IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received September 18, 2019, accepted October 23, 2019, date of publication October 31, 2019, date of current version November 12, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2950167

t-Intuitionistic Fuzzification of Lagrange's Theorem of t-Intuitionistic Fuzzy Subgroup

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This work was supported by the Research Center of the Center for Female Scientific and Medical Colleges, Deanship of Scientific Research, King Saud University.

ABSTRACT In this study, we propose the concept of t-intuitionistic fuzzy order of an element of a t-intuitionistic fuzzy subgroup (t-IFSG) of a finite group and examine different important algebraic properties of this phenomena. We also prove many useful algebraic aspects of this notion for a cyclic group. Moreover, we extend this ideology to define t-intuitionistic fuzzy order and index of a t-IFSG of group. In addition, we establish t-intuitionistic fuzzification of Langrange's theorem.

INDEX TERMS t-intuitionistic fuzzy subgroup (t-IFSG), t-intuitionistic fuzzy order of an element of t-IFSG, t-intuitionistic fuzzy order of t-IFSG, t-intuitionistic fuzzy quotient group, index of t-IFSG.

I. INTRODUCTION

The central idea to understand the Lagrange's theorem is the notion of a coset. A simpler way of seeing a potential link between Lagrange's theorem to real life is by showing a link from group theory to real life. The Lagrange's theorem is considered as an important tool of abstract algebra but step by step it can slowly be linked with the real world phenomena. This theorem also yields a very elegant proof of Fermat's Little Theorem, which is quite useful in cryptography and many other fields. The method to prove Wilson's Theorem shows another important significance of Lagrange's Theorem because one can view a prime order group as cyclic simply by virtue of this result. This theorem is a powerful tool to analyze finite groups; as it provides a precise overview about subgroups of any finite group. Lagrange's Theorem first appeared in the late 18^{th} century in connection with the problem of solving the equation of degree 5 or higher, and itsrelationship with symmetric functions. Lagrange stated his version of this theorem in 1770 even before the invention of the classical group theory, but the first complete proof was given by Pietro Abbati some 30 years later. For more details about the rich history of this remarkable theorem, we refer to [2], [26].

Vagueness is a pervasive part of the human experience. The real world is based neither on abstraction nor on precise measurements. This inaccuracy of calculation is quite a big challenge for a human brain. Many mathematical concepts have been developed as convenient tools to address this problem in which one of them is theory of fuzzy sets. Fuzzy logic is created on the theory of a set to reflect an uncertain knowledge. The intuitionistic fuzzy sets are very effective in a situation where description of a problem by a linguistic variable given in terms of a membership function only seems too rough. Due to the flexibility of intuitionistic fuzzy sets in handling uncertainty, this phenomena is considered as an efficient tool for more human consistent reasoning under the imperfectly defined facts and imprecise knowledge. This notion is infact a generalization of classical fuzzy sets as it provides an additional opportunity to present imperfect knowledge, leading to a more appropriate description of many real problems. These particular sets design suitable models in circumstances where we are faced with a human opinion that contains answers of the kind yes, no and does not apply. Another significance of this notion is that it allows a person to address the positive and the negative sides of an imprecise concept separately about a physical problem.

The concept of fuzzy sets was introduced by Zadeh [34] in 1965. In 1971, Roenfeld [25] started the investigation of fuzzy subgroups and found numerous essential properties of this concept. Atanassov [4] innovated the theory of intuitionistic fuzzy sets as a powerful extension of classical fuzzy sets. This particular theory has been a great source of inspiration for many mathematicians in various scientific fields like

The associate editor coordinating the review of this manuscript and approving it for publication was Giovanni Pau.

decision making problems [16] and medical diagnosis determination [10]. Ejegwa *et al.* [11] presented a comprehensive study on some selected models of IFSs in real life situations such as in diagnostic medicine and pattern recognition using Normalized Hamming distance measure. This notion was also applied in the academic career of the students [19] and for the selection of a school [22], [32]. In 2017, Garg and Rani [14] established consequences on the evidence measures for complex intuitionistic fuzzy sets. Atanassov [5] presented a comparative study between intuitionistic and type-1 fuzzy sets. The intuitionistic fuzzy soft module and its various operations were defined in [15]. Coung *et al.* [9] gave the idea of type-2 intuitionistic fuzzy sets in 2012. Biswas [8] proposed the idea of intuitionistic fuzzy subgroups. The algebraic features of intuitionistic fuzzy subgroup were analyzed in [28]. The authors [24], [27] introduced the notions of intuitionistic fuzzy topological group, intuitionistic fuzzy topological semi-group and intuitionistic fuzzy ideal topological spaces as gernalizations of intuitionistic fuzzy subgroup and fuzzy ideals. Many interesting results about intuitionistic fuzzy ideals and intuitionistic fuzzy prime ideals were presented in [6]. In 2016, Abbasizadeh and Davvaz [1] developed a link between algebraic hyper structures and intuitionistic fuzzy sets and presented the theories of intuitionistic fuzzy subpolygroup and intuitionistic fuzzy topological polygroup. In [20], a new type of intuitionistic fuzzy rings were introduced by using the concept of intuitionistic fuzzy space. In 2018, Yamin and Sharma [33] studied the theory of intuitionistic fuzzy rings. The concepts of intuitionistic fuzzy prime ideals, weakly completely prime ideals and completely prime ideals were presented in [17]. Alsarahead and Ahmad [3] defined complex intuitionistic fuzzy subring, intuitionistic π-fuzzy sets and homogeneous complex intuitionistic fuzzy subrings. A new concept of complex intuitionistic fuzzy subrings based on the notion of complex intuitionistic fuzzy subspace was presented by Husban *et al.* [18]. The theory of intuitionistic L-fuzzy subrings was established by Meena and Thomas [21] in 2011. Sharma and Kaur [30] interpreted the idea of intuitionistic fuzzy prime sub-module. The algebraic structure of hesitant intuitionistic fuzzy soft sets was studied in [31]. The concept of intuitionistic fuzzy hyperideals of a semi hyper-ring was analyzed in [12]. In 2016, Eyoh *et al.* [13] studied an approach based on a new interval type-2 intuitionistic fuzzy logic system of Takagi-Sugeno-Kang fuzzy inference. The study of intuitionistic M-fuzzy sub-bigroup of an M-bigroup was presented in [23]. The idea of t-intuitionistic fuzzy subgroup was introduced by Sharma [29] in 2012. The ideas of t- intuitionistic fuzzy subalgebra and t- intuitionistic fuzzy normal subalgebra of BG-algebras were proposed by Barbhuiya [7] in 2015.

An outline of this article is shaped as: The notions of t-intuitionistic fuzzy order of an element and t-intuitionistic fuzzy order of t-IFSG are defined in section 2 along with the many important algebraic characteristics of these phenomena. In section 3, we establish the fundamental properties of t-intuitionistic fuzzy order of an element of t-IFSG of a finite cyclic group. Section 4 deals with the concepts of t-intuitionistic fuzzy quotient group and the index of t-IFSG. In addition, we present t-intuitionistic fuzzification of Lagrange's Theorem.

II. t-INTUITIONISTIC FUZZY ORDER OF AN ELEMENT OF t-INTUITIONISTIC FUZZY SUBGROUP

We start this section with following three definitions, which we use in our main results.

Definition 1 [29]: A t-IFS A_t of a group G is called the t-intuitionistic fuzzy subgroup of G (t-IFSG) if $\mu_{A_t}(ab^{-1}) \geq \min\{\mu_{A_t}(a), \mu_{A_t}(b)\}$ and $\nu_{A_t}(ab^{-1}) \leq \max\{\nu_{A_t}(a), \nu_{A_t}(b)\}$, $\forall a, b \in G$.

Theorem 2: Let A_t be a t-IFSG of a group G and $a \in G$. Then $\mu_{A_t}(ab) = \mu_{A_t}(b)$ and $\nu_{A_t}(ab) = \nu_{A_t}(b)$ for all $b \in G$, if and only if $\mu_{A_t}(a) = \mu_{A_t}(e)$ and $\nu_{A_t}(a) = \nu_{A_t}(e)$.

Proof: Suppose that $\mu_{A_t}(ab) = \mu_{A_t}(b)$ and $\nu_{A_t}(ab) = \nu_{A_t}(b)$, $\forall b \in G$. By replacing b with e, we have required result.

Conversely, Let $\mu_{A_t}(a) = \mu_{A_t}(e)$. Since A_t is t-IFSG, therefore, $\mu_{A_t}(b) \leq \mu_{A_t}(e)$ and $\nu_{A_t}(b) \geq \nu_{A_t}(e)\ \forall b \in G$.

This means that $\mu_{A_t}(b) \leq \mu_{A_t}(a)\ \forall b \in G$.

Now $\mu_{A_t}(ab) \geq \min\{\mu_{A_t}(a), \mu_{A_t}(b)\}$. Therefore, we have

$$\mu_{A_t}(ab) \geq \mu_{A_t}(b) \quad \forall b \in G. \tag{2.1}$$

But $\mu_{A_t}(b) = \mu_{A_t}(a^{-1}ab) \geq \min\{\mu_{A_t}(a), \mu_{A_t}(ab)\}$. This shows that

$$\mu_{A_t}(b) \geq \mu_{A_t}(ab) \quad \forall b \in G. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\mu_{A_t}(ab) = \mu_{A_t}(b).$$

Similarly, we can show that $\nu_{A_t}(ab) = \nu_{A_t}(b)$.

Remark 3: It is important to note that if $A_t(a) = A_t(e)$. Then $A_t(ab) = A_t(ba)\ \forall b \in G$.

Definition 4 [29]: A t-IFSG A_t is called t-intuitionistic fuzzy normal subgroup (t-IFNSG) of G, if $\mu_{A_t}(a) = \mu_{A_t}(b^{-1}ab)$ and $\nu_{A_t}(a) = \nu_{A_t}(b^{-1}ab)$, for all $a, b \in G$.

The above definition can also be visualized as:

$$\mu_{A_t}(ab) = \mu_{A_t}(ba) \quad \text{and } \nu_{A_t}(ab) = \nu_{A_t}(ba).$$

Definition 5 [29]: Let $a, b \in G$, then a map $aA_t : G \to [0, 1]$ defined by

$$\mu_{aA_t}(b) = \mu_{A_t}(ba^{-1}), \quad \nu_{aA_t}(b) = \nu_{A_t}(ba^{-1})$$

is called the t-intuitionistic fuzzy left coset determined by a and A_t.

Next, we define the notion of t-intuitionistic fuzzy order of an element of t-IFSG. Moreover, we define the t-intuitionistic fuzzy order of t-IFSG and show that t-intuitionistic fuzzy order of any element and its inverse is the same. We prove some fundamental algebraic attributes of t-intuitionistic fuzzy order of an element of t-IFSG.

Definition 6: Consider a t-IFSG A_t of a finite group G and $a \in G$. Then the t-intuitionistic fuzzy order of the element $a \in A_t$ is denoted by $t - IFO_{A_t}(a)$ and is defined as:

$$t - IFO_{A_t}(a) = |S(a)|, \quad \text{where}$$
$$S(a) = \left\{c \in G : \mu_{A_t}(c) \geq \mu_{A_t}(a), \nu_{A_t}(c) \leq \nu_{A_t}(a)\right\}.$$

It is interesting to note that $t - IFO_{A_t}(e)$ may or may not be 1.

Let us now explain the above stated fact by an example.

Example 7: The symmetric group G of degree 3 is defined as:

$$G = \langle a, b : a^3 = b^2 = e, ba = a^2 b\rangle.$$

The IFSG A of G is defined as follows:

$$\mu_A(z) = \begin{cases} 1 & \text{if } z = e \\ 0.5 & \text{if } z \in \langle a\rangle - \{e\} \text{ and} \\ 0.35 & \text{otherwise} \end{cases}$$

$$\nu_A(z) = \begin{cases} 0 & \text{if } z = e \\ 0.4 & \text{if } z \in \langle a\rangle - \{e\} \\ 0.5 & \text{otherwise.} \end{cases}$$

The t-IFSG A_t of G for $t = 0.6$ is defined as:

$$\mu_{A_t}(z) = \begin{cases} 0.6 & z = e \\ 0.5 & \text{if } z \in \langle a\rangle - \{e\} \text{ and} \\ 0.35 & \text{otherwise} \end{cases}$$

$$\nu_{A_t}(z) = \begin{cases} 0.4 & \text{if } z \in \langle a\rangle \\ 0.5 & \text{otherwise.} \end{cases}$$

Clearly $t - IFO_{A_t}(e) = t - IFO_{A_t}(a) = t - IFO_{A_t}(a^2) = 3$ and $t - IFO_{A_t}(b) = t - IFO_{A_t}(ab) = t - IFO_{A_t}(a^2 b) = 6$.

The following theorem shows that $S(a)$ forms a subgroup of G.

Theorem 8: $S(a)$ is a subgroup of G.

Proof: Since $a \in S(a)$, therefore $S(a)$ is a non-empty set. In view of definition 6, for any two elements $y, z \in S(a)$, we have $\mu_{A_t}(y) \geq \mu_{A_t}(a)$, $\nu_{A_t}(y) \leq \nu_{A_t}(a)$ and $\mu_{A_t}(z) \geq \mu_{A_t}(a)$, $\nu_{A_t}(z) \leq \nu_{A_t}(a)$.

Since A_t is a t-IFSG, therefore $\mu_{A_t}(yz^{-1}) \geq \min\left\{\mu_{A_t}(y), \mu_{A_t}(z)\right\} \geq \mu_{A_t}(a)$ and $\nu_{A_t}(yz^{-1}) \leq \max\{\nu_{A_t}(y), \nu_{A_t}(z)\} \leq \nu_{A_t}(a)$, which implies that $\mu_{A_t}(yz^{-1}) \geq \mu_{A_t}(a)$ and $\nu_{A_t}(yz^{-1}) \leq \nu_{A_t}(a)$. Thus, $yz^{-1} \in S(a)$. Consequently, $S(a)$ is a subgroup of G.

Corollary 9: Let A_t be a t-IFSG of a group G then t-intuitionistic fuzzy order of every elements of A_t divides order of G.

Proof: In view of above theorem and Langrange's Theorem, one can easily prove that t-intuitionistic fuzzy order of an element of t-IFSG always divides order of group G.

The following result establishes a relationship between t-intuitionistic fuzzy order of identity and non-identity elements of A_t.

Theorem 10: Let A_t be any t-IFSG of a group G and $e \neq a \in G$. Then $t - IFO_{A_t}(e) \leq t - IFO_{A_t}(a)$.

Proof: Let $z \in S(e)$, then $\mu_{A_t}(z) = \mu_{A_t}(e)$ and $\nu_{A_t}(z) = \nu_{A_t}(e)$. This means that $\mu_{A_t}(z) \geq max(\mu_{A_t}(a))$ and $\nu_{A_t}(z) \leq \min(\nu_{A_t}(a))$, $\forall a \in G$. Thus, $z \in S(a)$. Consequently, $S(e) \subseteq S(a)$ and hence $t - IFO_{A_t}(e) \leq t - IFO_{A_t}(a)$.

Remark 11: Let A be a FSG of a group G then $FO_A(a)|O(a), \forall a \in G$.

The subsequent results show a relation between the order of an element of G and t-intuitionistic fuzzy order of an element of A_t.

Theorem 12: Let A_t be a t-IFSG and $a \in G$ then $O(a)$ divides $t - IFO_{A_t}(a)$.

Proof: Assume that $O(a) = k$ and consider a subgroup $H = \langle a : a^k = e\rangle$ of G. In view of definition 6, we get $a^2 \in S(a)$. Similarly, $a^3, a^4, \ldots, a^{k-1}, a^k \in S(a)$. This shows that $H \subseteq S(a)$. Consequently, H is a subgroup of $S(a)$ and hence $|H|$ divides $|S(a)|$. This means that $|H|$ divides $t - IFO_{A_t}(a)$. Therefore, $O(a)$ divides $t - IFO_{A_t}(a)$.

Remark 13: We know that if $FO_A(a)|O(a)$ and $O(a)|t - IFO_{A_t}(a)$, then obviously $FO_A(a)$ divides $t - IFO_{A_t}(a)$.

Definition 14: The t-intuitionistic fuzzy order of t-IFSG A_t of G is denoted by $t - IFO(A_t)$ and is obtained by computing the greatest common divisor of t-intuitionistic fuzzy order of every element of A_t.

Example 15: The t-intuitionistic fuzzy order A_t of S_3 is 3 (see example 2.7).

Theorem 16: Let A_t be a t-IFSG and $a \in G$ then $\mu_{A_t}(a^k) \geq \mu_{A_t}(a)$ and $\nu_{A_t}(a^k) \leq \nu_{A_t}(a)$, for any integer k.

Proof: By using induction on k, the result is trivial for $k = 0$ and 1. If $k = 2$ then

$$\begin{aligned} \mu_{A_t}(a^2) &\geq \mu_{A_t}(a.a) \\ &\geq \min\{\mu_{A_t}(a), \mu_{A_t}(a)\} \\ &= \mu_{A_t}(a). \end{aligned}$$

Let the statement be true for $n < k$.

Now

$$\begin{aligned} \mu_{A_t}(a^{n+1}) &= \mu_{A_t}(a^n.a) \\ &\geq \min\{\mu_{A_t}(a^n), \mu_{A_t}(a)\} \\ &= \mu_{A_t}(a), \end{aligned}$$

which completes the induction.

If $k < 0$ then

$$\begin{aligned} \mu_{A_t}(a^k) &= \mu_{A_t}(a^k)^{-1} \\ &= \mu_{A_t}(a^{-k}) \geq \mu_{A_t}(a). \end{aligned}$$

Similarly, $\nu_{A_t}(a^k) \leq \nu_{A_t}(a)$.

Remark 17: If $(O(a), k) = 1$ then $\mu_{A_t}(a^k) = \mu_{A_t}(a)$ and $\nu_{A_t}(a^k) = \nu_{A_t}(a)$, for any integer k.

Theorem 18: Let $t - IFO_{A_t}(a) = n$ and $(n, m) = 1, m, n \in \mathbb{Z}$ and $a \in G$. Then $\mu_{A_t}(a^m) = \mu_{A_t}(a)$ and $\nu_{A_t}(a^m) = \nu_{A_t}(a)$.

Proof: We know that if $(n, m) = 1$ then $nr + ms = 1$, for some $r, s \in \mathbf{Z}$. So we have

$$\begin{aligned}\mu_{A_t}(a) &= \mu_{A_t}(a^{nr+ms}) \\ &= \mu_{A_t}\left((a^n)^r(a^m)^s\right) \\ &\geq \min\{\mu_{A_t}((a^n)^r), \mu_{A_t}((a^m)^s)\} \\ &= \min\{\mu_{A_t}(e), \mu_{A_t}(a^m)\} \geq \mu_{A_t}(a^m).\end{aligned}$$

But $\mu_{A_t}(a^m) \geq \mu_{A_t}(a)$.

Consequently, $\mu_{A_t}(a^m) = \mu_{A_t}(a)$.

Similarly, we can prove $\nu_{A_t}(a^m) = \nu_{A_t}(a)$.

Theorem 19: Let $m, n \in \mathbf{Z}$ such that $\mu_{A_t}(a^m) = \mu_{A_t}(e)$ and $\nu_{A_t}(a^n) = \nu_{A_t}(e)$, for all $a \in G$. Then both m and n divide $t - IFO_{A_t}(a)$.

Proof: Let a be non-identity element and $t - IFO_{A_t}(a) = z$. Suppose $m \not| z$, then $(m, z) = 1$.

In view of theorem 18, we have $\mu_{A_t}(a^m) = \mu_{A_t}(a)$. But $\mu_{A_t}(a^m) = \mu_{A_t}(e)$, so $a = e$.

So, we reach at a contradiction and thus m divides $t - IFO_{A_t}(a)$.

Similarly, we can prove n divides $t - IFO_{A_t}(a)$.

Theorem 20: If $t - IFO_{A_t}(a) = n$ then $t - IFO_{A_t}(a^m) = \frac{t - IFO_{A_t}(a)}{(m,n)}$, for some integer m..

Proof: Assume that $t - IFO_{A_t}(a^m) = s$.

Consider

$$\begin{aligned}\mu_{A_t}\left((a^m)^{\frac{n}{d}}\right) &= \mu_{A_t}\left((a^n)^{\frac{m}{d}}\right) \\ &\geq \mu_{A_t}(e^{\frac{m}{d}}) \\ &= \mu_{A_t}(e).\end{aligned}$$

Similarly, $\nu_{A_t}\left((a^m)^{n/d}\right) = \nu_{A_t}(e)$.

By using theorem 19, we have $n|d$ divides s.

Moreover, since $(m, n) = d$ therefore $np + mq = d$, for some $p, q \in \mathbf{Z}$. Now

$$\begin{aligned}\mu_{A_t}\left(a^{sd}\right) &= \mu_{A_t}\left(a^{s(np+mq)}\right) \\ &= \mu_{A_t}\left(a^{snp}a^{smq}\right) \\ &\geq \min\{\mu_{A_t}((a^n)^{sp}), \mu_{A_t}((a^{ms})^q)\} \\ &\geq \min\{\mu_{A_t}(a^n), \mu_{A_t}(a^{ms})\} \\ &\geq \min\{\mu_{A_t}(a^n), \mu_{A_t}(a^m)^s\} \\ &= \min\{\mu_{A_t}(e), \mu_{A_t}(e)\} \\ &= \mu_{A_t}(e).\end{aligned}$$

Similarly, it can be proved that $\nu_{A_t}(a^{sd}) = \nu_{A_t}(e)$. By using theorem 19, we have $sd|n$ and hence $s = n|d$.

Theorem 21: Let A_t be t-IFSG of G and $a \in G$ then $t - IFO_{A_t}(a) = t - IFO_{A_t}(a^{-1})$.

Proof: Since A_t is t-IFSG, therefore, $\mu_{A_t}(a^{-1}) = \mu_{A_t}(a)$ and $\nu_{A_t}(a^{-1}) = \nu_{A_t}(a), \forall\, a \in G$. This means that $S(a^{-1}) = S(a)$ and so $\left|S(a^{-1})\right| = |S(a)|$. Also we know $t - IFO_{A_t}(x) = O\left(S(x)\right) \forall x$. Therefore $t - IFO_{A_t}(a^{-1}) = t - IFO_{A_t}(a)$.

In the following result, we establish an equivalent form of t-intuitionistic fuzzy order of an element of t-IFNSG.

Theorem 22: Let A_t be t-IFNSG of G and a be any fixed element of G. Then $t - IFO_{A_t}(a) = t - IFO_{A_t}(b^{-1}ab), \forall b \in G$.

Proof: In view of definition 4, we have $\mu_{A_t}(a) = \mu_{A_t}(b^{-1}ab)$ and $\nu_{A_t}(a) = \nu_{A_t}(b^{-1}ab)$. The application of definition 6 in the above relations yields that $S(a) = S(b^{-1}ab)$. Consequently,

$$t - IFO_{A_t}(a) = t - IFO_{A_t}(b^{-1}ab).$$

Theorem 23: Let A_t be t-IFNSG of a group G then $t - IFO_{A_t}(ab) = t - IFO_{A_t}(ba), \forall a, b \in G$.

Proof: Since

$$\begin{aligned}t - IFO_{A_t}(ab) &= t - IFO_{A_t}((b^{-1}b)(ab)) \\ &= t - IFO_{A_t}(b^{-1}(ba)b)\end{aligned}$$

Also by theorem 22, we have $t - IFO_{A_t}(b^{-1}(ba)b) = t - IFO_{A_t}(ba)$.

Thus, we have $t - IFO_{A_t}(ab) = t - IFO_{A_t}(ba)$.

Theorem 24: Let $t - IFO_{A_t}(a) = n, \forall a \in G$.

If $i \equiv j (\bmod n)$, where $i, j \in \mathbb{Z}$ then $t - IFO_{A_t}(a^i) = t - IFO_{A_t}(a^j)$.

Proof: Assume that $t - IFO_{A_t}(a^i) = r$ and $t - IFO_{A_t}(a^j) = s$. Since $i = j + kn$ for some $k \in \mathbf{Z}$, therefore

$$\begin{aligned}\mu_{A_t}\left((a^i)^s\right) &= \mu_{A_t}\left((a^{j+nk})^s\right) \\ &= \mu_{A_t}\left((a^j)^s(a^n)^{ks}\right) \\ &\geq \min\left\{\mu_{A_t}\left((a^j)^s\right), \mu_{A_t}\left((a^n)^{ks}\right)\right\} \\ &\geq \min\left\{\mu_{A_t}(e), \mu_{A_t}(a^n)\right\} \\ &= \min\left\{\mu_{A_t}(e), \mu_{A_t}(e)\right\} \\ &= \mu_{A_t}(e).\end{aligned}$$

Thus, $r|s$, similarly, we can prove $s|r$. Hence $t - IFO_{A_t}(a^i) = t - IFO_{A_t}(a^j)$.

Theorem 25: Let for all $a, b \in G$ $\left(t - IFO_{A_t}(a), t - IFO_{A_t}(b)\right) = 1, ab = ba$ and $A_t(ab) = A_t(e)$. Then $A_t(a) = A_t(b) = A_t(e)$.

Proof: Suppose that $t - IFO_{A_t}(a) = n$ and $t - IFO_{A_t}(b) = m$. The application of theorem 16 on the given condition yields that $\mu_{A_t}(e) = \mu_{A_t}(a^m b^m)$. By using theorem 19, we get $\mu_{A_t}(e) = \mu_{A_t}(a^m) = \mu_{A_t}(b^m)$. Now, we obtain the required result by applying the similar arguments for non-membership function ν_{A_t}.

Theorem 26: If $\left(t - IFO_{A_t}(a), t - IFO_{A_t}(b)\right) = 1$ and $ab = ba$ for all $a, b \in G$, then $t - IFO_{A_t}(ab) = \left[t - IFO_{A_t}(a)\right] \times \left[t - IFO_{A_t}(b)\right]$.

Proof: Suppose that $t - IFO_{A_t}(ab) = n, t - IFO_{A_t}(a) = r$ and $t - IFO_{A_t}(b) = s$. Now Consider

$$\begin{aligned}\mu_{A_t}\left((ab)^{rs}\right) &= \mu_{A_t}\left(a^{rs}b^{rs}\right) \\ &\geq \min\{\mu_{A_t}\left((a^s)^r\right), \mu_{A_t}\left((b^s)^r\right) \\ &\geq \mu_{A_t}(e).\end{aligned}$$

Likewise, $\nu_{A_t}((ab)^{rs}) = \nu_{A_t}(e)$.

In view of theorem 19, we obtained the following relation

$$rs|n \tag{2.3}$$

Since $(r, s) = 1$, therefore either $s|n$ or $r|n$.

Assume that $r|n$, then in view of theorem 18, we have

$$t - IFO_{A_t}(a^n) = \frac{r}{(r,n)}. \tag{2.4}$$

By using theorem 20, in the above relation for $t-IFO_{A_t}(b^n)$ establishes the following relation

$$t - IFO_{A_t}(b^n) = \frac{s}{(n,s)}. \tag{2.5}$$

Again from theorem 20 and equations (2.4), (2.5) we obtain

$$(t - IFO_{A_t}(a^n), t - IFO_{A_t}(b^n)) = 1.$$

From theorem 25 and equations (2.4), (2.5) we get $A_t(e) = A_t(a^n) = A_t(b^n)$. This means that

$$n|rs. \tag{2.6}$$

Using from (2.3) and (2.6), we have the required result.

Remark 27: Let A_t and B_t be two t-IFSG of a group G. If $A_t \subseteq B_t$ and $A_t(e) = B_t(e)$ then $t - IFO_{A_t}(a)|t - IFO_{B_t}(a)$, $\forall\, a \in G$.

Theorem 28: If A_t and B_t are any two t-IFSG of G such that $A_t \subseteq B_t$ and $A_t(e) = B_t(e)$, then $t-IFO(A_t)|t-IFO(B_t)$.

Proof: Since $t - IFO(A_t)$ and $t - IFO(B_t)$ are finite, therefore, t-intuitionistic fuzzy order of each element of A_t and B_t is finite. Let H and L be the sets consisting of t-intuitionistic fuzzy orders of the elements in A_t and B_t respectively. By remark 27, $t-IFO_{A_t}(a)$ divides $t-IFO_{B_t}(a)$ for all $a \in G$. Then greatest common divisor of all elements of H divides greatest common divisor of all elements of L. This shows that $t - IFO(A_t)$ divides $t - IFO(B_t)$.

III. PROPERTIES OF t-INTUITIONISTIC FUZZY ORDER OF AN ELEMENT of t-INTUITIONISTIC FUZZY SUBGROUP OF A FINITE CYCLIC GROUP

In this section, we investigate fundamental algebraic aspects of t-intuitionistic fuzzy order of an element of t-IFSG in a cyclic groups.

Lemma 29: Let A_t be a t-IFSG of a cyclic group G and a, b be any two generators of G then $t-IFO_{A_t}(a) = t-IFO_{A_t}(b)$.

Proof: Suppose that $O(G) = n$. Since a and b are generators of G, therefore $a^n = b^n = e$.

Since for some $m \in$ Z, we have $b = a^m$, therefore $(m, n) = 1$. Next, the application of theorem 18 yields that

$$t - IFO_{A_t}(a) = t - IFO_{A_t}(a^m) = t - IFO_{A_t}(b).$$

Theorem 30: Let A_t be a t-IFSG of a finite cyclic group G. The following statements hold for all $a, b \in G$:

1) If $O(a) = O(b)$ then $t - IFO_{A_t}(a) = t - IFO_{A_t}(b)$.
2) If $O(a)$ divides $O(b)$ then $t - IFO_{A_t}(b)$ divides $t - IFO_{A_t}(a)$.

Proof: Let z be a generator of G then $a = z^r$, $b = z^s$ and $t - IFO_{A_t}(z) = m$, where $r, s, m \in \mathbb{Z}$. By using lemma 29, m is independent of a particular choice of generator z of G. We know that $O(a) = {}^{n}/_{(r,n)}$ and $O(b) = {}^{n}/_{(s,n)}$. In view of theorem 20, we have $t - IFO_{A_t}(a) = {}^{m}/_{(r,m)}$ and $t - IFO_{A_t}(b) = {}^{m}/_{(s,m)}$. From theorem 12, we have $n|m$.

(i) Since $O(a) = O(b)$. This implies that $O(z^r) = O(z^s)$. This shows that $(r, n) = (s, n)$. From the above relation, we have $(r, m) = (s, m)$. Consequently, $t-IFO_{A_t}(a) = t - IFO_{A_t}(b)$.

(ii) Since $O(a)|O(b)$, so $(s, n)|(r, n)$. This implies that $(s, m)|(r, m)$. Also $n|m$, thus $t-IFO_{A_t}(b)|t-IFO_{A_t}(a)$.

Corollary 31: Let A_t be a t-IFSG of a cyclic group G of order n. If $t - IFO_{A_t}(a) = t - IFO_{A_t}(b)$, then $A_t(a) = A_t(b)$, $\forall\, a, b \in G$.

Corollary 32: For any t-IFSG A_t of a group G, if $O(a) = O(b)$ then $A_t(a) = A_t(b)$, $\forall a, b \in G$.

Corollary 33: Let A_t be a t-IFSG of a cyclic group G of order n. If $t - IFO_{A_t}(b)$ divides $t - IFO_{A_t}(a)$, then $\mu_{A_t}(b) \geq \mu_{A_t}(a)$ and $\nu_{A_t}(b) \leq \nu_{A_t}(a)$.

Theorem 34: Let A_t be a t-IFSG of a unit group $G$and H be a cyclic subgroup of G generated by z. For all $a, b \in H$, if $O(a)$ divides $O(b)$ then $\mu_{A_t}(a) \geq \mu_{A_t}(b)$ and $\nu_{A_t}(a) \leq \nu_{A_t}(b)$.

Proof: Suppose $O(a) = r$ and $O(b) = qr$ for some $q \in N$. Let $a = z^m$ and $b = z^n$ for some $m, n \in N$. It follows that $z^{mr} = e = z^{nqr}$. Thus, $a = b^q$. So $\mu_{A_t}(a) = \mu_{A_t}(b^q) \geq \mu_{A_t}(b)$. Similarly, we can prove $\nu_{A_t}(a) \leq \nu_{A_t}(b)$.

The following example shows that the above theorem is not valid for all $a, b \in G$.

Example 35: Consider t-IFSG A_t of U_{30} as follows:

$$\mu_{A_t}(z) = \begin{cases} 0.7 & \text{if } z = 1 \\ 0.6 & \text{if } z \in \{7, 13, 19\} \\ 0.4 & \text{if } z \in \{11, 17, 23, 29\} \end{cases} \quad \text{and}$$

$$\nu_{A_t}(z) = \begin{cases} 0.3 & \text{if } z \in \{1, 7, 13, 19\} \\ 0.5 & \text{if } z \in \{11, 17, 23, 29\}. \end{cases}$$

We know that $O(29) = 2$ and $O(13) = 4$ in U_{30}.

Clearly, $O(29)$ divides $O(13)$ but $\mu_{A_t}(13) > \mu_{A_t}(29)$ and $\nu_{A_t}(13) < \nu_{A_t}(29)$.

IV. t-INTUITIONISTIC FUZZIFICATION OF LAGRANGE'S THEOREM

In this section, we define the notion of t-intuitionistic fuzzy index of t-IFSG and present an approach to the t-intuitionistic fuzzification of Langrange's theorem of t-IFSG.

Theorem 36: Let A_t be a t-IFNSG of a finite group G and Ω be the set of all t-intuitionistic fuzzy left cosets of G by A_t. Then Ω is a group under composition

$$(aA_t) \circ (bA_t) = (ab)A_t \quad \forall a, b \in G.$$

Define a map $\bar{A}_t : \Omega \to [0, 1]$ by

$$\bar{A}_t(aA_t) = A_t(a), \quad \forall a \in G.$$

Then $\bar{A}_t$ is a t-IFSG of Ω.

Proof: Let $a, b, a_0, b_0 \in G$ such that

$$aA_t = a_0A_t \quad \text{and } bA_t = b_0A_t. \tag{4.1}$$

Then we must show that

$$(aA_t) \circ (bA_t) = (a_0A_t) \circ (b_0A_t),$$

that is, $(ab)A_t = (a_0b_0)A_t$.

In view of definition 5, we have

$$\mu_{(ab)A_t}(g) = \mu_{A_t}(gb^{-1}a^{-1}) \ \forall g \in G$$

and

$$\mu_{(a_0b_0)A_t}(g) = \mu_{A_t}(gb_0^{-1}a_0^{-1}) \ \forall g \in G.$$

Now

$$\begin{aligned} \mu_{A_t}(gb^{-1}a^{-1}) &= \mu_{A_t}(gb_0^{-1}b_0b^{-1}a^{-1}) \\ &= \mu_{A_t}(gb_0^{-1}a_0^{-1}a_0b_0b^{-1}a^{-1}) \\ &\geq \min\left[\mu_{A_t}(gb_0^{-1}a_0^{-1}), \mu_{A_t}(a_0b_0b^{-1}a^{-1})\right]. \end{aligned} \tag{4.2}$$

Now, the application of definition 5 in (4.1) gives that

$$\mu_{A_t}(ga^{-1}) = \mu_{A_t}(ga_0^{-1}) \ \forall g \in G \tag{4.3}$$

and

$$\mu_{A_t}(gb^{-1}) = \mu_{A_t}(gb_0^{-1}) \ \forall g \in G. \tag{4.4}$$

Now replace g by $a_0b_0b^{-1}$ in (4.3), we get

$$\mu_{A_t}(a_0b_0b^{-1}a^{-1}) = \mu_{A_t}(a_0b_0b^{-1}a_0^{-1})$$

Substitute g by b_0 in (4.4), we have

$$= \mu_{A_t}(b_0b^{-1}) = \mu_{A_t}(e)$$

But $\mu_{A_t}(e) \geq \mu_{A_t}(gb_0^{-1}a_0^{-1})$, since A_t is t-IFSG, therefore $\mu_{A_t}(u) \geq \mu_{A_t}(e)$ and $\nu_{A_t}(u) \leq \nu_{A_t}(e)$, $\forall u \in G$. Thus (4.2) now yields that

$$\mu_{A_t}(gb^{-1}a^{-1}) \geq \mu_{A_t}(gb_0^{-1}a_0^{-1}).$$

Similarly, $\mu_{A_t}(gb^{-1}a^{-1}) \leq \mu_{A_t}(gb_0^{-1}a_0^{-1})$.

This shows that $\mu_{A_t}(gb^{-1}a^{-1}) = \mu_{A_t}(gb_0^{-1}a_0^{-1})$.

Consequently, $\mu_{(ab)A_t} = \mu_{(a_0b_0)A_t} \ \forall g \in G$.

In the same way, we can show that

$$\nu_{(ab)A_t} = \nu_{(a_0b_0)A_t} \quad \forall g \in G.$$

This shows that the composition is well defined.

The composition is clearly associative and one can easily view the inverse of aA_t is $a^{-1}A_t$ for $a \in G$.

Hence it follows that Ω is a group.

Now, let $\bar{A}_t(aA_t), \bar{A}_t(bA_t) \in \bar{A}_t$ where $aA_t, bA_t \in \Omega$. Consider

$$\begin{aligned} \bar{\mu}_{A_t}(\mu_{aA_t} \circ \mu_{bA_t}) &= \bar{\mu}_{A_t}(\mu_{abA_t}) = \mu_{A_t}(ab) \\ &\geq \min\{\mu_{A_t}(a), \mu_{A_t}(b)\} \\ &= \min\{\bar{\mu}_{A_t}(\mu_{aA_t}), \bar{\mu}_{A_t}(\mu_{bA_t})\}. \end{aligned}$$

Similarly,

$$\bar{\nu}_{A_t}(\nu_{aA_t} \circ \nu_{bA_t}) \leq \max\{\bar{\nu}_{A_t}(\nu_{aA_t}), \bar{\nu}_{A_t}(\nu_{bA_t})\}.$$

Moreover,

$$\begin{aligned} \bar{\mu}_{A_t}(\mu_{a^{-1}A_t}) &= \mu_{A_t}(a^{-1}) \\ &= \mu_{A_t}(a) \\ &= \bar{\mu}_{A_t}(\mu_{aA_t}). \end{aligned}$$

Similarly, $\bar{\nu}_{A_t}(\nu_{a^{-1}A_t}) = \bar{\nu}_{A_t}(\nu_{aA_t})$.

This shows that $\bar{A}_t$ is a t-IFSG of Ω.

Definition 37: Let A_t be a t-IFNSG of a finite group G. Then $\bar{A}_t$ defined in theorem 36 is called the t-intuitionistic fuzzy quotient group determined byA_t.

In the following result, we establish a natural homomorphism between groups and Ω.

Theorem 38: Let A_t be a t-IFNSG of G and Ω be the collection of all t-intuitionistic fuzzy left cosets of G by A_t. Then there exist a natural homomorphism f from G to Ω defined by $f(a) = a\,A_t$, $\forall\, a \in G$ with Kernal $\{z \in G : \mu_{A_t}(z) = \mu_{A_t}(e), \nu_{A_t}(z) = \nu_{A_t}(e)\}$.

Proof: Let $a, b \in G$. Then

$$f(ab) = (ab)A_t = (aA_t) \circ (bA_t) = f(a)f(b).$$

Consequently, f is a homomorphism from G to Ω. Moreover,

$$\begin{aligned} Kerf &= \{z \in G : f(z) = A_t\} = \{z \in G : zA_t = A_t\}. \\ &= \{z \in G : (zA_t)(y) = A_t(y), \ \forall y \in G\}. \\ &= \left\{ \begin{array}{l} z \in G : \mu_{zA_t}(y) = \mu_{A_t}(y), \\ \nu_{zA_t}(y) = \nu_{A_t}(y), \ \forall y \in G \end{array} \right\}. \end{aligned}$$

In view of definition 5, we have

$$Kerf = \left\{ \begin{array}{l} z \in G : \mu_{A_t}(yz^{-1}) = \mu_{A_t}(y), \\ \nu_{A_t}(yz^{-1}) = \nu_{A_t}(y), \forall\, y \in G \end{array} \right\}.$$

The application of theorem 2 in the above relation yields that $\mu_{A_t}(z) = \mu_{A_t}(e)$ and $\nu_{A_t}(z) = \nu_{A_t}(e)$.

Consequently, $Kerf = \{z \in G : \mu_{A_t}(z) = \mu_{A_t}(e), \nu_{A_t}(z) = \nu_{A_t}(e)\}$.

Remark 39: Note that $|Kerf| = t - IFO(A_t)$.

Definition 40: The cardinality of the set Ω of all t- intuitionistic fuzzy left cosets of G by A_t is called the index of t-IFSG A_t and is denoted by $[G : A_t]$.

Theorem 41: (t-Intuitionistic Fuzzification of Lagrange's Theorem): Let G be a finite group and A_t denote t-IFSG of G. Then $[G : A_t]$ divides $O(G)$.

Proof: In view of theorem 38, we have a homomorphism f from G to Ω, where

$$\Omega = \{aA_t : a \in G\}.$$

where aA_t is defined in definition 5. Since G is finite, it is clear that Ω is also a finite set.

Define a subgroup K of G as follows

$$K = \{z \in G : zA_t = eA_t\} \tag{4.5}$$

By using theorem 38 in the above relation, we get $K = \{z \in G : A_t(z) = A_t(e)\}$.

The left decomposition of G as a disjoint union of cosets of G modulo K is given by:

$$G = a_1K \cup a_2K \cup \ldots \cup a_mK \tag{4.6}$$

where $a_1K = K$. Now, we show that corresponding to each coset a_iK given in (4.6), there is a t-intuitionistic fuzzy coset in Ω and also this correspondence is injective. Consider any coset a_iK. Let $k \in K$, then

$$\begin{aligned} f(a_ik) &= a_ikA_t \\ &= a_iA_tkA_t \\ &= a_iA_teA_t = a_iA_t. \end{aligned}$$

This means that f maps each element of a_iK into the t-intuitionistic fuzzy coset a_iA_t.

Now, we establish a natural correspondence $\hat{f}$ between the set $\{a_iK : 1 \le i \le m\}$ and the set Ω defined by

$$\hat{f}(a_iK) = a_iA_t, \quad 1 \le i \le m.$$

The correspondence $\hat{f}$ is one-to-one.

For this, let $a_jA_t = a_lA_t$, then $a_l^{-1}a_jA_t = eA_t$,

By using (4.5), we have $a_l^{-1}a_j \in K$,. This means that $a_jK = a_lK$ and hence $\hat{f}$ is one-to-one.

It is quite evident from the above discussion that $[G : K]$ and $[G : A_t]$ are equal. Since $[G : K]$ divides $O(G)$, therefore $[G : A_t]$ also divides $O(G)$.

Corollary 42: Let A_t be t-IFSG of a finite group G then $t - IFO(A_t)$ divides $O(G)$.

The index of t-IFSG A_t of a finite group G may be obtained from the following relation.

Remark 43: $[G : A_t] = O(G)/t - IFO(A_t)$.

We illustrate above algebraic fact by the following examples.

Example 44: Consider the finite presentation of the dihedral group of order 6 as follows:

$$D_3 = \langle a, b : a^3 = b^2 = 1, ab = b^2a\rangle.$$

The t-IFSG A_t of D_3 correspond to the value $t = 0.7$ is given by:

$$\mu_{A_{0.7}}(z) = \begin{cases} 0.7 & \text{if } z \in \{1, a, a^2\} \\ 0.5 & \text{otherwise} \end{cases} \text{ and}$$

$$\nu_{A_{0.7}}(z) = \begin{cases} 0.3 & \text{if } z \in \{1, a, a^2\} \\ 0.45 & \text{otherwise.} \end{cases}$$

The set of all 0.7-intuitionistic fuzzy left cosets of D_3 by $A_{0.7}$ is given by:

$$\Omega = \{A_{0.7}, sA_{0.7}\}.$$

This means that $[G : A_{0.7}] = Card(\Omega) = 2$.

Example 45: Consider a cyclic group of order 8, that is $C_8 = \langle a : a^8 = 1\rangle$.

The t-IFSG A_t of C_8 correspond to $t = 0.6$ is given by:

$$\mu_{A_{0.6}}(z) = \begin{cases} 0.6 & \text{if } z \in \{1, a^2, a^4, a^6\} \\ 0.5 & \text{otherwise} \end{cases} \text{ and}$$

$$\nu_{A_{0.6}}(z) = \begin{cases} 0.4 & \text{if } z \in \{1, a^2, a^4, a^6\} \\ 0.45 & \text{otherwise.} \end{cases}$$

The set of all 0.6-intuitionistic fuzzy left cosets of C_8 by $A_{0.6}$ is given by:

$$\Omega = \{A_{0.6}, aA_{0.6}\}.$$

This means that $[C_8 : A_{0.6}] = Card(\Omega) = 2$.

V. CONCLUSION

This paper revolves around the development of the theory to formulate the t-intuitionistic fuzzification of Lagrange's Theorem of t-IFSG of a finite group. In this work, we have introduced the concepts of t-intuitionistic fuzzy order of an element and t-intuitionistic fuzzy order of t-IFSG and have proved the fundamental algebraic attributes of these phenomena. Furthermore, we have developed many algebraic characteristics of t-intuitionistic fuzzy order of an elements of t-IFSG of a cyclic group. Moreover, we have proposed the idea of t-intuitionistic fuzzy quotient group of a group by its t-IFNSG and have presented an approach to the t-intuitionistic fuzzification of Lagrange's Theorem.

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