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Sequential Three-Way Decision of Tolerance-Based Multi-Granularity Fuzzy-Rough Sets

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ABSTRACT The existing three-way decision-making methods are classified into two classes-a single one-step three-way decision-making and a sequential, multi-step three-way decision-making, and this paper studies the latter. In multi-granularity model, sequential thresholds are used to represent its multi-level granularity. Firstly, the concept of similarity is introduced and tolerance-based fuzzy decision theory rough set is proposed to obtain the upper and lower approximation of decision class and the three-way decisions of the whole decision class. Secondly, the tolerance-based sequential three-way decision is extended to the case of multi-granularity, and the concepts of upper and lower approximation and its three-way decision in optimistic and pessimistic situations are proposed, and some related properties are verified. By adopting the aggressive and conservative strategy, we put forward the concepts of upper and lower approximation and related properties under pessimistic-optimistic conditions as well as sequential three-way decision, the validity of these methods is verified by an example. In optimistic-pessimistic situations, a counterexample is given to prove that it is unable to make decisions. On this basis, uncertainty measures, precision and roughness, are introduced, and some properties of them are studied. Finally, the paper analyses and proves the relationship between the above three models.

INDEX TERMS Decision class, lower approximation, multi-granularity, sequential three-way decision, tolerance-based fuzzy decision theory rough set, upper approximation.

I. INTRODUCTION

Three-way decision was originally proposed by Yao [1], [2], [3] to describe three regions of rough set. Yao [3] formalized a more general framework of three-way decision called the trisecting-and-acting model, which divides a universal set into three pair-wise disjoint parts and performs effective strategies on some or all of the parts. In three-way decision, three regions are called positive, negative and boundary regions, and can be interpreted in terms of decision rules, namely rules for acceptance, rules for non-commitment, and rules for rejection. Objects satisfying acceptance rules are put into positive region, non-commitment rules into boundary region and rejection rules into negative region [4]. In recent years, there have been some successful applications in many fields, such as investment management [5], cluster analysis [6], [7], face recognition [8], recommendation [9] and the others [10]– [12].

Sequential three-way decision is a cost-effective decision-making method [13]. Its goal is to get the required accuracy level at the minimum cost, especially for multi-granularity problems. The most classical sequential three-way decision [13] is mainly aimed at a series of attributes, using the multi-granularity structure of the universe to achieve a sequential, multi-step three-way decision. Qian *et al.* [14], [15] studied the sequential three-way decision based on multi-granularity on multiple thresholds and discussed the multi-granularity sequential three-way decision. Zhang *et al.* [16] studied the automatic encoder function of sequential three-way decision on multi-granularity model.

Fuzzy rough sets study the uncertainty and fuzziness of data sets. The combination of Zadeh's fuzzy set [17] and rough set provides a key way for the uncertainty reasoning of real data. The concept of fuzzy rough set surpasses the shortcomings of classical rough set method in every aspect. Nanda and Majumdar [18] specialize in the study of fuzzy rough sets. Skowron and Stepaniuk [19] studied the tolerance approximation space. Dubois and Prade [20], [21] discussed rough fuzzy sets and fuzzy rough sets and studied them together. Huang *et al.* [22] studied the intuitionistic fuzzy multi-granularity rough set. Under the dominance relation, Huang *et al.* [23] studied the rough set model in the

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intuitionistic fuzzy information system. Jensen and Shen [24] discussed the feature selection of fuzzy rough sets based on tolerance relation. Liang and Liu [25] studied three-way decision of intuitionistic fuzzy decision theory rough set. Anoop *et al.* [26] studied the attribute reduction method based on tolerance intuitionistic fuzzy rough sets.

Multi-granularity rough set was first proposed by Qian *et al.* [27] in 2010. Subsequently, more and more research in this area has attracted wide attention. Multi-granularity rough set model uses many different binary relations. Qian *et al.* [28] and Yang *et al.* [29] discussed incomplete multi-granularity rough set, Xu *et al.* [30], [31] constructed and discussed two types of multi-granularity tolerance rough set model, and constructed multi-granularity fuzzy rough set based on fuzzy approximation space. Qian and Liang [32] studied multi-granularity decision theory rough set. In the covering approximation space, Lin *et al.* [33] studied the multi-granularity covering fuzzy rough set. In [34],the generalized multi granularity and two quantity decision theory rough set is proposed. In [35] and [36], the dynamic game model has certain reference value for the study of this paper.

However, although there are many studies on sequential three-way decision, fuzzy rough set and multi-granularity rough set, there are not many studies combining them. In this paper, we combine fuzzy rough set with multi-granularity rough set under tolreance relation to study the sequential three-way decision. We discuss the optimistic multi-granularity sequential three-way decision-making, the pessimistic multi-granularity sequential three-way decision and study the pessimistic-optimistic multi-granularity sequential three-way decision by using appropriate aggregation strategy. The biggest difference with the previous research is that the multi-granularity model can not make three decisions under the optimistic-pessimistic situation. Finally, the relationship between these models and some properties are discussed.

The rest of the paper is organized as follows. Section 2 briefly introduces the preliminary notions considered in this study. Section 3 three-way decision of sequence based on tolerance relation is proposed. On this basis, combined with multi-granularity rough set, three-way decision of sequence based on tolerance multi-granularity fuzzy rough set is proposed, which includes optimistic, pessimistic and pessimistic-optimistic models, and some correlations between uncertainty measurement: accuracy and roughness are studied. Section 4 analyses the relationship among the three models mentioned above. Section 5 concludes this paper.

II. PRELIMINARIES

A. FUZZY DECISION SYSTEM

A Fuzzy decision system (FDS) is a quadruple $(U, C\bigcup D, V_F, F)$, where U is a finite non-empty set of objects, C is a finite non-empty set of conditional attributes, D is a set of decision attributes, and $C\bigcap D = \emptyset$, V_F is the collection of all fuzzy values, F is an information function that maps an object x in U to exactly one value v in V_F, such that $F(x, a) = \mu_a(x)$, where $\mu_a(x)$ is a membership grade of the object x for an attribute a.

B. TOLERANCE-BASED FUZZY-ROUGH SET

Let FDS is a fuzzy decision system. One of the widely used fuzzy similarity relation is defined as:

$$SIM_a(x_i, x_j) = 1 - \frac{|\ \mu_a(x_i) - \mu_a(x_j)\ |}{|\ \mu_{a_{max}} - \mu_{a_{min}}\ |}$$

where $\mu_a(x_i)$, $\mu_a(x_j)$ are membership grades of objects x_i, x_j and $\mu_{a_{max}}$, $\mu_{a_{min}}$ are maximum and minimum membership grades for an attribute a.

For a subset of attributes A, $(x_i, x_j) \in SIM_A^\delta$ iff $\prod_{a\in A} SIM_a(x_i, x_j) \geq \delta$.

Where δ is a similarity threshold, which required level of similarity for inclusion within tolerance classes. Tolerance classes are defined by fuzzy similarity relation as follows:

$$SIM_A^\delta(x_i) = \{x_j \in U \mid (x_i, x_j) \in SIM_A^\delta\}$$

For a decision class $D_j \in \pi_D$,π_D is the partition of decision classes. The lower and upper approximations of D_j with repect to a cover C_A are defined as follows:

$$\begin{aligned}\underline{A}^\delta(D_j) &= \{x_i \mid SIM_A^\delta(x_i) \subseteq D_j\}\\ &= \{x_i \mid P(D_j \mid SIM_A^\delta(x_i)) = 1\}\\ \overline{A^\delta}(D_j) &= \{x_i \mid SIM_A^\delta(x_i) \bigcap D_j \neq \emptyset\}\\ &= \{x_i \mid P(D_j \mid SIM_A^\delta(x_i)) > 0\}\end{aligned}$$

The lower and upper approximations of the partition π_D are the families of the lower and upper approximations of all the similarity classes of π_D.

$$\begin{aligned}\underline{A}^\delta(\pi_D) &= \{\underline{A}^\delta(D_1), \underline{A}^\delta(D_2)\cdots \underline{A}^\delta(D_m)\}\\ \overline{A^\delta}(\pi_D) &= \{\overline{A^\delta}(D_1), \overline{A^\delta}(D_2)\cdots \overline{A^\delta}(D_m)\}\end{aligned}$$

We have the lower and upper approximations for each decision class, a position, boundary, negative region of a partition π_D with respect to a cover C_A is defined as follow:

$$\begin{aligned}POS(\pi_D) &= \bigcup_{1\leq j\leq m} \underline{A}^\delta(D_j)\\ BND(\pi_D) &= \bigcup_{1\leq j\leq m} (\overline{A^\delta}(D_j) - \underline{A}^\delta(D_j))\\ NEG(\pi_D) &= U - \bigcup_{1\leq j\leq m} \overline{A^\delta}(D_j)\end{aligned}$$

C. TOLERANCE-BASED MULTI-GRANULARITY FUZZY-ROUGH SET

Definition 1: Let A_1, $A_2\cdots A_n$ are n granular structures and $\forall X \subseteq U$, tolerance-based optimistic multi-granularity fuzzy lower and upper approximation are defined as follows,

$$\underline{\sum_{i=1}^{n} A_i^O}(X) = \{x \in U \mid \bigvee_{i=1}^{n} SIM_{A_i}^{\delta}(x) \subseteq X\}$$

$$\overline{\sum_{i=1}^{n} A_i^O}(X) = \sim \underline{\sum_{i=1}^{n} A_i^O}(\sim X)$$

The pair $< \underline{\sum_{i=1}^{n} A_i^O}(X), \overline{\sum_{i=1}^{n} A_i^O}(X) >$ is called the tolerance-based optimistic multi-granularity fuzzy rough sets of X.

Definition 2: Let $A_1, A_2 \cdots A_n$ are n granular structures and $\forall X \subseteq U$, tolerance-based pessimistic multi-granularity fuzzy lower and upper approximation are defined as follows,

$$\underline{\sum_{i=1}^{n} A_i^P}(X) = \{x \in U \mid \bigwedge_{i=1}^{n} SIM_{A_i}^{\delta}(x) \subseteq X\}$$

$$\overline{\sum_{i=1}^{n} A_i^P}(X) = \sim \underline{\sum_{i=1}^{n} A_i^P}(\sim X)$$

The pair $< \underline{\sum_{i=1}^{n} A_i^P}(X), \overline{\sum_{i=1}^{n} A_i^P}(X) >$ is called the tolerance-based pessimistic multi-granularity fuzzy rough sets of X.

D. TOLERANCE-BASED FUZZY DECISION-THEORETIC ROUGH SETS

Suppose the set of states $\Omega = \{X, \sim X\}$, $A = \{a_P, a_B, a_N\}$ is the set of actions, where a_P, a_B, a_N represent the three actions in classifying an object x, deciding $x \in POS(X)$, deciding $x \in BND(X)$, and deciding $x \in NEG(X)$. λ_{PP}, λ_{BP} and λ_{NP} denote when an object belongs to X, the loss incurred for taking actions of a_P, a_B and a_N. Similarly, λ_{NN}, λ_{BN} and λ_{PN} denote the losses incurred for taking the corresponding actions when the object belongs to $\sim X$. By using the conditional probability, the Bayesian decision procedure can decide how to assign x to these three disjoint regions. The expected loss $R(a_* \mid SIM_A^{\delta}(x))$ associated with taking the individual action can be expressed as

$$\begin{aligned} R(a_P \mid SIM_A^{\delta}(x)) &= \lambda_{PP} P(X \mid SIM_A^{\delta}(x)) \\ &\quad + \lambda_{PN} P(\sim X \mid SIM_A^{\delta}(x)) \\ R(a_N \mid SIM_A^{\delta}(x)) &= \lambda_{NP} P(X \mid SIM_A^{\delta}(x)) \\ &\quad + \lambda_{NN} P(\sim X \mid SIM_A^{\delta}(x)) \\ R(a_B \mid SIM_A^{\delta}(x)) &= \lambda_{BP} P(X \mid SIM_A^{\delta}(x)) \\ &\quad + \lambda_{BN} P(\sim X \mid SIM_A^{\delta}(x)) \end{aligned}$$

When $0 \leq \lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$ and $0 \leq \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following three minimum-risk decision rules:

(P1) If $P(X \mid SIM_A^{\delta}(x)) \geq \alpha$ and $P(X \mid SIM_A^{\delta}(x)) \geq \beta$, decide $x \in POS(x)$.

(N1) If $P(X \mid SIM_A^{\delta}(x)) \leq \gamma$ and $P(X \mid SIM_A^{\delta}(x)) \leq \beta$, decide $x \in NEG(x)$.

(B1) If $P(X \mid SIM_A^{\delta}(x)) \leq \alpha$ and $P(X \mid SIM_A^{\delta}(x)) \geq \beta$, decide $x \in BND(x)$.

where

$$\begin{aligned} \alpha &= \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP}})) \\ \beta &= \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP}})) \\ \gamma &= \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP}})) \end{aligned}$$

If a loss function also satisfies the condition $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, then $\alpha \geq \gamma \geq \beta$.

When $\alpha > \beta$, we have $\alpha > \gamma > \beta$. $(P1) - (B1)$ can be re-expressed as follows:

(P2) If $P(X \mid SIM_A^{\delta}(x)) \geq \alpha$, decide $x \in POS(x)$.

(N2) If $P(X \mid SIM_A^{\delta}(x)) \leq \beta$, decide $x \in NEG(x)$.

(B2) If $\beta < P(X \mid SIM_A^{\delta}(x)) < \alpha$, decide $x \in BND(x)$.

Using these decision rules, the probabilistic lower and upper approximations of a decision class D_j with respect to a cover C_A can be defined by:

$$\underline{A^{\delta,\,(\alpha,\,\beta)}}(D_j) = \{x \in U \mid P(D_j \mid SIM_A^{\delta}(x)) \geq \alpha\}$$
$$\overline{A^{\delta,\,(\alpha,\,\beta)}}(D_j) = \{x \in U \mid P(D_j \mid SIM_A^{\delta}(x)) > \beta\}$$

In tolerance-based fuzzy decision-therotic rough set models, three probabilistic regions of a partition π_D with respect to a cover C_A can be defined as follows:

$$POS^{\alpha,\,\beta}(\pi_D) = \bigcup_{j=1}^{m} \underline{A^{\delta,\,(\alpha,\,\beta)}}(D_j)$$

$$BND^{\alpha,\,\beta}(\pi_D) = \bigcup_{j=1}^{m} (\overline{A^{\delta,\,(\alpha,\,\beta)}}(D_j) - \underline{A^{\delta,\,(\alpha,\,\beta)}}(D_j))$$

$$NEG^{\alpha,\,\beta}(\pi_D) = U - \bigcup_{j=1}^{m} \overline{A^{\delta,\,(\alpha,\,\beta)}}(D_j)$$

III. SEQUENTIAL THREE-WAY DECISIONS OF TOLERANCE-BASED MULTI-GRANULARITY FUZZY-ROUGH SET

A. TOLERANCE-BASED SEQUENTIAL THREE-WAY DECISIONS

In [15], the author discusses the sequential three-way decisions based on equivalence relation. In this section, we discuss sequential three-way decisions based on tolerance relation and give some properties.

Definition 3: For a fuzzy decision system F, a given decision class D_j^l, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \cdots, (\alpha^l, \beta^l)\}$, for a granular structure $A \subseteq C$, C is a finite non-empty set of conditional attributes. the (α^l, β^l) lower approximation and upper approximation are defined by

$$\underline{A^{\delta,\,(\alpha^l,\,\beta^l)}}(D_j^l) = \{x \in U^l \mid P(D_j^l \mid SIM_A^{\delta}(x)) \geq \alpha^l\}$$
$$\overline{A^{\delta,\,(\alpha^l,\,\beta^l)}}(D_j^l) = \{x \in U^l \mid P(D_j^l \mid SIM_A^{\delta}(x)) > \beta^l\}$$

TABLE 1. Fuzzy decision system.

U	a_1	a_2	a_3	a_4	D
x_1	0.5	0.4	0.8	1	1
x_2	0.7	0.3	0.6	0.2	2
x_3	0.8	0.5	1	0.7	1
x_4	0.4	0.5	0.9	0.2	1
x_5	0.3	0.6	0.5	0.7	2
x_6	1	0.8	0.6	0.4	1

where $U^1 = U, U^{l+1} = BND_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l)$ is the gradually reduced universe.

The pair $< \underline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j),\ \overline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j) >$ is called the lth-level lower and upper approximations induced by A with respect to D_j^l in U^l. Thus, we can obtain positive, boundary and negative regions as follows

$$POS_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l) = \{x \in U^l \mid P(D_j^l \mid SIM_A^\delta(x)) \geq \alpha^l\}$$
$$BND_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l) = \{x \in U^l \mid \beta^l < P(D_j^l \mid SIM_A^\delta(x)) < \alpha^l\}$$
$$NEG_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l) = \{x \in U^l \mid P(D_j^l \mid SIM_A^\delta(x)) \leq \beta^l\}$$

when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:

If $P(D_j^l \mid SIM_A^\delta(x)) \geq \alpha^l$, decision $x \in POS_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l)$

If $P(D_j^l \mid SIM_A^\delta(x)) \leq \beta^l$, decision $x \in NEG_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l)$

If $\beta^l < P(D_j^l \mid SIM_A^\delta(x)) < \alpha^l$, decision $x \in BND_{A^\delta}^{(\alpha^l,\ \beta^l)}(D_j^l)$

Proposition 1 Given (α^l, β^l)-lower approximation $\underline{A^{\delta,(\alpha^l,\beta^l)}}(D_j)$ and (α^l, β^l)-upper approximation $\overline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j)$, then

$$(1)\ \underline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l) \subseteq \overline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l)$$
$$(2)\ \underline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(U) = \overline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(U) = U$$

Note: $\underline{A^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l) \nsubseteq D_j^l$.

The following example shows that there is no inclusion relationship between them.

Example 1. Give a fuzzy decision system.

$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, a decision class $D_1^1 = \{x_1, x_3, x_4, x_6\}$,$A = \{a_1\}$, let $\delta = 0.57$, $(\alpha, \beta)^2 = \{(0.75, 0.5), (0.7, 0.68)\}$

$$SIM_{a_1}(x_1, x_1) = 1, SIM_{a_1}(x_1, x_2) = \frac{5}{7},$$
$$SIM_{a_1}(x_1, x_3) = \frac{4}{7}, SIM_{a_1}(x_1, x_4) = \frac{6}{7},$$
$$SIM_{a_1}(x_1, x_5) = \frac{5}{7}, SIM_{a_1}(x_1, x_6) = \frac{2}{7},$$
$$SIM_{a_1}(x_2, x_2) = 1, SIM_{a_1}(x_2, x_3) = \frac{6}{7},$$
$$SIM_{a_1}(x_2, x_4) = \frac{4}{7}, SIM_{a_1}(x_2, x_5) = \frac{3}{7},$$
$$SIM_{a_1}(x_2, x_6) = \frac{4}{7}, SIM_{a_1}(x_3, x_3) = 1,$$
$$SIM_{a_1}(x_3, x_4) = \frac{3}{7}, SIM_{a_1}(x_3, x_5) = \frac{2}{7},$$
$$SIM_{a_1}(x_3, x_6) = \frac{5}{7}, SIM_{a_1}(x_4, x_4) = 1,$$
$$SIM_{a_1}(x_4, x_5) = \frac{3}{7}, SIM_{a_1}(x_4, x_6) = \frac{1}{7},$$
$$SIM_{a_1}(x_5, x_5) = 1, SIM_{a_1}(x_5, x_6) = 0,$$
$$SIM_{a_1}(x_6, x_6) = 1,\ \text{then}$$
$$SIM_{a_1}^\delta(x_1) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$SIM_{a_1}^\delta(x_2) = \{x_1, x_2, x_3, x_4, x_6\}$$
$$SIM_{a_1}^\delta(x_3) = \{x_1, x_2, x_3, x_6\}$$
$$SIM_{a_1}^\delta(x_4) = \{x_1, x_2, x_4\}$$
$$SIM_{a_1}^\delta(x_5) = \{x_1, x_5\}$$
$$SIM_{a_1}^\delta(x_6) = \{x_2, x_3, x_6\}$$

we can obtain the conditional probabilities under a granular structure a_1 are computed as.

$$P(D_1^1 \mid SIM_{a_1}^\delta(x_1)) = \frac{3}{5}$$
$$P(D_1^1 \mid SIM_{a_1}^\delta(x_2)) = \frac{4}{5}$$
$$P(D_1^1 \mid SIM_{a_1}^\delta(x_3)) = \frac{3}{4}$$
$$P(D_1^1 \mid SIM_{a_1}^\delta(x_4)) = \frac{2}{3}$$
$$P(D_1^1 \mid SIM_{a_1}^\delta(x_5)) = \frac{1}{2}$$
$$P(D_1^1 \mid SIM_{a_1}^\delta(x_6)) = \frac{2}{3}$$

then $\underline{A^{0.57,(0.75^1,0.5^1)}}(D_1^1) = \{x_2, x_3\} \nsubseteq D_1^1$.

Example 2. (Continued example 1)

$$U^1 = U, D_1^1 = \{x_1, x_3, x_4, x_6\}$$
$$POS_{a_1}^{(\alpha^1,\ \beta^1)}(D_1^1) = \{x_2, x_3\},$$
$$BND_{a_1}^{(\alpha^1,\ \beta^1)}(D_1^1) = \{x_1, x_4, x_6\},$$
$$NEG_{a_1}^{(\alpha^1,\ \beta^1)}(D_1^1) = \{x_5\}$$
$$U^2 = \{x_1, x_4, x_6\}, D_1^2 = \{x_1, x_4, x_6\}$$
$$P(D_1^2 \mid SIM_{a_1}^\delta(x_1)) = \frac{2}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^\delta(x_4)) = \frac{1}{3}$$
$$P(D_1^2 \mid SIM_{a_1}^\delta(x_6)) = \frac{2}{3}$$
$$POS_{a_1}^{(\alpha^2,\ \beta^2)}(D_1^2) = \emptyset,$$
$$BND_{a_1}^{(\alpha^2,\ \beta^2)}(D_1^2) = \emptyset,$$
$$NEG_{a_1}^{(\alpha^2,\ \beta^2)}(D_1^2) = \{x_1, x_4, x_6\}$$
$$\text{Then} POS_{a_1}^{(\alpha,\ \beta)^2}(D_1) = POS_{a_1}^{(\alpha^1,\ \beta^1)}(D_1^1)) \bigcup POS_{a_1}^{(\alpha^2,\ \beta^2)}(D_1^2) = \{x_2, x_3\}$$
$$NEG_{a_1}^{(\alpha,\ \beta)^2}(D_1) = NEG_{a_1}^{(\alpha^1,\ \beta^1)}(D_1^1) \bigcup NEG_{a_1}^{(\alpha^2,\ \beta^2)}(D_1^2) = \{x_1, x_4, x_5, x_6\}$$
$$BND_{a_1}^{(\alpha,\ \beta)^2}(D_1) = \emptyset$$

The $POS_{a_1}^{(\alpha,\ \beta)^2}(D_1), NEG_{a_1}^{(\alpha,\ \beta)^2}(D_1), BND_{a_1}^{(\alpha,\ \beta)^2}(D_1)$ are sequential three-way decisions of D_1 under a single attribute a_1.

B. SEQUENTIAL THREE-WAY DECISIONS OF TOLERANCE-BASED MULTI-GRANULARITY FUZZY-ROUGH SET

In the previous section, we introduced sequential three-way decisions based on tolerance relation in single-granularity case. In this section, we will introduce sequential three-way decisions in multi-granularity case, optimistic, pessimistic and pessimistic-optimistic case, and gave a counter-example in optimistic-pessimistic case to prove that they could not make decision.

1) TOLERANCE-BASED OPTIMISTIC MULTI-GRANULARITY FUZZY-ROUGH SET SEQUENTIAL THREE-WAY DECISIONS

Definition 4: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the lower approximation and upper approximation of the tolerance-based optimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$\underline{\sum_{i=1}^{n} A_i^{(\delta, O, (\alpha^l, \beta^l))}}(D_j^l) = \{x \in U^l \mid \bigvee_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \geq \alpha^l\}$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta, O, (\alpha^l, \beta^l))}}(D_j^l) = \{x \in U^l \mid \bigvee_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) > \beta^l\}$$

where

$$U^1 = U, U^{l+1} = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

The pair

$$< \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l), \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) >$$

is called the lth-level tolerance-based optimistic multi-granularity fuzzy-rough set sequential three-way decisions.

According to the lower and upper approximations,the tolerance-based optimistic multi-granularity fuzzy-rough set sequential three-way decisions boundary region of D_j^l is

$$BND^{O,\ (\alpha^l,\ \beta^l)}(D_j^l) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

Proposition 2. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. then,

$$(1)\ \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \bigcup_{i=1}^{n} \underline{A_i^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l)$$

$$(2)\ \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \bigcup_{i=1}^{n} \overline{A_i^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l)$$

$$(3)\ \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(U) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(U) = U$$

(4) If $0 \leq \beta^l \leq \beta^{l'} < \alpha^{l'} \leq \alpha^l \leq 1,$ then

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l)$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

Similar to the classical three-way decisions, we can obtain the decision rules tie-broke:

(OP) If $\exists\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \geq \alpha^l\}$, decide $POS^{O,(\alpha^l,\ \beta^l)}(D_j^l)$

(ON) If $\forall\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \leq \beta^l\}$, decide $NEG^{O,(\alpha^l,\ \beta^l)}(D_j^l)$

(OB) Otherwise, decide $BND^{O,(\alpha^l,\ \beta^l)}(D_j^l)$

Definition 5: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the accuracy and roughness of the tolerance-based optimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$a_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = \frac{\left| \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) \right|}{\left| \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l) \right|}$$

$$\rho_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 1 - \alpha_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l)$$

Proposition 3. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. $a_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l), \rho_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l)$ are the accuracy and roughness measures, then,

$$(1)\ 0 \leq a_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l), \rho_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) \leq 1$$

$$(2)\ a_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 1 \Longleftrightarrow \rho_{\sum_{i=1}^{n} A_i}^{O,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 0$$

(3) $0 \leq \beta^{l} \leq \beta^{l'} < \alpha^{l'} \leq \alpha^{l} \leq 1, a^{O,\,\delta,\,(\alpha^{l},\,\beta^{l})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j})$
$\leq a^{O,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j}), \rho^{O,\,\delta,\,(\alpha^{l},\,\beta^{l})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j})$
$\geq \rho^{O,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j})$

Proof: (1), (2) are straightforward.

(3) By $\alpha^{l'} \leq \alpha^{l}$,

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l},\,\beta^{l}))}}(D^{l}_{j}) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l'},\,\beta^{l'}))}}(D^{l}_{j}),$$

then

$$\mid \underline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l},\,\beta^{l}))}}(D^{l}_{j}) \mid \leq \mid \underline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l'},\,\beta^{l'}))}}(D^{l}_{j}) \mid .$$

By

$$\beta^{l} \leq \beta^{l'}, \overline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l'},\,\beta^{l'}))}}(D^{l}_{j}) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l},\,\beta^{l}))}}(D^{l}_{j}),$$

then

$$\mid \overline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l'},\,\beta^{l'}))}}(D^{l}_{j}) \mid \leq \mid \overline{\sum_{i=1}^{n} A_i^{(\delta,\,O,\,(\alpha^{l},\,\beta^{l}))}}(D^{l}_{j}) \mid .$$

Thus,

$$a^{O,\,\delta,\,(\alpha^{l},\,\beta^{l})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j}) \leq a^{O,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j}),$$

so

$$\rho^{O,\,\delta,\,(\alpha^{l},\,\beta^{l})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j}) \geq \rho^{O,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D^{l}_{j}).$$

Example 3. (Continued example 1)

In example 1, we calculate the similarity classes and conditional probabilities of $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6$ under a_1. Next, we calculate the similarity class and conditional probability of $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6$ under a_2.

$U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, a decision class $D^1_1 = \{x_1, x_3, x_4, x_6\}$,$A = \{a_2\}$, let $\delta = 0.57$, $(\alpha, \beta)^2 = \{(0.75, 0.5), (0.7, 0.68)\}$

$$SIM_{a_2}(x_1, x_1) = 1, SIM_{a_2}(x_1, x_2) = \frac{4}{5},$$
$$SIM_{a_2}(x_1, x_3) = \frac{4}{5}, SIM_{a_2}(x_1, x_4) = \frac{4}{5},$$
$$SIM_{a_2}(x_1, x_5) = \frac{3}{5}, SIM_{a_2}(x_1, x_6) = \frac{1}{5},$$
$$SIM_{a_2}(x_2, x_2) = 1, SIM_{a_2}(x_2, x_3) = \frac{3}{5},$$
$$SIM_{a_2}(x_2, x_4) = \frac{3}{5}, SIM_{a_2}(x_2, x_5) = \frac{2}{5},$$
$$SIM_{a_2}(x_2, x_6) = 0, SIM_{a_2}(x_3, x_3) = 1,$$
$$SIM_{a_2}(x_3, x_4) = 1, SIM_{a_2}(x_3, x_5) = \frac{4}{5},$$
$$SIM_{a_2}(x_3, x_6) = \frac{2}{5}, SIM_{a_2}(x_4, x_4) = 1,$$
$$SIM_{a_2}(x_4, x_5) = \frac{4}{5}, SIM_{a_2}(x_4, x_6) = \frac{2}{5},$$
$$SIM_{a_2}(x_5, x_5) = 1, SIM_{a_2}(x_5, x_6) = \frac{3}{5},$$
$$SIM_{a_2}(x_6, x_6) = 1,\ \text{then}$$
$$SIM^{\delta}_{a_2}(x_1) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$SIM^{\delta}_{a_2}(x_2) = \{x_1, x_2, x_3, x_4\}$$
$$SIM^{\delta}_{a_2}(x_3) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$SIM^{\delta}_{a_2}(x_4) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$SIM^{\delta}_{a_2}(x_5) = \{x_1, x_3, x_4, x_5, x_6\}$$
$$SIM^{\delta}_{a_2}(x_6) = \{x_5, x_6\}$$

we can obtain the conditional probabilities under a granular structure a_1 are computed as.

$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_1)) = \frac{3}{5}$$
$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_2)) = \frac{3}{4}$$
$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_3)) = \frac{3}{5}$$
$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_4)) = \frac{3}{5}$$
$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_5)) = \frac{4}{5}$$
$$P(D^1_1 \mid SIM^{\delta}_{a_2}(x_6)) = \frac{1}{2}$$

(1) $U^1 = U$, then

$$\underline{\sum_{i=1}^{2} A_i^{(\delta,\,O,\,(\alpha^1,\,\beta^1))}}(D^1_1) = \{x_2, x_3, x_5\}$$
$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\,O,\,(\alpha^1,\,\beta^1))}}(D^1_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$
$$POS^{O,\,(\alpha^1,\,\beta^{l})}(D^1_1) = \{x_2, x_3, x_5\},$$
$$BND^{O,\,(\alpha^1,\,\beta^{l})}(D^1_1) = \{x_1, x_4, x_6\},$$
$$NEG^{O,\,(\alpha^1,\,\beta^{l})}(D^1_1) = \emptyset$$

(2) $U^2 = \{x_1, x_4, x_6\}, D^2_1 = \{x_1, x_4\}$

$$P(D^2_1 \mid SIM^{\delta}_{a_1}(x_1)) = \frac{2}{5}, P(D^2_1 \mid SIM^{\delta}_{a_2}(x_1)) = \frac{2}{5}$$
$$P(D^2_1 \mid SIM^{\delta}_{a_1}(x_4)) = \frac{2}{3}, P(D^2_1 \mid SIM^{\delta}_{a_2}(x_4)) = \frac{2}{5}$$
$$P(D^2_1 \mid SIM^{\delta}_{a_1}(x_6)) = 0, P(D^2_1 \mid SIM^{\delta}_{a_2}(x_6)) = 0$$

then $\underline{\sum_{i=1}^{2} A_i^{(\delta,\,O,\,(\alpha^2,\,\beta^2))}}(D^2_1) = \emptyset$

$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\,O,\,(\alpha^2,\,\beta^2))}}(D^2_1) = \emptyset$$

so $POS^{O,\,(\alpha^2,\,\beta^2)}(D^2_1) = \emptyset$,
$BND^{O,\,(\alpha^2,\,\beta^2)}(D^2_1) = \emptyset$,
$NEG^{O,\,(\alpha^2,\,\beta^2)}(D^2_1) = \{x_1, x_4, x_6\}$

Then

$POS^{O,\,(\alpha,\,\beta)^2}(D_1) = POS^{O,\,(\alpha^1,\,\beta^1)}(D_1^1)) \bigcup$

$POS^{O,\,(\alpha^2,\,\beta^2)}(D_1^2) = \{x_2, x_3, x_5\}$

$NEG^{O,\,(\alpha,\,\beta)^2}(D_1) = NEG^{O,\,(\alpha^1,\,\beta^1)}(D_1^1) \bigcup$

$NEG^{O,\,(\alpha^2,\,\beta^2)}(D_1^2) = \{x_1, x_4, x_6\}$

$BND^{O,\,(\alpha,\,\beta)^2}(D_1) = \emptyset$

The $POS^{O,\,(\alpha,\,\beta)^2}(D_1)$, $NEG^{O,\,(\alpha,\,\beta)^2}(D_1)$,

$BND^{O,\,(\alpha,\,\beta)^2}(D_1)$ are optimistic sequential three-way decisions of D_1 under two attributes a_1 and a_2.

2) TOLERANCE-BASED PESSIMISTIC MULTI-GRANULARITY FUZZY-ROUGH SET SEQUENTIAL THREE-WAY DECISIONS

Definition 6: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the lower approximation and upper approximation of the tolerance-based pessimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,P,(\alpha^l,\beta^l))}}(D_j^l) = \{x \in U^l \mid \bigwedge_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \geq \alpha^l\}$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,P,(\alpha^l,\beta^l))}}(D_j^l) = \{x \in U^l \mid \bigwedge_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) > \beta^l\}$$

where

$$U^1 = U,\ U^{l+1} = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

The pair

$$< \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l), \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) >$$

is called the lth-level tolerance-based pessimistic multi-granularity fuzzy-rough set sequential three-way decisions.

According to the lower and upper approximations,the tolerance-based pessimistic multi-granularity fuzzy-rough set sequential three-way decisions boundary region of D_j^l is

$$BND^{P,\,(\alpha^l,\,\beta^l)}(D_j^l) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

Proposition 4. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. then,

$$(1)\ \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \bigcap_{i=1}^{n} \underline{A_i^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l)$$

$$(2)\ \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \bigcap_{i=1}^{n} \overline{A_i^{\delta,\ (\alpha^l,\ \beta^l)}}(D_j^l)$$

$$(3)\ \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(U) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(U) = U$$

(4) If $0 \leq \beta^l \leq \beta^{l'} < \alpha^{l'} \leq \alpha^l \leq 1$, then

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l)$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

Similar to the classical three-way decisions, we can obtain the decision rules tie-broke:

(OP) If $\forall\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \geq \alpha^l\}$, decide $POS^{P,(\alpha^l,\ \beta^l)}(D_j^l)$

(ON) If $\exists\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \leq \beta^l\}$, decide $NEG^{P,(\alpha^l,\ \beta^l)}(D_j^l)$

(OB) Otherwise, decide $BND^{P,(\alpha^l,\ \beta^l)}(D_j^l)$

Definition 7: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the accuracy and roughness of the tolerance-based pessimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$a_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = \frac{\mid \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \mid}{\mid \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \mid}$$

$$\rho_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 1 - \alpha_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l)$$

Proposition 5. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. $a_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l)$, $\rho_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l)$ are the accuracy and roughness measures, then,

$$(1)\ 0 \leq a_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l), \rho_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) \leq 1$$

$$(2)\ a_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 1 \Longleftrightarrow \rho_{\sum_{i=1}^{n} A_i}^{P,\ \delta,\ (\alpha^l,\ \beta^l)}(D_j^l) = 0$$

$$(3)\ 0 \le \beta^l \le \beta^{l'} < \alpha^{l'} \le \alpha^l \le 1,\ a^{P,\ \delta,\ (\alpha^l,\ \beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l)$$
$$\le a^{P,\ \delta,\ (\alpha^{l'},\ \beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l),\ \rho^{P,\ \delta,\ (\alpha^l,\ \beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l)$$
$$\ge \rho^{P,\ \delta,\ (\alpha^{l'},\ \beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l)$$

Proof: (1), (2) are straightforward.

(3) By $\alpha^{l'} \le \alpha^l$,

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l),$$

then

$$\left| \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \right| \le \left| \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l) \right|.$$

By

$$\beta^l \le \beta^{l'},\ \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l),$$

then

$$\left| \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^{l'},\ \beta^{l'}))}}(D_j^l) \right| \le \left| \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \right|.$$

Thus,

$$a^{P,\ \delta,\ (\alpha^l,\ \beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \le a^{P,\ \delta,\ (\alpha^{l'},\ \beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l),$$

so

$$\rho^{P,\ \delta,\ (\alpha^l,\ \beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \ge \rho^{P,\ \delta,\ (\alpha^{l'},\ \beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l).$$

Example 4 calculates the sequential three-way decisions under pessimistic circumstances.

Example 4. (Continued example 3)

(1) $U^1 = U$, then

$$\underline{\sum_{i=1}^{2} A_i^{(\delta,\ P,\ (\alpha^1,\ \beta^1))}}(D_1^1) = \{x_2\}$$

$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\ P,\ (\alpha^1,\ \beta^1))}}(D_1^1) = \{x_1, x_2, x_3, x_4\}$$

$POS^{P,\ (\alpha^1,\ \beta^l}(D_1^1) = \{x_2\}, BND^{P,\ (\alpha^1,\ \beta^l}(D_1^1) = \{x_1, x_4, x_6\},$
$NEG^{P,\ (\alpha^1,\ \beta^l}(D_1^1) = \{x_5, x_6\}$

(2) $U^2 = \{x_1, x_3, x_4\}, D_1^2 = \{x_1, x_3, x_4\}$

$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_1)) = \frac{3}{5}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_1)) = \frac{3}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_3)) = \frac{1}{2}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_3)) = \frac{3}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_4)) = \frac{2}{3}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_4)) = \frac{3}{5}$$

then $\underline{\sum_{i=1}^{2} A_i^{(\delta,\ P,\ (\alpha^2,\ \beta^2))}}(D_1^2) = \emptyset$

$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\ P,\ (\alpha^2,\ \beta^2))}}(D_1^2) = \emptyset$$

so $POS^{P,\ (\alpha^2,\ \beta^2)}(D_1^2) = \emptyset,$
$BND^{P,\ (\alpha^2,\ \beta^2)}(D_1^2) = \emptyset,$
$NEG^{P,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_1, x_3, x_4\}$
Then $POS^{P,\ (\alpha,\ \beta)^2}(D_1) = POS^{P,\ (\alpha^1,\ \beta^l)}(D_1^1)) \bigcup$
$POS^{P,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_2\}$
$NEG^{P,\ (\alpha,\ \beta)^2}(D_1) = NEG^{P,\ (\alpha^1,\ \beta^l)}(D_1^1) \bigcup$
$NEG^{P,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_1, x_3, x_4, x_5, x_6\}$
$BND^{P,\ (\alpha,\ \beta)^2}(D_1) = \emptyset$

3) TOLERANCE-BASED PESSIMISTIC OPTIMISTIC MULTI-GRANULARITY FUZZY-ROUGH SET SEQUENTIAL THREE-WAY DECISIONS

For the optimistic and pessimistic three-way decisions, we can adopt the conservative strategy for the lower approximation and use the aggressive strategy for the upper approximations.

Definition 8: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the lower approximation and upper approximation of the tolerance-based pessimistic-optimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$\underline{\sum_{i=1}^{n} A_i^{(\delta, PO, (\alpha^l, \beta^l))}}(D_j^l) = \{x \in U^l \mid \bigwedge_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \ge \alpha^l\}$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta, PO, (\alpha^l, \beta^l))}}(D_j^l) = \{x \in U^l \mid \bigvee_{i=1}^{n} P(D_j^l \mid SIM_{A_i}^{\delta}(x)) > \beta^l\}$$

where

$$U^1 = U, U^{l+1} = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

The pair

$$< \underline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l), \overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l) >$$

is called the lth-level tolerance-based pessimistic-optimistic multi-granularity fuzzy-rough set sequential three-way decisions.

According to the lower and upper approximations,the tolerance-based pessimistic-optimistic multi-granularity fuzzy-rough set sequential three-way decisions boundary region of D_j^l is

$$BND^{PO,\,(\alpha^l,\,\beta^l)} = \overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l) - \underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l).$$

Proposition 6. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. then,

(1) $\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l) = \bigcap_{i=1}^{n} \underline{A_i^{\delta,\,(\alpha^l,\,\beta^l)}}(D_i^l)$

(2) $\overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l) = \bigcup_{i=1}^{n} \overline{A_i^{\delta,\,(\alpha^l,\,\beta^l)}}(D_i^l)$

(3) $\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(U) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(U) = U$

(4) If $0 \le \beta^l \le \beta^{l'} < \alpha^{l'} \le \alpha^l \le 1$, then

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l)$$

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l)$$

Similar to the classical three-way decisions, we can obtain the decision rules tie-broke:

(OP) If $\forall\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \ge \alpha^l\}$, decide $POS^{PO,(\alpha^l,\,\beta^l)}(D_j^l)$

(ON) If $\forall\ i \in \{1, 2, \cdots, n\}$ such that $P(D_j^l \mid SIM_{A_i}^{\delta}(x)) \le \beta^l\}$, decide $NEG^{PO,(\alpha^l,\,\beta^l)}(D_j^l)$

(OB) Otherwise, decide $BND^{PO,(\alpha^l,\,\beta^l)}(D_j^l)$

Definition 9: For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, the accuracy and roughness of the tolerance-based pessimistic-optimistic multi-granularity fuzzy-rough set sequential three-way decisions are defined as follows.

$$a^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) = \frac{\left|\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l)\right|}{\left|\overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l)\right|}$$

$$\rho^{O,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) = 1 - \alpha^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l)$$

Proposition 7. For a fuzzy decision system F, given n granular structures $GS = \{A_1, A_2, \cdots, A_n\}$, a given decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$. $a^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l)$, $\rho^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l)$ are the accuracy and roughness measures, then,

(1) $0 \le a^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l),\ \rho^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \le 1$

(2) $a^{O,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) = 1 \Longleftrightarrow \rho^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) = 0$

(3) $0 \le \beta^l \le \beta^{l'} < \alpha^{l'} \le \alpha^l \le 1$,

$$a^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \le a^{PO,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l)$$

$$\rho^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \ge \rho^{PO,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l)$$

Proof: (1), (2) are straightforward.

(3) By $\alpha^{l'} \le \alpha^l$,

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l),$$

then

$$\left|\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l)\right| \le \left|\underline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l)\right|.$$

By $\beta^l \le \beta^{l'}$,

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l),$$

then

$$\left|\overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^{l'},\,\beta^{l'}))}}(D_j^l) \le \overline{\sum_{i=1}^{n} A_i^{(\delta,\,PO,\,(\alpha^l,\,\beta^l))}}(D_j^l)\right|.$$

Thus,

$$a^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \le a^{PO,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l),$$

so

$$\rho^{PO,\,\delta,\,(\alpha^l,\,\beta^l)}_{\sum_{i=1}^{n} A_i}(D_j^l) \ge \rho^{PO,\,\delta,\,(\alpha^{l'},\,\beta^{l'})}_{\sum_{i=1}^{n} A_i}(D_j^l).$$

Example 5. (Continued example 3)

(1) $U^1 = U$, then

$$\underline{\sum_{i=1}^{2} A_i^{(\delta,\,PO,\,(\alpha^1,\,\beta^1))}}(D_1^1) = \{x_2\}$$

$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\,PO,\,(\alpha^1,\,\beta^1))}}(D_1^1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

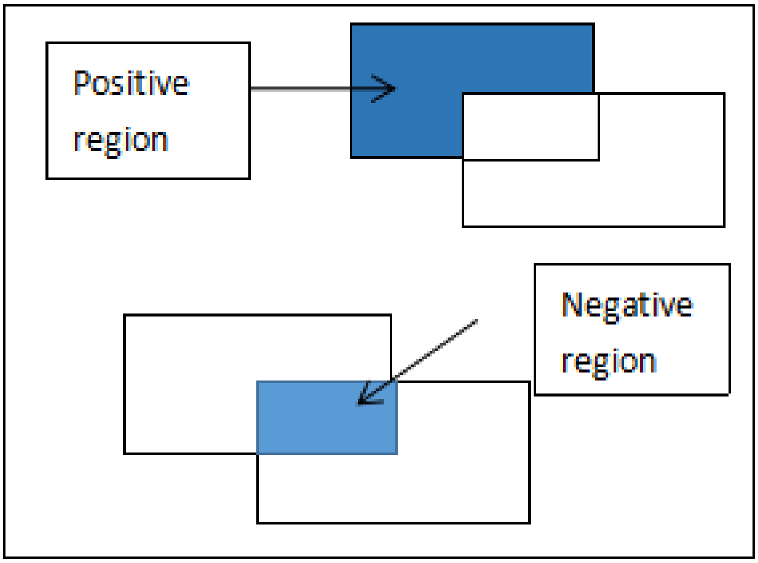

FIGURE 1. Optimistic multi-granulation three-way decisions.

$POS^{PO,\ (\alpha^1,\ \beta^l}(D_1^1) = \{x_2\}, BND^{PO,\ (\alpha^1,\ \beta^l}(D_1^1)$
$= \{x_1, x_3, x_4, x_5 x_6\},$
$NEG^{PO,\ (\alpha^1,\ \beta^l}(D_1^1) = \emptyset$
(2) $U^2 = \{x_1, x_3, x_4, x_5, x_6\}, D_1^2 = \{x_1, x_3, x_4, x_6\}$

$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_1)) = \frac{3}{5}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_1)) = \frac{3}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_3)) = \frac{3}{4}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_3)) = \frac{3}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_4)) = \frac{2}{3}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_4)) = \frac{3}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_5)) = \frac{1}{2}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_5)) = \frac{4}{5}$$
$$P(D_1^2 \mid SIM_{a_1}^{\delta}(x_6)) = \frac{2}{3}, P(D_1^2 \mid SIM_{a_2}^{\delta}(x_6)) = \frac{1}{2}$$

then $\underline{\sum_{i=1}^{2} A_i^{(\delta,\ PO,\ (\alpha^2,\ \beta^2))}}(D_1^2) = \emptyset$

$$\overline{\sum_{i=1}^{2} A_i^{(\delta,\ PO,\ (\alpha^2,\ \beta^2))}}(D_1^2) = \{x_3, x_5\}$$

so $POS^{PO,\ (\alpha^2,\ \beta^2)}(D_1^2) = \emptyset,$
$BND^{PO,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_3, x_5\},$
$NEG^{PO,\ (\alpha^1,\ \beta^l)}(D_1^2) = \{x_1, x_4, x_6\}$
Then $POS^{PO,\ (\alpha,\ \beta)^2}(D_1) = POS^{PO,\ (\alpha^1,\ \beta^l)}(D_1^1)) \bigcup$
$POS^{PO,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_2\}$
$NEG^{PO,\ (\alpha,\ \beta)^2}(D_1) = NEG^{PO,\ (\alpha^1,\ \beta^l)}(D_1^1) \bigcup$
$NEG^{PO,\ (\alpha^2,\ \beta^2)}(D_1^2) = \{x_1, x_4, x_6\}$
$BND^{PO,\ (\alpha,\ \beta)^2}(D_1) = \{x_3, x_5\}$

Similarly, we seem to use the aggregative strategy for the lower approximation and use the conservative strategy for the upper approximation. we define optimistic-pessimistic multi-granulation sequential three-way decisions in [34].

In [37], the author have been illustrated the optimistic-pessimistic multi-granulation sequential three-way decisions is right in the multi-granulation Pawlak rough set models. However, in tolerance-based multi-granularity fuzzy-rough set models, the model is not hold. Because for an object x satisfying $P(D_i^l \mid SIM_{A_i}^{\delta}) \geq \alpha^l$ for some $A_i \subseteq C$ and $P(D_i^l \mid SIM_{A_i}^{\delta}) \leq \beta^l$ for some $A_j \subseteq C$, we can not judge that $x \in POS^{OP,\ (\alpha^l,\ \beta^l)}(D_i^l)$ or $x \in NEG^{OP,\ (\alpha^l,\ \beta^l)}(D_i^l)$

Then, the following simple example to illustrate.

Example 6. (Continued example 2)

$$U^1 = U, D_1^1 = \{x_1, x_3, x_4, x_6\}, \text{ we can compute}$$
$$POS^{OP,\ (\alpha^1,\ \beta^1)}(D_1^1) = \{x_2, x_3, x_5\},$$
$$BND^{OP,\ (\alpha^1,\ \beta^1)}(D_1^1) = \{x_1, x_4\},$$
$$NEG^{OP,\ (\alpha^1,\ \beta^1)}(D_1^1) = \{x_5, x_6\}$$

we can obtain x_5 belong both $POS^{OP,\ (\alpha^1,\ \beta^1)}(D_1^1)$ and $NEG^{OP,\ (\alpha^1,\ \beta^1)}(D_1^1)$

In order to make readers understand the above theory more intuitively, Figure 1, Figure 2 and Figure 3 respectively show the positive and negative regions of multi-granularity three-way decision under optimistic, pessimistic and pessimistic-optimistic situations.

4) THE RELATIONSHIP AMONG THE THREE MODELS

In this section, we discuss the relationship among the three types of multi-granulation sequential three-way decisions, respectively, the optimistic multi-granulation sequential three-way decisions, the pessimistic multi-granulation sequential three-way decisions, and the pessimistic-optimistic multi-granulation sequential three-way decisions.

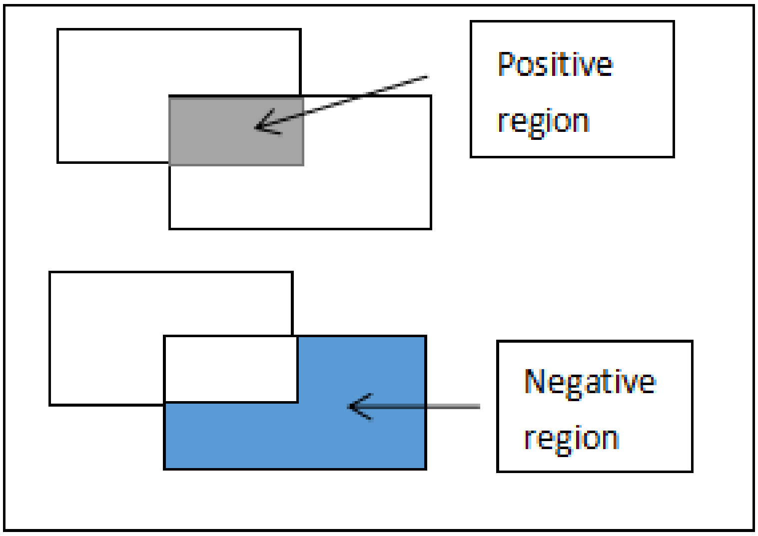

FIGURE 2. Pessimistic multi-granulation three-way decisions.

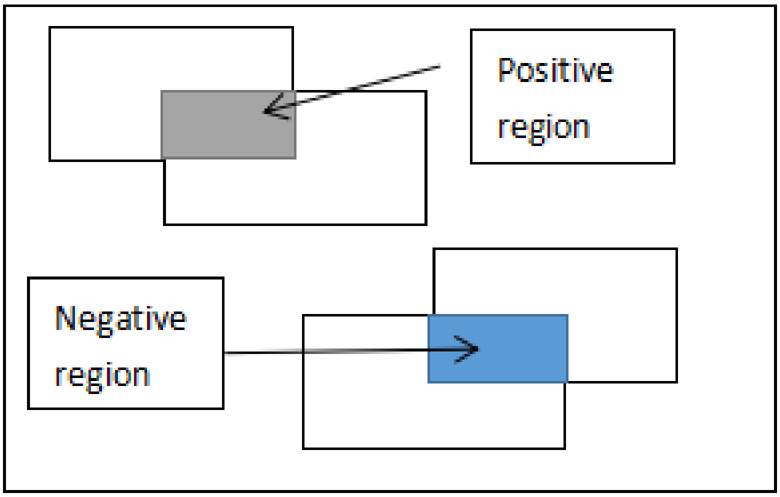

FIGURE 3. Pessimistic-Optimistic multi-granulation three-way decisions.

Theorem 1. For a fuzzy decision system F, let $A_1, A_2, \cdots, A_n$ are n granular structures a decision class D_j^l, and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, where $0 \le \beta^l < \alpha^l \le 1$. we have

$$(1)\ \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \underline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

$$\subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

$$(2)\ \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

$$= \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

Proof: (1) By Definitions 5 and 6, we can easily obtain

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \underline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

Furthermore, we proof

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

$$\forall x \in \underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \Longleftrightarrow x \in \{x \in U^l :$$

$$P(D_j^l \mid SIM_{A_1}^{\delta}(x)) \ge \alpha^l \bigwedge P(D_j^l \mid SIM_{A_2}^{\delta}(x)) \ge \alpha^l \bigwedge$$

$$\cdots \bigwedge P(D_j^l \mid SIM_{A_n}^{\delta}(x)) \ge \alpha^l\}$$

$$\Longrightarrow x \in \{x \in U^l : P(D_j^l \mid SIM_{A_1}^{\delta}(x)) \ge \alpha^l$$

$$\bigvee P(D_j^l \mid SIM_{A_2}^{\delta}(x)) \geq \alpha^l \bigvee \cdots \bigvee P(D_j^l \mid SIM_{A_n}^{\delta}(x)) \geq \alpha^l\}$$
$$\Longrightarrow x \in \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

from which we can obtain

$$\underline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \underline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

$$(2)\ \forall x \in \overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \Longleftrightarrow x \in \{x \in U^l :$$
$$P(D_j^l \mid SIM_{A_1}^{\delta}(x)) > \beta^l \bigwedge P(D_j^l \mid SIM_{A_2}^{\delta}(x)) > \beta^l$$
$$\bigwedge \cdots \bigwedge P(D_j^l \mid SIM_{A_n}^{\delta}(x)) > \beta^l\}$$
$$\Longrightarrow x \in \{x \in U^l : P(D_j^l \mid SIM_{A_1}^{\delta}(x)) > \beta^l$$
$$\bigvee P(D_j^l \mid SIM_{A_2}^{\delta}(x)) > \beta^l \bigvee \cdots \bigvee P(D_j^l \mid SIM_{A_n}^{\delta}(x)) > \beta^l\}$$
$$\Longrightarrow x \in \overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

from which we can obtain

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\ P,\ (\alpha^l,\ \beta^l))}}(D_j^l) \subseteq \overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l)$$

By Definitions 4 and 6, we can easily obtain

$$\overline{\sum_{i=1}^{n} A_i^{(\delta,\ PO,\ (\alpha^l,\ \beta^l))}}(D_j^l) = \overline{\sum_{i=1}^{n} A_i^{(\delta,\ O,\ (\alpha^l,\ \beta^l))}}(D_j^l).$$

Theorem 2. For a fuzzy decision system F, let $A_1, A_2, \cdots, A_n$ are n granular structures and the dynamic threshold parameter sequence $(\alpha,\ \beta)^l = \{(\alpha^1,\ \beta^1), (\alpha^2,\ \beta^2), \cdots, (\alpha^l,\ \beta^l)\}$, a decision partition π_D, we have

$$(1)\ POS^{P,\ (\alpha,\ \beta)^l}(\pi_D) = POS^{PO,\ (\alpha,\ \beta)^l}(\pi_D) \subseteq POS^{O,\ (\alpha,\ \beta)^l}(\pi_D)$$
$$(2)\ NEG^{PO,\ (\alpha,\ \beta)^l}(\pi_D) = NEG^{O,\ (\alpha,\ \beta)^l}(\pi_D) \subseteq NEG^{P,\ (\alpha,\ \beta)^l}(\pi_D)$$
$$(3)\ BND^{O,\ (\alpha,\ \beta)^l}(\pi_D) \subseteq BND^{PO,\ (\alpha,\ \beta)^l}(\pi_D)$$
$$BND^{P,\ (\alpha,\ \beta)^l}(\pi_D) \subseteq BND^{PO,\ (\alpha,\ \beta)^l}(\pi_D)$$

Proof: we can obtain them by lower and upper approximations of the optimistic multi-granulation sequential three-way decisions, the pessimistic multi-granulation sequential three-way decisions and the pessimistic-optimistic multi-granulation sequential three-way decisions.

IV. CONCLUSION

In this paper, we propose sequence three-way decision method based on tolerance relation of multi-granularity fuzzy rough sets. On this basis, the upper and lower approximation of decision class and its sequence three-way decision under optimistic, pessimistic and pessimistic-optimistic models are proposed, and some related properties of precision and roughness are studied. For each model, an example is given to verify its validity.For the optimistic-pessimistic model, counter-examples are given to show that it can not make decisions. Finally, the relationship and properties of the three models are analyzed.

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