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Chaotic Dynamics in Asymmetric Rock-Paper-Scissors Games

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ABSTRACT Evolutionary game dynamics is a combination of game theory and dynamical systems. Using dynamical theory, we investigate chaotic behavior in asymmetric Rock-Paper-Scissors games under imitative dynamics with two different populations. Our theoretical analysis and numerical simulations demonstrate that the dynamical system can give rise to chaotic behavior in zero-sum and positive-sum asymmetric games. However, chaos cannot occur in a negative-sum asymmetric game. In particular, the numerical simulations show that there is a limit cycle in a negative-sum asymmetric game.

INDEX TERMS Evolutionary game, imitative dynamics, chaos, rock-paper-scissors.

I. INTRODUCTION

The game Rock-Paper-Scissors(RPS) describes cyclic dominance among three competing species in social networks [1]–[5], economics [6], [7] and biological systems [8], [9]. It is a famous three-strategy game, in which rock crushes scissors, scissors cut paper, and paper wraps rock. There are many classical examples in reality, such as the mating strategies of side-blotched lizards [10], bacteriocin producing bacteria [11] and the overgrowth of marine sessile organisms [12]. Evolutionary game dynamics such as replicator dynamics [13], imitative dynamics [14], [15], best-response dynamics [16] can describe the evolution of the frequency of strategy in a population. At the same time, some complex dynamical behaviors such as stability, limit cycle and chaos are observed in evolutionary game dynamics [17], [18].

Many researchers have studied the RPS game by using the replicator equation, which was first proposed by Taylor in 1978 [19]. The RPS game has been studied under replicator dynamics in various differential equations including delayed equations [20] and mutational equations [21], [22]. To the best of our knowledge, imitative dynamics is a generalization of replicator dynamics and can be used in various fields. Wang *et al.* [23] investigated the cooperative promotion among imitative contributing players, in which players selected one of their neighbors to imitate according to their contribution. Hu *et al.* [24] discussed imitative dynamics with discrete delay and obtained two sufficient conditions for stability in imitative dynamics. The emphasis in these literatures is on the effects of delays and mutations on evolutionary dynamics. In reality, asymmetry is another relevant factor in the study of dynamical behaviors in evolutionary dynamics.

As asymmetry is a common phenomenon in social networks [25], [26], sports and biological systems [27]–[30], many researchers have studied asymmetric situations in evolutionary dynamics. In the area of social networks, Du *et al.* [31] considered the effects of asymmetric cost in complex networks and proved that asymmetric cost can promote cooperative behavior in evolutionary games. In the area of biological systems, He *et al.* [32] analyzed asymmetric game dynamics based on individuals' own volition and showed that asymmetric mechanisms could lead to more complex dynamics than occur in symmetric situation. In the field of sports, Misirlisory and Haggard [33] discussed the asymmetric predictability of football penalty shootouts, the results of this study could help teams better prepare for penalty shootouts. Recently, Hauert *et al.* [28] investigated asymmetric individuals with environmental feedback and found that asymmetric interactions could alter social

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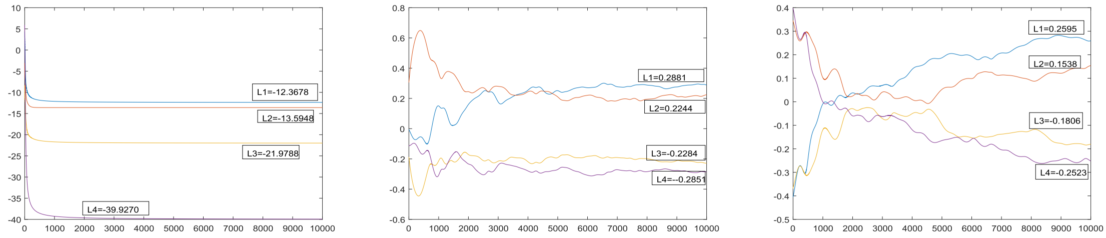
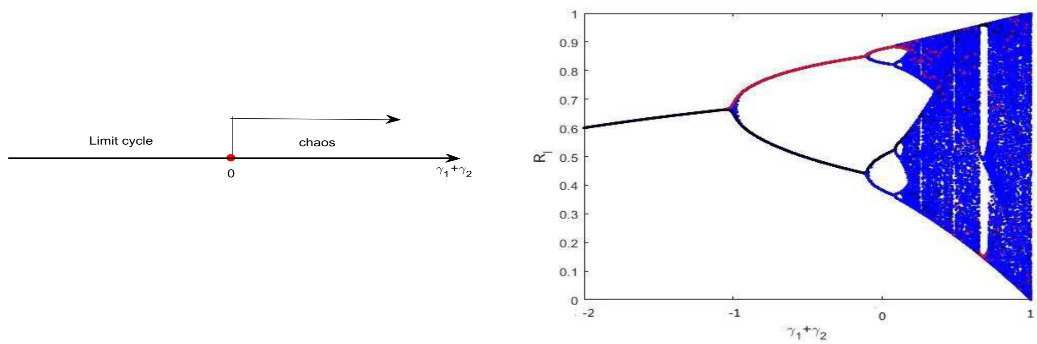
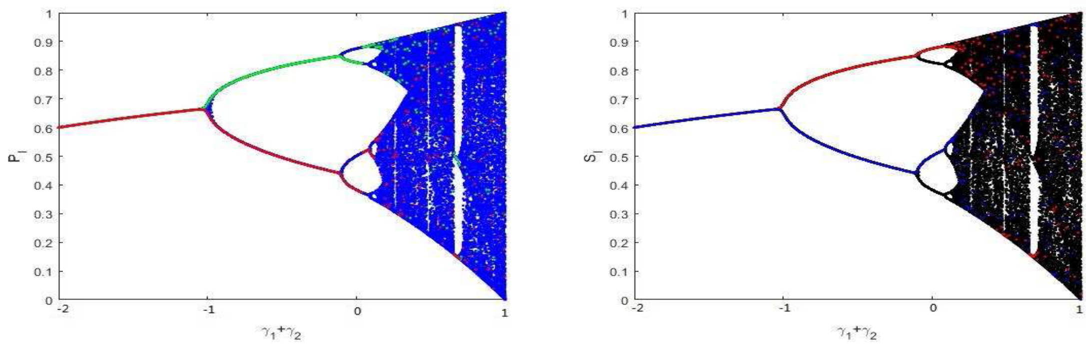


(a) $\gamma_1 + \gamma_2 < 0$ (b) $\gamma_1 + \gamma_2 = 0$ (c) $\gamma_1 + \gamma_2 > 0$

FIGURE 1. **The Lyapunov Exponent of system (2) for different sign of $\gamma_1 + \gamma_2$ at (0.45, 0.01, 0.45, 0.24).**

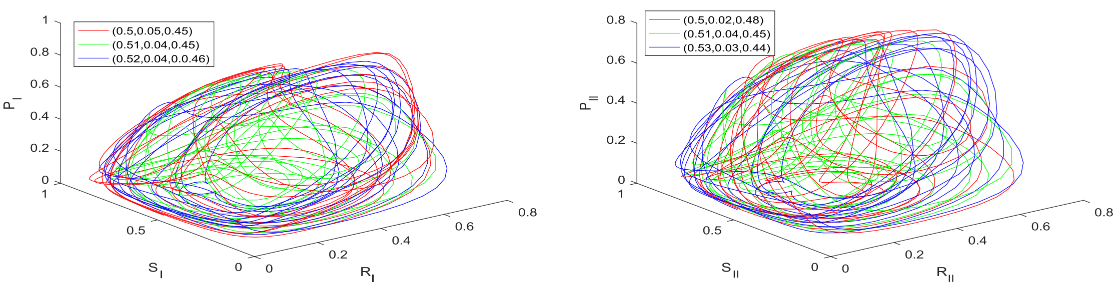


(a) Bifurcation value (b) R_I



(c) P_I (d) S_I

FIGURE 2. **Bifurcation diagrams for system (2) against variation of $\gamma_1 + \gamma_2$.**

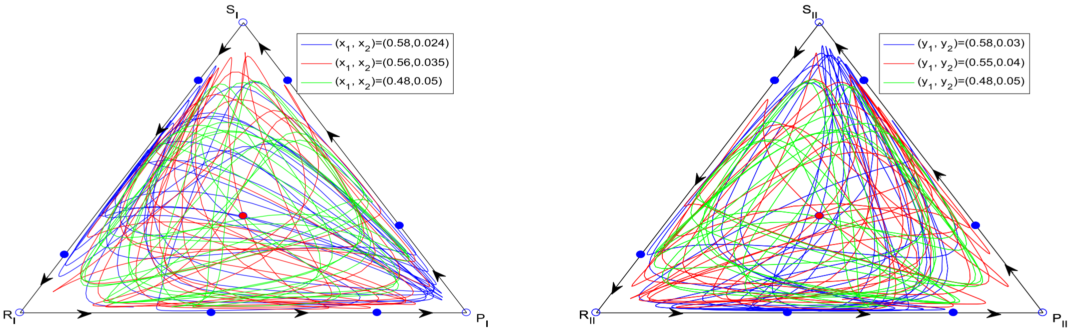


(a) Population I (b) Population II

FIGURE 3. **The chaotic behavior of of 3-dimension in each population when $\gamma_1 = 0.5, \gamma_2 = -0.5$.**

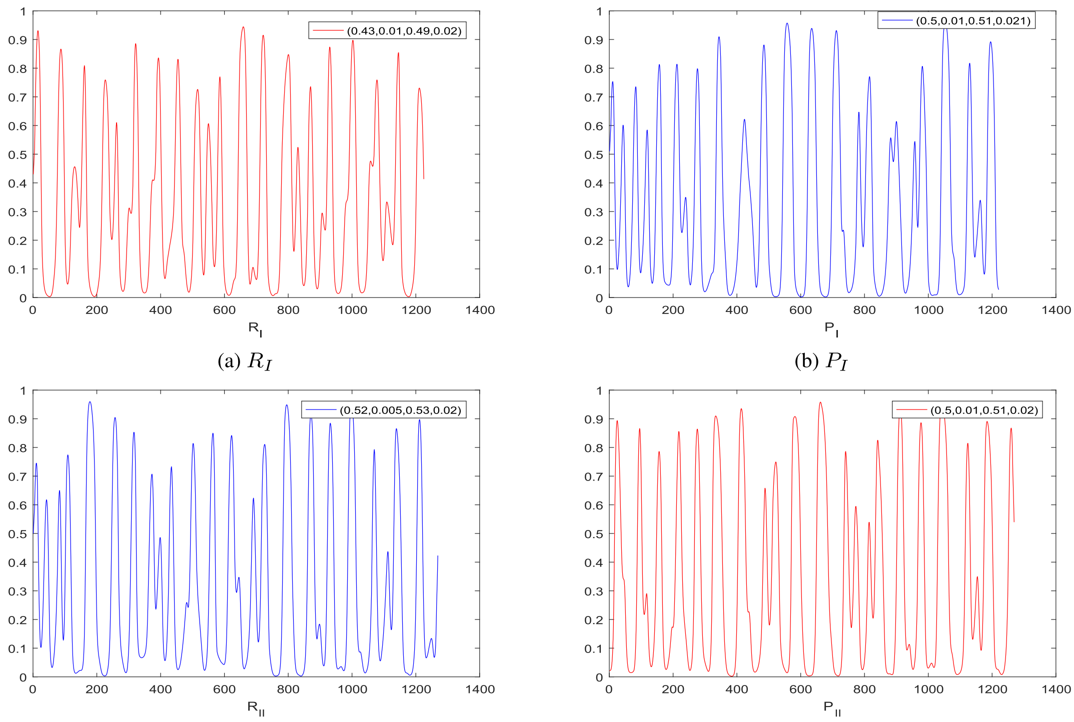
dilemmas and promote persistent periodicity in evolutionary games. The results of these studies clearly illustrate that asymmetry may change dynamical behaviors, and even leads to chaos.

Chaos is a type of quasistochastic behavior in deterministic nonlinear systems, and is an important natural phenomenon [34]–[40]. The presence of chaos is significant in evolutionary games. In 1988, Skyrms [41] gave



(a) Population I (b) Population II

FIGURE 4. The chaotic behavior of of 2-dimension in each population when $\gamma_1 = 0.5, \gamma_2 = -0.5$.



(a) R_I (b) P_I

(c) R_{II} (d) S_{II}

FIGURE 5. The solution trajectory of R_I, P_I, R_{II}, S_{II} when $\gamma_1 = 0.5, \gamma_2 = -0.5$.

examples of chaotic dynamics with four strategies to show that chaos could exist in a four-strategy game. In 2002, Sato *et al.* [42] investigated the chaotic behavior of RPS games in replicator dynamics and demonstrated the existence of Hamiltonian chaos in a zero-sum game. Then the author used game dynamics to describe collective behavior by using a discrete-time stochastic model and drew some relevant conclusion concerning stability and diversity in collective adaptation [43]. Subsequently, Aguiar and Castro [44] studied robust complex behavior in RPS games and provided analytical proof for the existence of chaotic switching and relative asymptotic stability. In addition, there has been some research on chaos in discrete time [45], [46]. In 2011, Salvetti *et al.* [45] analyzed chaotic behavior on a discrete-time version of replicator equation and discovered the uncertainty of the population caused by sensitivity to initial conditions.

Motivated by previous research, we aim to discuss the complex dynamics in asymmetric RPS under imitative dynamics. The major contributions of this paper can be summarized as follows: (i) there is a heterogeneous cycle on the boundary of a simplex; (ii) chaotic behavior is exhibited inside of the simplex in zero-sum and positive-sum asymmetric RPS games; and (iii) chaotic behavior cannot occur in negative-sum asymmetric RPS games.

The rest of this paper is organized as follows. Section 2 presents the imitative dynamics model and analyzes the stability of equilibria. Section 3 provides the dynamical analysis and describes the chaos in the imitative dynamics model. Section 4 offers conclusions and a discussions.

II. ASYMMETRIC ROCK-PAPER-SCISSORS GAME

A. DERIVATION

Let $G = \{N, S, U\}$ denote an asymmetric Rock-Paper-Scissors game, where N represents the players, S represents the strategy set, and U represents the payoff. Suppose that two populations (I and II) of agents are matched to play this game, and let their payoffs be denoted as U_I and U_{II}, respectively.

$$U_I: \begin{array}{c} \\ R_I \\ P_I \\ S_I \end{array} \begin{array}{c} \begin{array}{ccc} R_{II} & P_{II} & S_{II} \end{array} \\ \begin{pmatrix} \gamma_1 & -1 & 1 \\ 1 & \gamma_1 & -1 \\ -1 & 1 & \gamma_1 \end{pmatrix} \end{array},$$

$$U_{II}: \begin{array}{c} \\ R_{II} \\ P_{II} \\ S_{II} \end{array} \begin{array}{c} \begin{array}{ccc} R_I & P_I & S_I \end{array} \\ \begin{pmatrix} \gamma_2 & -1 & 1 \\ 1 & \gamma_2 & -1 \\ -1 & 1 & \gamma_2 \end{pmatrix} \end{array}, \quad (-1 < \gamma_1,\ \gamma_2 < 1).$$

In population I, the payoff matrix indicates that each strategy receives a payoff γ_1 when there is a tie; otherwise, the loser receives a payoff -1, while the winner receives a payoff of 1. In population II, the payoff matrix indicates that each strategy receives a payoff γ_2 when there is a tie; otherwise, the loser receives a payoff -1, while the winner receives a payoff of 1. The RPS game is an asymmetric game if $\gamma_1 \neq \gamma_2$. An asymmetric RPS game is called a zero-sum game if $\gamma_1 + \gamma_2 = 0$, a positive-sum game if $\gamma_1 + \gamma_2 > 0$, and a negative-sum game if $\gamma_1 + \gamma_2 < 0$.

Let $(\mathrm{x}, \mathrm{y}) = (x_1, x_2, x_3, y_1, y_2, y_3)$ be the frequency of $(R_I, P_I, S_I, R_{II}, P_{II}, S_{II})$ and $(f_1, f_2, f_3, g_1, g_2, g_3)$ be the expected payoff of $(R_I, P_I, S_I, R_{II}, P_{II}, S_{II})$ with

$$\sum_{i=1}^{3} x_i = 1, \quad f_i(\mathrm{x}) = \sum_{j=1}^{3} y_j a_{ij},$$
$$\sum_{i=1}^{3} y_i = 1, \quad g_i(\mathrm{y}) = \sum_{j=1}^{3} x_j b_{ij}, \quad (i, j = 1, 2, 3).$$

Here a_{ij} denotes the payoff of a S_i-individual plays against a S_j-individual in population I and b_{ij} denotes the payoff of a S_i-individual plays against a S_j-individual in population II.

The imitative dynamics of the asymmetric RPS game can be governed as follows,

$$\begin{cases} \dot{x}_i = x_i \sum_{j \neq i} x_j [F_{ij}(\mathrm{x}) - F_{ji}(\mathrm{x})], \\ \dot{y}_i = y_i \sum_{j \neq i} x_j [G_{ij}(\mathrm{y}) - G_{ji}(\mathrm{y})], \end{cases} \quad (i, j = 1, 2, 3) \quad (1)$$

where

$$F_{ij}(\mathrm{x}) = \frac{f_i(\mathrm{x}) - f_j(\mathrm{x})}{\sum_{i=1}^{3} f_i(\mathrm{x})}, \quad G_{ij}(\mathrm{x}) = \frac{g_i(\mathrm{y}) - g_j(\mathrm{y})}{\sum_{i=1}^{3} g_i(\mathrm{y})}.$$

Since $x_i, y_i (i = 1, 2, 3)$ are the frequencies of the three strategies, the region of interest is the three-dimensional simplex $\Delta_I,\ \Delta_{II}$ in R^3:

$$\Delta_I \equiv \{(x_1, x_2, x_3) \in R^3 : \sum_{i=1}^{3} x_i = 1, \quad x_i \geq 0, \ (i = 1, 2, 3)\},$$
$$\Delta_{II} \equiv \{(y_1, y_2, y_3) \in R^3 : \sum_{j=1}^{3} y_j = 1, \quad y_j \geq 0, \ (i = 1, 2, 3)\}.$$

Based on game theory, we know that the simplexes Δ_I, Δ_{II} are invariant sets. Furthermore, both the interior and the boundary of the simplex are invariant sets. We can eliminate x_3 (y_3) by $x_3 = 1 - x_1 - x_2$ ($y_3 = 1 - y_1 - y_2$), which gives the projection of $\sum$ into the $\Delta_I \times \Delta_{II}$ plane:

$$S \equiv \{(x_1, x_2), (y_1, y_2) \in R^2 : (x_1, x_2, 1 - x_1 - x_2), (y_1, y_2, 1 - y_1 - y_2) \in \textstyle\sum\}.$$

In this case, equations (1) can be written as:

$$\begin{cases} \frac{\dot{x}_1}{x_1} = [1 + \frac{1}{\gamma_1} - (1 + \frac{3}{\gamma_1}) y_1 - 2y_2] \cdot x_2 \\ \quad + [2y_1 + (1 - \frac{3}{\gamma_1}) y_2 + \frac{1}{\gamma_1} - 1] \cdot (1 - x_1), \\ \frac{\dot{x}_2}{x_2} = [1 - \frac{1}{\gamma_1} - 2y_1 + (\frac{3}{\gamma_1} - 1) y_2] \cdot x_1 \\ \quad + [(1 + \frac{3}{\gamma_1}) y_1 + 2y_2 - \frac{1}{\gamma_1} - 1] \cdot (1 - x_2), \\ \frac{\dot{y}_1}{y_1} = [1 + \frac{1}{\gamma_2} - (1 + \frac{3}{\gamma_2}) x_1 - 2x_2] \cdot y_2 \\ \quad + [2x_1 + (1 - \frac{3}{\gamma_2}) x_2 + \frac{1}{\gamma_2} - 1] \cdot (1 - y_1), \\ \frac{\dot{y}_2}{y_2} = [1 - \frac{1}{\gamma_2} - 2x_1 + (\frac{3}{\gamma_2} - 1) x_2] \cdot y_1 \\ \quad + [(1 + \frac{3}{\gamma_2}) x_1 + 2x_2 - \frac{1}{\gamma_2} - 1] \cdot (1 - y_2). \end{cases} \quad (2)$$

B. STABILITY OF EQUILIBRIA

In this subsection, we first formulate the equilibria in the imitative equation. Equations (2) shows the existence of 16 equilibria, which can be obtained as follows:

(i) At the vertices of the simplex: (9 equilibria)

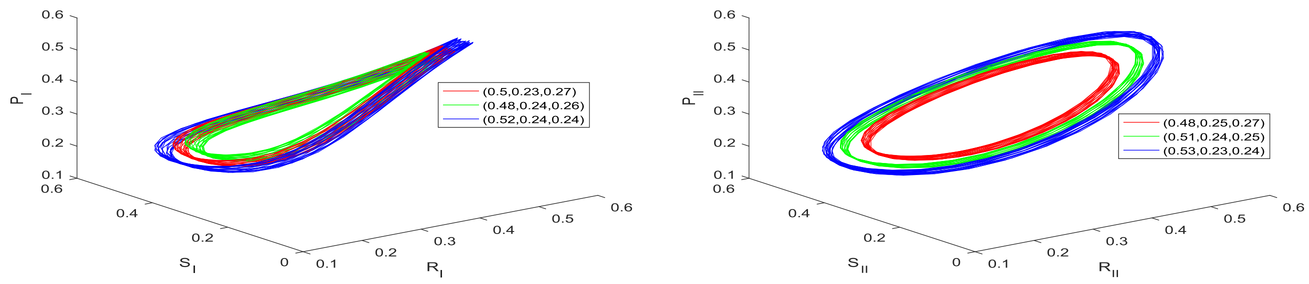
$$\{(1, 0, 1, 0), (0, 1, 0, 1), (0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\};$$

(ii) On the boundary of the simplex: (6 equilibria)

$$\Big\{(\frac{1+\gamma_2}{3+\gamma_2}, 0, 0, \frac{1-\gamma_1}{3-\gamma_1}), (\frac{2}{3+\gamma_2}, \frac{1+\gamma_2}{3+\gamma_2}, \frac{2}{3-\gamma_1}, 0), (0, \frac{2}{3+\gamma_2}, \frac{1-\gamma_1}{3-\gamma_1}, \frac{2}{3-\gamma_1}), (0, \frac{1-\gamma_2}{3-\gamma_2}, \frac{1+\gamma_1}{3+\gamma_1}, 0), (\frac{1-\gamma_2}{3-\gamma_2}, \frac{2}{3-\gamma_2}, 0, \frac{2}{3+\gamma_1}), (\frac{2}{3-\gamma_2}, 0, \frac{2}{3+\gamma_1}, \frac{1+\gamma_1}{3+\gamma_1})\Big\};$$

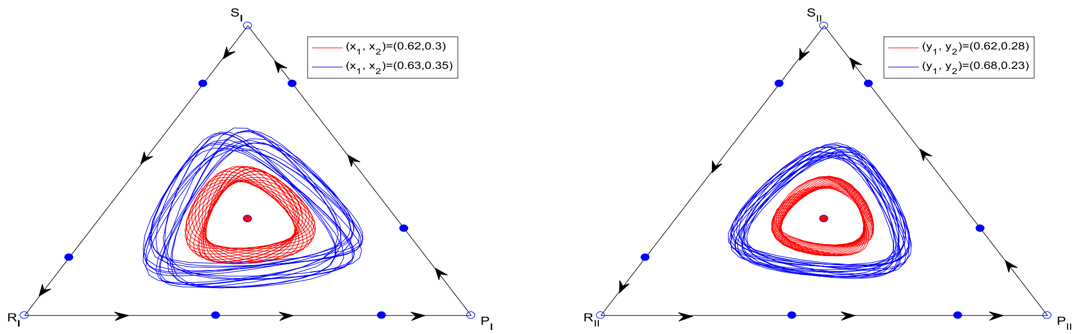
and (iii) In the interior of the simplex: (1 equilibrium)

$$(\mathrm{x}^*, \mathrm{y}^*) = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}.$$



(a) Population I (b) Population II

FIGURE 6. The periodic orbit of 3-dimension in each population when $\gamma_1 = -0.1, \gamma_2 = 0.05$.



(a) Population I (b) Population II

FIGURE 7. The periodic orbit of 2-dimension in each population when $\gamma_1 = -0.1, \gamma_2 = 0.05$.

III. MAIN RESULTS

In this section, we analysis the dynamical behavior of imitative dynamics (2).

A. LINEAR ANALYSIS

To discuss the stability of these equilibria, we linearize equations *(2)*. As a result, we can analyze the stability of each equilibrium through the eigenvalues of the Jacobian matrix. The eigenvalues of these equilibria are calculated in Table 1.

From Table 1, we can easily see that each equilibrium on the boundary of the simplex has two eigenvalues, which have different signs. In this case, it follows that there exists a heteroclinic cycle on the boundary of each simplex. Since the eigenvalues of the inner equilibrium z* are conjugated complex roots, the dynamic behavior of the interior of each simplex is complex.

B. COMPLEX DYNAMICS

In this subsection, we discuss the dynamical behavior in the interior of of each simplex. Since the dynamics cannot be predicted solely from the eigenvalues at z*, we analyze the complex dynamics through the different signs of $\gamma_1 + \gamma_2$.

Next, we give a theorem to illustrate the complex dynamical behaviors in imitative dynamics (2).

Theorem 1: The following three statements hold for the asymmetric RPS game under imitative dynamics (2):

(i) There exists a heteroclinic cycle on the boundary of the simplex.

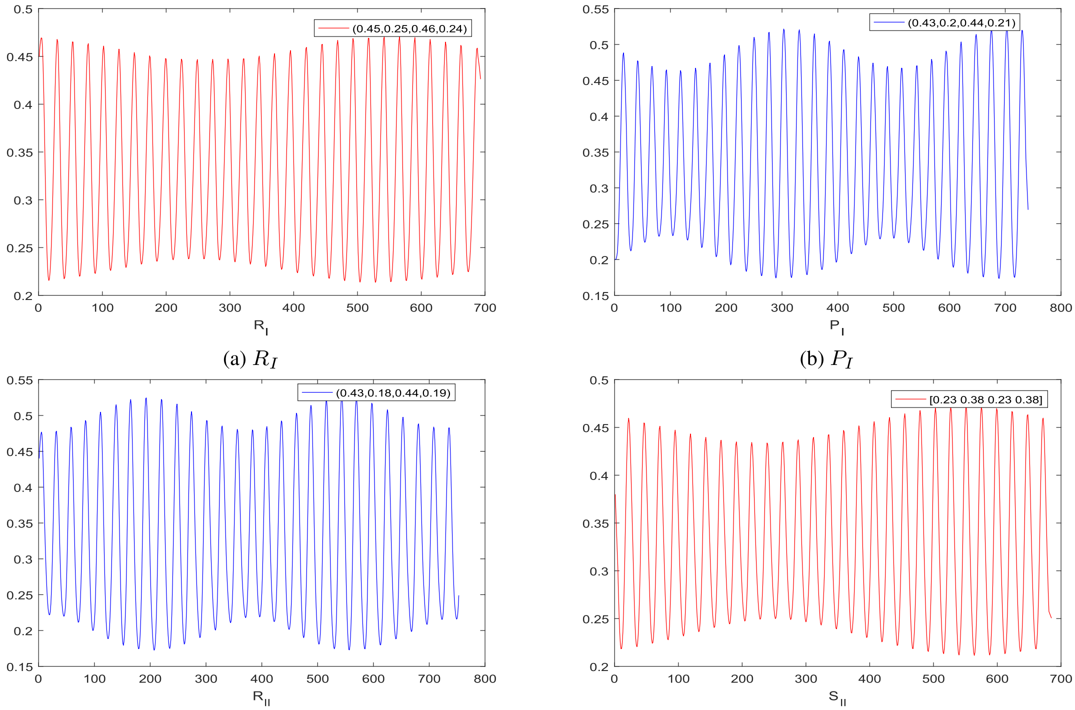
(ii) Chaotic behavior is displayed in the interior of each simplex when $\gamma_1 + \gamma_2 \geq 0$.

TABLE 1. The equilibria and their eigenvalues.

Equilibria	Eigenvalues
$(1,0,1,0)$ $(0,1,0,1)$ $(0,0,0,0)$	$\lambda_1 = \frac{1-\gamma_1}{\gamma_1},\ \lambda_2 = \frac{-1-\gamma_1}{\gamma_1},\ \lambda_3 = \frac{1-\gamma_2}{\gamma_2},\ \lambda_4 = \frac{-1-\gamma_2}{\gamma_2}$
$(1,0,0,1)$ $(0,1,0,0)$ $(0,0,1,0)$	$\lambda_1 = \frac{2}{\gamma_1},\ \lambda_2 = \frac{2}{\gamma_2},\ \lambda_3 = \frac{\gamma_2-1}{\gamma_1},\ \lambda_4 = \frac{1+\gamma_2}{\gamma_2}$
$(1,0,0,0)$ $(0,1,1,0)$ $(0,0,0,1)$	$\lambda_1 = \frac{2}{\gamma_1},\ \lambda_2 = \frac{1+\gamma_1}{\gamma_1},\ \lambda_3 = -\frac{2}{\gamma_2},\ \lambda_4 = \frac{\gamma_2-1}{\gamma_2}$
x_1 x_2 x_3 x_4 x_5 x_6	$\lambda_1 = \frac{3+\gamma_2^2}{\gamma_2^2+3\gamma_2},\ \lambda_2 = -\frac{3+\gamma_1^2}{\gamma_1^2+3\gamma_1}$ $\lambda_{3,4} = \pm\frac{2\sqrt{-\gamma_1\gamma_2(\gamma_1-1)(\gamma_1-3)(\gamma_2+1)(\gamma_2+3)}}{\gamma_1^2\gamma_2^2-3\gamma_1^2\gamma_2+3\gamma_1\gamma_2^2+9\gamma_1\gamma_2}$
$(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\lambda_{1,2} = \pm\sqrt{\frac{\sqrt{3}(\gamma_1+\gamma_2)i+(\gamma_1\gamma_2-3)}{9\gamma_1\gamma_2}}$, $\lambda_{3,4} = \pm\sqrt{\frac{\sqrt{3}(\gamma_1+\gamma_2)i+(3-\gamma_1\gamma_2)}{9\gamma_1\gamma_2}}$

(iii) Chaotic behavior cannot occur in the interior of any simplex when $\gamma_1 + \gamma_2 < 0$.

Proof: First, we consider the boundary of the simplex in the asymmetric RPS game. Since the boundary of the simplex is an invariant set and each equilibrium on the boundary of



(a) R_I (b) P_I (c) R_{II} (d) S_{II}

FIGURE 8. The solution trajectory of R_I, P_I, R_{II}, S_{II} when $\gamma_1 = -0.1, \gamma_2 = 0.05$.

simplex has two eigenvalues with different signs as shown in Table 1, it follows that a heteroclinic cycle is exists on the boundary of the simplex.

Second, we consider the interior of each simplex in the asymmetric RPS game. The Lyapunov Exponent(LE) in continuous system is:

$$L_i = \lim_{t\to\infty} \ln \frac{||\delta x_i(x_0, t)||}{||\delta x(x_0, t)||},$$

where x_0 is a center, $||\delta x(x_0, t)||$ is radius of n-sphere ellipsoid, $||\delta x_i(x_0, t)||$ is semimajor axis of ellipsoid.

The approximate formula of LE as follows,

$$\begin{cases} L_1 = -\frac{\ln d}{T} + \frac{1}{NT}\sum_{k=1}^{N} \ln ||V_1^{(k)}||, \\ L_2 = -\frac{\ln d}{T} + \frac{1}{NT}\sum_{k=1}^{N} \ln ||V_2^{(k)}||, \\ L_3 = -\frac{\ln d}{T} + \frac{1}{NT}\sum_{k=1}^{N} \ln ||V_3^{(k)}||, \\ L_4 = -\frac{\ln d}{T} + \frac{1}{NT}\sum_{k=1}^{N} \ln ||V_4^{(k)}||. \end{cases} \quad (3)$$

The LE of system (2) can be calculated through approximate formula (3) in Table 2.

From the results shown in Figure 1 and Table 2, we see that there are two positive LEs in system (2) when $\gamma_1 + \gamma_2 = 0$. This outcome implies that chaotic behavior is shown in the interior of each simplex. At the same time, we can see that there are four negative LEs in system (2) when $\gamma_1 + \gamma_2 < 0$, which implies that chaotic behavior does not emerge in this case. □

TABLE 2. Lyapunov Exponent for different parameters.

γ_1, γ_2	LE	(0.3,0.15,0.3,0.25)	(0.5,0.25,0.5,0.15)	(0.7,0.15,0.7,0.2)
$\gamma_1 = 0.5$, $\gamma_2 = -0.5$	L1	0.275	0.2881	0.32
	L2	0.242	0.2244	0.1508
	L3	-0.2374	-0.2284	-0.142
	L4	-0.2796	-0.2851	-0.3288
$\gamma_1 = -0.1$, $\gamma_2 = 0.05$	L1	-8.9845	-12.3457	-12.3678
	L2	-18.9985	-13.5647	-13.5948
	L3	-19.9308	-21.9604	-21.9788
	L4	-39.9355	-39.873	-39.927
$\gamma_1 = 0.1$, $\gamma_2 = -0.05$	L1	0.2136	0.276	0.2595
	L2	0.1587	0.1901	0.1538
	L3	-0.1677	-0.207	-0.1806
	L4	-0.2183	-0.2709	-0.2523

Remark 1: The sensitivity of results in Theorem 1 with respect to parameters γ_1 and γ_2 in system (2) can be described as follows (see Figure 2). This outcome implies that $\gamma_1+\gamma_2 = 0$ is the critical value, i.e. there is a limit cycle in the interior of simplex when $\gamma_1 + \gamma_2 < 0$, and chaotic behavior emerges when $\gamma_1 + \gamma_2 \geq 0$.

Remark 2: The interpretation of the system (2) when it evolves to a chaotic behavior is that the interior of the trajectory of system (2) in unpredictable, i.e. the evolution results of the mixed strategy in asymmetric RPS game is unpredictable. Since the payoff of draw is different in two populations ($\gamma_1 \neq \gamma_2$), the evolutionary strategy stability of asymmetric RPS game cannot be reached. In this case, the chaotic behavior can be evolved in this asymmetric RPS game.

IV. NUMERICAL SIMULATIONS

In this section, we give two simulations for imitative dynamics with zero-sum and nonzero-sum games.

Example 1: In the case of a zero-sum game, we take

$$\gamma_1 = 0.5, \quad \gamma_2 = -0.5,$$

in equation *(2)* and choose initial values:

(0.3,0.15,0.3,0.25), (0.5,0.25,0.5,0.15), (0.7,0.15,0.7,0.2).

The dynamical behavior is presented in the following figures. Figure 3 shows the 3-dimensional dynamical behavior in each population for the initial values chosen above. Figure 4 shows 2-dimensional dynamical behavior in each population, and Figure 5 shows the frequency of each strategy in the asymmetric RPS game. Since the strategy state (x, y) is four-dimensional, it is drawn in two pieces, with x represented on the left-hand side of population I and y represented on the right-hand side of population II.

Example *1* shows that chaos emerges in the interior of each simplex when $\gamma_1 + \gamma_2 = 0$, which is consistent with the result in *Theorem 1* (ii).

Example 2: In the case of a nonzero-sum game, we take

$$\gamma_1 = -0.1, \quad \gamma_2 = 0.05,$$

in equation *(2)* and choose initial values:

(0.25, 0.15, 0.28, 0.25), (0.45, 0.2, 0.45, 0.15), (0.65, 0.15, 0.65, 0.2).

The dynamical behavior is presented in the following figures. Figure 6 shows the 3-dimensional dynamical behavior in each population, Figure 7 shows 2-dimensional dynamical behavior in each population, and Figure 8 shows the frequency of each strategy in the asymmetric RPS game.

Example *2* shows that the trajectories approach a limit cycle in the interior of each simplex when $\gamma_1 + \gamma_2 < 0$, which expands the result in *Theorem 1* (iii).

V. CONCLUSION

In this paper, the complex dynamics of asymmetric RPS games under imitative dynamics have been investigated. On the one hand, the results obtained are different from those for the symmetric RPS game under imitative dynamics, i.e., chaos emerges in the asymmetric RPS game, while it cannot occur in the symmetric case. On the other hand, the results are more complicated than in replicator dynamics. Some numerical examples have also been given to illustrate the effectiveness of our results.

In game theory, the choice of strategy is difficult because of the presence of chaos. To predict the behavior of an opponent, players must know more information about the opponent. In this case, chaos is important in evolutionary game dynamics for more equitable interactions.

If we increase the number of interaction players (i.e., from two players to three players or more), the chaotic behaviors may become much more complicated. As an extension of this work, we plan to explore the evolutionary dynamics of a game with N players.

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