm{w} = (w_1, w_2, \cdots, w_n)$, the closeness coefficient CC_i of each alternative $a_i \in A$ is determined as follows:

$$CC_i = \frac{\sum_{j=1}^{n} D(\widehat{p}_{ij}, \widehat{p}_{\#j}) \cdot w_j}{\sum_{j=1}^{n} \left(D(\widehat{p}_{ij}, \widehat{p}_{*j}) + D(\widehat{p}_{ij}, \widehat{p}_{\#j})\right) \cdot w_j}. \tag{37}$$

Let ε denote a sufficiently small number, in which $0 < \varepsilon \leq 1/n$. By aggregating the closeness coefficient $CC_n i$ for all $a_i \in A$ as the maximal objective, the following PF TOPSIS model is constructed to solve the unknown weights as follows:

PF TOPSIS Model

$$\max \left\{\sum_{i=1}^{m} CC_i\right\}$$

$$\text{s.t.} \begin{cases} CC_i = \dfrac{\sum_{j=1}^{n} D(\widehat{p}_{ij}, \widehat{p}_{\#j}) \cdot w_j}{\sum_{j=1}^{n} \left(D(\widehat{p}_{ij}, \widehat{p}_{*j}) + D(\widehat{p}_{ij}, \widehat{p}_{\#j})\right) \cdot w_j} \\ (i = 1, 2, \cdots, m), \\ \sum_{j=1}^{n} w_j = 1, \\ w_j \geq \varepsilon (j = 1, 2, \cdots, n). \end{cases} \tag{38}$$

It is noted that the above problem is a nonlinear programming model. Based on the optimal weight vector $\bar{w}$ and the Hamming distances $D(\widehat{p}_{ij}, \widehat{p}_{*j})$ and $D(\widehat{p}_{ij}, \widehat{p}_{\#j})$, the optimal closeness coefficient $\overline{CC}_i$ for each alternative a_i is derived as follows:

$$\overline{CC}_i = \frac{\sum_{j=1}^{n} D(\widehat{p}_{ij}, \widehat{p}_{\#j}) \cdot \bar{w}_j}{\sum_{j=1}^{n} \left(D(\widehat{p}_{ij}, \widehat{p}_{*j}) + D(\widehat{p}_{ij}, \widehat{p}_{\#j})\right) \cdot \bar{w}_j}. \tag{39}$$

The priority ranking of the alternatives is acquired based on the decreasing order of the $\overline{CC}_i$ values.

Consider the GDM problem of railway project investment. To establish the collective PF decision matrix $\widehat{\mathrm{P}}$, this paper employed the OWA operator to fuse the three PF decision matrices P^1, P^2, and P^3. Here, $\varpi = (\varpi_1, \varpi_2, \varpi_3) = (0.2429, 0.5142, 0.2429)$ that is derived by the normal distribution based method proposed by Xu [42]. The determination results of the collective PF evaluative rating $\widehat{p}_{ij}$ are indicated in Table 7. This table also presents the positive-ideal rating $\widehat{p}_{*j}$ and the negative-ideal rating $\widehat{p}_{\#j}$ for each criterion.

TABLE 7. Collective PF evaluative ratings based on OWA operators.

c_j	$\widehat{p}_{1j}$ in $\widehat{\mathrm{P}}$	$\widehat{p}_{3j}$ in $\widehat{\mathrm{P}}$
c_1	(0.5786, 0.4786; 0.7508, 0.5600)	(0.5786, 0.5514; 0.7992, 0.5153)
c_2	(0.5571, 0.7057; 0.8991, 0.4254)	(0.4271, 0.3514; 0.5531, 0.5617)
c_3	(0.4757, 0.4271; 0.6393, 0.5342)	(0.8514, 0.3571; 0.9233, 0.7472)
c_4	(0.7057, 0.6028; 0.9281, 0.5499)	(0.8757, 0.1514; 0.8887, 0.8910)
c_5	(0.7057, 0.6271; 0.9441, 0.5375)	(0.6028, 0.3028; 0.6746, 0.7036)
c_6	(0.3757, 0.7543; 0.8427, 0.2942)	(0.5757, 0.5271; 0.7806, 0.5280)
	$\widehat{p}_{2j}$ in $\widehat{\mathrm{P}}$	$\widehat{p}_{4j}$ in $\widehat{\mathrm{P}}$
c_1	(0.8757, 0.3514; 0.9436, 0.7571)	(0.6814, 0.7300; 0.9986, 0.4781)
c_2	(0.7757, 0.4543; 0.8989, 0.6627)	(0.5057, 0.2271; 0.5543, 0.7312)
c_3	(0.7514, 0.5543; 0.9337, 0.5954)	(0.3271, 0.5300; 0.6228, 0.3521)
c_4	(0.9000, 0.2271; 0.9282, 0.8426)	(0.7543, 0.5328; 0.9235, 0.6085)
c_5	(0.8514, 0.2514; 0.8878, 0.8172)	(0.6271, 0.4543; 0.7744, 0.6009)
c_6	(0.8514, 0.3028; 0.9037, 0.7824)	(0.6543, 0.5543; 0.8575, 0.5526)
	$\widehat{p}_{*j}$	$\widehat{p}_{\#j}$
c_1	(0.8757, 0.3514; 0.9436, 0.7571)	(0.5786, 0.7300; 0.9314, 0.4267)
c_2	(0.7757, 0.2271; 0.8083, 0.8187)	(0.4271, 0.7057; 0.8249, 0.3465)
c_3	(0.8514, 0.3571; 0.9233, 0.7472)	(0.3271, 0.5543; 0.6436, 0.3394)
c_4	(0.9000, 0.1514; 0.9126, 0.8939)	(0.7057, 0.6028; 0.9281, 0.5499)
c_5	(0.8514, 0.2514; 0.8878, 0.8172)	(0.6028, 0.6271; 0.8699, 0.4874)
c_6	(0.8514, 0.3028; 0.9037, 0.7824)	(0.3757, 0.7543; 0.8427, 0.2942)

This paper computed the Hamming distances $D(\widehat{p}_{ij}, \widehat{p}_{*j})$ and $D(\widehat{p}_{ij}, \widehat{p}_{\#j})$ and designated the non-zero boundary condition $\varepsilon = 0.025$. Based on the PF TOPSIS model in (38), the following nonlinear programming model was established:

$$\max \{CC_1 + CC_2 + CC_3 + CC_4\}$$

subject to:

$$\begin{aligned} CC_1 = {} & (0.2470w_1 + 0.1116w_2 + 0.1481w_3 + 0.0000w_4 \\ & + 0.1064w_5 + 0.0000w_6) \big/ (0.6008w_1 + 0.5403w_2 \\ & + 0.5515w_3 + 0.3416w_4 + 0.4197w_5 + 0.5519w_6), \\ CC_2 = {} & (0.3982w_1 + 0.3849w_2 + 0.3904w_3 + 0.3056w_4 \end{aligned}$$

$$+0.3509w_5+0.5519w_6)\big/(0.3982w_1+0.5401w_2+0.5608w_3+0.3418w_4+0.3509w_5+0.5519w_6),$$

$$CC_3=(0.1821w_1+0.3214w_2+0.5478w_3+0.3407w_4+0.2731w_5+0.2720w_6)\big/(0.5508w_1+0.6866w_2+0.5478w_3+0.3704w_4+0.5520w_5+0.6191w_6),$$

$$CC_4=(0.1035w_1+0.4258w_2+0.0218w_3+0.0725w_4+0.1625w_5+0.2774w_6)\big/(0.4694w_1+0.6857w_2+0.5654w_3+0.3416w_4+0.4556w_5+0.5519w_6),$$

$$\sum_{j=1}^{6} w_j=1,\quad w_j\geq 0.025 \text{ for } j=1,2,\cdots,6.$$

Solving the above PF TOPSIS model, the optimal weight vector was acquired as follows: $\bar{w}=(\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_6)=(0.0250,\ 0.0250,\ 0.0250,\ 0.0250, 0.8750,\ 0.0250)$. Moreover, the optimal closeness coefficients were derived as follows: $\overline{CC}_1=0.2449$, $\overline{CC}_2=0.9753$, $\overline{CC}_3=0.5079$, and $\overline{CC}_4=0.3550$, which yields the ultimate priority ranking $a_2\succ a_3\succ a_4\succ a_1$ and the best compromise solution a_2.

The ranking results generated by the PF TOPSIS model are concordant with those by the proposed model and the other comparative approaches. However, the model in (38) is a nonlinear programming model that is inherently much more difficult to optimize. Furthermore, the solution results of the optimal weights (i.e., $\bar{w}_1=\bar{w}_2=\bar{w}_3=\bar{w}_4=\bar{w}_6=\varepsilon$) are undesirable and dissatisfactory and cannot be easily accepted by the decision makers. Recall that the parametric PF LINMAP model and the benchmark-based approach rendered the optimal weight vectors (0.0250, 0.3019, 0.2633, 0.2050, 0.1798, 0.0250) and (0.0250, 0.1957, 0.1918, 0.4357, 0.1268, 0.0250), respectively, under the parameter settings of $h=0.4$, $\varepsilon=0.025$, and $\eta=0.2$. Nevertheless, except for $\bar{w}_5=0.8750$, the PF TOPSIS model yielded the identical weights of the remaining criteria, i.e., $\bar{w}_j=\varepsilon=0.0250$ for $j\in\{1,2,3,4,6\}$. The three decision makers may be dissatisfied by these inactive optimal weights because these weights are equal to the non-zero boundary condition. Thus, the solution quality generated by the PF TOPSIS model is lower than those produced by the parametric PF LINMAP model and the benchmark-based version. In addition, the PF TOPSIS model belongs to nonlinear problems and is intrinsically more difficult to solve. Therefore, the usefulness and superiority of the proposed parametric PF LINMAP methodology has been demonstrated through the comparisons with the well-known and widely used TOPSIS approach in PF contexts.

In the nutshell, the merits of applying the parametric PF LINMAP model over the PF TOPSIS approach have been examined and explained in the final comparative analysis. On the whole, this paper has conducted the four comparative studies consisting of the influences of distinct reference points on the application results, the effects of application outcomes under various settings of the weighting parameter, the contrasts among the feasible ranges about relevant parameters, and the comparisons of the application results with the PF TOPSIS method. The comparative results have validated the effectiveness and advantages of the proposed methodology in solving multiple criteria GDM problems within complex PF environments.

VII. CONCLUSION

This paper has developed a novel parametric PF LINMAP methodology to manage GDM problems in the PF context. This paper has presented the PF closeness index to identify the PF closeness-based order relation between PF evaluative ratings. Combining the weight vector of criteria, this paper has proposed the comprehensive closeness measure for the sake of the objective order relation between alternatives. According to the contrast results between subjective preference relations and objective order relations, the individual order consistency and inconsistency indices have been defined to determine individual goodness of fit and poorness of fit, respectively. To formulate a PF LINMAP model, this paper has introduced some useful parameters, including the lowest acceptable level towards the difference between goodness and poorness of fit, the non-zero boundary condition for criterion weights, and the weighting parameter for transforming a bi-objective optimization model into an effective linear programming model. To maximize the total collective comprehensive closeness measure and minimize the collective poorness of fit, this paper has established the parametric PF LINMAP model to solve for the optimal weight vector and individual degrees of violation. Correspondingly, the optimal collective comprehensive closeness measures can be acquired to determine the ultimate priority ranking of alternatives and the best compromise solution. The proposed PF LINMAP methodology has the capability to adjust the optimal collective comprehensive closeness measures with the controlling parameters. Furthermore, the practical application and comparative analyses have demonstrated the potentials on the real-world GDM problems with information uncertainties.

This paper provides two recommendations for future research: an integration with data envelopment analysis (DEA) and an extension to interval-valued PF decision environments. First, Liu et al. [43] employed DEA cross-efficiency with intuitionistic fuzzy preference relations to propose a novel GDM approach. In a similar manner, Liu *et al.* [44] established an integrated GDM approach with interval fuzzy preference relations using DEA and stochastic simulation. Based on their results, the DEA-based approach can avoid information distortion during GDM processes and acquire more credible decision-making results. Motivated by Liu *et al.* [43], [44], this paper suggests an integrated approach by combining DEA and the proposed parametric PF LINMAP model to avoid possible information loss and distortion issues in GDM practices. Second, the proposed methodology focuses on the PF decision environment. By extending developed concepts and techniques to interval-valued PF contexts, another prospective research direction can be made on the development of an interval-valued PF LINMAP methodology. That is, this paper suggests to propound a parametric interval-valued PF LINMAP model to cope with more sophisticated information characterized by interval-valued Pythagorean membership grades.

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Data Availability Statement:

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of Interest Statement:

The author declares that there is no conflict of interest regarding the publication of this paper.

Ethical Approval:

This article does not contain any studies with human participants or animals performed by the author.

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