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The Probabilistic Interval-Valued Hesitant Pythagorean Fuzzy Set and Its Application in Selecting Processes of Project Private Partner

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ABSTRACT The expression of fuzzy information under multi-attribute decision making (MADM) is constantly expanded by scholars to solve the problem of uncertain decision making in various application fields. Such as select project private partners, which is difficult to make an appropriate select in a complex and changeable environments. Although the aggregation operator under interval-valued hesitant Pythagorean fuzzy environment is an effective method to solve uncertain decision-making problems, there are still some drawbacks in aggregating operators that do not consider the information loss. In this paper, we develop Hamacher operations and Choquet integral-based method to solve select project private partner problem under probabilistic interval-valued hesitant Pythagorean fuzzy information, which could express decision-makers' preference information more flexibly and consider the significance and the correlations among the elements. Firstly, we define the probabilistic interval-valued hesitant Pythagorean fuzzy set (PIVHPFS) as an extended mathematical expression of fuzzy sets (FS). Afterward, the Hamacher algorithm concepts are given under the PIVHPFS environment. Besides, we utilize Hamacher operations and Choquet interval-based method to develop the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator. At the same time, some definitions and theorems based on PIVHPFHCIG operator are proposed. After that, we utilize the PIVHPFHCIG operator to develop an approach to solve the MADM problems under the PIVHPFS situation. The new method is feasible to overcome the drawback of information loss, and it is more reasonable for obtaining a better decision result. Finally, the introduction of the best project private partner selecting problem proves the effectiveness and feasibility of PIVHPFS, and the comparison between PIVHPFS and other similar techniques decision methods are also provided.

INDEX TERMS Multi-attribute decision making (MADM), probabilistic interval-valued hesitant pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator, Hamacher algorithm, Choquet integral-based.

I. INTRODUCTION

Project cooperation is a win-win model, which can ease the tension in the fund chain during the project construction process. The selection of the project's private partners is in the preparatory stage of the project and plays a decisive role in the completion of the whole project. If the project leader makes a mistake in the selection of a private partner, it will bring many potential risks to the entire project. The decision making in selecting processes of project private partner includes qualitative and quantitative attributes. In the real-world decision-making process, the evaluation of qualitative attributes of various enterprises by relevant experts is fuzzy and uncertain, which is difficult to determine subjectively with accurate numerical values. The fuzzy multi-attribute decision making (FMADM) is perfect to avoid this problem. FMADM is widely applied in many fields, such as human resource evaluation, green supplier selection and

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military management, etc., especially under the circumstance of uncertain and complex decision making. It usually has three steps as selection, evaluation and decision. In 1965, Zadeh [1] applied fuzzy set (FS) to solve the problem of incomplete, fuzzy and imperfect in the step of evaluation in MADM; however, it is not able to present the attitude of decision-makers. Therefore, in 1986, Atanassov [2] introduced intuitionistic fuzzy set (IFS) to consider the membership and non-membership degree, this had given a more complete description of the fuzzy and uncertainty in MADM. In 2010, Torra [3] introduced hesitant fuzzy set (HFS) which enabled that one evaluation information may have a set of different possible values, this provide a huge advantage in depicting hesitation preference of decision-makers. More recently, the Pythagorean fuzzy set (PFS) has been applied by Yager [4] is that fulfills the condition is only required that the square sum of membership and non-membership degree should be equal or less than one, rather than the sum of them should be equal or less than one. Beyond and dispute, the PFS have more ability than IFS and HFS to express evaluation information for future MADM problems. Based on the theory of PFS, Khan and Abdullah [5] introduced interval-valued Pythagorean fuzzy Choquet integral average (IVPFCIA) operator and extended the concept of traditional grey relational analysis (GRA) method to solve MADM problem. Khan *et al.* [6] proposed a new extension of classical VIKOR method for MADM problems under PFS information. Khan *et al.* [7] based on fuzzy measures, an interval-valued Pythagorean fuzzy Choquet integral geometric (IVPFCIG) operator is investigated for MAGDM problems. Khan *et al.* [8] proposed a novel approach based on TOPSIS method and the maximizing deviation method for solving MADM problems under PFS information and the information about attribute weights is incomplete. Khan *et al.* [9] introduced the concept of Pythagorean hesitant fuzzy set (PHFS) and define score and accuracy degree of PHFS, and developed maximizing deviation method for solving MADM problems. Khan *et al.* [10] developed several Pythagorean hesitant fuzzy Choquet integral operators under PFS information with fuzzy measure for MADM problems. Khan [11] used the Choquet integral and Einstein operations to develop Pythagorean fuzzy aggregation operators for MAGDM problems.

The research on aggregation operators is very rich. Firstly, Atanassov [12] researched the order operation of the IFS and the sum and product operations in 1994. Then, De *et al.* [13] extend the sum and product of IFS to scalar multiplication and exponentiation. These aggregation operators are based on algebraic sum and algebraic product. After that, Wang and Liu [14] developed the new IFS aggregation operators based on Einstein t-conorm and t-norm. Hamacher t-conorm and t-norm [15], which include the parameter which provides the available choices to decision-makers during the aggregation operators process and hence makes it more general and more flexible to model the MADM than others. Garg [16] defined intuitionistic fuzzy Hamacher interaction weighted averaging and geometric aggregation operators with entropy weight to solve MADM problems. In the real-world decision-making process, the evaluation information among MADM process frequently cannot be expressed accurately by specific value. In 1989, Atanassov and Gargov [17] applied interval-valued intuitionistic fuzzy set (IVIFS) to overcome the drawback. The essence of interval-valued is to replace the real numbers value of membership and non-membership degree by intervals. After that, many researchers do well in extending different interval-valued fuzzy sets for fuzzy MADM. In order to identify the uncertainty in MADM, Chen *et al.* [18] extended HFS to interval-valued hesitant fuzzy set (IVHFS) in which the membership of elements to a given set is not precisely defined, but consists of several interval values. At the same time, a decision-making method based on interval-valued hesitant preference relation is developed to consider the differences of opinions between individual decision-makers. Garg [19] introduced interval-valued Pythagorean fuzzy set (IVPFS) and two new aggregation operators, and developed an improved accuracy function under IVPFS environment by considering the unknown hesitation degree.

In most cases, it is assumed that all elements in different FS are independent. In 1974, Sugeno [20] introduced fuzzy measure to simulate the interactions phenomena among decision criteria. It proved that it is not sufficient to solve the MADM and multi-attribute group decision making (MAGDM) by assuming the independence between characters. Therefore, the Choquet integral [21] is widely applied in MADM and MAGDM problems, many researchers have extended it to solve the decision-making problems under different fuzzy environments. Xu [22] established the intuitionistic fuzzy set operator which may not only consider the ranks when calculators are concentrated but also point out the dependence between characters. Wei *et al.* [23] had formed up some algorithm and functions based on hesitant fuzzy Choquet integral in order to solve the problem of multi-attribute decision-making problem under the circumstance of hesitant fuzzy situation. Peng and Yang [24] considered the importance and correlation between the factors and applied Pythagorean fuzzy Choquet integral operators in MADM. Joshi and Kumar [25] developed an improved intuitionistic fuzzy Choquet integral operator for MADM Problems. Pasi *et al.* [26] proposed a multi-criteria decision-making approach based on the Choquet integral for assessing the credibility of user-generated content. Abdullah and Mohd [27] considered the interactions among criteria, using Choquet integral operators to develop an innovation to the Pythagorean fuzzy Hamacher operator.

The uncertainty in personal decision making is an important influence factor of the final decision. Although HFS may present the opinion of decision-makers, it will fail to represent the completed decision information since the information loss problem. Thus, some researchers introduced probability into fuzzy sets. Zhang *et al.* [28] applied the probabilistic hesitant fuzzy set and its calculation rules. Wang and Li [29] avoided the inconsistence in decision making by probabilistic hesitant fuzzy information methods based on correlation parameters. It is no doubt that researchers had already improved a lot in the description of decision making by hesitant fuzzy sets. Zhai *et al.* [30] defined the expand mathematical prescription for PIVHFS, they also established some new measurement models based on PIVIHFSs. In order to extend the HPFS, Li *et al.* [31] established probabilistic hesitant multi-attribute decision-making model based on PHFS and BWM. Farhadinia and Xu [32] defined adjusted PLTS and transformed it into OWHFE. Li and Wang [33] solved the group decision-making problem by forming up the agreement between the PHFPR and predicted consistence. Zhou and Xu [34] introduced uncertain probabilistic hesitant fuzzy elements (UPHFEs) and also expanded it into uncertain probabilistic hesitant fuzzy preference relationships (UPHFPRs). Li and Wang [35] defined the probabilistic hesitant fuzzy preference relation (PHFPR) on the basis of expected multiplicative consistency transitivity in multi-criteria decision-making.

On the one hand, aggregating information in the decision-making process is very important. On the other hand, although many effective methods have been applied, the information loss and inaccuracy problems are still existing in decision-making process. However, all of the scholars fail to solve MADM problems based on Hamacher algorithm and Choquet integral-based method under the probabilistic interval-valued hesitant Pythagorean fuzzy environment. We can see that most of the existing hesitant Pythagorean fuzzy or interval-valued hesitant Pythagorean fuzzy aggregation operators are based on algebraic product and algebraic sum, which decision-makers often are limited in rational. Lacking of attitudinal character parameter decision-makers is unable to express optimistic or pessimistic attitudes towards evaluation information. The hesitancy and probability have been not combined in IVPFS environment at the same time to describe uncertain information in real-world decision-making problems. To overcome this limitation, and motivated from the above-mentioned idea, we define a new aggregate operator in this paper, it is called as probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator. The contributions of this paper are as follows:

(1) This paper expands the concept of IVHPFS to PIVHPFS and formulated the score function and accuracy function of PIVHPHE, ways to compare and rank two PIVHPHEs.
(2) This paper utilizes Hamacher operations and Choquet interval-based method to develop the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator, where the correlations among the elements are considered.
(3) This paper utilizes the PIVHPFHCIG operator to develop an approach to select the best project private partner under PIVHPFS situation, which achieves higher economic efficiency and value for money.

To do so, the remainder of this paper is organized as follows: Section II reviews some previous researches such as IVHPFSs, Choquet integral and Hamacher algorithm. In Section III, we define the concept and Hamacher algorithm of PIVHPFS. In addition, its score function and comparison method are applied. In Section IV, we propose the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator and its several basic theorems. After that, based on Choquet interval-based method and PIVHPFHCIG operator, a new MADM approach is proposed in Section V. In Section VI, the example of selecting best project private partner is applied to illustrated and test the method based on PIVHPFHCIG operator, as well as some necessary comparison with existing method under IVPFS and HPFS environment. Finally, in Section VII, the essay is concluded.

II. PRELIMINARIES

In this section the interval-valued hesitant Pythagorean fuzzy set (IVHPFS), Choquet integral and Hamacher algorithm will be introduced.

A. INTERVAL-VALUE HESITATION PYTHAGOREAN FUZZY SET (IVHPFS)

Definition 1 [36]: Let X be a finite set, the interval-valued hesitant Pythagorean fuzzy set (IVHPFS) $\tilde{P}$ can be defined as:

$$\tilde{P} = \left\{ \left\langle x_i, \left(\tilde{h}_{\tilde{P}}(x_i), \tilde{g}_{\tilde{P}}(x_i) \right) | x_i \in X \right\rangle \right\}, \tag{1}$$

where $i = 1, 2, \cdots, n$, $\tilde{h}_{\tilde{P}}(x_i), \tilde{g}_{\tilde{P}}(x_i)$ are the intervals belong to [0,1], they represent all the probable IVHPFS membership and non-membership degrees in $\tilde{P}$. Given $x_i \in X$, $\tilde{P} = \left(\tilde{h}_{\tilde{P}}(x_i), \tilde{g}_{\tilde{P}}(x_i) \right)$ is identified as a interval-valued hesitant Pythagorean fuzzy element (IVHPFE),

$$\left(\tilde{h}_{\tilde{P}}, \tilde{g}_{\tilde{P}} \right) = \left\{ \tilde{\gamma} | \tilde{\gamma} \in \tilde{h}_{\tilde{P}}(x_i), \tilde{g}_{\tilde{P}}(x_i) \right\} \tag{2}$$

where $\tilde{\gamma} = \left(\left[\tilde{u}^L, \tilde{u}^U \right], \left[\tilde{v}^L, \tilde{v}^U \right] \right)$ is the interval-valued, and $\tilde{u}^L = \inf \tilde{h}_{\tilde{P}}(x_i), \tilde{u}^U = \sup \tilde{h}_{\tilde{P}}(x_i)$, $\tilde{v}^L = \inf \tilde{h}_{\tilde{P}}(x_i), \tilde{v}^U = \sup \tilde{h}_{\tilde{P}}(x_i)$ and $0 \leq \left(\tilde{u}^U \right)^2 + \left(\tilde{v}^U \right)^2 \leq 1$.

There are two special forms of IVHPFSs:

(1) If $\tilde{u}^U = \tilde{u}^L, \tilde{v}^U = \tilde{v}^L$, then IVHPFSs is reduced to the HPFSs.
(2) If $u^U + v^U \leq 1$, then IVHPFSs is reduced IVHIFS.

Definition 2 [37]: Let X be a finite set, and $\tilde{P} = \left\langle \left[\tilde{u}^L, \tilde{u}^U \right], \left[\tilde{v}^L, \tilde{v}^U \right] \right\rangle$ be a finite IVPFS correlated with X, the score function is defined as: $S(\tilde{P}) = \frac{1}{2}((\tilde{u}^L)^2 + (\tilde{u}^U)^2 - (\tilde{v}^L)^2 - (\tilde{v}^U)^2)$, and the accuracy function is identified as: $H(\tilde{P}) = \frac{1}{2}((\tilde{u}^L)^2 + (\tilde{u}^U)^2 + (\tilde{v}^L)^2 + (\tilde{v}^U)^2)$.

B. FUZZY MEASURE AND CHOQUET INTEGRAL

When the characteristics of the candidate plans are being gathered, the additivity is not followed. In order to solve the problem, the fuzzy measure was applied by replacing the additivity measure with a monotonic metric.

Definition 3 [20]: Let X be a finite set and ρ be a set function $\rho : P(X) \to [0,1]$. $P(X)$ is the power set of X, there exist a fuzzy measure ρ on X if it satisfies:

(1) $\rho(\emptyset) = 0$, $\rho(X) = 1$;
(2) $\forall B, C \subseteq X$, if $B \subseteq C$, then $\rho(B) \leq \rho(C)$;
(3) $\rho(B \cup C) = \rho(B) + \rho(C) + \tau\rho(B)\rho(C)$, for all $B, C \subseteq X$ and $B \cap C = \emptyset$, where $\tau > -1$.

In the MADM process, fuzzy measure ρ is able to describe the relationship between characteristics since ρ is a non-negative and non-addable set function. Let X be the characteristic set, and the importance of sub characteristic set B and C can be expressed by their fuzzy measure $\rho(B), \rho(C)$.

Let C be finite set, then $\cup_{i=1}^{n} c_i = C$, the measure definition of ρ under τ can be expressed as:

$$\rho(C) = \begin{cases} \frac{1}{\tau}\left(\prod_{i=1}^{n}(1+\tau\rho(c_i)) - 1\right), & \tau \neq 0 \\ \sum_{i=1}^{n}\rho(c_i), & \tau = 0. \end{cases} \tag{3}$$

where $c_i \cap c_j = \emptyset$, $i \neq j$, the sub set with only one element c_i, $\rho(c_i)$ is called the fuzzy measure and $\rho_i = \rho(c_i)$. When $\rho(C) = 1$, then

$$\tau = \prod_{i=1}^{n}(1+\tau\rho(c_i)) - 1. \tag{4}$$

Definition 4 [21]: Let f be the positive real number function in X, ρ is the fuzzy measure in X, then the discrete Choquet integral for f according to ρ is:

$$C_\rho\left(f_{c(1)}, f_{c(2)}, \cdots, f_{c(n)}\right) = \sum_{i=1}^{n} f_{c(i)}\left[\rho\left(A_{(i)}\right) - \rho\left(A_{(i+1)}\right)\right] \tag{5}$$

where (i) is the substitute in $f_{c(i)}$ which lead to $f_{c(1)} \leq f_{c(2)} \leq \cdots \leq f_{c(n)}$, $A_{(i)} = \{c(i), c(i+1), \cdots, c(n)\}$, and $A_{(n+1)} = 0$.

C. HAMACHER ALGORITHM

$t-$norms and $t-$conorms are applied in fuzzy set identification, which is very significant in MADM.

Definition 5 [38]: Generalized $t-$norm and $t-$conorm are introducted by Hamacher, this includes Hamacher product and Hamacher sum, and the definitions are:

Hamacher product $\otimes$ is the $t-$norm, Hamacher sum $\oplus$ is the $t-$conorm:

$$T(a,b) = a \otimes b = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)} \tag{6}$$

$$T*(a,b) = a \oplus b = \frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab} \tag{7}$$

where $\gamma > 0$, especially, when $\gamma = 1$, then Hamacher $t-$norm and $t-$conorm equal to algebraic $t-$norm and $t-$conorm:

$$T(a,b) = a \otimes b = ab \tag{8}$$

$$T*(a,b) = a \oplus b = a+b-ab \tag{9}$$

when $\gamma = 2$, then Hamacher $t-$norm and $t-$conorm are equal to Einstein $t-$norm and $t-$conorm:

$$T(a,b) = a \otimes b = \frac{ab}{1+(1-a)(1-b)} \tag{10}$$

$$T*(a,b) = a \oplus b = \frac{a+b}{1+ab} \tag{11}$$

III. PROBABILISTIC INTERVAL-VALUED HESITANT PYTHAGOREAN FUZZY SET (PIVHPFS)

Definition 6: Let X be a finite set, then a Probabilistic interval-valued hesitant Pythagorean fuzzy set (PIVHPFS) associated with X can be identified as:

$$S = \left\{\left\langle x, \left[\left(\left[\tilde{u}^L, \tilde{u}^U\right], \left[\tilde{v}^L, \tilde{v}^U\right]\right), p\right] | x \in X\right\rangle\right\} \tag{12}$$

S is formed up by a series of probabilistic interval-valued hesitant Pythagorean fuzzy elements (PIVHPFE) which can be expressed as $\left[\left(\left[\tilde{u}^L(x), \tilde{u}^U(x)\right], \left[\tilde{v}^L(x), \tilde{v}^U(x)\right]\right), p\right]$, these elements include both membership and non-membership degrees. Obviously, PIVHPFE is a special situation of PIVHPFS, similar to the relationship between a fuzzy number and fuzzy set. Every PIVHPFE is formed up by a group of interval-valued Pythagorean fuzzy number (IVPHFN) and the probability, where the probability is used to present the possible degree of its corresponding IVPHFN. When $\left[\left(\left[\tilde{u}^L, \tilde{u}^U\right], \left[\tilde{v}^L, \tilde{v}^U\right]\right), p\right]$ is a finite PIVHPFE, it can be identified as $[([\tilde{u}_{i_l}^L(x), \tilde{u}_{i_l}^U(x)], [\tilde{v}_{i_l}^L(x), \tilde{v}_{i_l}^U(x)]), p_{i_l}]$, where $l = 1, 2, \cdots, L(PIVHPFE)$ are positive integer that express the number of elements that contained by PIVHPFE and with the probability that $p_{i_l} \in [0,1]$, $\sum_{l=1}^{L(PIVHPFE)} p_{i_l} \leq 1$. The vagueness interval $\left[\tilde{\pi}_{i_l}^L(x), \tilde{\pi}_{i_l}^U(x)\right]$ can be calculated by $\tilde{\pi}_{i_l}^L(x) = \sqrt{1-\left(\tilde{u}_{i_l}^U(x)\right)^2 - \left(\tilde{v}_{i_l}^U(x)\right)^2}$ and $\tilde{\pi}_{i_l}^U(x) = \sqrt{1-\left(\tilde{u}_{i_l}^L(x)\right)^2 - \left(\tilde{v}_{i_l}^L(x)\right)^2}$.

Example 1: Let X be a finite set, there are two PIVHPFEs on X, $\langle[([0.3, 0.4], [0.4, 0.5]), 0.9]\rangle$ and $\langle[([0.1, 0.2], [0.3, 0.4]), 0.3], [([0.2, 0.3], [0.3, 0.4]), 0.7]\rangle$, each PIVHPFE independently depicts the entire uncertainty space. The first PIVHPFE of PIVHPFS uses only one probabilistic IVHPFN to characterize the entire uncertain environment, where $\sum_{i=1}^{L(PIVHPFE_2)} p_i = 0.9 < 1$; another PIVHPFE uses two probabilistic IVHPFNs to characterize the entire uncertainty space, where $\sum_{i=1}^{L(PIVHPFE_1)} p_i = 0.3 + 0.7 = 1$.

Let $L(PIVHPFE_1)$ and $L(PIVHPFE_2)$ be the number of elements of $PIVHPFE_1$ and $PIVHPFE_2$. let $L(PIVHPFE_1) = L(PIVHPFE_2)$ for calculation purpose, assume that the number of elements of PIVHFE is $L(PIVHPFE)$.

In order to compare the size of different PIVHPHEs, it is necessary to include membership degree, non-membership degree and hesitation degree, $\left(\left[\tilde{u}_{i_l}^L(x), \tilde{u}_{i_l}^U(x)\right], \left[\tilde{v}_{i_l}^L(x), \tilde{v}_{i_l}^U(x)\right], \left[\tilde{\pi}_{i_l}^L(x), \tilde{\pi}_{i_l}^U(x)\right]\right)$, to define the following score function and exact function.

Definition 7: Let X be a finite set, and $\tilde{P}_i = \left[\left(\left[\tilde{u}_{i_l}^L(x), \tilde{u}_{i_l}^U(x)\right], \left[\tilde{v}_{i_l}^L(x), \tilde{v}_{i_l}^U(x)\right]\right), p_{i_l}\right]$ be a collection of PIVHPHEs associated with X, where $i = 1, 2, \cdots, n, l = 1, 2, \cdots, L(PIVHPFE)$.

The PIVHPHE's score function $\tilde{P}_i$ can be formulated as:

$$S\left(\tilde{P}_i\right) = \sum_{l=1}^{L(PIVHPFE)} p_{i_l} \times [(\tilde{u}_{i_l}^L(x) - \tilde{v}_{i_l}^U(x)) + (\tilde{u}_{i_l}^U(x) - \tilde{v}_{i_l}^L(x))]/2 \quad (13)$$

The PIVHPHE's accuracy function $\tilde{P}_i$ can be defined as:

$$H\left(\tilde{P}_i\right) = \sum_{l=1}^{L(PIVHPFE)} p_{i_l} \times \frac{2 - \tilde{\pi}_{i_l}^L(x) - \tilde{\pi}_{i_l}^U(x)}{2} \quad (14)$$

Definition 8: Let X be a finite set, for any two $\tilde{P}_i = \left[\left(\left[\tilde{u}_{i_l}^L(x), \tilde{u}_{i_l}^U(x)\right], \left[\tilde{v}_{i_l}^L(x), \tilde{v}_{i_l}^U(x)\right]\right), p_{i_l}\right]$ be the finite PIVHPHEs associated with X, where $i = 1, 2, l = 1, 2, \cdots, L(PIVHPFE)$, then:

(1) If $S\left(\tilde{P}_1\right) > S\left(\tilde{P}_2\right)$, then $\tilde{P}_1 \succ \tilde{P}_2$, that is, $\tilde{P}_1$ is greater than $\tilde{P}_2$;
(2) If $S\left(\tilde{P}_1\right) < S\left(\tilde{P}_2\right)$, then $\tilde{P}_1 \prec \tilde{P}_2$, that is, $\tilde{P}_1$ is less than $\tilde{P}_2$;
(3) If $S\left(\tilde{P}_1\right) = S\left(\tilde{P}_2\right)$, then
   (i) If $H\left(\tilde{P}_1\right) > H\left(\tilde{P}_2\right)$, then $\tilde{P}_1 \succ \tilde{P}_2$, that is, $\tilde{P}_1$ is greater than $\tilde{P}_2$;
   (ii) If $H\left(\tilde{P}_1\right) < H\left(\tilde{P}_2\right)$, then $\tilde{P}_1 \prec \tilde{P}_2$, that is, $\tilde{P}_1$ is less than $\tilde{P}_2$;
   (iii) If $H\left(\tilde{P}_1\right) = H\left(\tilde{P}_2\right)$, then $\tilde{P}_1 \sim \tilde{P}_2$, that is, $\tilde{P}_1$ is equals $\tilde{P}_2$.

Definition 9: Let X be a finite set, and for any $\tilde{P}_i = \left\langle \tilde{u}_{i_l}(x), \tilde{v}_{i_l}(x), p_{i_l} \right\rangle$ be the finite PHPFE associated with X, where $i = 1, 2, \cdots, n, l = 1, 2, \cdots, L(PHPFE)$, $S\left(\tilde{P}_i\right) = \sum_{l=1}^{L(PIVHPFE)} p_{i_l} \times \left(\tilde{u}_{i_l}(x) - \tilde{v}_{i_l}(x)\right)$ is called the score function and $H\left(\tilde{P}_i\right) = \sum_{l=1}^{L(PIVHPFE)} p_{i_l} \times \left(\tilde{u}_{i_l}(x) + \tilde{v}_{i_l}(x)\right)$ is called the accuracy function.

Definition 10: Let X be a finite set, for any two $\tilde{P}_i = \left[\left(\left[\tilde{u}_{i_l}^L(x), \tilde{u}_{i_l}^U(x)\right], \left[\tilde{v}_{i_l}^L(x), \tilde{v}_{i_l}^U(x)\right]\right), p_{i_l}\right]$ be the finite PIVHPHEs associated with X, where $i = 1, 2, l = 1, 2, \cdots, L(PIVHPFE)$, $\gamma \in (0, +\infty), \lambda > 0$, the probabilistic interval-valued hesitant Pythagorean Hamacher fuzzy (PIVHPHF) operation can be identified as follows:

(1)

$$\tilde{P}_1 \oplus \tilde{P}_2 = \langle[\sqrt{\frac{\left(\tilde{u}_{1_l}^L\right)^2 + \left(\tilde{u}_{2_l}^L\right)^2 - \left(\tilde{u}_{1_l}^L\right)^2\left(\tilde{u}_{2_l}^L\right)^2 - (1-\gamma)\left(\tilde{u}_{1_l}^L\right)^2\left(\tilde{u}_{2_l}^L\right)^2}{\gamma + (1-\gamma)\left(1 - \left(\tilde{u}_{1_l}^L\right)^2\left(\tilde{u}_{2_l}^L\right)^2\right)}},$$
$$\sqrt{\frac{\left(\tilde{u}_{1_l}^U\right)^2 + \left(\tilde{u}_{2_l}^U\right)^2 - \left(\tilde{u}_{1_l}^U\right)^2\left(\tilde{u}_{2_l}^U\right)^2 - (1-\gamma)\left(\tilde{u}_{1_l}^U\right)^2\left(\tilde{u}_{2_l}^U\right)^2}{\gamma + (1-\gamma)\left(1 - \left(\tilde{u}_{1_l}^U\right)^2\left(\tilde{u}_{2_l}^U\right)^2\right)}}],$$
$$[\frac{\tilde{v}_{1_l}^L \tilde{v}_{2_l}^L}{\sqrt{\gamma + (1-\gamma)\left(\left(\tilde{v}_{1_l}^L\right)^2 + \left(\tilde{v}_{2_l}^L\right)^2 - \left(\tilde{v}_{1_l}^L\right)^2\left(\tilde{v}_{2_l}^L\right)^2\right)}},$$
$$\frac{\tilde{v}_{1_l}^U \tilde{v}_{2_l}^U}{\sqrt{\gamma + (1-\gamma)\left(\left(\tilde{v}_{1_l}^U\right)^2 + \left(\tilde{v}_{2_l}^U\right)^2 - \left(\tilde{v}_{1_l}^U\right)^2\left(\tilde{v}_{2_l}^U\right)^2\right)}}],$$
$$\overline{p_{1_l} + p_{2_l}}\rangle \quad (15)$$

(2)

$$\tilde{P}_1 \otimes \tilde{P}_2 = \langle[\frac{\tilde{u}_{1_l}^L \tilde{u}_{2_l}^L}{\sqrt{\gamma + (1-\gamma)\left(\left(\tilde{u}_{1_l}^L\right)^2 + \left(\tilde{u}_{2_l}^L\right)^2 - \left(\tilde{u}_{1_l}^L\right)^2\left(\tilde{u}_{2_l}^L\right)^2\right)}},$$
$$\frac{\tilde{u}_{1_l}^U \tilde{u}_{2_l}^U}{\sqrt{\gamma + (1-\gamma)\left(\left(\tilde{u}_{1_l}^U\right)^2 + \left(\tilde{u}_{2_l}^U\right)^2 - \left(\tilde{u}_{1_l}^U\right)^2\left(\tilde{u}_{2_l}^U\right)^2\right)}}],$$
$$[\sqrt{\frac{\left(\tilde{v}_{1_l}^L\right)^2 + \left(\tilde{v}_{2_l}^L\right)^2 - \left(\tilde{v}_{1_l}^L\right)^2\left(\tilde{v}_{2_l}^L\right)^2 - (1-\gamma)\left(\tilde{v}_{1_l}^L\right)^2\left(\tilde{v}_{2_l}^L\right)^2}{\gamma + (1-\gamma)\left(1 - \left(\tilde{v}_{1_l}^L\right)^2\left(\tilde{v}_{2_l}^L\right)^2\right)}},$$
$$\sqrt{\frac{\left(\tilde{v}_{1_l}^U\right)^2 + \left(\tilde{v}_{2_l}^U\right)^2 - \left(\tilde{v}_{1_l}^U\right)^2\left(\tilde{v}_{2kl}^U\right)^2 - (1-\gamma)\left(\tilde{v}_{1_l}^U\right)^2\left(\tilde{v}_{2_l}^U\right)^2}{\gamma + (1-\gamma)\left(1 - \left(\tilde{v}_{1_l}^U\right)^2\left(\tilde{v}_{2_l}^U\right)^2\right)}}],$$
$$\overline{p_{1_l} + p_{2_l}} > \quad (16)$$

(3)

$$\lambda\tilde{P} = \langle[\sqrt{\frac{\left(1 + (\gamma - 1)\left(\tilde{u}_l^L\right)^2\right)^\lambda - \left(1 - \left(\tilde{u}_l^L\right)^2\right)^\lambda}{\left(1 + (\gamma - 1)\left(\tilde{u}_l^L\right)^2\right)^\lambda + (\gamma - 1)\left(1 - \left(\tilde{u}_l^L\right)^2\right)^\lambda}},$$
$$\sqrt{\frac{\left(1 + (\gamma - 1)\left(\tilde{u}_l^U\right)^2\right)^\lambda - \left(1 - \left(\tilde{u}_l^U\right)^2\right)^\lambda}{\left(1 + (\gamma - 1)\left(\tilde{u}_l^U\right)^2\right)^\lambda + (\gamma - 1)\left(1 - \left(\tilde{u}_l^U\right)^2\right)^\lambda}}],$$
$$[\frac{\sqrt{\gamma}\left(\tilde{v}_l^L\right)^\lambda}{\sqrt{\left(1 + (\gamma - 1)\left(1 - \left(\tilde{v}_l^L\right)^2\right)\right)^\lambda + (\gamma - 1)\left(\tilde{v}_l^L\right)^{2\lambda}}},$$
$$\frac{\sqrt{\gamma}\left(\tilde{v}_l^U\right)^\lambda}{\sqrt{\left(1 + (\gamma - 1)\left(1 - \left(\tilde{v}_l^U\right)^2\right)\right)^\lambda + (\gamma - 1)\left(\tilde{v}_l^U\right)^{2\lambda}}}], \quad p\rangle \quad (17)$$

(4)

$$\tilde{P}^{\lambda} = \langle [\frac{\sqrt{\gamma}\left(\tilde{u}_l^L\right)^{\lambda}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_l^L\right)^2\right)\right)^{\lambda}+(\gamma-1)\left(\tilde{u}_l^L\right)^{2\lambda}}}, \frac{\sqrt{\gamma}\left(\tilde{u}_l^U\right)^{\lambda}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_l^U\right)^2\right)\right)^{\lambda}+(\gamma-1)\left(\tilde{u}_l^U\right)^{2\lambda}}}], [\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_l^L\right)^2\right)^{\lambda}-\left(1-\left(\tilde{v}_l^L\right)^2\right)^{\lambda}}{\left(1+(\gamma-1)\left(\tilde{v}_l^L\right)^2\right)^{\lambda}+(\gamma-1)\left(1-\left(\tilde{v}_l^L\right)^2\right)^{\lambda}}}, \sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_l^U\right)^2\right)^{\lambda}-\left(1-\left(\tilde{v}_l^U\right)^2\right)^{\lambda}}{\left(1+(\gamma-1)\left(\tilde{v}_l^U\right)^2\right)^{\lambda}+(\gamma-1)\left(1-\left(\tilde{v}_l^U\right)^2\right)^{\lambda}}}], \quad p\rangle \tag{18}$$

where $\overline{p_{1_l}+p_{2_l}} = (p_{1_l}+p_{2_l})/((\sum_{l=1}^{L(PIVHPFE)} p_{1_l}) + (\sum_{l=1}^{L(PIVHPFE)} p_{2_l}))$, $l = 1, 2, \cdots, L(PIVHPFE_1)$, $k = 1, 2, \cdots, L(PIVHPFE_2)$, and $\sum_{k=1}^{L(PIVHPFE)} \overline{p_1^{(k)}+p_2^{(k)}} = 1$.

Obviously, when $\gamma = 1$, the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher operation degenerates into a classical probabilistic interval-valued hesitant Pythagorean fuzzy operation; when $\gamma = 2$, the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher operation degenerates into rate interval-valued hesitant Pythagorean fuzzy Einstein operation.

Theorem 1: The probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher product is a $t-$norm, and the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher sum is a $t-$conorm.

Theorem 2: Let $\tilde{P}, \tilde{P}_1, \tilde{P}_2$ be three PIVHPFEs, and $\lambda, \lambda_1, \lambda_2 > 0$, then the Hamacher operations of PIVHPFS can be expressed as follows:

(1) $\tilde{P}_1 \oplus \tilde{P}_2 = \tilde{P}_2 \oplus \tilde{P}_1$;
(2) $\tilde{P}_1 \otimes \tilde{P}_2 = \tilde{P}_2 \otimes \tilde{P}_1$;
(3) $\lambda\left(\tilde{P}_1 \oplus \tilde{P}_2\right) = \lambda\tilde{P}_1 \oplus \lambda\tilde{P}_2$;
(4) $\left(\tilde{P}_1 \otimes \tilde{P}_2\right)^{\lambda} = \tilde{P}_1^{\lambda} \otimes \tilde{P}_1^{\lambda}$;
(5) $\lambda_1\tilde{P} \oplus \lambda_2\tilde{P} = (\lambda_1+\lambda_2)\tilde{P}$;
(6) $\tilde{P}^{\lambda_1} \otimes \tilde{P}^{\lambda_2} = \tilde{P}^{\lambda_1+\lambda_2}$.

It is easy to prove that above PIVHPHF operation rules proposed satisfy the *Theorem 1* and *Theorem 2*. *Theorem 1* as an extended mathematical expression of $t-$norm and $t-$conorm for our proposed approaches. *Theorem 2* provides the rules of operation for our proposed algorithm, which plays a significant role in the aggregation operator.

IV. PIVHPHS HAMACHER CHOQUET INTEGRAL GEOMETRIC MEAN (PIVHPFHCIG) OPERATOR

In the real-world MADM process, in order to reflect the information aggregation and importance of decision-makers completely, PIVHPFE will be applied to express the evaluation information under the programming characteristics. After that, the information aggregation operator will be applied to aggregating the comprehensive characteristics of information. Therefore, based on *Definition 10*, the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator will be introduced to investigate the related characteristics.

Definition 11: Let X be a finite set, and for any $\tilde{P}_i = \langle \tilde{u}_{i_l}(x), \tilde{v}_{i_l}(x), p_{i_l}\rangle$ be the finite PIVHPFE associated with X, where $i = 1, 2, \cdots, n$, $l = 1, 2, \cdots, L(PIVHPFE)$, ρ is the fuzzy measure, and the PIVHPFHCIG operator can be identified as follows:

$$\begin{aligned} &PIVHPFHCIG_{\rho}(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_n) \\ &\quad = \mathop{\otimes}_{i=1}^{n} \tilde{P}_i^{\rho(A_{(i)})-\rho(A_{(i+1)})} \\ &\quad = (\tilde{P}_1^{\rho(A_{(1)})-\rho(A_{(2)})}) \otimes (\tilde{P}_2^{\rho(A_{(2)})-\rho(A_{(3)})}) \\ &\qquad \otimes \cdots \otimes (\tilde{P}_n^{\rho(A_{(n)})-\rho(A_{(n+1)})}), \end{aligned} \tag{19}$$

where (i) is a permutation of $\tilde{P}_i$, which lead to $\tilde{P}_{(1)} \le \tilde{P}_{(2)} \le \cdots \le \tilde{P}_{(n)}$, $A_{(i)} = \left(c_{(i)}, c_{(i+1)}, \cdots, c_{(n)}\right)$, and $A_{(n+1)} = 0$.

On the Hamacher operation laws (1) to (6) of PIVHPFS, the following theorem can be derived.

Theorem 3: Let X be a finite set, and for any $\tilde{P}_i = \langle \tilde{u}_{i_l}(x), \tilde{v}_{i_l}(x), p_{i_l}\rangle$ be a collection of PIVHPFEs associated with X, where $i = 1, 2, \cdots, n$, $l = 1, 2, \cdots, L(PIVHPFE)$, ρ is the fuzzy measure, then shown as follows:

$$\begin{aligned} &PIVHPFHCIG_{\rho}\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_n\right) \\ &= \langle(([\sqrt{\gamma}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^L\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})} \\ &\quad /(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^L\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})} \\ &\quad +(\gamma-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^L\right)^{2\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}, \\ &\quad \sqrt{\gamma}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^U\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}/ \\ &\quad /(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^U\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})} \\ &\quad +(\gamma-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^U\right)^{2\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}], \\ &\quad [(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})} \\ &\quad -\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2} \\ &\quad /(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})} \end{aligned}$$

$$+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2},$$
$$(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}],$$
$$\overline{\sum_{i=1}^{n}p_{i_l}}\rangle\rangle \tag{20}$$

where $\overline{\sum_{i=1}^{n}p_{i_l}}=\sum_{i=1}^{n}p_{i_l}/\sum_{l=1}^{L(PIVHPFE)}\sum_{i=1}^{n}p_{i_l}$.

Proof of Theorem 3: By operation laws of PIVHPFS and the mathematical induction, we have the following:

(1) When $n=2$, according to the PIVHPFHCIG operator on *Definition 11*, then

$$PIVHPFHCIG_{\rho}\left(\tilde{P}_1,\tilde{P}_2\right)$$
$$=\tilde{P}_1^{\rho_{1,2}}\otimes\tilde{P}_2^{\rho_{2,3}}$$
$$=\langle\langle[\frac{\sqrt{\gamma}\left(\tilde{u}_{1_l}^{L}\right)^{\rho_{1,2}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{1_l}^{L}\right)^2\right)\right)^{\rho_{1,2}}+(\gamma-1)\left(\tilde{u}_{1_l}^{L}\right)^{2\rho_{1,2}}}},$$
$$\frac{\sqrt{\gamma}\left(\tilde{u}_{1_l}^{U}\right)^{\rho_{1,2}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{1_l}^{U}\right)^2\right)\right)^{\rho_{1,2}}+(\gamma-1)\left(\tilde{u}_{1_l}^{U}\right)^{2\rho_{1,2}}}}],$$
$$[\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_{1_l}^{L}\right)^2\right)^{\rho_{1,2}}-\left(1-\left(\tilde{v}_{1_l}^{L}\right)^2\right)^{\rho_{1,2}}}{\left(1+(\gamma-1)\left(\tilde{v}_{1_l}^{L}\right)^2\right)^{\rho_{1,2}}+(\gamma-1)\left(1-\left(\tilde{v}_{1_l}^{L}\right)^2\right)^{\rho_{1,2}}}},$$
$$\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_{1_l}^{U}\right)^2\right)^{\rho_{1,2}}-\left(1-\left(\tilde{v}_{1_l}^{U}\right)^2\right)^{\rho_{1,2}}}{\left(1+(\gamma-1)\left(\tilde{v}_{1_l}^{U}\right)^2\right)^{\rho_{1,2}}+(\gamma-1)\left(1-\left(\tilde{v}_{1_l}^{U}\right)^2\right)^{\rho_{1,2}}}}],$$
$$p_1\rangle\rangle$$
$$\otimes\langle\langle[\frac{\sqrt{\gamma}\left(\tilde{u}_{2_k}^{L}\right)^{\rho_{2,3}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{2_k}^{L}\right)^2\right)\right)^{\rho_{2,3}}+(\gamma-1)\left(\tilde{u}_{2_k}^{L}\right)^{2\rho_{2,3}}}},$$
$$\frac{\sqrt{\gamma}\left(\tilde{u}_{2_k}^{U}\right)^{\rho_{2,3}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{2_k}^{U}\right)^2\right)\right)^{\rho_{2,3}}+(\gamma-1)\left(\tilde{u}_{2_k}^{U}\right)^{2\rho_{2,3}}}}],$$
$$[\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_{2_k}^{L}\right)^2\right)^{\rho_{2,3}}-\left(1-\left(\tilde{v}_{2_k}^{L}\right)^2\right)^{\rho_{2,3}}}{\left(1+(\gamma-1)\left(\tilde{v}_{2_k}^{L}\right)^2\right)^{\rho_{2,3}}+(\gamma-1)\left(1-\left(\tilde{v}_{2_k}^{L}\right)^2\right)^{\rho_{2,3}}}},$$
$$\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}_{2_k}^{U}\right)^2\right)^{\rho_{2,3}}-\left(1-\left(\tilde{v}_{2_k}^{U}\right)^2\right)^{\rho_{2,3}}}{\left(1+(\gamma-1)\left(\tilde{v}_{2_k}^{U}\right)^2\right)^{\rho_{2,3}}+(\gamma-1)\left(1-\left(\tilde{v}_{2_k}^{U}\right)^2\right)^{\rho_{2,3}}}}],$$
$$p_{2_k}\rangle\rangle$$
$$=\langle\langle[\frac{\sqrt{\gamma}\prod_{i=1}^{2}\left(\tilde{u}_{i_l}^{L}\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{2}\left(\tilde{u}_{i_l}^{L}\right)^{2\rho_{i,i+1}}}},$$
$$\frac{\sqrt{\gamma}\prod_{i=1}^{2}\left(\tilde{u}_{i_l}^{U}\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^{U}\right)^2\right)\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{2}\left(\tilde{u}_{i_l}^{U}\right)^{2\rho_{i,i+1}}}}],$$
$$[\sqrt{\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho_{i,i+1}}-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho_{i,i+1}}}},$$
$$\sqrt{\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho_{i,i+1}}-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho_{i,i+1}}}}],$$
$$\overline{\sum_{i=1}^{2}p_{i_l}}\rangle\rangle$$

where $\rho\left(A_{(i)}\right)-\rho\left(A_{(i+1)}\right)=\rho_{i,i+1}$, $l=1,2,\cdots,L\left(PIVHPFE\right)$ and $\overline{\sum_{i=1}^{n}p_{i_l}}=\sum_{i=1}^{n}p_{i_l}/\sum_{l=1}^{L(PIVHPFE)}\sum_{i=1}^{n}p_{i_l}$. Therefore when $n=2$, it is correct.

(2) Assuming that $n=q$, the theorem is true, and the following proves that $n=k+1$ is also true.

The proof ends. There are proved by mathematical induction the *Theorem 3* is correct, which Provides methods and ways for information aggregation.

Theorem 4: Let $\tilde{P}_i=\left[\left(\left[\tilde{u}_{i_l}^{L},\tilde{u}_{i_l}^{U}\right],\left[\tilde{v}_{i_l}^{L},\tilde{v}_{i_l}^{U}\right]\right),p_{i_l}\right]$ be a collection of PIVHPFEs, where $i=1,2,\cdots,n$, $l=1,2,\cdots,L\left(PIVHPFE\right)$, ρ is the fuzzy measures,then

(1) *(Idempotency):* If all $\tilde{P}_i$ are equal, i.e., $\tilde{P}_i=\tilde{P}=[([\tilde{u}_{i_l}^{L},\tilde{u}_{i_l}^{U}],[\tilde{v}_{i_l}^{L},\tilde{v}_{i_l}^{U}],p_{i_l})]$ are PIVHPFEs, where $l=1,2,\cdots,L\left(PIVHPFE\right)$, then

$$PIVHPFHCIG_{\rho}\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)=\tilde{P}.$$

Proof: According to *Theorem 3*, if $\tilde{P}_i = \tilde{P}$, for all $i\,(i = 1, 2, \cdots, n)$, then

$$PIVHPFHCIG_\rho\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_n\right) = \langle([\sqrt{\gamma}\left(\tilde{u}^L_{i_l}\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})} /(\left(1+(\gamma-1)\left(1-\left(\tilde{u}^L_{i_l}\right)^2\right)\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})} + (\gamma-1)\left(\tilde{u}^L_{i_l}\right)^{\sum_{i=1}^{n}2\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}, \sqrt{\gamma}\left(\tilde{u}^U_{i_l}\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\begin{aligned}
PIVHPFHCIG_\rho\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_k, \tilde{P}_{k+1}\right) &= PIVHPFHCIG_\rho\left(\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_k\right) \otimes \tilde{P}_{k+1}\\
&= \langle([\frac{\sqrt{\gamma}\prod_{i=1}^{k}\left(\tilde{u}^L_{i_l}\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(1-\left(\tilde{u}^L_{i_l}\right)^2\right)\right)^{\rho_{i,i+1}} + (\gamma-1)\prod_{i=1}^{k}\left(\tilde{u}^L_{i_l}\right)^{2\rho_{i,i+1}}}},\\
&\quad \frac{\sqrt{\gamma}\prod_{i=1}^{k}\left(\tilde{u}^U_{i_l}\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(1-\left(\tilde{u}^U_{i_l}\right)^2\right)\right)^{\rho_{i,i+1}} + (\gamma-1)\prod_{i=1}^{k}\left(\tilde{u}^U_{i_l}\right)^{2\rho_{i,i+1}}}}],\\
&\quad [\sqrt{\frac{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(\tilde{v}^L_{i_l}\right)^2\right)^{\rho_{i,i+1}} - \prod_{i=1}^{k}\left(1-\left(\tilde{v}^L_{i_l}\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(\tilde{v}^L_{i_l}\right)^2\right)^{\rho_{i,i+1}} + (\gamma-1)\prod_{i=1}^{k}\left(1-\left(\tilde{v}^L_{i_l}\right)^2\right)^{\rho_{i,i+1}}}},\\
&\quad \sqrt{\frac{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(\tilde{v}^U_{i_l}\right)^2\right)^{\rho_{i,i+1}} - \prod_{i=1}^{k}\left(1-\left(\tilde{v}^U_{i_l}\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{k}\left(1+(\gamma-1)\left(\tilde{v}^U_{i_l}\right)^2\right)^{\rho_{i,i+1}} + (\gamma-1)\prod_{i=1}^{k}\left(1-\left(\tilde{v}^U_{i_l}\right)^2\right)^{\rho_{i,i+1}}}}],\\
&\quad \overline{\sum_{i=1}^{k} p_{i_l}})\rangle\\
&\quad \otimes \langle([\frac{\sqrt{\gamma}\left(\tilde{u}^L_{(k+1)_l}\right)^{\rho_{k+1,k+2}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}^L_{(k+1)_l}\right)^2\right)\right)^{\rho_{k+1,k+2}} + (\gamma-1)\left(\tilde{u}^L_{(k+1)_l}\right)^{2\rho_{k+1,k+2}}}},\\
&\quad \frac{\sqrt{\gamma}\left(\tilde{u}^U_{(k+1)_l}\right)^{\rho_{k+1,k+2}}}{\sqrt{\left(1+(\gamma-1)\left(1-\left(\tilde{u}^U_{(k+1)_l}\right)^2\right)\right)^{\rho_{k+1,k+2}} + (\gamma-1)\left(\tilde{u}^U_{(k+1)_l}\right)^{2\rho_{k+1,k+2}}}}],\\
&\quad [\sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}^L_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}} - \left(1-\left(\tilde{v}^L_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}}}{\left(1+(\gamma-1)\left(\tilde{v}^L_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}} + (\gamma-1)\left(1-\left(\tilde{v}^L_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}}}},\\
&\quad \sqrt{\frac{\left(1+(\gamma-1)\left(\tilde{v}^U_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}} - \left(1-\left(\tilde{v}^U_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}}}{\left(1+(\gamma-1)\left(\tilde{v}^U_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}} + (\gamma-1)\left(1-\left(\tilde{v}^U_{(k+1)_l}\right)^2\right)^{\rho_{k+1,k+2}}}}],\\
&\quad p^{(k)}_{(k+1)_l})\rangle
\end{aligned}$$

$$/(\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^U\right)^2\right)\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(\gamma-1)\left(\tilde{u}_{i_l}^U\right)^{\sum_{i=1}^{n}2\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}],$$

$$[(\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$-\left(1-\left(\tilde{v}_{i_l}^L\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$

$$/(\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(\gamma-1)\left(1-\left(\tilde{v}_{i_l}^L\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2},$$

$$(\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^U\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$-\left(1-\left(\tilde{v}_{i_l}^U\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$

$$/(\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^U\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(\gamma-1)\left(1-\left(\tilde{v}_{i_l}^U\right)^2\right)^{\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}],$$

$$\overline{\sum_{i=1}^{n}p_{i_l}})\rangle$$

$$=\tilde{P}$$

Since, $\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})=1$, $\overline{\sum_{i=1}^{n}p_{i_l}}=\sum_{i=1}^{n}p_{i_l}/\sum_{l=1}^{L(PIVHPFE)}\sum_{i=1}^{n}p_{i_l}=p$, so $PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)=\tilde{P}$.

(2) *(Monotonicity):* Let $\bar{P}_i=[([\bar{u}_{i_l}^L,\bar{u}_{i_l}^U],[\bar{v}_{i_l}^L,\bar{v}_{i_l}^U],\bar{p}_{i_l})]$ be a collection of PIVHPFEs, where $i=1,2,l=1,2,\cdots,L\,(PIVHPFE)$.

if $\tilde{u}_{i_l}^L\le\bar{u}_{i_l}^L$, $\tilde{u}_{i_l}^U\le\bar{u}_{i_l}^U$, $\tilde{v}_{i_l}^L\ge\bar{v}_{i_l}^L$, $\tilde{v}_{i_l}^U\ge\bar{v}_{i_l}^U$ for all i and l, then

$$PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)\le PIVHPFHCIG_\rho\left(\bar{P}_1,\bar{P}_2,\cdots,\bar{P}_n\right).$$

Proof: Since, $A_{(i+1)}\subseteq A_{(i)}$, then $\rho\left(A_{(i)}\right)-\rho\left(A_{(i+1)}\right)\ge 0$. For all $i\,(i=1,2,\cdots,n)$, we have $\tilde{P}_i\le\bar{P}_i$, so

$$\sqrt{\gamma}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^L\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^L\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(\gamma-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^L\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$

$$\le\sqrt{\gamma}\prod_{i=1}^{n}\left(\bar{u}_{i_l}^L\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$=\langle(([\frac{\sqrt{\gamma}\prod_{i=1}^{k+1}\left(\tilde{u}_{i_l}^L\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^L\right)^2\right)\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{k+1}\left(\tilde{u}_{i_l}^L\right)^{2\rho_{i,i+1}}}},$$

$$\frac{\sqrt{\gamma}\prod_{i=1}^{k+1}\left(\tilde{u}_{i_l}^U\right)^{\rho_{i,i+1}}}{\sqrt{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^U\right)^2\right)\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{k+1}\left(\tilde{u}_{i_l}^U\right)^{2\rho_{i,i+1}}}}],$$

$$[\sqrt{\frac{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho_{i,i+1}}-\prod_{i=1}^{k+1}\left(1-\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{k+1}\left(1-\left(\tilde{v}_{i_l}^L\right)^2\right)^{\rho_{i,i+1}}}},$$

$$\sqrt{\frac{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^U\right)^2\right)^{\rho_{i,i+1}}-\prod_{i=1}^{k+1}\left(1-\left(\tilde{v}_{i_l}^U\right)^2\right)^{\rho_{i,i+1}}}{\prod_{i=1}^{k+1}\left(1+(\gamma-1)\left(\tilde{v}_{i_l}^U\right)^2\right)^{\rho_{i,i+1}}+(\gamma-1)\prod_{i=1}^{k+1}\left(1-\left(\tilde{v}_{i_l}^U\right)^2\right)^{\rho_{i,i+1}}}}],$$

$$\overline{\sum_{i=1}^{k+1}p_{i_l}})\rangle$$

$$/(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\bar{u}_{i_l}^{L}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}\left(\bar{u}_{i_l}^{L}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2};$$
$$\sqrt{\gamma}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$/(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\tilde{u}_{i_l}^{U}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$\leq\sqrt{\gamma}\prod_{i=1}^{n}\left(\bar{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$/(\prod_{i=1}^{n}\left(1+(\gamma-1)\left(1-\left(\bar{u}_{i_l}^{U}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}\left(\bar{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2};$$
$$(\prod_{i=1}^{n}(1+(\gamma-1)(\tilde{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}(1-(\tilde{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}(1+(\gamma-1)(\tilde{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}(1-(\tilde{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$\geq(\prod_{i=1}^{n}(1+(\gamma-1)(\bar{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}(1-(\bar{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}(1+(\gamma-1)(\bar{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}(1-(\bar{v}_{i_l}^{L})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2};$$
$$(\prod_{i=1}^{n}(1+(\gamma-1)(\tilde{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}(1-(\tilde{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}(1+(\gamma-1)(\tilde{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}(1-(\tilde{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$\geq(\prod_{i=1}^{n}(1+(\gamma-1)(\bar{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}(1-(\bar{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}(1+(\gamma-1)(\bar{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(\gamma-1)\prod_{i=1}^{n}(1-(\bar{v}_{i_l}^{U})^2)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2};$$

Thus by *Theorem 3* and *Definition 7* we have,

$$PIVHPFHCIG_{\rho}\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)\leq PIVHPFHCIG_{\rho}\left(\bar{P}_1,\bar{P}_2,\cdots,\bar{P}_3\right).$$

(3) *(Boundedness):* Let $\tilde{P}^-$ and $\tilde{P}^+$ be two PIVHPFEs, and $\tilde{P}^-=[([\tilde{u}_{i_l}^{L-},\tilde{u}_{i_l}^{U-}],[\tilde{v}_{i_l}^{L+},\tilde{v}_{i_l}^{U+}]),p_{i_l}]$, $\tilde{P}^+=[([\tilde{u}_{i_l}^{L+},\tilde{u}_{i_l}^{U+}],[\tilde{v}_{i_l}^{L-},\tilde{v}_{i_l}^{U-}),p_{i_l}]$, where $i=1,2,\cdots,n$, $l=1,2,\cdots,L$ *(PIVHPFE)*, if $\tilde{u}_{i_l}^{L-}=\min_i\left\{\tilde{u}_{i_l}^{L}\right\}$, $\tilde{u}_{i_l}^{U-}=\min_i\left\{\tilde{u}_{i_l}^{U}\right\}$, $\tilde{v}_{i_l}^{L+}=\max_i\left\{\tilde{v}_{i_l}^{L}\right\}$, $\tilde{v}_{i_l}^{U+}=\max_i\left\{\tilde{v}_{i_l}^{U}\right\}$, $\tilde{u}_{i_l}^{L+}=\max_i\left\{\tilde{u}_{i_l}^{L}\right\}$, $\tilde{u}_{i_l}^{U+}=\max_i\left\{\tilde{u}_{i_l}^{U}\right\}$, $\tilde{v}_{i_l}^{L-}=\min_i\left\{\tilde{v}_{i_l}^{L}\right\}$ and $\tilde{v}_{i_l}^{U-}=\min_i\left\{\tilde{v}_{i_l}^{U}\right\}$, then

$$\tilde{P}^-\leq PIVHPFHCIG_{\rho}\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)\leq\tilde{P}^+.$$

Proof: Since $A_{(i+1)}\subseteq A_{(i)}$, $\rho\left(A_{(i)}\right)-\rho\left(A_{(i+1)}\right)\geq 0$. For any $\tilde{P}_i\,(i=1,2,\cdots,n)$ and $\sum_{i=1}^{n}\rho(A_{(i)})-\rho(A_{(i+1)})=1$.

Let $f(x)=\frac{1+(r-1)(1-x)}{x}, x\in[0,1]$, then $f'(x)=\left(\frac{1+(r-1)(1-x)}{x}\right)'=\frac{-r}{x^2}<0$. Then $f(x)$ is an decreasing function. Since for all i, $\tilde{u}_{i_l}^{L-}\leq\tilde{u}_{i_l}^{L}\leq\tilde{u}_{i_l}^{L+}$, then $f\left(\tilde{u}_{i_l}^{L+}\right)\leq f\left(\tilde{u}_{i_l}^{L}\right)\leq f\left(\tilde{u}_{i_l}^{L-}\right)$, i.e.,

$$\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L+}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L+}\right)^2}\leq\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\leq\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L-}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L-}\right)^2}.$$

We have,

$$\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L+}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L+}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}\leq\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L-}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L-}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\Leftrightarrow \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L+}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L+}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L-}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L-}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\Leftrightarrow \frac{r}{\left(\tilde{u}_{i_l}^{L+}\right)^2}-(r-1)$$

$$\le \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \frac{r}{\left(\tilde{u}_{i_l}^{L-}\right)^2}-(r-1)$$

$$\Leftrightarrow \frac{r}{\left(\tilde{u}_{i_l}^{L+}\right)^2}$$

$$\le \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}+(r-1)$$

$$\le \frac{r}{\left(\tilde{u}_{i_l}^{L-}\right)^2}$$

$$\Leftrightarrow \frac{\left(\tilde{u}_{i_l}^{L-}\right)^2}{r}$$

$$\le \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)}{\left(\tilde{u}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}+(r-1)}$$

$$\le \frac{\left(\tilde{u}_{i_l}^{L+}\right)^2}{r}$$

$$\Leftrightarrow \frac{\left(\tilde{u}_{i_l}^{L-}\right)^2}{r} \le \prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{2\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}) \le \frac{\left(\tilde{u}_{i_l}^{L+}\right)^2}{r}$$

$$\Leftrightarrow \left(\tilde{u}_{i_l}^{L-}\right)^2 \le r\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{2\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}) \le \left(\tilde{u}_{i_l}^{L+}\right)^2$$

$$\Leftrightarrow \tilde{u}_{i_l}^{L-} \le \sqrt{r}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{L}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{L}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2} \le \tilde{u}_{i_l}^{L+}$$

The same as above, we have,

$$\tilde{u}_{i_l}^{U-} \le \sqrt{r}\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(1-\left(\tilde{u}_{i_l}^{U}\right)^2\right)\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(\tilde{u}_{i_l}^{U}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2} \le \tilde{u}_{i_l}^{U+}.$$

Let $g(y) = \frac{1+(r-1)y}{1-y}, y \in [0,1]$, then $g(y) = \left(\frac{1+(r-1)y}{1-y}\right)' = \frac{r}{(1-y)^2} > 0$, thus $g(y)$ is an increasing function. Since for all i, $g\left(\tilde{v}_{i_l}^{L+}\right) \le g\left(\tilde{v}_{i_l}^{L}\right) \le g\left(\tilde{v}_{i_l}^{L-}\right)$, i.e.,

$$\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2} \le \frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2} \le \frac{1+(r-1)\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2},$$

We have,

$$\left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\le \left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\Leftrightarrow \left(1+\frac{r\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\leq \left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\leq \left(1+\frac{r\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\Leftrightarrow \prod_{i=1}^{n}\left(1+\frac{r\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\leq \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\leq \prod_{i=1}^{n}\left(1+\frac{r\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\Leftrightarrow 1+\frac{r\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}$$

$$\leq \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$\leq 1+\frac{r\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}$$

$$\Leftrightarrow r+\frac{r\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}$$

$$\leq \prod_{i=1}^{n}\left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}+(r-1)$$

$$\leq r+\frac{r\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}$$

$$\Leftrightarrow \frac{1}{r+\frac{r\left(\tilde{v}_{i_l}^{L-}\right)^2}{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}}$$

$$\leq \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2}{1-\left(\tilde{v}_{i_l}^{L}\right)^2}\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}+(r-1)}$$

$$\leq \frac{1}{r+\frac{r\left(\tilde{v}_{i_l}^{L+}\right)^2}{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}}$$

$$\Leftrightarrow \frac{1-\left(\tilde{v}_{i_l}^{L-}\right)^2}{r} \leq \prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})$$

$$\leq \frac{1-\left(\tilde{v}_{i_l}^{L+}\right)^2}{r}$$

$$\Leftrightarrow 1-\left(\tilde{v}_{i_l}^{L-}\right)^2 \leq r\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})$$

$$\leq 1-\left(\tilde{v}_{i_l}^{L+}\right)^2$$

$$\Leftrightarrow \left(\tilde{v}_{i_l}^{L+}\right)^2 \leq 1-r\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}) \leq \left(\tilde{v}_{i_l}^{L-}\right)^2$$

$$\Leftrightarrow \left(\tilde{v}_{i_l}^{L+}\right)^2 \leq (\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})$$

$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}) \leq \left(\tilde{v}_{i_l}^{L-}\right)^2$$

$$\Leftrightarrow \tilde{v}_{i_l}^{L+} \leq (\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$

$$-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{L}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}\leq \tilde{v}_{i_l}^{L^-}.$$

The same as above, we have,

$$\Leftrightarrow \tilde{v}_{i_l}^{U^+}\leq (\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$-\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}$$
$$/(\prod_{i=1}^{n}\left(1+(r-1)\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})}$$
$$+(r-1)\prod_{i=1}^{n}\left(1-\left(\tilde{v}_{i_l}^{U}\right)^2\right)^{\rho(A_{(i)})-\rho(A_{(i+1)})})^{1/2}\leq \tilde{v}_{i_l}^{L^-}$$

Thus according to *Theorem 3* and *Definition 7* we have,

$$\tilde{P}^-\leq PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)\leq \tilde{P}^+.$$

(4) *(Permutation invariance):* Let $\tilde{Q}_i$ be a any substitution on $\tilde{P}_i\ (i=1,2,\cdots,n)$, then

$$PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)=PIVHPFHCIG_\rho\left(\tilde{Q}_1,\tilde{Q}_2,\cdots,\tilde{Q}_n\right).$$

Proof: Thus, according to *Idempotency* in *Theorem 4*, we have

$$PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)=\mathop{\otimes}_{i=1}^{n}\tilde{P}_i^{\rho(A_{(i)})-\rho(A_{(i+1)})}=\mathop{\otimes}_{i=1}^{n}\tilde{Q}_i^{\rho(A_{(i)})-\rho(A_{(i+1)})}=PIVHPFHCIG_\rho\left(\tilde{Q}_1,\tilde{Q}_2,\cdots,\tilde{Q}_n\right)$$

Hence,

$$PIVHPFHCIG_\rho\left(\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_n\right)=PIVHPFHCIG_\rho\left(\tilde{Q}_1,\tilde{Q}_2,\cdots,\tilde{Q}_n\right).$$

Theorem 4 are well proved according to the probabilistic interval-valued hesitant Pythagorean Hamacher fuzzy (PIVHPHF) operation which have been identified in *Definition 10*, which ensures the scientific and effective operation of the algorithm.

V. THE MULTI-ATTRIBUTE DECISION MAKING BASED ON PIVHPFHCIG OPERATOR

In this section, a aggregation operator method is established based on the previous algorithm and also the Choquet integral is applied to solve the problem of the correlation between characteristics. This provide a possibility of solve MADM problems under PIVHPFS environment.

For a MADM problem, let $A=\{A_1,A_2,\cdots,A_m\}$ be a set of alternatives, and $C=\{C_1,C_2,\ldots,C_n\}$ be a set of attributes. The alternatives A_i under finite attributes set C_j will be evaluated by PIVHPFS value $P_{ij}(i=1,2,\cdots,m,j=1,2,\cdots,n)$ and then the PIVHPFS decision making matrix will be estimated to select the best strategy.

For the sake of select the best alternative, an algorithm based on PIVHPFHCIG operator is provided and the key steps of the algorithm are given as follows:

Step 1: Basing on the evaluation information, the PIVHPFS decision making matrix $M=\left(P_{ij}\right)_{m\times n}$ is established.

Step 2: The decision making matrix will be standardized unless all the characteristics are interest type.

$$P_{ij}=\begin{cases}P_{ij}, & C_j \text{ is benefit type;}\\ P_{ij}^c, & C_j \text{ is cos t type.}\end{cases}$$

Step 3: The fuzzy measure of $C=\{C_1,C_2,\ldots,C_n\}$ should be identified by satisfy the parameter τ.

Step 4: Based on $M=\left(P_{ij}\right)_{m\times n}$ every alternative $A_i\ (i=1,2,\cdots,m)$ can be integrated by PIVHPFHCIG operator shown in Eq. (19) and then the comprehensive attributes value $P_i\ (i=1,2,\cdots,m)$ of alternatives A_i can be estimated.

Step 5: Utilize Eqs. (13) and (14) to obtain the score function and accuracy function for each alternative A_i.

Step 6: The candidate strategy will be ranked by PIVHPFE comparison rules shown in *Definition 8*, and the best alternative will be selected.

In the actual decision making, the individual decision-maker is the main concern, therefore, the geometric operator is selected.

VI. AN APPLICATION EXAMPLE

In the previous hesitant Pythagorean fuzzy information aggregation analysis, it is always assuming that the attributes are independent from each other. Based on this assumption, combined with probabilistic and interval-valued to form up the new fuzzy set and applying Hamacher algorithm to aggregating operator, the decision-making method is more accurate and rational than before. In this section, the example data (cited from Wang *et al.* [36]) is provided to illustrate the application of the proposed approach in Section VI.

A. THE DECISION PROCEDURE OF THE PROPOSED MADM METHOD

The Mulan Avenue and Loess Highway Upgrade Project, with a total length of 25.32 km, will be expanded according to the standard of Class I Highway and Urban Main Road, with a pavement width of 50 to 64 meters, six lanes in both directions and a design speed of 60 km/h. The whole project consists of two medium bridges and two small bridges, with supporting the construction of traffic engineering, power and telecommunication engineering and greening engineering,

TABLE 1. The decision matrix given by PIVHPFS.

	C_1	C_2	C_3	C_4
A_1	$\left\langle \begin{array}{l}([0.3,0.4],[0.4,0.5],0.5),\\([0.3,0.4],[0.2,0.3],0.5)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.1,0.4],[0.4,0.5],0.4),\\([0.2,0.3],[0.2,0.4],0.6)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.3,0.5],[0.2,0.4],0.3),\\([0.2,0.4],[0.3,0.4],0.7)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.3,0.4],[0.4,0.5],0.6),\\([0.2,0.3],[0.2,0.4],0.4)\end{array}\right\rangle$
A_2	$\left\langle \begin{array}{l}([0.1,0.4],[0.3,0.5],0.4),\\([0.3,0.4],[0.1,0.5],0.6)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.2,0.4],[0.3,0.5],0.3),\\([0.2,0.4],[0.3,0.5],0.6)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.1,0.3],[0.2,0.4],0.45),\\([0.5,0.7],[0.4,0.6],0.5)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.1,0.4],[0.2,0.5],0.7),\\([0.3,0.5],[0.2,0.4],0.3)\end{array}\right\rangle$
A_3	$\left\langle \begin{array}{l}([0.5,0.6],[0.4,0.7],0.3),\\([0.2,0.3],[0.3,0.5],0.7)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.2,0.5],[0.3,0.6],0.4),\\([0.3,0.5],[0.2,0.5],0.6)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.2,0.3],[0.1,0.3],0.65),\\([0.2,0.4],[0.3,0.5],0.3)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.2,0.5],[0.3,0.5],0.4),\\([0.3,0.5],[0.2,0.5],0.6)\end{array}\right\rangle$
A_4	$\left\langle \begin{array}{l}([0.3,0.5],[0.1,0.4],0.4),\\([0.3,0.5],[0.4,0.6],0.6)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.4,0.6],[0.3,0.5],0.2),\\([0.3,0.4],[0.2,0.4],0.8)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.1,0.3],[0.2,0.4],0.8),\\([0.3,0.5],[0.2,0.5],0.2)\end{array}\right\rangle$	$\left\langle \begin{array}{l}([0.2,0.4],[0.2,0.3],0.7),\\([0.4,0.6],[0.3,0.5],0.3)\end{array}\right\rangle$

with a total investment of 1.498 billion yuan. The project is planned to be a cooperation model. The government chooses social capital through public bidding. Four enterprises have submitted bidding plans for the project.

In order to select the best project private partner, four experts are organized to evaluate the indicators of the four enterprises respectively. We assume that the four private partners $A_i\,(i=1,2,3,4)$, there are four attributes need to be considered: C_1 is the quality, C_2 is the return rate, C_3 is the delivery rate, C_4 is the service and technique. Based on the acknowledge and experience of the experts, the evaluation information $\tilde{p}_{ij}\,(i,j=1,2,\cdots,4)$ under the probabilistic interval-valued hesitant Pythagorean fuzzy environment are given. Decision-making matrix $M=\left(\tilde{P}_{ij}\right)_{4\times 4}$ is established as:

Step 1: Establish the PIVHPFS decision matrix shown in Table 1.

Step 2: All the attributes are benefit type, therefore there is no need to standardize the decision matrix $M=\left(\tilde{P}_{ij}\right)_{4\times 4}$.

Step 3: Identify the fuzzy measures of all the attributes are $\rho(c_1)=0.4$, $\rho(c_2)=0.25$, $\rho(c_3)=0.37$, $\rho(c_4)=0.2$. Use Eq. (4), we have,

$$\begin{aligned}\tau &= \prod\nolimits_{i=1}^{n}(1+\tau\rho(c_i))-1\\ &=(1+0.4\tau)(1+0.25\tau)(1+0.37\tau)(1+0.2\tau)-1\end{aligned}$$

Use MATLAB to solve for τ, we get $\tau=-0.44$. Utilize Eq. (3), we obtain,

$$\begin{aligned}\rho(c_1,c_2) &= \frac{1}{\tau}\left(\prod\nolimits_{i=1}^{n}(1+\tau\rho(c_i))-1\right)\\ &=\frac{1}{-0.44}\left[(1-0.44\times 0.4)(1-0.44\times 0.25)-1\right]\\ &=0.6,\end{aligned}$$

The same as above, $\rho(c_1,c_3)=0.7$, $\rho(c_1,c_4)=0.56$, $\rho(c_2,c_3)=0.58$, $\rho(c_2,c_4)=0.43$, $\rho(c_3,c_4)=0.54$, $\rho(c_1,c_2,c_3)=0.88$, $\rho(c_1,c_2,c_4)=0.75$, $\rho(c_1,c_3,c_4)=0.84$, $\rho(c_2,c_3,c_4)=0.73$, $\rho(c_1,c_2,c_3,c_4)=1$.

Then utilize the Eq. (5) to calculate the Choquet integral:

$$\begin{aligned}\rho(A_1)-\rho(A_2) &= \rho(c_1,c_2,c_3,c_4)-\rho(c_2,c_3,c_4)\\ &=1-0.73=0.27,\\ \rho(A_2)-\rho(A_3) &= \rho(c_2,c_3,c_4)-\rho(c_3,c_4)\\ &=0.73-0.54=0.19,\\ \rho(A_3)-\rho(A_4) &= \rho(c_3,c_4)-\rho(c_4)\\ &=0.54-0.2=0.34,\\ \rho(A_4)-\rho(A_5) &= \rho(c_4)=0.2.\end{aligned}$$

Step 4: Utilize the PIVHPFHCIG operator(where $\gamma=0.5$) shown in Eq. (20) to get the comprehensive value of A_i.

$$\tilde{P}_1=\left\langle\begin{array}{l}([0.2349,0.3956],[0.3490,0.4701],0.4500),\\([0.2174,0.3360],[0.2397,0.3767],0.5500)\end{array}\right\rangle,$$

$$\tilde{P}_2=\left\langle\begin{array}{l}([0.1132,0.3404],[0.2520,0.4701],0.4805),\\([0.3106,0.4468],[0.2921,0.5241],0.5195)\end{array}\right\rangle,$$

$$\tilde{P}_3=\left\langle\begin{array}{l}([0.2450,0.3996],[0.2906,0.5543],0.4430),\\([0.2276,0.3726],[0.2664,0.5000],0.5570)\end{array}\right\rangle,$$

$$\tilde{P}_4=\left\langle\begin{array}{l}([0.3025,0.4440],[0.2908,0.5169],0.5250),\\([0.3025,0.4440],[0.2908,0.5169],0.4750)\end{array}\right\rangle.$$

Step 5: Utilize the Eq. (13) to calculate the score function,we obtain the value of $S_i\,(i=1,2,3,4)$:

$$\begin{aligned}S_1&=-0.0597, & S_2&=-0.0798,\\ S_3&=-0.0907, & S_4&=-0.0306.\end{aligned}$$

Step 6: From the score function it is can be found that $-0.0306>-0.0597>-0.0798>-0.0907$, therefore the ranking of strategies is $A_4\succ A_1\succ A_2\succ A_3$, which means A_4 is the best alternative that should be selected through analyzing the Choquet integral-based method of PIVHPFS in selecting processes of project private partner.

B. THE INFLUENCE OF γ IN PIVHPFHCIG OPERATOR AGGREGATING

In order to test the influence of attitudinal character parameter γ in PIVHPFHCIG operator aggregating, different values of $\gamma=0.5,0.8,1,2,5,8,10,20$ are given by decision-makers. With MATLAB, the ranking of the strategies can be calculated fast, with the ranking based on different γ, the inner rules can be observed and analyzed. All the comprehensive value, score function, and rankings will be listed in Table 2.

From Table 2, when $\gamma=0.5,0.8,1,2$, the ranking of the strategies is $A_4\succ A_1\succ A_2\succ A_3$. However, when parameter $\gamma=5,8,10,20$, the ranking of strategies changed to $A_4\succ A_2\succ A_1\succ A_3$. It illustrates that parameter γ has significant influence in the score function and ranking. Although the ranking will be changed with γ, in a certain value scope, the best strategy will not change and keep being A_4 it proved that PIVHPFHCIG operator is stable. It is easy

TABLE 2. The score function and ranking under different γ.

γ	Score values A_i $(i = 1, 2, 3, 4)$	Ranking result
0.5	$S_1 = -0.0597, S_2 = -0.0798, S_3 = -0.0907, S_4 = -0.0306.$	$A_4 \succ A_1 \succ A_2 \succ A_3$
0.8	$S_1 = -0.0466, S_2 = -0.0604, S_3 = -0.0733, S_4 = -0.0052.$	$A_4 \succ A_1 \succ A_2 \succ A_3$
1	$S_1 = -0.0417, S_2 = -0.0527, S_3 = -0.0664, S_4 = 0.0048.$	$A_4 \succ A_1 \succ A_2 \succ A_3$
2	$S_1 = -0.0304, S_2 = -0.0337, S_3 = -0.0497, S_4 = 0.0287.$	$A_4 \succ A_1 \succ A_2 \succ A_3$
5	$S_1 = -0.0215, S_2 = -0.0177, S_3 = -0.0351, S_4 = 0.0477.$	$A_4 \succ A_2 \succ A_1 \succ A_3$
8	$S_1 = -0.0184, S_2 = -0.0120, S_3 = -0.0297, S_4 = 0.0541.$	$A_4 \succ A_2 \succ A_1 \succ A_3$
10	$S_1 = -0.0171, S_2 = -0.0098, S_3 = -0.0274, S_4 = 0.0565.$	$A_4 \succ A_2 \succ A_1 \succ A_3$
20	$S_1 = -0.0138, S_2 = -0.0039, S_3 = -0.0214, S_4 = 0.0623.$	$A_4 \succ A_2 \succ A_1 \succ A_3$

TABLE 3. The decision matrix given by PHPFS.

	C_1	C_2	C_3	C_4
A_1	$\left\langle \begin{array}{l}(0.35, 0.45), 0.5,\\(0.35, 0.25), 0.5\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.25, 0.4), 0.4,\\(0.25, 0.3), 0.6\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.4, 0.3), 0.3,\\(0.3, 0.35), 0.7\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.35, 0.45), 0.6,\\(0.25, 0.3), 0.4\end{array}\right\rangle$
A_2	$\left\langle \begin{array}{l}(0.25, 0.4), 0.4,\\(0.35, 0.3), 0.6\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.3, 0.4), 0.3,\\(0.35, 0.45), 0.6\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.2, 0.3), 0.45,\\(0.6, 0.5), 0.5\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.25, 0.35), 0.7,\\(0.4, 0.3), 0.3\end{array}\right\rangle$
A_3	$\left\langle \begin{array}{l}(0.5, 0.55), 0.3,\\(0.25, 0.4), 0.7\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.35, 0.45), 0.4,\\(0.4, 0.35), 0.6\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.25, 0.2), 0.65,\\(0.3, 0.4), 0.3\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.35, 0.4), 0.4,\\(0.4, 0.35), 0.6\end{array}\right\rangle$
A_4	$\left\langle \begin{array}{l}(0.4, 0.25), 0.4,\\(0.4, 0.5), 0.6\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.5, 0.4), 0.2,\\(0.35, 0.3), 0.8\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.2, 0.3), 0.8,\\(0.4, 0.35), 0.2\end{array}\right\rangle$	$\left\langle \begin{array}{l}(0.3, 0.25), 0.7,\\(0.5, 0.4), 0.3\end{array}\right\rangle$

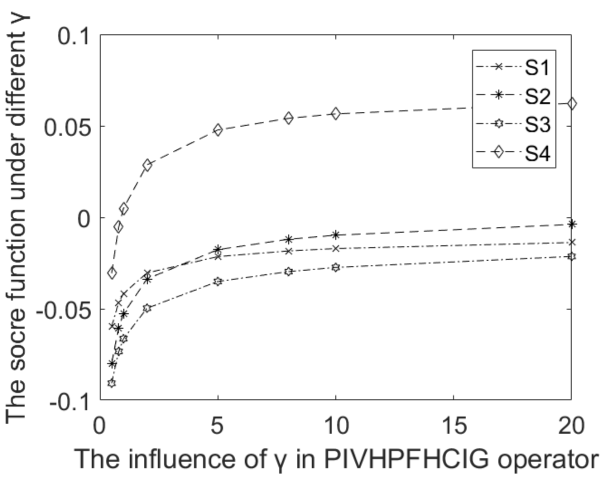

FIGURE 1. The influence of a collect of γ in score function based on PIVHPFHCLG operator.

to see in Figure 1 that score values obtained by the PIVHPFHCIG operator are almost in the trend of increase as the parameter γ increase. There is no influence on the best project private partner for parameter $\gamma = 5, 8, 10, 20$.

It can be ground-based on the above discussion, although the change of parameter γ has a certain influence in selecting the best project private partner, the result is well stability. If consider the simplicity of the calculation, the alter γ may be reduced by a small positive integer such as 1 or 2. Due to the use of MATLAB, we can easily calculate the ranking under different alter values, which we can research the dynamic changes of the best project private partner and find its internal variation law.

C. COMPARISON WITH THE EXITING OPERATOR

To illustrate the rationality and reliability of the method under PIVHPFS environment, two methods with the PHPFS and IVHPFS are compared with the proposed method. In order to ensure the accuracy and the effectiveness of the comparison, basing on the same original data, the Choquet geometric operator is applied to estimate the decision making models under different fuzzy sets situation.

The PIVHPFS are replaced by PHPFS, the way is to use the medium value of the interval-valued to replace the interval-valued of membership degree (non-membership). Then the new decision matrix is established by PHPFS shown in Table 3.

The comprehensive value of A_i can be calculated by PHPFHCIG operator (with $\gamma = 0.5$):

$$\tilde{P}_1 = \langle (0.3242, 0.3979), 0.450, (0.2793, 0.3839), 0.550 \rangle,$$
$$\tilde{P}_2 = \langle (0.2330, 0.3599), 0.481, (0.3932, 0.4129), 0.519 \rangle,$$
$$\tilde{P}_3 = \langle (0.3227, 0.4185), 0.443, (0.3034, 0.3817), 0.557 \rangle,$$
$$\tilde{P}_4 = \langle (0.2942, 0.3024), 0.525, (0.3770, 0.4018), 0.475 \rangle.$$

The score function for each strategy is $S_1 = -0.0907$, $S_2 = -0.0712$, $S_3 = -0.0860$, $S_4 = -0.0161$.

From the score function it is can be found that $-0.0161 > -0.0712 > -0.0860 > -0.0907$, which means that $A_4 \succ A_2 \succ A_3 \succ A_1$, A_4 will be selected as the best alternative. It is the same choice as PIVHPFS.

Similarly, PIVHPFS will be replaced with IVPFS by instead the membership (non-membership) degrees of interval-valued that is formed up by the maximum and minimum degrees of membership (non-membership) shown in Table 4.

The comprehensive value of A_i can be calculated by IVPFHCIG operator (with $\gamma = 0.5$):

$$\tilde{P}_1 = \langle [0.2063, 0.4472], [0.2274, 0.4086] \rangle,$$
$$\tilde{P}_2 = \langle [0.1756, 0.3579], [0.2921, 0.5603] \rangle,$$
$$\tilde{P}_3 = \langle [0.2147, 0.3996], [0.2397, 0.4701] \rangle,$$

TABLE 4. The decision matrix given by IVPFS.

	C_1	C_2	C_3	C_4
A_1	$\langle [0.3, 0.5], [0.2, 0.4] \rangle$	$\langle [0.1, 0.5], [0.2, 0.3] \rangle$	$\langle [0.2, 0.5], [0.3, 0.4] \rangle$	$\langle [0.3, 0.5], [0.1, 0.5] \rangle$
A_2	$\langle [0.1, 0.5], [0.1, 0.4] \rangle$	$\langle [0.3, 0.4], [0.3, 0.4] \rangle$	$\langle [0.2, 0.3], [0.4, 0.7] \rangle$	$\langle [0.2, 0.4], [0.2, 0.5] \rangle$
A_3	$\langle [0.4, 0.6], [0.2, 0.5] \rangle$	$\langle [0.3, 0.5], [0.2, 0.5] \rangle$	$\langle [0.1, 0.3], [0.3, 0.4] \rangle$	$\langle [0.3, 0.5], [0.2, 0.5] \rangle$
A_4	$\langle [0.1, 0.5], [0.4, 0.5] \rangle$	$\langle [0.3, 0.6], [0.2, 0.4] \rangle$	$\langle [0.2, 0.3], [0.2, 0.5] \rangle$	$\langle [0.2, 0.4], [0.3, 0.6] \rangle$

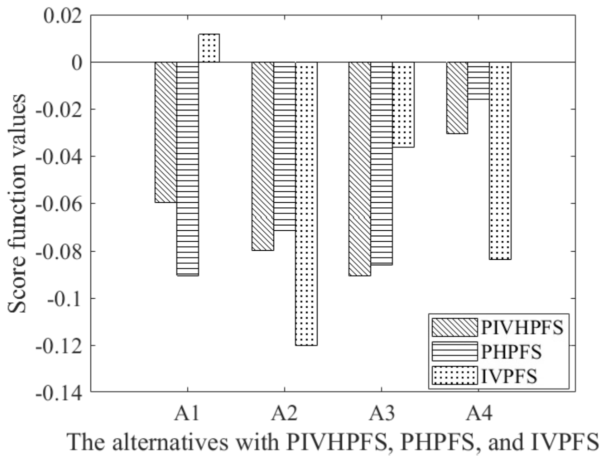

FIGURE 2. The scores of the alternatives based on PIVHPFS, PHPFS, and IVPFS.

$$\tilde{P}_4 = \langle [0.1756, 0.3807], [0.2908, 0.5087] \rangle.$$

Utilize the *Definition 2* to calculate the score function for each strategy is $S_1 = 0.0119$, $S_2 = -0.1202$, $S_3 = -0.0363$, $S_4 = -0.0838$.

From the score function it is can be found that $0.0119 > -0.0363 > -0.0838 > -0.1202$, which means that $A_1 \succ A_3 \succ A_4 \succ A_2$, A_1 will be selected as the best alternative. It is obviously different with PIVHPFS and PHPFS.

According to compared with the existing mentioned comparison, the result of rankings of the alternatives is shown in Figure 2. Although the best project private partner is the same A_4 for both PIVHPFHCIG and PHPFHCIG, the ranking of the alternatives is different from the A_1 based on IVPFHCIG. we choose the same value of parameter γ, the decision-makers have the same optimistic attitude or pessimistic attitude in the face of MADM information. PHPFHCIG has considered probabilistic hesitant fuzzy information, but has depicted the value of membership and non-membership degree with real numbers, which still can't adapt well to the uncertainty in decision-making process. IVPFHCIG hasn't combined probabilistic in the decision process, which ignored the risk of information loss from aggregation operators. In comparison, PIVHPFHCIG has an advantage in dealing with these issues.

D. COMPARATIVE ANALYSIS

This result illustrated that the MADM method based on PIVHPFS environment is accurate and rational. Compared with the existing Pythagorean hesitant fuzzy MCDM method, the advantages of PIVHPFHCIG operator are present as follows:

(1) Considering the correlation between characteristics, and Choquet integral satisfies the multi-attribute decision making better.

(2) Our proposed approach is more flexible and dynamic since the Hamacher aggregation operator is more generalized in parameter γ application. Decision-makers are able to select the parameters γ themselves, which reflect the risk attitude preference of decision-makers. Comparing with existing hesitant fuzzy aggregation to obtain static fixed evaluation resulted, it may reflect the inner rules.

(3) With considering the probabilistic in the MADM process, the approach under PIVHPFS and PHPFS environment may avoid the information loss, and reflect the significance of each attribute.

Although they have selected the same project private partner A_4, in a way, PIVHPFS is more effective than PHPFS in complex uncertain decision making for the real number value is replaced by interval-valued. The shortcoming of the technique under IVPFS environment which selects A_1 as a partner is obvious, it has a quite wide gap with PIVHPFS and PHPFS. The effectiveness also improved by PIVHPFS and the interval-valued make the complex decision more rational. Those shortages will be improved in the future.

VII. CONCLUSION

In this paper, we extend Pythagorean fuzzy set (PFS) to probabilistic interval-valued hesitant Pythagorean fuzzy set (PIVHPFS). As a new operational law, it considers the information loss, hesitant and uncertainty for aggregating different preference opinions of decision-makers during the MADM process. Furthermore, we develop the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator, which reflects the risk attitude preference of decision-makers and consider the phenomenon of the interaction among the MADM. Based on the PIVHPFHCIG operator, we extend the Choquet integral-based method to solve the MADM problem. we performed some comparisons with similar existing research, which illustrates the rationality and effectiveness of the proposed techniques and provides a good complement to the existing work on PFS. The proposed techniques differ from existing approaches for MADM problems, which not only develop the method under probabilistic interval-valued hesitant Pythagorean fuzzy set (PIVHPFS) environment rather than interval-valued Pythagorean fuzzy set (IVPFS) or hesitant Pythagorean fuzzy set (HPFS) but also consider the correlations among the elements in MADM process. Nevertheless, the probabilistic information combine with different fuzzy sets should be attached great attention. In the future, we will extend probabilistic to other fuzzy sets. For instance, dual hesitant sets [39], linguistic dual hesitant fuzzy sets [40], Neutrosophic

Hesitant Fuzzy sets [41], Mixed-discrete Z-numbers [42], Pythagorean fuzzy Hamacher Prioritized operators [43], dual hesitant Pythagorean fuzzy sets [44], generalized Dice similarity measures of PFS [45], Pythagorean Fuzzy Hamacher Power Aggregation Operators [46], etc.

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