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Multidimensional Hierarchical Interpolation Method on Sparse Grids for the Absorption Problem

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ABSTRACT The numerical integration of multidimensional functions using some variables of the sparse grid method for the absorption problem is presented in this paper. The multivariate quadrature expressions are constructed by combining tensor of suited one dimensional formula. We develop a multidimensional adaptive quadrature algorithm for the implementation of sparse grid based on a hierarchical basis. Furthermore, we obtain a new error bound at each sparse grid point. The numerical examples are shown to demonstrate the efficiency of our algorithm for the absorption problem and confirm the theoretical estimates.

INDEX TERMS Multidimensional systems, interpolation, grid computing, error analysis, convergence of numerical methods.

I. INTRODUCTION

Multivariate integrals arise in many scientific and engineering application fields such as statistical mechanics, the valuation of financial derivatives etc. Integration over products of the unit sphere is equivalent to multiple integration over the unit sphere. Such multiple integrals can be approximated in a number of ways, including Monte Carlo methods. Conventional approaches for such approximation are usually limited by the trouble problem ''curse of dimension'' [1]. Furthermore, computational complexity research reveals that the cost of computation grows exponentially as the dimensions of the problems increase [2]. There are a lot of methods for numerical computation of multi-dimensional functions such as Quasi-Monte Carlo approaches [3], lattice rules [4] and computation methods with artificial neural networks [5]. Each of these approaches has a particular complexity that is independent of the problem's dimension. The application of sparse grid techniques might offer a promising way out.

Sparse grid techniques were firstly introduced by Smolyak [6] to overcome the curse of dimension to a certain extent. In this method, multivariate quadrature formulas are constructed using combinations of tensor products of suited 1-D formulas. In this way, the numerical accuracy obtain independent of dimension of the problem up to logarithmic factors. During recent years, sparse grid method has become a popular approximation tool for high dimensional problems. This method has been applied to different computational tasks by several authors, such as the rectangle rule [7], the Clenshaw-Curtis rule [8], the Gauss rules [9] and so on. Further some applications of sparse grid method are the stochastic simulations [10], wavelet analysis [11], high-dimensional elliptic problems [12], [13], reaction-diffusion systems [14], integral equations [15], optimal control problems [16], and the Fourier transformation [17]. The sparse grid method has intrinsic distributed structures, so parallel implementation becomes straightforward, in comparison with the other numerical methods.

It is worth pointing out that a different approach to sparse grid approximations is the so-called combination technique. It represents a sparse grid as a superposition of much coarser full grids. The problem is solved on each of these full grid independently in the sparse grid space. Under certain conditions, it is shown that the combination technique is equal to the sparse grid solution [14]. The reference [14] indicates the combination technique is promising, but it only focuses on numerical results and error analysis is ignored. It is well known that error analysis is very important for numerical

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computation. The current paper differs from [14] in that it focus on error analysis while [14] focuses on numerical results. Furthermore, we do obtain some results on the upper bound of approximation error.

The main contributions of this paper are: 1) We develop a novel algorithm for implementing sparse grid method based on the hierarchical polynomials for high dimensional numerical integration, the characteristics method addressed in this paper is causality free and has wonderful parallelism. It can be easily integrated in a parallel computer with any grid, such as sparse grids to mitigate the curse of dimensionality. 2) We examine the algorithm theoretically, and prove an upper bound for the approximation error and approximate it numerically. 3) We also examine the performance of the algorithm in practice, via two numerical examples with different dimensions. It is shown that the computation results using sparse grid method on the absorption problem are much more accurate than the results using Quasi-Monte Carlo integration approach in [3].

The rest of this paper is organized as follows. We construct the hierarchical bases and sparse grid in Section 2. The algorithm and error bound on sparse grids for multidimensional integration are obtained in Section 3. Two numerical examples are shown to demonstrate the accuracy, efficiency, and applications of the proposed method in Section 4.

II. HIERARCHICAL BASES AND SPARSE GRID

There are two different interpolation problems: The first one is called scattered data interpolation, the information, a vector of the form (x_i, y_i) is given and fixed. This problem is to find a smooth function f, or a polynomial f of minimal degree, such that $f(x_i) = y_i$, for $i = 1, 2, \cdots, n$. The second problem is how to select interpolation points $x_i \in \mathbb{R}^d$, which achieve a good approximation. The sparse grid interpolation problem is an optimal recovery problem.

Assume that the approximate smooth function $f : [-1, 1]^d \to \mathbb{R}$, contains finitely many function values. This kind of multivariate approximation is part of the solution of operator equations using Galerkin method.

A. ONE DIMENSIONAL HIERARCHICAL BASES

Let $\mathcal{U}^i$ be a scheme that uses N_i grid points X^i in $[-1, 1]$, where

$$X^i = \{x_1^i, x_2^i, \cdots, x_{N_i}^i\}$$

then $\{\mathcal{U}^i\}$ is a sequence of functions for quadrature. As a convention, we always set $X^0 = \emptyset$ and $\mathcal{U}^0$ to be the zero function.

If $X^{i-1} \subset X^i$, the grids $\{X^i\}$ are called nested grids. the grid points can be rearranged in this way

$$X = X^1 \cup (X^2 - X^1) \cup (X^3 - X^2) \cup \cdots = \{x_0, x_1, x_2, \cdots\}$$

with $\{x_j, j \in \mathcal{I}^i\} = X^i$, where $\mathcal{I}^i = \{0, 1, \cdots, N_i - 1\}$.

Suppose that $\omega(x) > 0, x \in [-1, 1]$ is a weight function, $V_1 \subset V_2 \subset \cdots \subset V_i \cdots$ are a set of finite dimensional spaces in $L^2_\omega([-1, 1])$, and $\{\varphi_k(x), k = 0, 1, \cdots\}$ is a sequence of basis functions of $L^2_\omega([-1, 1])$. V_i and $\varphi_k(x)$ satisfy the following relationship

$$V_i = span\{\varphi_k, k \in \mathcal{I}^i\}.$$

Then the coefficients $\{b_k^i\}$ can be determined by

$$f(x_j^i) = \sum_{k \in \mathcal{I}^i} b_k^i \varphi(x_j^i), \quad j = 1, \cdots, N_i. \tag{1}$$

The fast transforms between $\{b_k^i, k \in \mathcal{I}^i\}$ and $\{f(x_j^i), x_j^i \in X^i\}$ can be obtained when the basis functions $\{\varphi_k\}$ are Chebyshev polynomials or Fourier series. If a set of basis functions $\{\phi_k\}$ can be found with $V_i = span\{\phi_k, k \in \mathcal{I}^i\}$ and

$$\phi_k(x_j) = 0, \quad for\ any\ j \in \mathcal{I}^i,\ k \notin \mathcal{I}^i, \tag{2}$$

the basis functions $\{\phi_k, k \in \mathcal{I}^i\}$ are called a group of hierarchical bases. The expansion coefficients $\{b_j^i\}$ don't rely on the level of sparse grid, which is one of the most important property of hierarchical bases. The relative result has been obtained as follows

Lemma 1 [12]: Hierarchical bases usually exist for nested schemes in one-dimensional case. Furthermore, a sequence of hierarchical bases is obtained by

$$\phi_k(x) = \varphi_k(x) + \sum_{l \in \mathcal{I}^i} c_{k,l} \varphi_l(x), \quad k \in \Delta\mathcal{I}^{i+1},\ i = 0, 1, 2, \cdots, \tag{3}$$

where $\Delta\mathcal{I}^{i+1} = \mathcal{I}^{i+1} - \mathcal{I}^i$, and $c_{k,l} = -\varphi_k(x_j)A_{jl}$ with $A = (A_{jl})_{l,j \in \mathcal{I}^i}$ being the inverse matrix of $B = (\varphi_l(x_j))_{l,j \in \mathcal{I}^i}$.

B. MULTIDIMENSIONAL HIERARCHICAL INTERPOLATION ON SPARSE GRIDS

Consider the one-dimensional scheme $\mathcal{U}^i$, we define $\mathcal{U}^0 = 0$, $\Delta^i = \mathcal{U}^i - \mathcal{U}^{i-1}$, then the d-dimensional sparse grid by Smolyak algorithm is

$$A(q, d) = \mathcal{U}_d^q = \sum_{|\mathbf{i}| \le q} (\Delta^{i_1} \otimes \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}), \tag{4}$$

where $\mathbf{i} = (i_1, i_2, \cdots, i_d)$ represents multiple indices of grid levels, $\otimes$ is a tensor product operation, and $|\mathbf{i}| = i_1 + i_2 + \cdots + i_d$ for $\mathbf{i} \in \mathbb{N}^d$. For integers $q \ge d$, it is shown that (4) is equal to

$$A(q, d) = \sum_{q-d+1 \le |\mathbf{i}| \le q} (-1)^{q-|\mathbf{i}|} \binom{d-1}{q-|\mathbf{i}|} (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_d}), \tag{5}$$

where $\binom{d-1}{q-|\mathbf{i}|}$ is binomial coefficient of selecting $(q - |\mathbf{i}|)$ elements from an $(d-1)$ element set. It is clear that Smolyak formulas $A(q, d)$ are linear combination of tensor product operators. In order to compute $A(q, d)$, function values are only needed to obtain at the sparse grid in the set

$$H(q, d) = \bigcup_{q-d+1 \le |\mathbf{i}| \le q} (X^{i_1} \times \cdots \times X^{i_d}), \tag{6}$$

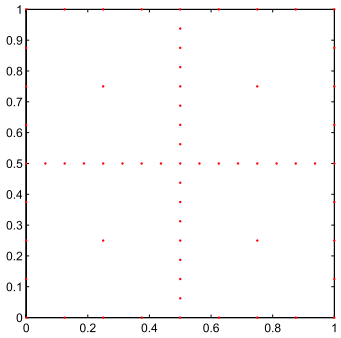

FIGURE 1. Sparse grids in $[0, 1]^2$ with $q = 6$.

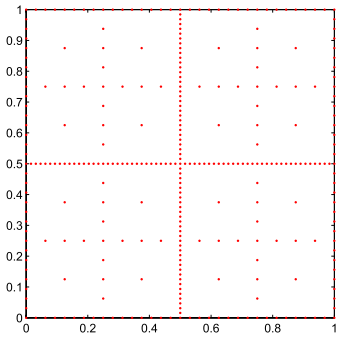

FIGURE 2. Sparse grids in $[0, 1]^2$ with $q = 8$.

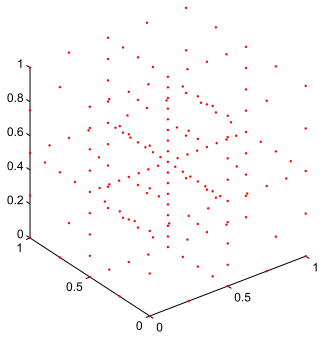

FIGURE 3. Sparse grids in $[0, 1]^3$ with $q = 6$.

where $X^i = \{x_1^i, x_2^i, \cdots, x_{N_i}^i\} \subset [-1, 1]$ indicates the set of points used by $\mathcal{U}^i$.

Then the numbers N_i of knots which are used in the formulas $\mathcal{U}^i$ can be specified. To get nested sets of points, i.e., $X^i \subset X^{i+1}$ and thereby $H(q, d) \subset H(q + 1, d)$, N_i can be selected as

$$\begin{cases} N_1 = 1, & X^1 = \left\{\frac{1}{2}\right\} \\ N_i = 2^{i-1} + 1, & X^i = \left\{\frac{k-1}{2^{i-1}} | k = 1, 2, \cdots, N_i\right\}. \end{cases} \tag{7}$$

It is well known that sparse grids have a hierarchical structure. The set of grid points contains several layers of subsets for each variable. These subsets have a telescope structure.

Four plots of 2-D and 3-D sparse grids are shown in Figure 1-4 for $q = 6$ and 8. If $q = 8$, the sparse points has 385 grid points whereas the corresponding tensor grid, has

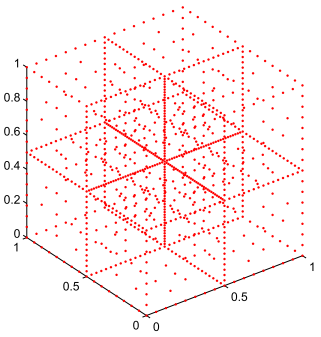

FIGURE 4. Sparse grids in $[0, 1]^3$ with $q = 8$.

TABLE 1. Number of points in the spars grid.

d	levels$(q-d)$	poly. dim.	sparse points	tensor points
2	1	3	5	4
	2	6	13	9
	3	10	29	16
	4	15	65	25
10	1	11	21	1024
	2	66	221	59045
	3	286	1581	1048576
20	1	21	41	1048576
	2	231	841	3.5e+9
50	1	51	101	1.1e+16
	2	1326	5101	7.2e+23

$(2^6+1)^2 = 4225$ points. The difference becomes increasingly significant for higher dimensions.

Smolyak formulas based on polynomial interpolation at the extrema of the Chebyshev polynomials are investigated in this paper. For any choice of $N_i > 1$ these knots are given by

$$x_j^i = -cos\frac{\pi(j-1)}{N_i - 1}, \quad j = 1, 2, \cdots, N_i \tag{8}$$

Furthermore, if $N_i = 1$, then $x_1^i = 0$. The function $\varphi(x_j^i)$ in (1) are characterized by the demand that $\mathcal{U}^i$ reproduces all polynomials of degree less than N_i.

It is also important to choose $N_1 = 1$ if we want to optimal recovery for relatively large d, because the number of points in other cases used by $A(q, d)$ increases very fast with d. The number of points in the sparse grid is shown in Table 1.

For sparse grids, a novel algorithm can be designed with the properties of hierarchical bases as follows. Let $f \in C([-1, 1]^d)$, for $d = 1$, the interpolation operator $\mathcal{U}^i$ will be obtained as

$$\mathcal{U}^i(f) = \sum_{k \in \mathcal{I}^i} b_k^i \varphi_k(x)$$

where b_k^i is deduced from (1). Further, we have

$$\mathcal{U}^i(f) = \sum_{k \in \mathcal{I}^i} b_k^i \phi_k(x), \quad \Delta^i(f)(x) = \sum_{k \in \Delta\mathcal{I}^i} b_k^i \phi_k(x).$$

where $\Delta\mathcal{I}^i = \mathcal{I}^i - \mathcal{I}^{i-1}$ Therefore, (4) can be rewritten as

$$\mathcal{U}_d^q(f)(\mathbf{x}) = \sum_{d \le \mathbf{i} \le q} \sum_{\mathbf{k} \in \Delta\mathcal{I}^{i_1} \times \cdots \times \Delta\mathcal{I}^{i_d}} b_{k_1, \cdots, k_d} \phi_{k_1}(x_1)$$
$$\cdots \phi_{k_d}(x_d) = \sum_{\mathbf{k} \in \mathcal{I}_d^q} b_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{x}), \tag{9}$$

where $\mathbf{x} = (x_1, \cdots, x_d)$, $\phi_{\mathbf{k}}(\mathbf{x}) = \phi_{k_1}(x_1) \cdots \phi_{k_d}(x_d)$, and

$$\mathcal{I}_d^q = \bigcup_{d \le |\mathbf{i}| \le q} \Delta\mathcal{I}^{i_1} \times \cdots \times \Delta\mathcal{I}^{i_d}. \tag{10}$$

We can determine the expansion coefficients $\{b_{\mathbf{k}}, \mathbf{k} \in \mathcal{I}_d^q\}$ by

$$f(\mathbf{x}_j) = \sum_{\mathbf{k} \in \mathcal{I}_d^q} b_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{x}_j) \ \ \forall \mathbf{j} \in \mathcal{I}_d^q. \tag{11}$$

Hence, an interpolation operator $A(q, d)$ is defined that maps the function values on the grid $H(q, d)$ onto the space $V(q, d) = span\{\phi_{\mathbf{k}}, \mathbf{k} \in \mathcal{I}_d^q\}$.

III. ALGORITHM AND ERROR ANALYSIS

In this section, we will describe the new algorithm in detail for the transformation between $\{f(\mathbf{x}_j)\}$ and $\{b_{\mathbf{k}}\}$ in (11), and we will show the error estimation of this method.

A. ALGORITHM

First of all, let's consider the transformation on the two dimensional tensor product grid $X^1 \times X^2 = \{(x_{j_1}, x_{j_2}), j_1 \in \mathcal{I}^1, j_2 \in \mathcal{I}^2\}$. Then we have

$$f(\mathbf{x}_j) = \sum_{\mathbf{k} \in \mathcal{I}^1 \times \mathcal{I}^1} b_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{x}_j) \ \ \forall \mathbf{j} \in \mathcal{I}^1 \times \mathcal{I}^1.$$

The coefficient $\{b_{\mathbf{k}}, \mathbf{k} \in \mathcal{I}^1 \times \mathcal{I}^1\}$ can be obtained by two dimensional transforms on $\{f(\mathbf{x}_j), \mathbf{j} \in \mathcal{I}^1 \times \mathcal{I}^1\}$, one along the first dimensional, and another along the second dimension, i.e.,

$$b'_{k_1,j_2} = \sum_{j_1 \in \mathcal{I}^1} f(x_{j_1}, x_{j_2}) T_{j_1,k_1}, \quad b_{k_1,k_2} = \sum_{j_2 \in \mathcal{I}^1} b'_{k_1,j_2} T_{j_2,k_2}$$

or

$$b''_{j_1,k_2} = \sum_{j_2 \in \mathcal{I}^1} f(x_{j_1}, x_{j_2}) T_{j_2,k_2}, \quad b_{k_1,k_2} = \sum_{j_1 \in \mathcal{I}^1} b''_{j_1,k_2} T_{j_1,k_1},$$

where T is the inverse matrix of $(\phi_k(x_j))_{k,j}$. According to property (2), the values of b'_{j_1,k_2} and b''_{k_1,j_2} don't rely on grid level. So all the coefficient $\{b_{\mathbf{k}}, \mathbf{k} \in \mathcal{I}_2^5\}$ (cf. Fig.1(b)) can be computed. We can easily extend the procedure to d-dimensional cases. The exact procedures are shown in Algorithm 1 as follows.

Algorithm 1 Fast Forward Transform on Sparse Grid

Input: $q, d, X_d^q, \{f(\mathbf{x_j}), \mathbf{j} \in \mathcal{I}_d^q\}$.
Output: $\{b_{\mathbf{k}}, \mathbf{k} \in \mathcal{I}_d^q | f(\mathbf{x_j}) = \sum_{\mathbf{k} \in \mathcal{I}_d^q} b_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{x_j})\}$

```
b_j ← f(x_j) for j ∈ I_d^q
for d' = 1 to d do
  for all i' = (i_1, ··· , i_{d'-1}, i_{d'+1}, ··· , i_d) do
  one dimensional transform along the d'dimension.
  end for
end for
```

B. ERROR ANALYSIS

For sparse grid's error estimation, the work of Peherstorfer [6] and Bungartz [1] is referred for the reader. It is well known that the size of sparse grids increases with the dimension d on the order of

$$O(N(log(N)^{d-1}),$$

that is in contrast to the size of the corresponding dense grid

$$O(N^d),$$

where the measurement level of refinement of the sparse grid is $N = 2^{q-d}$ and $q - d$. It is obvious that the accuracy has relationship with the number of grid points. For instance, here is a well known fact that an upper bound of interpolation error satisfies

$$\| e \|_{L^2} = O(N^{-2}(log(N))^{d-1}).$$

A small price in terms of accuracy with sparse grid is paid compared with a dense grid which is $O(N^{-2})$.

Suppose that $\mathbb{P}(k, d)$ is the polynomial's space in d dimensions of total degree at most k, then we can get the facts from [18].

Lemma 2: The formula $A(q, d)$ is exact on

$$E(q, d) = \sum_{\mathbf{i}|=q} \left(\mathbb{P}(N_{i_1} - 1, 1) \bigotimes \cdots \bigotimes \mathbb{P}(N_{i_d} - 1, 1)\right), \tag{12}$$

and

$$dim\mathbb{P}(k, d) = \binom{k+d}{d} \approx \frac{1}{k!} d^k \tag{13}$$

Lemma 3: The formula $A(d+k, d)$ is exact for all polynomials of degree k, and in the sequel, if let $d \to \infty$ and fix k, then the number of knots used by $A(q, d)$ are approximated to $\frac{2^k}{k!} d^k$.

The proof of the two lemmas can be found in [18].

In fact, if $\{a_1, a_2, \cdots, a_k\}$ is an arbitrary set of real numbers, then a polynomial of degree k is expressible in terms of the generalized monomials. This fact has the following analog.

Theorem 1: Let $a_1, a_2, \cdots, a_k$ be k arbitrary real numbers, and let p be any polynomial of degree k. Then there exist constants $A_0, A_1, \cdots, A_k$ such that

$$p(s) = A_0 + A_1 \binom{s+a_1}{1} + \cdots + A_k \binom{s+a_k}{k} \tag{14}$$

identically in s.

Proof: Evidently the statement of the theorem is true for $k = 0$. We proceed by induction with respect to k and assume that the theorem is true for some nonnegative integer $k - 1$. If

$$p(s) = c_0 s^k + c_1 s^{n-1} + \cdots + c_k$$

then the polynomial

$$q(s) = p(s) - c_0 k! \binom{s+a_k}{k} \tag{15}$$

TABLE 2. Computational results for the test inegral.

	Trapez		Chenshaw		Patterson	
$q-d$	Points	Error	Points	Error	Points	Error
1	1	2.608e-001	1	2.608e-001	1	608e-001
2	13	1.399e+000	13	8.456e-001	13	9.557e-003
3	85	2.671e-001	85	2.588e-001	97	1.029e-003
4	389	4.872e-001	389	2.341e-001	545	2.603e-004
5	1457	1.119e-001	1457	3.246e-002	2561	2.853e-005
6	4865	1.262e-001	4865	2.808e-002	10625	1.683e-006
7	15121	7.919e-002	15121	1.865e-003	40193	3.510e-008

is of degree $k-1$. since the leading coefficient of $\binom{s+a_k}{k}$ is

$$\frac{s^k}{k!}.$$

By the induction hypothesis, q can be represented in the form

$$q(s) = A_0 + A_1\binom{s+a_1}{1} + \cdots + A_k\binom{s+a_{k-1}}{k-1}$$

Solving (15)for $p(s)$ we obtain a representation of the desired form (14), where $A_k = c_0 k!$. □

Let's define the space

$$V_d^k = \{f : [-1,1]^d \to \mathbb{R} | D^\alpha f \text{ continuous if } \alpha_i \le k \text{ for all } i\}$$

with the norm

$$\| f \|_{W^{k,\infty}} = \max\{\| \frac{\partial^{|\mathbf{i}|}}{\partial x_1^{i_1}\cdots\partial x_d^{i_d}} f \|_{L^\infty} | 1 \le i_1, \cdots, i_d \le k\},$$

then the interpolation of f on a sparse grid satisfies

$$\| e \|_{W^{k,\infty}} = O(\frac{(logM)^{(k+2)(d-1)+1}}{M^k}),$$

where M represents the sparse grid points. Furthermore, let's define

$$\theta_i = \max_{\mathbf{x}} \sum_{j=1}^{N_i} |\varphi_j^i(x_j^i)|$$

for $i \ge 1$. Note that the number of sparse grids is the Lebesgue constant for polynomial interpolations. So we can define

$$S_l = \sum_{\mathbf{i}=l} \theta_{i_1}\theta_{i_2}\cdots\theta_{i_d}.$$

Then we will prove an upper bound for error e in the following theorem.

Theorem 2: Assume that $\epsilon > 0$ is an upper bound of the numerical error at each grid point, then we have

$$\| e \|_{L^\infty} < \epsilon \sum_{l=q-d+1}^{q} \binom{d-1}{q-l} S_l \tag{16}$$

Proof: It is obvious that

$$\sum_{|\mathbf{i}|=l} \sum_{1\le\mathbf{k}\le N_\mathbf{i}} |\varphi_{k_1}^{i_1} \otimes \cdots \otimes \varphi_{k_d}^{i_d}|$$

$$= \sum_{\mathbf{i}=l} \sum_{k_1=1}^{N_1} \cdots \sum_{k_1=d}^{N_d} |\varphi_{k_1}^{i_1} \otimes \cdots \otimes \varphi_{k_d}^{i_d}|$$

$$= \sum_{\mathbf{i}=l} (\sum_{k_1=1}^{N_1} |\varphi_{k_1}^{i_1}|) \otimes \cdots \otimes (\sum_{k_d=1}^{N_d} |\varphi_{k_d}^{i_d}|)$$

$$\le \sum_{\mathbf{i}=l} \theta_{i_1}\theta_{i_2}\cdots\theta_{i_d}$$

for any $x \in [0,1]^d$ and any integer $l > d-1$, we obtain

$$\sum_{|\mathbf{i}|=l} \sum_{1\le\mathbf{k}\le N_\mathbf{i}} |\varphi_{k_1}^{i_1} \otimes \cdots \otimes \varphi_{k_d}^{i_d}| \le S_l. \tag{17}$$

Then the upper bound (16) is a corollary of equation (17). □

IV. NUMERICAL EXAMPLES

In this section, we investigate two numerical examples, both of which are based on sparse grid method by applying them to two different systems, where they have exact solutions. In particular, we examine the computational efficiency and accuracy of the new method and compare them with other existing methods.

A. SIX DIMENSIONAL INTEGRATION PROBLEM

Once we have computed a sparse grid interpolation of an objective function, we can compute the integral value of it for the given range. Consider the following six dimensional integral problem which is from [15] as follows

$$f(x) = \int_{[0,1]^d} (1+\frac{1}{d})^d \prod_{i=1}^{d} (x_i)^{\frac{1}{d}} dx. \tag{18}$$

The exact value of the integral is equal to 1. we reproduce the results for Smolyak quadrature formulas with the trapezoidal rule (trapez), the Clenshaw-Curits (clenshaw) formulas, and the Gauss-Patterson (patterson) formulas as univariate basis integration routines for dimension $d = 6$ in Table 2.

It is shown in Table 2 that the Patterson formula performs best if the ratio of error to sparse grid points is considered. The Clenshaw-Curits and trapezoidal rules perform worst probably because the function evaluations in the origin but not location are required. The superiority of Patterson formulas over Clenshaw-Curtis formulas in sparse grid decreases with rising dimension d. This can be accounted to the fact that the Clenshaw-Curtis formulas requires less function evaluations for the same classical polynomial exactness for $q < 2d$.

B. ABSORPTION PROBLEM

Monte Carlo methods are always used to solve integral equations associated with transport. The behavior of quasi-random

sequences in this setting was studied by Sarkar and Prasa in [19], where a fairly simple one dimensional absorption problem was investigated. In order to illustrate a higher-dimensional case, we consider the same absorption problem from Morokoff in [3], given by the integral equation

$$y(x) = x + \int_x^1 \gamma y(z)dz \tag{19}$$

which describes particles traveling through a one dimensional slab of length one. In each step the particle travels a distance that is uniformly distributed on [0,1]. This may cause it to exit the slab; otherwise, it may be absorbed with probability $1-\gamma$ before the next step. In the equation, x describes the current position of the particle, and $y(x)$ gives the probability that the particle will eventually leave the slab given that it has already made it to x. The quantity of interest to compute is then $y(0)$, the probability that a particle entering the slab will leave the slab. The problem's solution is

$$y(x) = \frac{1}{\gamma} - \frac{1-\gamma}{\gamma} e^{\gamma(1-x)}. \tag{20}$$

The solution can also be rewritten by the infinite dimensional integral over the unit cube

$$y(x) = \int_{[0,1]^\infty} \sum_{n=0}^{\infty} F_n(x, \mathbf{z})d\mathbf{z}, \tag{21}$$

with

$$F_n(x, \mathbf{z}) = \gamma^n \prod_{j=1}^{n} \theta\big((1-x) - \sum_{j=1}^{n} z_j\big)\theta\big(\sum_{j=1}^{n+1} z_j - (1-x)\big), \tag{22}$$

where θ is the Heaviside function

$$\theta(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0. \end{cases} \tag{23}$$

Two alternate representations are given in [3], the first being an infinite integral with an integrand with a jump, and the second one with a smooth integrand. The sparse grid method in this paper doesn't work well for the first representation, since it is a non-smooth function where the discontinuities are not parallel to the coordinate direction. However, in case of the second representation, we can compute the very accurate results using sparse grid method.

In this frame work, the solution of the absorption problem can be represented as

$$\tilde{y}(x) = \int_{[0,1]^d} \sum_{n=0}^{d-1} F_n(x, \mathbf{z})d\mathbf{z}, \tag{24}$$

But the integrand is discontinuous in this formulation, which has a negative effect on the performance of the sparse grid method. Alternatively, we can replace the functional $F_n(x, \mathbf{z})$ by $F'_n(x, \mathbf{z})$ in the infinite-dimensional integral, and the same solution can be obtained, the more information of this result can be found in [3]. Here $F'_n(x, \mathbf{z})$ is given by

$$F'_n(x, \mathbf{z}) = \gamma^n (1-x)^n \Big(\prod_{j=1}^{n-1} (z_j)^{n-j}\Big)\Big(1 - (1-x)\prod_{j=1}^{n} z_j\Big). \tag{25}$$

Then we can perform an analogous truncation at finite dimension d. Here the solution of the absorption problem is approximated using sparse grid method with $d = 20$, $\gamma = 0.5$, and the initial particle position $x = 0$. For comparison, we also compute the integral using quasi-Monte Carlo method in [3] with the same number of points. Table 3 shows the computational results of using the two different methods respectively. It is clearly shown that the error of sparse grid method is less than quasi-Monte Carlo method.

TABLE 3. Computational results for the absorption problem.

Points	Sparse grid's error	quasi-Monte Carlo's error
41	4.622e-004	1.668e-002
87	5.606e-006	7.017e-003
177	6.010e-007	4.990e-003
367	1.566e-007	4.995e-003
739	3.893e-008	4.295e-003
1531	2.461e-008	2.234e-003
3085	1.061e-009	1.244e-003
6181	2.750e-009	1.337e-003
12393	1.335e-009	4.173e-004
24795	3.006e-010	4.940e-004
49739	1.791e-010	2.042e-004

V. CONCLUSIONS AND DISCUSSIONS

The sparse grid approach is one of the popular methods recently used to reduce the computational cost associated with spatial discretization in solving high dimensional integration problems. In this paper, we proposed an efficient algorithm for the implementation of sparse grid method based on a nested quadrature. A novel approach for the transformation between the values at each sparse grid and the coefficients of expansion in the hierarchical bases is developed by using the properties of hierarchical bases. Furthermore, another advantage of our method is that the accuracy of the interpolated solution can be numerically checked pointwise, and a theorem on the error upper bound is proved. Numerical examples have verified the effectiveness and feasibility of our schemes. It is shown that the sparse grid method is more accurate and efficient than Quasi-Monte Carlo integration approach in [3]. The algorithm proposed in the paper is essential to solve a few of high dimensional integration problems at a reasonable cost. Further progress on acceleration can be achieved by a parallelization of the method based on sparse grid which is subject to future work.

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