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New Bounds for All Types of Multi-state Consecutive k-Out-of-r-From-n: F System Reliability

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ABSTRACT A multi-state consecutive k-out-of-r-from-n: F system (named MS C(k,r,n:F)) comprises of n linearly ordered components. The MS C(k,r,n:F) and its components have more than 2 states: 0, 1, …,H. The state of MS C(k, r, n:F) is less than j if any r consecutive components contains k_l or more, for all $j \leq l \leq H$. This system is a model for a large number of applications. The existing methods for system-reliability evaluation are suitable for some special cases only. In this paper, we suggest bounds for MS C(k, r, n:F) reliability. These bounds are suitable for all system types whether in case of equal or unequal components probabilities. The suggested bounds are examined by previously published examples, when available. Also, illustration examples for the new bounds and system modelling are presented. Furthermore, we studied the cases that make the proposed bounds are sharp.

INDEX TERMS Boole-Bonferroni bounds, upper bound of hunter-worsley, multi-state consecutive k-out-of-r-from-n: F system, system reliability.

NOTATION

MS: multi-state.
C(k, r, n:F): consecutive-k-out-of-r-from-n: F system.
n: number of system components.
r: number of consecutive components, $r \leq n$.
N: number of all r-consecutive components, $N = n - r + 1$.
H: highest state for the components and system.
k_j: minimum required number of components that have a state less than j, $k_j \leq r$.
k_j^G: minimum required number of components that have a state greater than or equal j, $k_j^G = r\text{-}k_j{+}1$.
k: the vector of k_j-s.
k^G: the vector of k_j^G-s.
x_i: the state of the component i, $x_i \in \{0,1,2,\ldots,H\}$.
x: the state vector of system components, $x = (x_1, x_2,\ldots,x_n)$.
$\phi(x)$: the structure function of system state, $\phi(x) \in \{0, 1, 2, \ldots, H\}$.
$p_{i,j}$: $\Pr\{x_i = j\}, \sum_{b=0}^{H} p_{i,b} = 1$.
$Q_{i,j}$: $\Pr\{x_i < j\}, Q_{i,j} = 1 - \sum_{b=j}^{H} p_{i,b}$.
$A_{i,j}$: Event that at least k_l components in state below l, for all $j \leq l \leq H$, from the r consecutive components: i, i+1,…,$i + r$-1.
$S_{1,j}$: $\sum_{\tau_1} \Pr(A_{\tau_1,j})$, for $1 \leq \tau_1 \leq N$.
$S_{2,j}$: $\sum_{\tau_1,\tau_2} \Pr(A_{\tau_1,j}A_{\tau_2,j})$, for $1 \leq \tau_1 < \tau_2 \leq N$.
F_j: $\Pr\{\phi(x) < j\}$.
R_j: $\Pr\{\phi(x) \geq j\}$.
UB$_j$: the upper bounds of R_j.
LB$_j$: the lower bounds of R_j.
$\lfloor z \rfloor$: the lower integer part of z.
δ_j: number of components with state $< j$ from r consecutive components.
E_j: the maximum error at the level j.

I. INTRODUCTION

Reliability indicators are very importance in determining direction and determining procedures for the design and development of systems or suggesting the most efficient method of system maintenance. One of these systems is

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C(k,r,n:F), that was studied by many papers (e.g. Malinowski and Preuss [1], Tong [2], Griffith [3], Papastavridis and Koutras [4], Habib and Szántai [5], Eryilmaz *et al.* [6], and Zhu *et al.* [7]). This system comprises of n linearly arranged components and fails if any$r$-consecutive components contains at least k failed components. When $k = r$, the system is named consecutive-k-out-of-n: F system (e.g. Eryilmaz [8], Dăuş and Beiu [9], and Zhang *et al.* [10]), when $r = n$, the system is named k-out-of-n: F system (e.g. Pham [11], and Eryilmaz [12], [27]). In the last few years, there are many systems were generalized to multi-state systems which give more elasticity for modelling the equipment conditions. Such as MS k-out-n: systems [14]–[17], MS consecutive k-out-n: systems [2], [18], [19], and MS consecutive-k-out-of-r-from-n: F systems (named MS C(k,r,n:F)) [20]–[22]. These systems and its components have more than 2 states: 0, 1, …,H. In this article we will study the reliability of MS C(k,r,n:F). The MS C(k,r,n:F) is a system with redundancy, so that it is a model for a large number of applications [23]–[25] such as series of microwave towers systems, inspection procedures, mobile communications, oil pipeline, radar detection, sliding window detection probabilities, telecommunication, and sampling in statistical quality control. Therefore, evaluation the reliability of such systems is very important. The reliability of MS C(k,r,n:F) was evaluated for some special cases only, decreasing, increasing and constant types [20]–[22]. As we will see in the following section, the non-monotone MS C(k,r,n:F) type is the general case of the other types, but the reliability of this type is not existed. Furthermore, when the components are non i.i.d., the reliability of MS C(k,r,n:F) is not found. There are many systems have non i.i.d. components. Such systems are general cases of i.i.d. components-systems. But in other side, calculating their reliability is more complex. So that, some recent papers concerned generalization many models to the case where the components are non i.i.d. For example, the generalized multi-state k-out-of-n: G system [16], the phased mission parallel systems [26], and the traditional linear consecutive system [27].

In this paper, we will give a description of MS C(k,r,n:F) and some practical examples in section II. The state of MS C(k,r,n:F) is less than j if at least one event occurred of the dependent events: $A_{1,j}, A_{2,j}, \ldots, A_{N,j}$. This means that evaluation the system reliability is determined exactly by calculation the probability of the union of these events, which is very difficult. But we can evaluate its bounds using the upper bound of Hunter-Worsley and the second order Boole-Bonferroni bounds. These bounds depend on estimation the individual event probabilities $\Pr(A_{i,j})$'s and the pairwise event probabilities $\Pr(A_{i,j} \bigcap A_{l,j})$'s. The theoretical background of Boole-Bonferroni and Hunter-Worsley bounds will be given in sections III and IV. Calculation the probabilities $\Pr(A_{i,j})$ and $\Pr(A_{i,j} \bigcap A_{l,j})$ in section V will enable us to propose new bounds for all system types of MS C(k,r,n:F), decreasing, increasing, constant and non-monotone systems. The suggested bounds are suitable for i.i.d. or non i.i.d. components. An illustrative example of how to calculate these bounds will be provided. In the numerical results section, we tested the proposed bounds with previously published examples, when available. Moreover, we give other examples of general cases of MS C(k,r,n:F), which are not found previously, as well as studying the cases where the proposed bounds are sharp.

II. DESCRIPTION OF MS C(k, r, n:F)

The MS C(k,r,n:F) comprises of n linearly ordered components. The possible states of MS C(k,r,n:F) and its components are: 0, 1, …,H. The state of MS C(k,r,n:F) is less than j if at least k_l components out of any r-consecutive components, where k_l is the minimum number of the components which are in state below l for all $j \le l \le H$. In other words, $\phi(x) < j$ if at least one event occurred of the dependent events: $A_{1,j}, A_{2,j}, \ldots, A_{N,j}$. The event $A_{i,j}$ occurs if:

$$\begin{aligned} \delta_j &\ge k_j, \\ \delta_{j+1} &\ge k_{j+1}, \\ \delta_{j+2} &\ge k_{j+2}, \\ &\vdots \\ \text{and } \delta_H &\ge k_H. \end{aligned}$$

The values of $k = (k_1, k_2, k_3, \ldots, k_H)$ vector categorize the MS C(k,r,n:F) to 4 cases:

Case 1: When $k_1 \ge k_2 \ge k_3 \ge \ldots \ge k_H$, the system is named a decreasing MS C(k,r,n:F). The exact reliability of the decreasing MS C(k,r,n:F) was evaluated by Habib et al. [20].

Case 2: When $k_1 \le k_2 \le k_3 \le \ldots \le k_H$, the system is named an increasing MS C(k,r,n:F). The lower and upper bounds of increasing MS C(k,r,n:F) with equal component probability were evaluated by Radwan *et al.* [22]. But there is no any algorithm for the reliability of increasing MS C(k,r,n:F) with unequal component probability.

Case 3: When $k_1 = k_2 = k_3 = \ldots = k_H$, the system is named a constant MS C(k,r,n:F), which is a special case of decreasing or increasing MS C(k,r,n:F).

Case 4: When the values of k vector are not ordered in descending, constant, or ascending order, the system is called a non-monotone MS C(k,r,n:F). This case is the general case for the other cases. The reliability of this case is not found previously. In addition, evaluation the reliability of this case is more complex than other cases.

Note that, likewise as in the binary system, the MS C(k^G,r,n:G) and the MS C(k,r,n:F) are considered as mirror images for each other. Furthermore, the increasing MS C(k^G,r,n:G) is a decreasing MS C(k,r,n:F). The following two examples will be given to illustrate system modelling.

Example 1: Radar Detection System

For a radar detection system which consists of 10 linearly ordered radar stations. The detection levels of this system are:

1. Non detection level (state 0).
2. Low detection level (state 1).

3. Medium detection level (state 2).
4. Good detection level (state 3).

The detection levels of each station are:

1. In the first time, good detection level (state 3).
2. After some time, medium detection level (state 2).
3. After more time, low detection level (state 1).
4. The station not works, non-detection level (state 0).

The system state is evaluated as follows:

1. If any 6 consecutive stations contain at least 3 stations in state less than 1, at least 4 stations in state less than 2 and at least 5 stations in state less than 3, then the system state is less than 1.
2. If any 6 consecutive stations contain at least 4 stations in state less than 2 and at least 5 stations in state less than 3, then the system state is less than 2.
3. If any 6 consecutive stations contain at least 5 stations in state less than 3, then the system state is less than 3.

Such a system can be represented by an increasing MS C(k, r =6, n =10:F) with $k(3, 4, 5)$.

Example 2: Surveillance Cameras System

For a surveillance cameras system which consists of 25 linearly ordered cameras. The surveillance levels of the system are:

1. Non surveillance level (state 0).
2. Low surveillance level (state 1).
3. Medium surveillance level (state 2).
4. Good surveillance level (state 3).

The surveillance levels of each camera are:

1. In the first time, good surveillance level (state 3).
2. After some time, medium surveillance level (state 2).
3. After more time, low surveillance level (state 1).
4. The camera not works, non-surveillance level (state 0).

The system state is evaluated as follows:

1. If any 5 consecutive cameras contain at least 3 cameras in state less than 1, then the system state is less than 1.
2. If any 5 consecutive cameras contain at least 4 cameras in state less than 2, then the system state is less than 2.
3. If any 5 consecutive cameras contain at least 2 cameras in state less than 3, then the system state is less than 3.

Such a system can be represented by a non-monotone MS C(k, r =5, n =25:F) with $k(3, 4, 2)$.

III. BOUNDS OF BOOLE-BONFERRONI

The Boole-Bonferroni bounds are used for estimation the probability of the union of the N dependent events. These bounds were studied and improved by many papers [28]–[34]. The technique of these bounds is based on solving of the linear programming problem.

The following expected value proved by Prékopa [35]:

$$E\left[\binom{\mu}{i}\right] = \sum_{l=1}^{N} \binom{l}{i} b_l = S_{i,j},\ i = 1, 2, \ldots, N \qquad (1)$$

where μ designates the number of those events which occur (among the events $A_{1,j}, A_{2,j}, \ldots, A_{N,j}$), $b_l = \Pr(\mu = l)$ and

$$\binom{l}{i} = 0,\ \text{if } i > l.$$

The value $S_{i,j}$ called the i^{th} binomial moment of μ.

By consideration $b_1, b_2 \ldots, b_N$ as variables and estimate $S_{1,j}, S_{2,j}, \ldots, S_{V,j}$; $V < N$, then we have the following linear programming problems:

$$\begin{aligned}
&\text{Minimize } \{b_1 + b_2 + \ldots + b_V + \ldots + b_N\}\\
&\text{Subject to :}\\
&b_1 + \binom{2}{1} b_2 + \ldots + \binom{V}{1} b_V + \ldots + \binom{N}{1} b_N = S_{1,j}\\
&b_2 + \ldots + \binom{V}{2} b_V + \ldots + \binom{N}{2} b_N = S_{2,j}\\
&\ddots \ \ddots \ \vdots\\
&b_V + \ldots + \binom{N}{V} b_N = S_{V,j}\\
&b_1 \geq 0, b_2 \geq 0, \ldots, b_V \geq 0, \ldots, b_N \geq 0
\end{aligned} \qquad (2)$$

$$\begin{aligned}
&\text{Maximize } \{b_1 + b_2 + \ldots + b_V + \ldots + b_N\}\\
&\text{Subject to :}\\
&b_1 + \binom{2}{1} b_2 + \ldots + \binom{V}{1} b_V + \ldots + \binom{N}{1} b_N = S_{1,j}\\
&b_2 + \ldots + \binom{V}{2} b_V + \ldots + \binom{N}{2} b_N = S_{2,j}\\
&\ddots \ \ddots \ \vdots\\
&b_V + \ldots + \binom{N}{V} b_N = S_{V,j}\\
&b_1 \geq 0, b_2 \geq 0, \ldots, b_V \geq 0, \ldots, b_N \geq 0
\end{aligned} \qquad (3)$$

Solving of these problems will give us the best possible of lower and upper bounds for:

$$\Pr(A_{1,j} + \ldots + A_{N,j}) = \Pr(\mu \geq 1) \qquad (4)$$

These bounds are named Boole-Bonferroni bounds. When V =2, the solutions of problems (2) and (3) give us the second order of Boole-Bonferroni bounds as follows:

$$\begin{aligned}
\frac{2S_{1,j}}{u_j + 1} - \frac{2S_{2,j}}{u_j(u_j + 1)} &\leq \Pr(A_{1,j} + \ldots + A_{N,j})\\
&\leq S_{1,j} - \frac{2S_{2,j}}{N},
\end{aligned} \qquad (5)$$

where:

$$u_j = 1 + \left\lfloor \frac{2S_{2,j}}{S_{1,j}} \right\rfloor. \qquad (6)$$

Evaluation these bounds depends on estimation the binomial moments $S_{1,j}$ and $S_{2,j}$, $j = 1, 2, 3, \ldots, H$.

IV. UPPER BOUND OF HUNTER-WORSLEY

Hunter [36], Worsley [37] derived an effective upper bound using the binomial moment $S_{1,j}$ and some of the specific probabilities involved in $S_{2,j}$. Hunter–Worsley upper bound can be used for estimation the probability of the union of the N dependent events. This upper bound can be calculated quickly and always sharper than the Boole-Bonferroni upper bound in (5). The upper bound of Hunter–Worsley is given by:

$$\Pr(A_{1,j}+\ldots+A_{N,j}) \leq S_{1,j} - \sum_{a\in T} \Pr(A_{a,j}A_{a+1,j}), \quad (7)$$

where $T = \{1, 2, \ldots, N-1\}$.

V. PROPOSED BOUNDS FOR MS C(k, r, n:F)

Using the definition of $A_{i,j}$, $i = 1, 2, \ldots, N$, the state of MS C(k,r,n:F) is less than j, if at least one event occurred of the dependent events: $A_{1,j}, A_{2,j}, \ldots, A_{N,j}$. Then

$$F_j = \Pr\left\{\bigcup_{i\in\Omega} A_{i,j}\right\} \text{ for all } \Omega = \{1, 2, \ldots, N\} \quad (8)$$

Evaluation the exact value of F_j is very difficult. Therefore, we can give an approximation for MS C(k,r,n:F) reliability using Boole-Bonferroni bounds as follows:

$$\frac{2S_{1,j}}{u_j+1} - \frac{2S_{2,j}}{u_j(u_j+1)} \leq F_j \leq S_{1,j} - \frac{2S_{2,j}}{N}, \quad (9)$$

where:

$$u_j = 1 + \left\lfloor \frac{2S_{2,j}}{S_{1,j}} \right\rfloor. \quad (10)$$

Using Hunter–Worsley upper bound we have:

$$F_j \leq S_{1,j} - \sum_{a\in T} \Pr(A_{a,j}A_{a+1,j}), \quad (11)$$

where $T = \{1, 2, \ldots, N-1\}$.

In order to evaluate these bounds, we need to calculate the binomial moments $S_{1,j}$ and $S_{2,j}$. Therefore, we suggest the following formulae for evaluation $S_{1,j}$ and $S_{2,j}$, which are the main result in this paper.

A. EVALUATION THE BINOMIAL MOMENT $S_{1,j}$

The binomial moment $S_{1,j}$ is defined by:

$$S_{1,j} = \Pr(A_{1,j}) + \Pr(A_{2,j}) + \ldots + \Pr(A_{N,j}), \quad (12)$$

where

$$\Pr(A_{a,j}) = \prod_{i=a}^{a+r-1} \sum_{y_i=x_i}^{1} Q_{i,j}^{y_i}.\beta_{i,j}, \quad (13)$$

$$x_i = \max(0, k_j - (a+r-1-i) - \sum_{u=a}^{i-1} y_u), \quad (14)$$

$$\beta_{i,j} = \begin{cases} \prod_{g=0}^{H-j-1} \sum_{t_{i,g}=0}^{\min(T_{i,g},T'_{i,g},\ldots,T'_{i,H-j-1})} p_{i,H-g}^{t_{i,g}}.p_{i,j}^{T_{i,H-j}}; & j < H \\ p_{i,j}^{1-y_i}; j = H \end{cases} \quad (15)$$

$$T_{i,g} = 1 - y_i - \sum_{v=0}^{g-1} t_{i,v}, \quad (16)$$

$$T'_{i,g} = r - k_{H-g} - \sum_{u=a}^{i-1} \sum_{v=0}^{g} t_{u,v}. \quad (17)$$

B. EVALUATION THE BINOMIAL MOMENT $S_{2,j}$

The binomial moment $S_{2,j}$ is defined by:

$$S_{2,j} = \sum_{a,b} \Pr(A_{a,j}A_{b,j}),\ 1 \leq a < b \leq N$$
$$= \sum_{a=1}^{N-1} \sum_{b=a+1}^{N} \Pr(A_{a,j}A_{b,j}), \quad (18)$$

where $\Pr(A_{a,j}A_{b,j})$, $1 \leq a < b \leq N$, is calculated through two cases:

Case 1: If b-a >r-1, then:

$$\Pr(A_{a,j}A_{b,j}) = \Pr(A_{a,j}) \times \Pr(A_{b,j}). \quad (19)$$

Case 2: If b-a ≤r-1, then:

$$\Pr(A_{a,j}A_{b,j}) = \prod_{i=a}^{b+r-1} \sum_{y_i=\max(0,k_j-r-d_i+i+1)}^{1} Q_{i,j}^{y_i}.\alpha_{i,j}, \quad (20)$$

$$d_i = \begin{cases} a + \sum_{u=a}^{i-1} y_u;\ i < a+r \\ b + \sum_{u=b}^{i-1} y_u;\ i \geq a+r \end{cases} \quad (21)$$

$$\alpha_{i,j} = \begin{cases} \prod_{g=0}^{H-j-1} \sum_{t_{i,g}=0}^{\min(T_{i,g},T'_{i,g},\ldots,T'_{i,H-j-1})} p_{i,H-g}^{t_{i,g}}.p_{i,j}^{T_{i,H-j}}; & j < H \\ p_{i,j}^{1-y_i}; j = H \end{cases} \quad (22)$$

$$T_{i,g} = 1 - y_i - \sum_{v=0}^{g-1} t_{i,v}, \quad (23)$$

$$T'_{i,g} = \begin{cases} r - k_{H-g} - \sum_{u=a}^{i-1} \sum_{v=0}^{g} t_{u,v};\ i < a+r \\ r - k_{H-g} - \sum_{u=b}^{i-1} \sum_{v=0}^{g} t_{u,v};\ i \geq a+r \end{cases} \quad (24)$$

After estimation the lower bound and upper bounds of F_j, we can get the lower bound and upper bound of R_j by:

$$\mathrm{LB}_j = 1 - \text{(upper bound of } F_j), \quad (25)$$
$$\mathrm{UB}_j = 1 - \text{(lower bound of } F_j). \quad (26)$$

Furthermore, the Approximate value of R_j is given by:

$$\hat{R}_j = \frac{LB_j + UB_j}{2}. \quad (27)$$

The maximum error is:

$$E_j = UB_j - \hat{R}_j = \hat{R}_j - LB_j. \quad (28)$$

Example 3:

Consider an increasing MS C(k,4,6:F) with k=(2,3,4) and the unequal components probabilities are given in Table 1.

TABLE 1. The probabilities of 20 components ($p_{i,j}$).

i \ j	0	1	2	3
1	0.12	0.14	0.25	0.49
2	0.15	0.23	0.33	0.29
3	0.08	0.19	0.5	0.23
4	0.06	0.35	0.41	0.18
5	0.1	0.2	0.24	0.46
6	0.12	0.14	0.28	0.46
7	0.09	0.19	0.33	0.39
8	0.11	0.16	0.34	0.39
9	0.12	0.23	0.28	0.37
10	0.18	0.24	0.37	0.21
11	0.06	0.27	0.39	0.28
12	0.19	0.21	0.33	0.27
13	0.09	0.15	0.28	0.48
14	0.13	0.15	0.26	0.46
15	0.12	0.22	0.44	0.22
16	0.16	0.14	0.3	0.4
17	0.17	0.22	0.29	0.32
18	0.2	0.3	0.25	0.25
19	0.06	0.3	0.3	0.34
20	0.09	0.31	0.39	0.21

The system state is:

- Less than 1, if any 4 consecutive components contain $\delta_1 \geq 2$, $\delta_2 \geq 3$ and $\delta_3 \geq 4$.
- Less than 2, if any 4 consecutive components contain $\delta_2 \geq 3$ and $\delta_3 \geq 4$.
- Less than 3, if any 4 consecutive components contain $\delta_3 \geq 4$.

Furthermore, all the possible states of the events $A_{1,j}, A_{2,j}, A_{3,j}, A_{1,j}A_{2,j}$ and $A_{1,j}A_{3,j}$ for all $j = 1, 2, 3$ are given in Table 2 and Table 3, which mentioned for illustration only.

For state 3:

$$\begin{aligned}
\Pr(A_{1,3}) &= \sum_{y_1=1}^{1} Q_{1,3}^{y_1} \cdot \beta_{1,3} \sum_{y_2=1}^{1} Q_{2,3}^{y_2} \cdot \beta_{2,3} \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \\
&\quad \cdot \beta_{3,3} \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \cdot \beta_{4,3} \\
&= 0.228630 \\
\Pr(A_{2,3}) &= \sum_{y_2=1}^{1} Q_{2,3}^{y_2} \cdot \beta_{2,3} \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \cdot \beta_{3,3} \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \\
&\quad \cdot \beta_{4,3} \sum_{y_5=1}^{1} Q_{5,3}^{y_5} \cdot \beta_{5,3} \\
&= 0.242079 \\
\Pr(A_{3,3}) &= \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \cdot \beta_{3,3} \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \cdot \beta_{4,3} \sum_{y_5=1}^{1} Q_{5,3}^{y_5} \\
&\quad \cdot \beta_{5,3} \sum_{y_6=1}^{1} Q_{6,3}^{y_6} \cdot \beta_{6,3} \\
&= 0.184116 \\
S_{1,3} &= \Pr(A_{1,3}) + \Pr(A_{2,3}) + \Pr(A_{3,3}) = 0.654825 \\
\Pr(A_{1,3}A_{2,3}) &= \sum_{y_1=1}^{1} Q_{1,3}^{y_1} \cdot \alpha_{1,3} \cdot \sum_{y_2=1}^{1} Q_{2,3}^{y_2} \cdot \alpha_{2,3} \cdot \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \\
&\quad \cdot \alpha_{3,3} \cdot \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \cdot \alpha_{4,3} \\
&\quad \cdot \sum_{y_5=1}^{1} Q_{5,3}^{y_5} \cdot \alpha_{5,3} = 0.123460 \\
\Pr(A_{2,3}A_{3,3}) &= \sum_{y_2=1}^{1} Q_{2,3}^{y_2} \cdot \alpha_{2,3} \cdot \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \cdot \alpha_{3,3} \\
&\quad \cdot \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \cdot \alpha_{4,3} \\
&\quad \cdot \sum_{y_5=1}^{1} Q_{5,3}^{y_5} \cdot \alpha_{5,3} \cdot \sum_{y_6=1}^{1} Q_{6,3}^{y_6} \cdot \alpha_{6,3} = 0.130723 \\
\Pr(A_{1,3}A_{3,3}) &= \sum_{y_1=1}^{1} Q_{1,3}^{y_1} \cdot \alpha_{1,3} \cdot \sum_{y_2=1}^{1} Q_{2,3}^{y_2} \cdot \alpha_{2,3} \\
&\quad \cdot \sum_{y_3=1}^{1} Q_{3,3}^{y_3} \cdot \alpha_{3,3} \cdot \sum_{y_4=1}^{1} Q_{4,3}^{y_4} \cdot \alpha_{4,3} \\
&\quad \cdot \sum_{y_5=1}^{1} Q_{5,3}^{y_5} \cdot \alpha_{5,3} \cdot \sum_{y_6=1}^{1} Q_{6,3}^{y_6} \cdot \alpha_{6,3} = 0.066668 \\
S_{2,3} &= \Pr(A_{1,3}A_{2,3}) + \Pr(A_{2,3}A_{3,3}) + \Pr(A_{1,3}A_{3,3}) \\
&= 0.320851 \\
i &= 1 + \left\lfloor \frac{2S_{2,3}}{S_{1,3}} \right\rfloor = 1
\end{aligned}$$

Using Boole-Bonferroni bounds, $0.333974 \leq F_3 \leq 0.440924$.

Using Hunter–Worsley upper bound, $F_3 \leq 0.400642$.

Then $F_3 = 0.367308 \pm 0.033334$

For state 2:

$$\begin{aligned}
\Pr(A_{1,2}) &= \sum_{y_1=0}^{1} Q_{1,2}^{y_1} \cdot \beta_{1,2} \sum_{y_2=1-y_1}^{1} Q_{2,2}^{y_2} \\
&\quad \cdot \beta_{2,2} \sum_{y_3=2-y_1-y_2}^{1} Q_{3,2}^{y_3} \cdot \beta_{3,2} \\
&\quad \cdot \sum_{y_4=3-y_1-y_2-y_3}^{1} Q_{4,2}^{y_4} \cdot \beta_{4,2} = 0.062143 \\
\Pr(A_{2,2}) &= \sum_{y_2=0}^{1} Q_{2,2}^{y_2} \cdot \beta_{2,2} \sum_{y_3=1-y_2}^{1} Q_{3,2}^{y_3}
\end{aligned}$$

TABLE 2. The possible states for states 2 and 3 in example 3.

State 2			State 3		
$A_{1,2}$	$A_{1,2}A_{2,2}$	$A_{1,2}A_{3,2}$	$A_{1,3}$	$A_{1,3}A_{2,3}$	$A_{1,3}A_{3,3}$
(●,●,●,●)	(●,●,●,●,●)	(●,●,●,●,●,●)	(*,*,*,*)	(*,*,*,*,*)	(*,*,*,*,*,*)
(●,●,●,2)	(●,●,●,●,2)	(●,●,●,●,●,2)			
(●,●,2,●)	(●,●,●,2,●)	(●,●,●,●,2,●)			
(●,2,●,●)	(●,●,2,●,●)	(●,●,●,2,●,●)			
(2,●,●,●)	(●,2,●,●,●)	(●,●,2,●,●,●)			
	(2,●,●,●,●)	(●,2,●,●,●,●)			
	(2,●,●,●,2)	(●,2,●,●,●,2)			
		(●,2,●,●,2,●)			
		(2,●,●,●,●,●)			
		(2,●,●,●,●,2)			
		(2,●,●,●,2,●)			

● : represent state 0 or 1
* : represent state 0, 1 or 2

TABLE 3. The possible states for state 1 in example 3.

$A_{1,1}$	$A_{1,1}A_{2,1}$		$A_{1,1}A_{3,1}$				
(0,0,0,0)	(0,0,0,0,0)	(1,0,0,0,0)	(0,0,0,0,0,0)	(0,1,0,0,1,0)	(0,2,0,1,0,1)	(1,0,2,0,0,0)	(2,0,0,1,1,0)
(0,0,0,1)	(0,0,0,0,1)	(1,0,0,0,1)	(0,0,0,0,0,1)	(0,1,0,0,1,1)	(0,2,0,1,0,2)	(1,0,2,0,0,1)	(2,0,0,1,2,0)
(0,0,0,2)	(0,0,0,0,2)	(1,0,0,0,2)	(0,0,0,0,0,2)	(0,1,0,0,1,2)	(0,2,0,1,1,0)	(1,0,2,0,1,0)	(2,0,1,0,0,0)
(0,0,1,0)	(0,0,0,1,0)	(1,0,0,1,0)	(0,0,0,0,1,0)	(0,1,0,0,2,0)	(0,2,0,1,2,0)	(1,1,0,0,0,0)	(2,0,1,0,0,1)
(0,0,1,1)	(0,0,0,1,1)	(1,0,0,1,1)	(0,0,0,0,1,1)	(0,1,0,0,2,1)	(0,2,1,0,0,0)	(1,1,0,0,0,1)	(2,0,1,0,0,2)
(0,0,1,2)	(0,0,0,1,2)	(1,0,0,1,2)	(0,0,0,0,1,2)	(0,1,0,1,0,0)	(0,2,1,0,0,1)	(1,1,0,0,0,2)	(2,0,1,0,1,0)
(0,0,2,0)	(0,0,0,2,0)	(1,0,0,2,0)	(0,0,0,0,2,0)	(0,1,0,1,0,1)	(0,2,1,0,0,2)	(1,1,0,0,1,0)	(2,0,1,0,2,0)
(0,0,2,1)	(0,0,0,2,1)	(1,0,0,2,1)	(0,0,0,0,2,1)	(0,1,0,1,0,2)	(0,2,1,0,1,0)	(1,1,0,0,1,1)	(2,1,0,0,0,0)
(0,1,0,0)	(0,0,1,0,0)	(1,0,1,0,0)	(0,0,0,1,0,0)	(0,1,0,1,1,0)	(0,2,1,0,2,0)	(1,1,0,0,1,2)	(2,1,0,0,0,1)
(0,1,0,1)	(0,0,1,0,1)	(1,0,1,0,1)	(0,0,0,1,0,1)	(0,1,0,1,2,0)	(1,0,0,0,0,0)	(1,1,0,0,2,0)	(2,1,0,0,0,2)
(0,1,0,2)	(0,0,1,0,2)	(1,0,1,0,2)	(0,0,0,1,0,2)	(0,1,0,2,0,0)	(1,0,0,0,0,1)	(1,1,0,0,2,1)	(2,1,0,0,1,0)
(0,1,1,0)	(0,0,1,1,0)	(1,0,2,0,0)	(0,0,0,1,1,0)	(0,1,0,2,0,1)	(1,0,0,0,0,2)	(1,2,0,0,0,0)	(2,1,0,0,1,1)
(0,1,2,0)	(0,0,1,2,0)	(1,0,2,0,1)	(0,0,0,1,2,0)	(0,1,0,2,1,0)	(1,0,0,0,1,0)	(1,2,0,0,0,1)	(2,1,0,0,1,2)
(0,2,0,0)	(0,0,2,0,0)	(1,1,0,0,0)	(0,0,0,2,0,0)	(0,1,1,0,0,0)	(1,0,0,0,1,1)	(1,2,0,0,0,2)	(2,1,0,0,2,0)
(0,2,0,1)	(0,0,2,0,1)	(1,1,0,0,1)	(0,0,0,2,0,1)	(0,1,1,0,0,1)	(1,0,0,0,1,2)	(1,2,0,0,1,0)	(2,1,0,0,2,1)
(0,2,1,0)	(0,0,2,1,0)	(1,1,0,0,2)	(0,0,0,2,1,0)	(0,1,1,0,0,2)	(1,0,0,0,2,0)	(1,2,0,0,1,1)	
(1,0,0,0)	(0,1,0,0,0)	(1,2,0,0,0)	(0,0,1,0,0,0)	(0,1,1,0,1,0)	(1,0,0,0,2,1)	(1,2,0,0,1,2)	
(1,0,0,1)	(0,1,0,0,1)	(1,2,0,0,1)	(0,0,1,0,0,1)	(0,1,1,0,2,0)	(1,0,0,1,0,0)	(1,2,0,0,2,0)	
(1,0,0,2)	(0,1,0,0,2)	(2,0,0,0,0)	(0,0,1,0,0,2)	(0,1,2,0,0,0)	(1,0,0,1,0,1)	(1,2,0,0,2,1)	
(1,0,1,0)	(0,1,0,1,0)	(2,0,0,0,1)	(0,0,1,0,1,0)	(0,1,2,0,0,1)	(1,0,0,1,0,2)	(2,0,0,0,0,0)	
(1,0,2,0)	(0,1,0,2,0)	(2,0,0,0,2)	(0,0,1,0,2,0)	(0,1,2,0,1,0)	(1,0,0,1,1,0)	(2,0,0,0,0,1)	
(1,1,0,0)	(0,1,1,0,0)	(2,0,0,1,0)	(0,0,1,1,0,0)	(0,2,0,0,0,0)	(1,0,0,1,2,0)	(2,0,0,0,0,2)	
(1,2,0,0)	(0,1,2,0,0)	(2,0,0,1,1)	(0,0,1,2,0,0)	(0,2,0,0,0,1)	(1,0,0,2,0,0)	(2,0,0,0,1,0)	
(2,0,0,0)	(0,2,0,0,0)	(2,0,0,1,2)	(0,0,2,0,0,0)	(0,2,0,0,0,2)	(1,0,0,2,0,1)	(2,0,0,0,1,1)	
(2,0,0,1)	(0,2,0,0,1)	(2,0,1,0,0)	(0,0,2,0,0,1)	(0,2,0,0,1,0)	(1,0,0,2,1,0)	(2,0,0,0,1,2)	
(2,0,1,0)	(0,2,0,1,0)	(2,0,1,0,1)	(0,0,2,0,1,0)	(0,2,0,0,1,1)	(1,0,1,0,0,0)	(2,0,0,0,2,0)	
(2,1,0,0)	(0,2,1,0,0)	(2,0,1,0,2)	(0,0,2,1,0,0)	(0,2,0,0,1,2)	(1,0,1,0,0,1)	(2,0,0,0,2,1)	
		(2,1,0,0,0)	(0,1,0,0,0,0)	(0,2,0,0,2,0)	(1,0,1,0,0,2)	(2,0,0,1,0,0)	
		(2,1,0,0,1)	(0,1,0,0,0,1)	(0,2,0,0,2,1)	(1,0,1,0,1,0)	(2,0,0,1,0,1)	
		(2,1,0,0,2)	(0,1,0,0,0,2)	(0,2,0,1,0,0)	(1,0,1,0,2,0)	(2,0,0,1,0,2)	

$$\cdot \ \beta_{3,2} \sum_{y_4=2-y_2-y_3}^{1} Q_{4,2}^{y_4} \cdot \beta_{4,2}$$

$$\cdot \sum_{y_5=3-y_2-y_3-y_4}^{1} Q_{5,2}^{y_5} \cdot \beta_{5,2} = 0.069665$$

$$\Pr(A_{3,2}) = \sum_{y_3=0}^{1} Q_{3,2}^{y_3} \cdot \beta_{3,2} \sum_{y_4=1-y_2}^{1} Q_{4,2}^{y_4}$$

$$\cdot \ \beta_{4,2} \sum_{y_5=2-y_3-y_4}^{1} Q_{5,2}^{y_5} \cdot \beta_{5,2}$$

$$\cdot \sum_{y_6=3-y_3-y_4-y_5}^{1} Q_{6,2}^{y_6} \cdot \beta_{6,2} = 0.049466$$

$$S_{1,2} = \Pr(A_{1,2}) + \Pr(A_{2,2}) + \Pr(A_{3,2}) = 0.181273$$

$$\Pr(A_{1,2}A_{2,2}) = \sum_{y_1=0}^{1} Q_{1,2}^{y_1} \cdot \alpha_{1,2} \cdot \sum_{y_2=1-y_1}^{1} Q_{2,2}^{y_2} \cdot \alpha_{2,2}$$

$$\cdot \sum_{y_3=2-y_1-y_2}^{1} Q_{3,2}^{y_3} \cdot \alpha_{3,2}$$
$$\cdot \sum_{y_4=3-y_1-y_2-y_3}^{1} Q_{4,2}^{y_4} \cdot \alpha_{4,2} \cdot \sum_{y_5=3-y_2-y_3-y_4}^{1} Q_{5,2}^{y_5}$$
$$\cdot \alpha_{5,2} = 0.023792$$

$$\Pr(A_{2,2}A_{3,2}) = \sum_{y_2=0}^{1} Q_{2,2}^{y_2} \cdot \alpha_{2,2} \cdot \sum_{y_3=1-y_2}^{1} Q_{3,2}^{y_3} \cdot \alpha_{3,2}$$
$$\cdot \sum_{y_4=2-y_2-y_3}^{1} Q_{4,2}^{y_4} \cdot \alpha_{4,2}$$
$$\cdot \sum_{y_5=3-y_2-y_3-y_4}^{1} Q_{1,2}^{y_5} \cdot \alpha_{1,2} \cdot \sum_{y_6=3-y_3-y_4-y_5}^{1} Q_{6,2}^{y_6}$$
$$\cdot \alpha_{6,2} = 0.024715$$

$$\Pr(A_{1,2}A_{3,2}) = \sum_{y_1=0}^{1} Q_{1,2}^{y_1} \cdot \alpha_{1,2} \cdot \sum_{y_2=1-y_1}^{1} Q_{2,2}^{y_2} \cdot \alpha_{2,2}$$
$$\cdot \sum_{y_3=2-y_1-y_2}^{1} Q_{3,2}^{y_3} \cdot \alpha_{3,2}$$
$$\cdot \sum_{y_4=3-y_1-y_2-y_3}^{1} Q_{4,2}^{y_4} \cdot \alpha_{4,2}$$
$$\cdot \sum_{y_5=2-y_3-y_4}^{1} Q_{5,2}^{y_5} \cdot \alpha_{5,2}$$
$$\cdot \sum_{y_6=3-y_3-y_4-y_5}^{1} Q_{6,2}^{y_6} \cdot \alpha_{6,2} = 0.009378$$

$$S_{2,2} = \Pr(A_{1,2}A_{2,2}) + \Pr(A_{2,2}A_{3,2}) + \Pr(A_{1,2}A_{3,2})$$
$$= 0.057885$$
$$i = 1 + \left\lfloor \frac{2S_{2,2}}{S_{1,2}} \right\rfloor = 1$$

Using Boole-Bonferroni bounds, $0.123388 \leq F_2 \leq 0.142683$.

Using Hunter–Worsley upper bound, $F_2 \leq 0.132767$.

Then $F_2 = 0.128077 \pm 0.004689$

For state 1:

$$\Pr(A_{1,1}) = \sum_{y_1=0}^{1} Q_{1,1}^{y_1} \cdot \beta_{1,1} \sum_{y_2=0}^{1} Q_{2,1}^{y_2}$$
$$\cdot \beta_{2,1} \sum_{y_3=\max(0,1-y_1-y_2)}^{1} Q_{3,1}^{y_3} \cdot \beta_{3,1}$$
$$\cdot \sum_{y_4=\max(0,2-y_1-y_2-y_3)}^{1} Q_{4,1}^{y_4} \cdot \beta_{4,1} = 0.016936$$

$$\Pr(A_{2,1}) = \sum_{y_2=0}^{1} Q_{2,1}^{y_2} \cdot \beta_{2,1} \sum_{y_3=0}^{1} Q_{3,1}^{y_3}$$
$$\cdot \beta_{3,1} \sum_{y_4=\max(0,1-y_2-y_3)}^{1} Q_{4,1}^{y_4} \cdot \beta_{4,1}$$
$$\cdot \sum_{y_5=\max(0,2-y_2-y_3-y_4)}^{1} Q_{5,1}^{y_5} \cdot \beta_{5,1} = 0.015919$$

$$\Pr(A_{3,1}) = \sum_{y_3=0}^{1} Q_{3,1}^{y_3} \cdot \beta_{3,1} \sum_{y_4=0}^{1} Q_{4,1}^{y_4}$$
$$\cdot \beta_{4,1} \sum_{y_5=\max(0,1-y_3-y_4)}^{1} Q_{5,1}^{y_5} \cdot \beta_{5,1}$$
$$\cdot \sum_{y_6=\max(0,2-y_3-y_4-y_5)}^{1} Q_{6,1}^{y_6} \cdot \beta_{6,1} = 0.012258$$

$$S_{1,1} = \Pr(A_{1,1}) + \Pr(A_{2,1}) + \Pr(A_{3,1}) = 0.045113$$

$$\Pr(A_{1,1}A_{2,1}) = \sum_{y_1=0}^{1} Q_{1,1}^{y_1} \cdot \alpha_{1,1} \cdot \sum_{y_2=0}^{1} Q_{2,1}^{y_2}$$
$$\cdot \alpha_{2,1} \cdot \sum_{y_3=\max(0,1-y_1-y_2)}^{1} Q_{3,1}^{y_3} \cdot \alpha_{3,1}$$
$$\cdot \sum_{y_4=\max(0,2-y_1-y_2-y_3)}^{1} Q_{4,1}^{y_4} \cdot \alpha_{4,1}$$
$$\cdot \sum_{y_5=\max(0,2-y_2-y_3-y_4)}^{1} Q_{5,2}^{y_5} \cdot \alpha_{5,2}$$
$$= 0.004013$$

$$\Pr(A_{2,1}A_{3,1}) = \sum_{y_2=0}^{1} Q_{2,1}^{y_2} \cdot \alpha_{2,1} \cdot \sum_{y_3=0}^{1} Q_{3,1}^{y_3} \cdot \alpha_{3,1}$$
$$\cdot \sum_{y_4=\max(0,1-y_2-y_3)}^{1} Q_{4,1}^{y_4} \cdot \alpha_{4,1}$$
$$\cdot \sum_{y_5=\max(0,2-y_2-y_3-y_4)}^{1} Q_{5,1}^{y_5} \cdot \alpha_{5,1}$$
$$\cdot \sum_{y_6=\max(0,2-y_3-y_4-y_5)}^{1} Q_{6,2}^{y_6} \cdot \alpha_{6,2}$$
$$= 0.00391$$

$$\Pr(A_{1,1}A_{3,1}) = \sum_{y_1=0}^{1} Q_{1,1}^{y_1} \cdot \alpha_{1,1} \cdot \sum_{y_2=0}^{1} Q_{2,1}^{y_2} \cdot \alpha_{2,1}$$
$$\cdot \sum_{y_3=\max(0,1-y_1-y_2)}^{1} Q_{3,1}^{y_3} \cdot \alpha_{3,1}$$
$$\cdot \sum_{y_4=\max(0,2-y_1-y_2-y_3)}^{1} Q_{4,1}^{y_4} \cdot \alpha_{4,1}$$
$$\cdot \sum_{y_5=\max(0,1-y_3-y_4)}^{1} Q_{5,1}^{y_5} \cdot \alpha_{5,1}$$

TABLE 4. An example from [22], $n = 30$, $r = 15$, $k_1 = 7$, $k_2 = 9$, $k_3 = 11$, $k_4 = 13$, $p_0 = 0.1$, $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.2$, $p_4 = 0.4$.

Bounds	S_1- S_2 based	Hunter-Worsley	$\hat{R}_j$	E_j	Time (second)
UB_1	**0.999824**	-	0.999777	0.000047	0.000546
LB_1	0.999628	**0.999729**			
UB_2	**0.998870**	-	0.998491	0.000380	
LB_2	0.996924	**0.998111**			
UB_3	**0.983682**	-	0.976557	0.007126	
LB_3	0.943906	**0.969431**			
UB_4	**0.927776**	-	0.893282	0.034494	
LB_4	0.701955	**0.858788**			

TABLE 5. A non-monotone MS C(k, r= 14, n=20:F), with $k_1 = 9$, $k_2 = 7$, $k_3 = 11$, $p_0 = 0.1$, $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.4$.

Bounds	S_1- S_2 based	Hunter-Worsley	$\hat{R}_j$	E_j	Time (second)
UB_1	**0.999996**	-	0.999996	0.000001	0.000092
LB_1	0.999994	**0.999995**			
UB_2	**0.931434**	-	0.909707	0.021726	
LB_2	0.843692	**0.887981**			
UB_3	**0.816489**	-	0.766402	0.050086	
LB_3	0.599991	**0.716316**			

$$\cdot \sum_{y_6=\max(0,2-y_3-y_4-y_5)}^{1} Q_{6,2}^{y_6} \cdot \alpha_{6,2} = 0.001184$$

$$S_{2,1} = \Pr(A_{1,1}A_{2,1}) + \Pr(A_{2,1}A_{3,1}) + \Pr(A_{1,1}A_{3,1}) = 0.009095$$

$$i = 1 + \left\lfloor \frac{2S_{2,1}}{S_{1,1}} \right\rfloor = 1$$

Using Boole-Bonferroni bounds,
$0.036017 \leq F_1 \leq 0.039049$.
Using Hunter–Worsley upper bound, $F_1 \leq 0.037201$.
Then $F_1 = 0.036609 \pm 0.000592$

VI. NUMERICAL RESULTS

The numerical results of the proposed bounds were performed on Intel Core i5 with a 2.3 GHz CPU and 6 GB of RAM under Windows 10 operating system using VISUAL BASIC Program. The execution time is calculated per seconds for all examples in this section. As we will see in the Tables 4–7, the execution time is small, which demonstrates the effectiveness of the proposed bounds. Using our new bounds, we obtained the same results of all published examples of increasing MS C(k,r,n:F) [22] when the components are i.i.d., binary C(k,r,n:F) when H =1 [5], and generalized multi-state k-out-of-n: G system [16] when r=n. All of these systems are special cases of our bounds. For instance, Table 4 contains the same results for an example given by Radwan *et al.* [22]. This example evaluates the reliability of increasing MS C(k, r, n:F) when the components are i.i.d. In the Tables 5–7, we evaluated the reliability of MS C(k,r,n:F) for some new cases, which is not found previously, using the proposed bounds.

The numerical results in Table 5 are given for a non-monotone MS C(k, r= 14, n=20:F) when the components are i.i.d.

TABLE 6. An increasing MS C(k, r=12, n=20:F).

k_j	Bounds	S_1- S_2 based	Hunter-Worsley	$\hat{R}_j$	E_j	Time (second)
k_1=5, k_2=7, k_3=10	UB_1	**0.993232**	-	0.991087	0.002144	0.323671
	LB_1	0.983520	**0.988943**			
	UB_2	**0.925234**	-	0.895495	0.029739	
	LB_2	0.799708	**0.865756**			
	UB_3	**0.738726**	-	0.640687	0.098039	
	LB_3	0.290894	**0.542648**			
k_1=7, k_2=9, k_3=11	UB_1	**0.999918**	-	0.999900	0.000017	0.261302
	LB_1	0.999845	**0.999883**			
	UB_2	**0.994089**	-	0.992417	0.001672	
	LB_2	0.986975	**0.990746**			
	UB_3	**0.908125**	-	0.876188	0.031937	
	LB_3	0.750382	**0.844252**			

TABLE 7. A non-monotone MS C(k,r=13,n:F) with k_1 =9, k_2 =7, k_3 =11.

n	Bounds	S_1- S_2 based	Hunter-Worsley	$\hat{R}_j$	E_j	Time (second)
n=15	UB_1	**0.999998**	-	0.999998	0.000000	0.220103
	LB_1	0.999998	**0.999998**			
	UB_2	**0.954617**	-	0.950171	0.004446	
	LB_2	0.942018	**0.945725**			
	UB_3	**0.851446**	-	0.841014	0.010432	
	LB_3	0.820227	**0.830582**			
n=20	UB_1	**0.999998**	-	0.999997	0.000001	0.272001
	LB_1	0.999995	**0.999997**			
	UB_2	**0.928449**	-	0.902231	0.026218	
	LB_2	0.821123	**0.876013**			
	UB_3	**0.815647**	-	0.754557	0.061090	
	LB_3	0.544558	**0.693467**			

Table 6 contains numerical results for an increasing MS C(k, r=12, n=20:F) when the components are non i.i.d. The components probabilities for Tables 6 and 7 are given in Table 1. The results in the first half of Table 6 were calculated when k_1 =5, k_2 =7, k_3 =10 and the results in the second half were calculated when k_1 =7, k_2 =9, k_3 =11. By comparing the results in this table, it is clear that, the increasing of k_j values leads to the decreasing of E_j values. For example, $E_1 = 0.002144$ when k_1 =5, and $E_1 = 0.000017$ when k_1 =7. Thus, we can conclude that the new bounds are sharp when the difference between k and r decreases.

In Table 7, we evaluated the numerical results for a non-monotone MS C(k, r=13, n:F) when $k_1 = 9$, $k_2 = 7$, $k_3 = 11$, and the components are non i.i.d. The results in the first half of this table were performed when $n = 15$ and the results in the second half were performed when $n = 20$. Comparing the results between the two halves of this table shows that the decreasing of n value leads to the decreasing ofE_j values. For instance, $E_2 = 0.026218$ when n =20, and $E_2 = 0.004446$ when n =15. So, we can conclude that the suggested bounds are sharp when the difference between r and n decreases.

In all examples in the Tables 4–7, we note that the maximum error at the level j less than or equal the maximum error at the level j+1 (i.e. $E_j \leq E_{j+1}$). This indicates that the proposed bounds are sharp when the state of the required level decreases.

VII. CONCLUSION

The existing bounds for MS C(k,r,n:F) reliability are suitable for some special cases: decreasing, increasing, and constant MS C(k,r,n:F) with equal components probabilities only. In this paper, we suggested new bounds for general cases of MS C(k,r,n:F). The reliability of these cases is not found previously. These bounds are suitable for all system types: decreasing, increasing, constant and non-monotone systems, whether the components are i.i.d. or non i.i.d. The suggested bounds were examined by previously published examples, when available. Moreover, many examples are given for illustration the new bounds and system modelling. The new bounds become sharp when the combination of system parameters k_j, r, n and $p_{i,j}$ makes the system reliability increases. Therefore, there are many cases that make the proposed bounds are sharper such as:

1. Decreasing the difference between r and n; or
2. Decreasing the difference between k_j values and r; or
3. Increasing the probabilities of the components; or
4. Decreasing the state of the required level.

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