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Statistical Properties of Average Kendall's Tau Under Multivariate Contaminated Gaussian Model

HUADONG LAI AND WEICHAO XU, (Member, IEEE)
Department of Automatic Control, School of Automation, Guangdong University of Technology, Guangzhou 510006, China
Corresponding author: Weichao Xu (wcxu@gdut.edu.cn)

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ABSTRACT In this paper, we derived the analytical forms of the expectation and variance of average Kendall's tau (AKT) under a specific multivariate contaminated Gaussian model (MCGM), which can simulate a scenario where the multi-channel noise exhibits an impulsive manner. For a better understanding of AKT, we compared AKT to other two classical concordance correlation coefficients, i.e., Kendall's concordance coefficient (KCC), and the average Pearson's product moment correlation coefficient (APPMCC) with respect to the root mean squared error (RMSE). We also applied AKT, KCC and APPMCC to the problem of multi-channel random signal detection. Monte Carlo simulations not only validated our theoretical findings, but also revealed the advantage of AKT over KCC and APPMCC in terms of the receiver operating characteristic (ROC) curves.

INDEX TERMS Average Kendall's tau (AKT), multivariate contaminated Gaussian model (MCGM), root mean squared error (RMSE), multi-channel signal detection, receiver operating characteristic (ROC) curve.

I. INTRODUCTION

Correlation analysis has been widely utilized in a number of sub-areas in signal processing [1]–[4]. Being interpreted as the strength of the statistical relationship between two random variables, correlation should be large and positive if there is a high probability that large (small) values of one variable are associated with large (small) values of another; and it should be large (small) and negative if the direction reverses [5]. In the two-channel cases, correlation coefficients might be the most popular tools in the literature, including the classical Pearson's product moment correlation coefficient (PPMCC), Spearman's rho (SR) and Kendall's tau (KT) [6]. However, in practice, we often encounter multi-channel scenarios, such as measuring the strength of association among high dimensional random vectors [7], and detecting the presence of a common signal in multi-channels [8]–[10]. Under these situations, it is natural to formulate metrics through averaging the correlation coefficients for all pairs of signals at hand. In parallel to the aforementioned classical correlation coefficients, three concordance coefficients have been proposed, i.e., the average Pearson's product moment correlation coefficient (APPMCC) [11], average Spearman's rho (usually named Kendall's concordance coefficient (KCC) [12], [13], and average Kendall's tau (AKT) [14]–[20].

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There are many advantages and disadvantages of APPMCC, KCC and AKT. APPMCC is optimal when the signals obey the multivariate Gaussian model. However, theoretical and empirical evidences suggest that many communication and radar systems suffer from noise with impulsive characteristics, that is, the distribution of the noise has a tail heavier than that of Gaussian distribution. In other words, the majority of the multi-channel noise might follow a multivariate normal distribution, but there exists a tiny fraction of outliers with very large variance (impulsive noise) [21]–[23]. In such case, the noise can be well modeled by the so called multivariate contaminated Gaussian model (MCGM) [24], [25]. Under this circumstance, as shown in the simulation studies later on, the performance of APPMCC will deteriorate severely, and thus becomes impractical. On the other hand, we have shown in our previous work [26]–[28] that KT and SR are robust against impulsive noise modeled by bivariate contaminated Gaussian model. However, the properties of the corresponding multi-channel version, i.e., AKT and KCC under MCGM, are still unknown to the best of our knowledge.

Motivated by such unsatisfactory situation, in this work we focus on the statistical properties of AKT under the MCGM. The major contribution is threefold. Firstly, we establish the analytic expressions of the expectation and variance of AKT under a specific MCGM emulating impulsive noise that frequently encountered in practice. Secondly, we reveal the superiority of AKT over APPMCC and KCC in terms of the root mean squared error (RMSE) under the MCGM. Finally, we demonstrate the robustness of AKT against impulsive noise by an example of multi-channel random signal detection in the presence of additive impulsive noise.

The remainder of this paper is structured as follows. Section II presents the definitions of three concordance coefficients, the MCGM, as well as the some auxiliary results. In Section III, we establish the theoretical results with regard to the expectation and variance of AKT under the MCGM. Section IV verifies our theoretical findings under the MCGM. The comparative results of AKT with APPMCC and KCC are also provided in the same section, in terms of the root mean squared error (RMSE) and receiver operating characteristic (ROC) curves. Finally in Section IV-D, we summarize our main findings and our conclusion on AKT.

For convenience of later development, we use $\mathbb{E}(\cdot)$, $\mathbb{V}(\cdot)$, $\mathbb{C}(\cdot)$ and $\mathrm{corr}(\cdot,\cdot)$ to denote the expectation, variance, covariance and correlation of (between) random variables, respectively. The symbol $n^{[k]}$ stands for $n(n-1)\cdots(n-k+1)$ with k being a positive integer. The bivariate Gaussian distribution are denoted by $\mathcal{N}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. The notation $\mathrm{P}_m^0(Z_1, \cdots, Z_m) \triangleq \mathbb{P}(Z_1 > 0, \cdots, Z_m > 0)$ represents the positive orthant probabilities with respect to an m-dimensional normal random vector $[Z_1 \cdots Z_m]$. The symbol $R(\varrho_{rs})_{m\times m}$ represents the correlation matrix where $\varrho_{rs} \triangleq \mathrm{corr}(Z_r, Z_s)$ with $r, s = 1, \ldots, m$. We will also use $\mathrm{P}_m^0(R)$ to denote $\mathrm{P}_m^0(Z_1, \cdots, Z_m)$ for compactness. All other notation is to be defined in the text where it first enters.

II. METHODS

In this section, we present the definitions of three concordance coefficients and a particular MCGM employed to emulate the heavy-tailed impulsive noise mentioned in the previous section. Moreover, some auxiliary results are also established for ease of further theoretical analysis.

A. DEFINITIONS OF PPMCC, SR AND KT

Denote by $\mathbf{X}$ a data matrix with size $m \times n$, where m is the number of channels and n is the length of signal in each channel. Let

$$X^i \triangleq [X_1^i \; X_2^i \; \cdots \; X_n^i]$$

be the ith row of the matrix $\mathbf{X}$. Sorting X^i in ascending order generates a new vector

$$X^{(i)} \triangleq [X_{(1)}^i \; X_{(2)}^i \; \cdots \; X_{(n)}^i]$$

where $[X_{(1)}^i \; X_{(2)}^i \; \cdots \; X_{(n)}^i]$ is called the order statistics of X^i [29]. Suppose that X_j^i is at the kth position in $X^{(i)}$. The number k is termed the rank of X_j^i [30] and is denoted by Q_j^i. Let $\bar{X^i}$ and $\bar{X^j}$ represent the mathematical mean of X^i and X^j. Then, three classical correlation coefficients, i.e., PPMCC (r_P below) [11], SR (r_S below) [30], and KT (r_K below) [30] of X^i and X^j can be defined as

$$r_{\mathcal{P}}(X^i, X^j) \triangleq \frac{\sum_{a=1}^{n}(X_a^i - \bar{X^i})(X_a^j - \bar{X^j})}{\left[\sum_{a=1}^{n}(X_a^i - \bar{X^i})^2 \sum_{a=1}^{n}(X_a^j - \bar{X^j})^2\right]^{\frac{1}{2}}} \tag{1}$$

$$r_{\mathcal{S}}(X^i, X^j) \triangleq 1 - \frac{6\sum_{a=1}^{n}(Q_a^i - Q_a^j)^2}{n(n^2-1)} \tag{2}$$

$$r_{\mathcal{K}}(X^i, X^j) \triangleq \frac{\sum_{a=1}^{n}\sum_{b=1}^{n}\mathrm{sgn}(X_a^i - X_b^i)\,\mathrm{sgn}(X_a^j - X_b^j)}{n(n-1)}. \tag{3}$$

B. DEFINITION OF APPMCC, KCC AND AKT

For any two channel signal from the data matrix $\mathbf{X}$, we can obtain one of three classical correlation coefficients. Then, corresponding to the definitions of (1) − (3), we can obtain three correlation matrices, each with size $m\times m$. By averaging these matrices without the diagonal elements, three concordance coefficients, namely APPMCC ($\omega_{\mathcal{P}}$ below), KCC ($\omega_{\mathcal{S}}$ below) and AKT ($\omega_{\mathcal{K}}$ below), repsectively, can be defined as [7]:

$$\omega_{\mathcal{P}}(\mathbf{X}) \triangleq \frac{\sum_{i\neq j}^{m} r_{\mathcal{P}}(X^i, X^j)}{m(m-1)} \tag{4}$$

$$\omega_{\mathcal{S}}(\mathbf{X}) \triangleq \frac{\sum_{i\neq j}^{m} r_{\mathcal{S}}(X^i, X^j)}{m(m-1)} \tag{5}$$

$$\omega_{\mathcal{K}}(\mathbf{X}) \triangleq \frac{\sum_{i\neq j}^{m} r_{\mathcal{K}}(X^i, X^j)}{m(m-1)}. \tag{6}$$

C. MULTIVARIATE CONTAMINATED GAUSSIAN MODEL

Let $\{X_a^i, X_a^j\}_{a=1}^n$ be independent and identically distribution (i.i.d.) data pairs from the ith and jth rows of the data matrix $\mathbf{X}$. Assume that the joint probability density function (pdf) of X^i and X^j obeys the following form [24]

$$(1-\varepsilon)\mathcal{N}(\mu_i, \mu_j, \sigma_i^2, \sigma_j^2, \rho_{ij}) + \varepsilon\mathcal{N}(\mu_i', \mu_j', \lambda_i^2\sigma_i^2, \lambda_j^2\sigma_j^2, \rho_{ij}') \tag{7}$$

where $0 \leq \varepsilon \ll 1$, $\mu_i = \mu_i'$, $\mu_j = \mu_j'$, $\lambda_i \gg 1$, and $\lambda_j \gg 1$. Under this model, the parameters ρ_{ij} represents association information of interest; whereas the parameters ε and ρ_{ij}' represents undesired interferences. Our purpose is thus to investigate the robustness of AKT against the interference parameters. For convenience, in the following we use $\phi(x^i, x^j)$ and $\psi(x^i, x^j)$ to denote the pdfs of the two bivariate Gaussian components in (7), respectively.

D. AUXILIARY RESULTS

Lemma 1: Assume that Z_1, Z_2, Z_3, Z_4 follow a quadrivariate normal distribution with zero means and correlation matrix $R = (\varrho_{rs})_{4\times4}$. Define

$$H(t) \triangleq \begin{cases} 1 & t > 0 \\ 0 & t \le 0 \end{cases} \tag{8}$$

Then, the orthant probabilities can be defined as

$$\mathrm{P}_1^0(Z_1) = \mathbb{E}\{H(Z_1)\} = \frac{1}{2} \tag{9}$$

$$\mathrm{P}_2^0(Z_1, Z_2) = \mathbb{E}\{H(Z_1)H(Z_2)\} = \frac{1}{4}\left(1 + \frac{2}{\pi}\sin^{-1}\varrho_{12}\right) \tag{10}$$

$$\mathrm{P}_3^0(Z_1, Z_2, Z_3) = \mathbb{E}\{H(Z_1)H(Z_2)H(Z_3)\} = \frac{1}{8}\left(1 + \frac{2}{\pi}\sum_{r=1}^{2}\sum_{s=r+1}^{3}\sin^{-1}\varrho_{rs}\right) \tag{11}$$

$$\mathrm{P}_4^0(Z_1, Z_2, Z_3, Z_4) = \mathbb{E}\{H(Z_1)H(Z_2)H(Z_3)H(Z_4)\} = \frac{1}{16}\left[1 + \frac{2}{\pi}\sum_{r=1}^{3}\sum_{s=r+1}^{4}\sin^{-1}\varrho_{rs} + W(R)\right] \tag{12}$$

where

$$W(R) = \sum_{\ell=2}^{4}\frac{4}{\pi^2}\int_0^1 \frac{\varrho_{1\ell}}{[1-\varrho_{1\ell}^2u^2]^{\frac{1}{2}}}\sin^{-1}\left[\frac{\alpha_\ell(u)}{\beta_\ell(u)\gamma_\ell(u)}\right]du \tag{13}$$

with

$$\begin{aligned}
\alpha_2 &= \varrho_{34} - \varrho_{23}\varrho_{24} - [\varrho_{13}\varrho_{14} + \varrho_{12}(\varrho_{12}\varrho_{34} - \varrho_{14}\varrho_{23} - \varrho_{13}\varrho_{24})]u^2 \\
\alpha_3 &= \varrho_{24} - \varrho_{23}\varrho_{34} - [\varrho_{12}\varrho_{14} + \varrho_{13}(\varrho_{13}\varrho_{24} - \varrho_{14}\varrho_{23} - \varrho_{12}\varrho_{34})]u^2 \\
\alpha_4 &= \varrho_{23} - \varrho_{24}\varrho_{34} - [\varrho_{12}\varrho_{13} + \varrho_{14}(\varrho_{14}\varrho_{23} - \varrho_{13}\varrho_{24} - \varrho_{12}\varrho_{34})]u^2 \\
\beta_2 &= \beta_3 = [1 - \varrho_{23}^2 - (\varrho_{12}^2 + \varrho_{13}^2 - 2\varrho_{12}\varrho_{13}\varrho_{23})u^2]^{\frac{1}{2}} \\
\gamma_2 &= \beta_4 = [1 - \varrho_{24}^2 - (\varrho_{12}^2 + \varrho_{14}^2 - 2\varrho_{12}\varrho_{14}\varrho_{24})u^2]^{\frac{1}{2}} \\
\gamma_3 &= \gamma_4 = [1 - \varrho_{34}^2 - (\varrho_{13}^2 + \varrho_{14}^2 - 2\varrho_{13}\varrho_{14}\varrho_{34})u^2]^{\frac{1}{2}}
\end{aligned}$$

Proof: It is trivial to obtain the first statement (9). The results (10)–(12) have been established in [31]–[33], respectively. □

III. STATISTICAL PROPERTIES OF AKT UNDER MULTIVARIATE CONTAMINATED GAUSSIAN MODEL

In this section, we establish our major theoretical results with the assistance of Lemma 1. Specifically, we derive the expectation and variance of AKT under the multivariate contaminated Gaussian model (7).

Theorem 1: Let $\{X_k^i, X_k^j\}_{k=1}^n$ be i.i.d. samples generated from the model (7). Then, the expectation of AKT is

$$\mathbb{E}(\omega_\mathcal{K}) = \frac{2}{m(m-1)\pi}\sum_{i\neq j=1}^{m}\sum^{m}\Big[(1-\varepsilon)^2\sin^{-1}\rho_{ij} + 2\varepsilon(1-\varepsilon)\sin^{-1}\frac{\rho_{ij} + \lambda_i\lambda_j\rho'_{ij}}{\sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}} + \varepsilon^2\sin^{-1}\rho'_{ij}\Big]. \tag{14}$$

Proof: See Appendix A. □

Corollary 1: When $\lambda_i \to \infty$ and $\lambda_j \to \infty$, the expectation of AKT can be simplified as

$$\mathbb{E}(\omega_\mathcal{K}) = \frac{2}{m(m-1)\pi}\sum_{i\neq j=1}^{m}\sum^{m}[(1-\varepsilon)^2\sin^{-1}\rho_{ij} + \varepsilon(2-\varepsilon)\sin^{-1}\rho'_{ij}] \tag{15}$$

Moreover, for $\varepsilon = 0$, which means that the multivariate contaminated Gaussian model reduces to the multivariate Gaussian model, it follows that

$$\mathbb{E}(\omega_\mathcal{K})|_{\varepsilon=0} = \frac{2}{m(m-1)\pi}\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij} \tag{16}$$

Proof: Letting $\lambda_i \to \infty$ and $\lambda_j \to \infty$ in (14) along with some simple algebras, we can easily obtain the expression of (15). Substituting $\varepsilon = 0$ into (15) produces the statement of (16). □

Theorem 2: Let $\{\xi_\ell^i, \xi_\ell^j\}_{\ell=1}^n$ and $\{\zeta_\ell^i, \zeta_\ell^j\}_{\ell=1}^n$ be n i.i.d. samples generated from the bivariate normal distribution ϕ and ψ, respectively. Then, under the MCGM (7), the variance of AKT is

$$\mathbb{V}(\omega_\mathcal{K}) = \frac{16(I_1 + I_2 + I_3)}{m^2(m-1)^2n^2(n-1)^2} - [\mathbb{E}(\omega_\mathcal{K}) + 1]^2 \tag{17}$$

where

$$I_1 = 2\sum_{i\neq j=1}^{m}\sum^{m}\sum_{\ell=1}^{28}\alpha_\ell\beta_\ell\,\mathrm{P}_4^0(A_\ell) \tag{18}$$

$$I_2 = 4\sum_{i\neq j\neq k=1}^{m}\sum^{m}\sum^{m}\sum_{\ell=1}^{28}\alpha'_\ell\beta'_\ell\,\mathrm{P}_4^0(B_\ell) \tag{19}$$

$$I_3 = \sum_{i\neq j\neq k\neq l=1}^{m}\sum^{m}\sum^{m}\sum^{m}\sum_{\ell=1}^{32}\alpha''_\ell\beta''_\ell\,\mathrm{P}_4^0(C_\ell) \tag{20}$$

with α_ℓ-, β_ℓ-, and A_ℓ-terms in (18) defined in Table 1, α'_ℓ-, β'_ℓ-, and B_ℓ-terms in (19) defined in Table 2, α''_ℓ-, β''_ℓ-, and C_ℓ-terms in (20) defined in Table 3, and $\mathbb{E}(\omega_\mathcal{K})$ defined by (14).

Proof: See Appendix B. □

Remark 1: Because of the complicated integrals involved in the calculation of P_4^0-terms in (18)–(20), the variance of AKT can not be expressed into elementary functions in general. However, simplifications are available for various particular cases, as demonstrated in the following corollaries.

Corollary 2: When $\lambda_i \to \infty$ and $\lambda_j \to \infty$, the variance of AKT can be simplified as

$$
\begin{aligned}
\mathbb{V}(\omega_{\mathcal{K}}) = & \frac{8(n-2)(1+8\varepsilon)(1-\varepsilon)^2}{9n(n-1)m(m-1)} + \frac{4}{n(n-1)m(m-1)} \\
& + \frac{\Lambda_1}{n(n-1)m^2(m-1)^2} + \frac{16(\Lambda_2+\Lambda_3)}{n(n-1)m^2(m-1)^2\pi} \\
& - \frac{32(n-2)(1-\varepsilon)^3}{n(n-1)m^2(m-1)^2\pi^2}\sum_{i\neq j=1}^{m}\sum^{m}\left(\sin^{-1}\frac{\rho_{ij}}{2}\right)^2 \\
& + \frac{16(1-\varepsilon)^3\left[(2n-3)\varepsilon-1\right]}{n(n-1)m^2(m-1)^2\pi^2}\sum_{i\neq j=1}^{m}\sum^{m}\left(\sin^{-1}\rho_{ij}\right)^2 \\
& - \frac{8(2n-3)(1-\varepsilon)^4}{n(n-1)m^2(m-1)^2\pi^2} \\
& \times \left[\left(\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij}\right)^2 - 2\sum_{i\neq j=1}^{m}\sum^{m}\left(\sin^{-1}\rho_{ij}\right)^2\right] \\
& - \frac{16\varepsilon(1-\varepsilon)^2\left[(2n-3)(1-\varepsilon)+1\right]}{n(n-1)m^2(m-1)^2\pi^2} \\
& \times \sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij}\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho'_{ij} \\
& - \frac{8\varepsilon^2\left[2(n-2)(\varepsilon^2-3\varepsilon+3)+(2-\varepsilon)^2\right]}{n(n-1)m^2(m-1)^2\pi^2} \\
& \times \left(\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho'_{ij}\right)^2
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
\Lambda_1 = & \sum_{i\neq j\neq k\neq l=1}^{m}\sum^{m}\sum^{m}\sum^{m}\Big\{2(1-\varepsilon)^2W(C_1)+2\varepsilon^2W(C_2) \\
& + 4\varepsilon(1-\varepsilon)W(C_3) + 4(n-2)\left[(1-\varepsilon)^3W(C_5)+\varepsilon^3W(C_6)\right] \\
& + 4(n-2)(1-\varepsilon)\varepsilon\left[(1-\varepsilon)W(C_7)+2\varepsilon W(C_{11})\right]\Big\} \\
& + 16(n-2)\sum_{i\neq j\neq k=1}^{m}\sum^{m}\sum^{m}\Big[(1-\varepsilon)^3W(B_5)+\varepsilon^3W(B_6) \\
& \quad + 2(1-\varepsilon)\varepsilon^2W(B_{11})\Big] \\
& + 8(n-2)\varepsilon^2\sum_{i\neq j=1}^{m}\sum^{m}\left[\varepsilon W(A_6)+2(1-\varepsilon)W(A_{11})\right]
\end{aligned}
$$

$$
\Lambda_2 = (1-\varepsilon)^2(m-2)\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij}
$$

and

$$
\Lambda_3 = \varepsilon(m-2)\left[2(n-2)(1-\varepsilon)^2+2-\varepsilon\right]\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho'_{ij}.
$$

Proof: Letting $\lambda_i \to \infty$ and $\lambda_j \to \infty$ in (17), we can arrive at (21) with some tedious but straightforward algebra. □

Corollary 3: When $\varepsilon = 0$, viz., the multivariate contaminated Gaussian model reduces to pure Gaussian model, the variance of AKT becomes

$$
\mathbb{V}(\omega_{\mathcal{K}})|_{\varepsilon=0} = \frac{\Delta_1-\Delta_2+\Delta_3+\Delta_4}{m^2(m-1)^2n(n-1)} + \frac{4(2n+5)}{9m(m-1)n(n-1)} \tag{22}
$$

where

$$
\begin{aligned}
\Delta_1 &= \frac{16(m-2)}{\pi}\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij} \\
\Delta_2 &= \frac{8(2n-3)}{\pi^2}\left(\sum_{i\neq j=1}^{m}\sum^{m}\sin^{-1}\rho_{ij}\right)^2 \\
\Delta_3 &= \frac{32(n-2)}{\pi^2}\sum_{i\neq j=1}^{m}\sum^{m}\left[\left(\sin^{-1}\rho_{ij}\right)^2-\left(\sin^{-1}\frac{\rho_{ij}}{2}\right)^2\right] \\
\Delta_4 &= \sum_{i\neq j\neq k\neq l=1}^{m}\sum^{m}\sum^{m}\sum^{m}\left[2W(C_1)+4(n-2)W(C_5)\right] \\
& \quad + 16(n-2)\sum_{i\neq j\neq k=1}^{m}\sum^{m}\sum^{m}W(B_5).
\end{aligned}
$$

Moreover, when $\rho_{ij} = \rho$, the above expression (22) can be further simplified as

$$
\begin{aligned}
\mathbb{V}(\omega_{\mathcal{K}})|_{\varepsilon=0,\rho_{ij}=\rho} = & \frac{\mathcal{W}}{m(m-1)n(n-1)} \\
& + \frac{4(2n+5)}{9m(m-1)n(n-1)} \\
& + \frac{16(m-2)\sin^{-1}\rho}{m(m-1)n(n-1)\pi} \\
& - \frac{8(2n-3)\left(\sin^{-1}\rho\right)^2}{n(n-1)\pi^2} \\
& + \frac{32(n-2)\left[(\sin^{-1}\rho)^2-(\sin^{-1}\frac{\rho}{2})^2\right]}{m(m-1)n(n-1)\pi^2}
\end{aligned} \tag{23}
$$

where

$$
\mathcal{W} = (m-2)(m-3)\left[2W(C_1)+4(n-2)W(C_5)\right] + 16(m-2)(n-2)W(B_5).
$$

Furthermore,

$$
\mathbb{V}(\omega_{\mathcal{K}})|_{\varepsilon=0,\rho=0} = \frac{4(2n+5)}{9m(m-1)n(n-1)} \tag{24}
$$

$$
\mathbb{V}(\omega_{\mathcal{K}})|_{\varepsilon=0,\rho=1} = 0. \tag{25}
$$

Proof: Substituting $\varepsilon = 0$ into (17) and using (10)–(12) along with some tedious derivations, we can obtain the expression (22). Similarly, the results (23), (24) and (25) follow readily by replacing the parameters in (22) with the corresponding counterparts. □

TABLE 1. Quantities for evaluation of $\mathbb{E}(H_{ij}^2)$ in (18).

				Representive term				Correlation coefficients					
ℓ	C.M.[1]	β_ℓ	α_ℓ	Z_1	Z_2	Z_3	Z_4	ϱ_{12}	ϱ_{13}	ϱ_{14}	ϱ_{23}	ϱ_{24}	ϱ_{34}
1	A_1	$(1-\varepsilon)^2$	$n^{[2]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	ρ_{ij}	1	ρ_{ij}	ρ_{ij}	1	ρ_{ij}
2	A_2	ε^2	$n^{[2]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	ρ'_{ij}	1	ρ'_{ij}	ρ'_{ij}	1	ρ'_{ij}
3	A_3	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
4	A_4	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
5	A_5	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^i-\xi_1^i$	$\xi_3^j-\xi_1^j$	ρ_{ij}	$-\frac{1}{2}$	$-\frac{\rho_{ij}}{2}$	$-\frac{\rho_{ij}}{2}$	$-\frac{1}{2}$	ρ_{ij}
6	A_6	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_1^i$	$\zeta_3^j-\zeta_1^j$	ρ'_{ij}	$-\frac{1}{2}$	$-\frac{\rho'_{ij}}{2}$	$-\frac{\rho'_{ij}}{2}$	$-\frac{1}{2}$	ρ'_{ij}
7	A_7	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^i-\zeta_1^i$	$\xi_3^j-\zeta_1^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i^2}{1+\lambda_i^2}$	$-\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_j^2}{1+\lambda_j^2}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
8	A_8	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_3^i-\xi_1^i$	$\xi_3^j-\xi_1^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{1}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{ij}
9	A_9	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^i-\xi_1^i$	$\zeta_3^j-\xi_1^j$	ρ_{ij}	$-\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{1}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
10	A_{10}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^i-\xi_1^i$	$\zeta_3^j-\xi_1^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{1}{1+\lambda_i^2}$	$-\frac{\rho_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{ij}}{\lambda_{ij}}$	$-\frac{1}{1+\lambda_j^2}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
11	A_{11}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^i-\zeta_1^i$	$\zeta_3^j-\zeta_1^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\lambda_j}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{ij}
12	A_{12}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\zeta_1^i$	$\xi_3^j-\zeta_1^j$	ρ'_{ij}	$-\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_j}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
13	A_{13}	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\xi_3^i$	$\xi_1^j-\xi_3^j$	ρ_{ij}	$\frac{1}{2}$	$\frac{\rho_{ij}}{2}$	$\frac{\rho_{ij}}{2}$	$\frac{1}{2}$	ρ_{ij}
14	A_{14}	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\zeta_3^i$	$\zeta_1^j-\zeta_3^j$	ρ'_{ij}	$\frac{1}{2}$	$\frac{\rho'_{ij}}{2}$	$\frac{\rho'_{ij}}{2}$	$\frac{1}{2}$	ρ'_{ij}
15	A_{15}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\xi_3^i$	$\zeta_1^j-\xi_3^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i^2}{1+\lambda_i^2}$	$\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_j^2}{1+\lambda_j^2}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
16	A_{16}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\xi_3^i$	$\xi_1^j-\xi_3^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{1}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{ij}
17	A_{17}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\zeta_3^i$	$\xi_1^j-\zeta_3^j$	ρ_{ij}	$\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{1}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
18	A_{18}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\zeta_3^i$	$\xi_1^j-\zeta_3^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{1}{1+\lambda_i^2}$	$\frac{\rho_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ij}}{\lambda_{ij}}$	$\frac{1}{1+\lambda_j^2}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
19	A_{19}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\zeta_3^i$	$\zeta_1^j-\zeta_3^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\lambda_j}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{ij}
20	A_{20}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\xi_3^i$	$\zeta_1^j-\xi_3^j$	ρ'_{ij}	$\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_j}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
21	A_{21}	$(1-\varepsilon)^4$	$n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^j-\xi_4^j$	ρ_{ij}	0	0	0	0	ρ_{ij}
22	A_{22}	ε^4	$n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_4^i$	$\zeta_3^j-\zeta_4^j$	ρ'_{ij}	0	0	0	0	ρ'_{ij}
23	A_{23}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^j-\xi_4^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ_{ij}
24	A_{24}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^i-\xi_4^i$	$\zeta_3^j-\xi_4^j$	ρ_{ij}	0	0	0	0	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
25	A_{25}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_4^i$	$\zeta_3^j-\zeta_4^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ'_{ij}
26	A_{26}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\zeta_4^i$	$\xi_3^j-\zeta_4^j$	ρ'_{ij}	0	0	0	0	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$
27	A_{27}	$(1-\varepsilon)^2\varepsilon^2$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^j-\xi_4^j$	ρ'_{ij}	0	0	0	0	ρ_{ij}
28	A_{28}	$(1-\varepsilon)^2\varepsilon^2$	$4n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^i-\xi_4^i$	$\zeta_3^j-\xi_4^j$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$

1 C.M. is an abbreviation of correlation Matrix. The notation A_ℓ represents correlation coefficient matrix $R(\varrho_{rs})_{4\times 4}$, where $\varrho_{rs} \triangleq \mathrm{corr}(Z_r, Z_s)$, for $r, s = 1, \cdots, 4$.

2 In the columns containing $\varrho(\cdot)$, $\lambda_{ij} \triangleq \sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}$.

IV. RESULTS AND DISCUSSION

In this section, we 1) verify the correctness of Theorems 1 and 2 by Monte Carlo simulations, 2) compare AKT with APPMCC and KCC, in terms of RMSE, and 3) present an example of multi-channel signal detection in the presence of additive impulsive noise (under the MCGM). In the sequel, the notation $h = h_1(\Delta h)h_2$ stands for a list of h varying from h_1 to h_2 with an increment of Δh. The number of Monte Carlo trials is set to be 10^5 for purpose of accuracy. For convenience, we set all the parameters ρ_{ij} and ρ'_{ij} equal to ρ and ρ', respectively, unless otherwise stated.

TABLE 2. Quantities for evaluation of $\mathbb{E}(H_{ij}H_{ik})$ in (19).

				Representive term				Correlation coefficients					
ℓ	C.M.[1]	β'_ℓ	α'_ℓ	Z_1	Z_2	Z_3	Z_4	ϱ_{12}	ϱ_{13}	ϱ_{14}	ϱ_{23}	ϱ_{24}	ϱ_{34}
1	B_1	$(1-\varepsilon)^2$	$n^{[2]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\xi_2^i$	$\xi_1^k-\xi_2^k$	ρ_{ij}	1	ρ_{ik}	ρ_{ij}	ρ_{jk}	ρ_{ik}
2	B_2	ε^2	$n^{[2]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^k-\zeta_2^k$	ρ'_{ij}	1	ρ'_{ik}	ρ'_{ij}	ρ'_{jk}	ρ'_{ik}
3	B_3	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\xi_2^i$	$\zeta_1^k-\xi_2^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
4	B_4	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\zeta_2^i$	$\xi_1^k-\zeta_2^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	1	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
5	B_5	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^i-\xi_1^i$	$\xi_3^k-\xi_1^k$	ρ_{ij}	$-\frac{1}{2}$	$-\frac{\rho_{ik}}{2}$	$-\frac{\rho_{ij}}{2}$	$-\frac{\rho_{jk}}{2}$	ρ_{ik}
6	B_6	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_1^i$	$\zeta_3^k-\zeta_1^k$	ρ'_{ij}	$-\frac{1}{2}$	$-\frac{\rho'_{ik}}{2}$	$-\frac{\rho'_{ij}}{2}$	$-\frac{\rho'_{jk}}{2}$	ρ'_{ik}
7	B_7	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^i-\zeta_1^i$	$\xi_3^k-\zeta_1^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i^2}{1+\lambda_i^2}$	$-\frac{\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$-\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
8	B_8	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_3^i-\xi_1^i$	$\xi_3^k-\xi_1^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\rho_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{ik}
9	B_9	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^i-\xi_1^i$	$\zeta_3^k-\xi_1^k$	ρ_{ij}	$-\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
10	B_{10}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^i-\xi_1^i$	$\zeta_3^k-\xi_1^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{1}{1+\lambda_i^2}$	$-\frac{\rho_{ik}}{\lambda_{ik}}$	$-\frac{\rho_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
11	B_{11}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^i-\zeta_1^i$	$\zeta_3^k-\zeta_1^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_i\rho'_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\lambda_j\rho'_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{ik}
12	B_{12}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\zeta_1^i$	$\xi_3^k-\zeta_1^k$	ρ'_{ij}	$-\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_k\rho'_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_k\rho'_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
13	B_{13}	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\xi_3^i$	$\xi_1^k-\xi_3^k$	ρ_{ij}	$\frac{1}{2}$	$\frac{\rho_{ik}}{2}$	$\frac{\rho_{ij}}{2}$	$\frac{\rho_{jk}}{2}$	ρ_{ik}
14	B_{14}	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\zeta_3^i$	$\zeta_1^k-\zeta_3^k$	ρ'_{ij}	$\frac{1}{2}$	$\frac{\rho'_{ik}}{2}$	$\frac{\rho'_{ij}}{2}$	$\frac{\rho'_{jk}}{2}$	ρ'_{ik}
15	B_{15}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\xi_3^i$	$\zeta_1^k-\xi_3^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i^2}{1+\lambda_i^2}$	$\frac{\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
16	B_{16}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\xi_3^i$	$\xi_1^k-\xi_3^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{ik}
17	B_{17}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^i-\zeta_3^i$	$\xi_1^k-\zeta_3^k$	ρ_{ij}	$\frac{1}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
18	B_{18}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^i-\zeta_3^i$	$\xi_1^k-\zeta_3^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{1}{1+\lambda_i^2}$	$\frac{\rho_{ik}}{\lambda_{ik}}$	$\frac{\rho_{ij}}{\lambda_{ij}}$	$\frac{\rho_{jk}}{\lambda_{jk}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
19	B_{19}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^i-\zeta_3^i$	$\zeta_1^k-\zeta_3^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_i\rho'_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_j\rho'_{ij}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\lambda_j\rho'_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{ik}
20	B_{20}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^i-\xi_3^i$	$\zeta_1^k-\xi_3^k$	ρ'_{ij}	$\frac{\lambda_i}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_k\rho'_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\lambda_i\rho'_{ij}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_k\rho'_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
21	B_{21}	$(1-\varepsilon)^4$	$n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^k-\xi_4^k$	ρ_{ij}	0	0	0	0	ρ_{ik}
22	B_{22}	ε^4	$n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_4^i$	$\zeta_3^k-\zeta_4^k$	ρ'_{ij}	0	0	0	0	ρ'_{ik}
23	B_{23}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^k-\xi_4^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ_{ik}
24	B_{24}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^i-\xi_4^i$	$\zeta_3^k-\xi_4^k$	ρ_{ij}	0	0	0	0	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
25	B_{25}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^i-\zeta_4^i$	$\zeta_3^k-\zeta_4^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ'_{ik}
26	B_{26}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\zeta_4^i$	$\xi_3^k-\zeta_4^k$	ρ'_{ij}	0	0	0	0	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$
27	B_{27}	$(1-\varepsilon)^2\varepsilon^2$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^i-\xi_4^i$	$\xi_3^k-\xi_4^k$	ρ'_{ij}	0	0	0	0	ρ_{ik}
28	B_{28}	$(1-\varepsilon)^2\varepsilon^2$	$4n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^i-\xi_4^i$	$\zeta_3^k-\xi_4^k$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$

1 The notation B_ℓ represents correlation coefficient matrix $R(\varrho_{rs})_{4\times 4}$ with each element $\varrho_{rs} \triangleq \mathrm{corr}(Z_r, Z_s), r, s = 1, \cdots, 4$.

2 In the columns containing $\varrho(\cdot)$, $\lambda_{ij} = \sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}$.

A. VERIFICATION OF THEOREM 1 AND THEOREM 2

The correctness of the expectation and variance of AKT under MCGM (7) for small samples are verified in FIGURE 1 and FIGURE 2, respectively. In these two figures, the parameters are set to be $\varepsilon \in \{0.02, 0.08\}$, $\rho' \in \{0, 1\}$, $\rho = 0(0.1)0.9$ and $m \in \{2, 3, 4\}$. Good agreements are observed between the simulation results and the theoretical counterparts. Moreover, we also verify the correctness of Theorems 1 and 2 in some scenarios where ρ_{ij} and ρ'_{ij} are unequal. From TABLE 4, good agreements between simulation results and theoretical counterparts are again observed.

B. COMPARATIVE RESULTS OF RMSE

For $\varepsilon = 0$ and $\rho_{ij} = \rho$, the MCGM degenerates to a Multivariate Gaussian Model with all the correlation coefficients being ρ. In this case, it is of interest to estimate the parent correlation coefficient ρ based on the three concordance correlation coefficients, namely $\omega_{\mathcal{P}}$, $\omega_{\mathcal{K}}$ and $\omega_{\mathcal{S}}$ in (4)−(6),

TABLE 3. Quantities for evaluation of $\mathbb{E}(H_{ij}H_{kl})$ in (20).

ℓ	C.M.[1]	β''_ℓ	α''_ℓ	Representive term Z_1	Z_2	Z_3	Z_4	Correlation coefficients ϱ_{12}	ϱ_{13}	ϱ_{14}	ϱ_{23}	ϱ_{24}	ϱ_{34}
1	C_1	$(1-\varepsilon)^2$	$n^{[2]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^k-\xi_2^k$	$\xi_1^l-\xi_2^l$	ρ_{ij}	ρ_{ik}	ρ_{il}	ρ_{jk}	ρ_{jl}	ρ_{kl}
2	C_2	ε^2	$n^{[2]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^k-\zeta_2^k$	$\zeta_1^l-\zeta_2^l$	ρ'_{ij}	ρ'_{ik}	ρ'_{il}	ρ'_{jk}	ρ'_{jl}	ρ'_{kl}
3	C_3	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^k-\xi_2^k$	$\zeta_1^l-\xi_2^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\rho_{il}+\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{jl}+\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
4	C_4	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^k-\zeta_2^k$	$\xi_1^l-\zeta_2^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\rho_{il}+\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\rho_{jl}+\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
5	C_5	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^k-\xi_1^k$	$\xi_3^l-\xi_1^l$	ρ_{ij}	$-\frac{\rho_{ik}}{2}$	$-\frac{\rho_{il}}{2}$	$-\frac{\rho_{jk}}{2}$	$-\frac{\rho_{jl}}{2}$	ρ_{kl}
6	C_6	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^k-\zeta_1^k$	$\zeta_3^l-\zeta_1^l$	ρ'_{ij}	$-\frac{\rho'_{ik}}{2}$	$-\frac{\rho'_{il}}{2}$	$-\frac{\rho'_{jk}}{2}$	$-\frac{\rho'_{jl}}{2}$	ρ'_{kl}
7	C_7	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^k-\zeta_1^k$	$\xi_3^l-\zeta_1^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$-\frac{\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$-\frac{\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$-\frac{\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
8	C_8	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_3^k-\xi_1^k$	$\xi_3^l-\xi_1^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{il}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\rho_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\rho_{jl}}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{kl}
9	C_9	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^k-\xi_1^k$	$\zeta_3^l-\xi_1^l$	ρ_{ij}	$-\frac{\rho_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\rho_{il}}{\sqrt{2(1+\lambda_l^2)}}$	$-\frac{\rho_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\rho_{jl}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
10	C_{10}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^k-\xi_1^k$	$\zeta_3^l-\xi_1^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{ik}}{\lambda_{ik}}$	$-\frac{\rho_{il}}{\lambda_{il}}$	$-\frac{\rho_{jk}}{\lambda_{jk}}$	$-\frac{\rho_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
11	C_{11}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^k-\zeta_1^k$	$\zeta_3^l-\zeta_1^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\lambda_i\rho'_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_i\rho'_{il}}{\sqrt{2(1+\lambda_i^2)}}$	$-\frac{\lambda_j\rho'_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	$-\frac{\lambda_j\rho'_{jl}}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{kl}
12	C_{12}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^k-\zeta_1^k$	$\xi_3^l-\zeta_1^l$	ρ'_{ij}	$-\frac{\lambda_k\rho'_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\lambda_l\rho'_{il}}{\sqrt{2(1+\lambda_l^2)}}$	$-\frac{\lambda_k\rho'_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$-\frac{\lambda_l\rho'_{jl}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
13	C_{13}	$(1-\varepsilon)^3$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^k-\xi_3^k$	$\xi_1^l-\xi_3^l$	ρ_{ij}	$\frac{\rho_{ik}}{2}$	$\frac{\rho_{il}}{2}$	$\frac{\rho_{jk}}{2}$	$\frac{\rho_{jl}}{2}$	ρ_{kl}
14	C_{14}	ε^3	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^k-\zeta_3^k$	$\zeta_1^l-\zeta_3^l$	ρ'_{ij}	$\frac{\rho'_{ik}}{2}$	$\frac{\rho'_{il}}{2}$	$\frac{\rho'_{jk}}{2}$	$\frac{\rho'_{jl}}{2}$	ρ'_{kl}
15	C_{15}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^k-\xi_3^k$	$\zeta_1^l-\xi_3^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$\frac{\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$\frac{\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$\frac{\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
16	C_{16}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^k-\xi_3^k$	$\xi_1^l-\xi_3^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{il}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\rho_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\rho_{jl}}{\sqrt{2(1+\lambda_j^2)}}$	ρ_{kl}
17	C_{17}	$(1-\varepsilon)^2\varepsilon$	$2n^{[3]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^k-\zeta_3^k$	$\xi_1^l-\zeta_3^l$	ρ_{ij}	$\frac{\rho_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{il}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\rho_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\rho_{jl}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
18	C_{18}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\xi_1^k-\zeta_3^k$	$\xi_1^l-\zeta_3^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\rho_{ik}}{\lambda_{ik}}$	$\frac{\rho_{il}}{\lambda_{il}}$	$\frac{\rho_{jk}}{\lambda_{jk}}$	$\frac{\rho_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
19	C_{19}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_1^k-\zeta_3^k$	$\zeta_1^l-\zeta_3^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$\frac{\lambda_i\rho'_{ik}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_i\rho'_{il}}{\sqrt{2(1+\lambda_i^2)}}$	$\frac{\lambda_j\rho'_{jk}}{\sqrt{2(1+\lambda_j^2)}}$	$\frac{\lambda_j\rho'_{jl}}{\sqrt{2(1+\lambda_j^2)}}$	ρ'_{kl}
20	C_{20}	$(1-\varepsilon)\varepsilon^2$	$2n^{[3]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^k-\xi_3^k$	$\zeta_1^l-\xi_3^l$	ρ'_{ij}	$\frac{\lambda_k\rho'_{ik}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\lambda_l\rho'_{il}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\lambda_k\rho'_{jk}}{\sqrt{2(1+\lambda_k^2)}}$	$\frac{\lambda_l\rho'_{jl}}{\sqrt{2(1+\lambda_l^2)}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
21	C_{21}	$(1-\varepsilon)^4$	$n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_3^k-\xi_4^k$	$\xi_3^l-\xi_4^l$	ρ_{ij}	0	0	0	0	ρ_{kl}
22	C_{22}	ε^4	$n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_3^k-\zeta_4^k$	$\zeta_3^l-\zeta_4^l$	ρ'_{ij}	0	0	0	0	ρ'_{kl}
23	C_{23}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_3^k-\xi_4^k$	$\xi_3^l-\xi_4^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ_{kl}
24	C_{24}	$(1-\varepsilon)^3\varepsilon$	$2n^{[4]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\zeta_3^k-\xi_4^k$	$\zeta_3^l-\xi_4^l$	ρ_{ij}	0	0	0	0	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
25	C_{25}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_3^k-\zeta_4^k$	$\zeta_3^l-\zeta_4^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	ρ'_{kl}
26	C_{26}	$(1-\varepsilon)\varepsilon^3$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^k-\zeta_4^k$	$\xi_3^l-\zeta_4^l$	ρ'_{ij}	0	0	0	0	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
27	C_{27}	$(1-\varepsilon)^2\varepsilon^2$	$2n^{[4]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\xi_3^k-\xi_4^k$	$\xi_3^l-\xi_4^l$	ρ'_{ij}	0	0	0	0	ρ_{kl}
28	C_{28}	$(1-\varepsilon)^2\varepsilon^2$	$4n^{[4]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\zeta_3^k-\xi_4^k$	$\zeta_3^l-\xi_4^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	0	0	0	0	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
29	C_{29}	$(1-\varepsilon)^2$	$n^{[2]}$	$\xi_1^i-\xi_2^i$	$\xi_1^j-\xi_2^j$	$\xi_1^k-\xi_2^k$	$\xi_1^l-\xi_2^l$	ρ_{ij}	$-\rho_{ik}$	$-\rho_{il}$	$-\rho_{jk}$	$-\rho_{jl}$	ρ_{kl}
30	C_{30}	ε^2	$n^{[2]}$	$\zeta_1^i-\zeta_2^i$	$\zeta_1^j-\zeta_2^j$	$\zeta_1^k-\zeta_2^k$	$\zeta_1^l-\zeta_2^l$	ρ'_{ij}	$-\rho'_{ik}$	$-\rho'_{il}$	$-\rho'_{jk}$	$-\rho'_{jl}$	ρ'_{kl}
31	C_{31}	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\zeta_1^i-\xi_2^i$	$\zeta_1^j-\xi_2^j$	$\xi_1^k-\zeta_2^k$	$\xi_l^1-\zeta_2^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$-\frac{\rho_{il}+\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$-\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$-\frac{\rho_{jl}+\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$
32	C_{32}	$\varepsilon(1-\varepsilon)$	$n^{[2]}$	$\xi_1^i-\zeta_2^i$	$\xi_1^j-\zeta_2^j$	$\zeta_1^k-\xi_2^k$	$\zeta_1^l-\xi_2^l$	$\frac{\rho_{ij}+\lambda_i\lambda_j\rho'_{ij}}{\lambda_{ij}}$	$-\frac{\rho_{ik}+\lambda_i\lambda_k\rho'_{ik}}{\lambda_{ik}}$	$-\frac{\rho_{il}+\lambda_i\lambda_l\rho'_{il}}{\lambda_{il}}$	$-\frac{\rho_{jk}+\lambda_j\lambda_k\rho'_{jk}}{\lambda_{jk}}$	$-\frac{\rho_{jl}+\lambda_j\lambda_l\rho'_{jl}}{\lambda_{jl}}$	$\frac{\rho_{kl}+\lambda_k\lambda_l\rho'_{kl}}{\lambda_{kl}}$

1 The notation C_ℓ represents correlation coefficient matrix $R(\varrho_{rs})_{4\times 4}$ with each element $\varrho_{rs} \triangleq \mathrm{corr}(Z_r, Z_s)$, $r, s = 1, \cdots, 4$.
2 In the columns containing $\varrho(\cdot)$, $\lambda_{ij} = \sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}$.

respectively. It is well known that [26], as n large,

$$\mathbb{E}(r_{\mathcal{P}}(X^i, X^j)) = \rho, \tag{26}$$

$$\mathbb{E}(r_{\mathcal{S}}(X^i, X^j)) = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}, \tag{27}$$

$$\mathbb{E}(r_{\mathcal{K}}(X^i, X^j)) = \frac{2}{\pi}\sin^{-1}\rho. \tag{28}$$

Then, from (4)−(6), we have

$$\mathbb{E}(\omega_{\mathcal{P}}) = \rho, \tag{29}$$

$$\mathbb{E}(\omega_{\mathcal{S}}) = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}, \tag{30}$$

$$\mathbb{E}(\omega_{\mathcal{K}}) = \frac{2}{\pi}\sin^{-1}\rho. \tag{31}$$

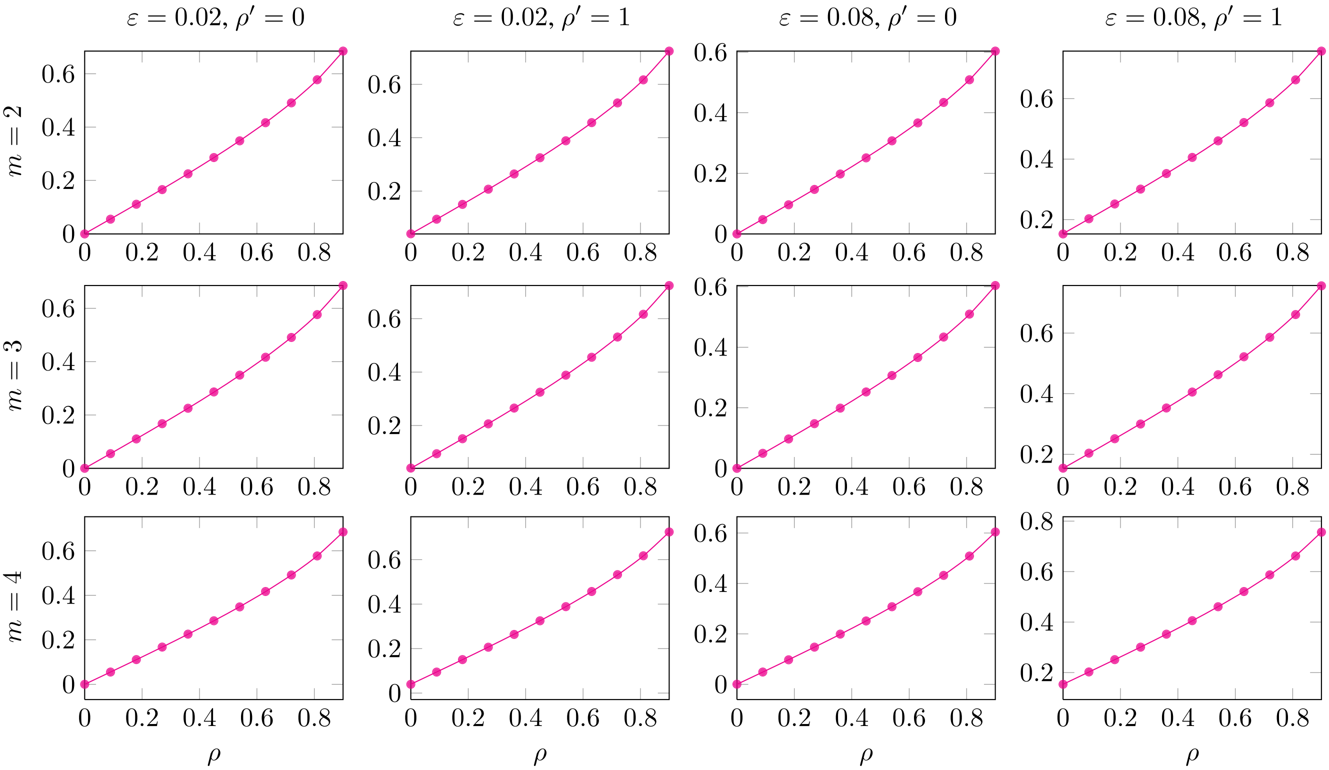


FIGURE 1. Verification of (14) in Theorem 1 for $n = 10$, $\lambda_i = \lambda_j = 10^5$. From top to bottom, each row corresponds to a different number of channels $m \in \{2, 3, 4\}$, respectively; whereas from left to right, each column corresponds to theoretical and observed values of $\mathbb{E}(\omega_{\mathcal{K}})$ with different parameter settings. Good agreements are observed between theoretical results (solid lines) and simulation counterparts (circles).

TABLE 4. Verification of mean and variance of AKT for distinct ρ_{ij} and ρ'_{ij}, where $n = 10$, $\varepsilon = 0.08$, $\lambda_i = 100$, $\lambda_j = 200$.

corr.matrix(ρ_{ij})	corr.matrix(ρ'_{ij})	(14)	$\mathbb{E}(\omega_{\mathcal{K}})\|_{\text{sim}}$ [1]	(17)	$\mathbb{V}(\omega_{\mathcal{K}})\|_{\text{sim}}$ [2]
$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix}$	0.3224	0.3223	0.0665	0.0667
$\begin{pmatrix} 1 & 0.1 & 0.2 \\ 0.1 & 1 & 0.3 \\ 0.2 & 0.3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.5 & 0.6 \\ 0.5 & 1 & 0.1 \\ 0.6 & 0.1 & 1 \end{pmatrix}$	0.1502	0.1501	0.0339	0.0341
$\begin{pmatrix} 1 & 0.3 & 0.2 & 0.4 \\ 0.3 & 1 & 0.5 & 0.7 \\ 0.2 & 0.5 & 1 & 0.6 \\ 0.4 & 0.7 & 0.6 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.4 & 0.6 & 0.7 \\ 0.4 & 1 & 0.8 & 0.5 \\ 0.6 & 0.8 & 1 & 0.3 \\ 0.7 & 0.5 & 0.3 & 1 \end{pmatrix}$	0.3153	0.3155	0.0257	0.0255

[1] $\mathbb{E}(\omega_{\mathcal{K}})\,|_{\text{sim}}$ stands for the expectation of AKT by simulation
[2] $\mathbb{V}(\omega_{\mathcal{K}})\,|_{\text{sim}}$ stands for the variance of AKT by simulation

Then, it is reasonable to construct three estimators of ρ by inverting (29)–(31), namely,

$$\hat{\rho}_{\mathcal{P}} \triangleq \omega_{\mathcal{P}}, \tag{32}$$

$$\hat{\rho}_{\mathcal{S}} \triangleq 2\sin\left(\frac{\pi}{6}\omega_{\mathcal{S}}\right), \tag{33}$$

$$\hat{\rho}_{\mathcal{K}} \triangleq \sin(\frac{\pi}{2}\omega_{\mathcal{K}}). \tag{34}$$

Given the definitions of $\hat{\rho}$, it is of interest to compare their performance in terms of RMSE, which is

$$\text{RMSE} \triangleq \sqrt{\mathbb{E}(\hat{\rho}-\rho)^2}. \tag{35}$$

TABLE 5 lists the results of RMSE for $\hat{\rho}_{\mathcal{P}}$, $\hat{\rho}_{\mathcal{S}}$ and $\hat{\rho}_{\mathcal{K}}$ defined in (32)–(34), respectively, under the MCGM

$$(1-\varepsilon)\mathcal{N}(0, 0, 1, 1, \rho) + \varepsilon\mathcal{N}(0, 0, 10^{10}, 10^{10}, \rho') \tag{36}$$

where $\varepsilon \in \{0.02, 0.04, 0.06, 0.08\}$, $\rho' \in \{0, 0.9\}$ and $\rho = 0(0.1)0.9$, and the number of channels $m = 4$. It appears that 1) the RMSEs of $\hat{\rho}_{\mathcal{P}}$ are much larger than $\hat{\rho}_{\mathcal{S}}$ and $\hat{\rho}_{\mathcal{K}}$, meaning its poor performance under the MCGM, 2) the RMSEs of $\hat{\rho}_{\mathcal{S}}$ and $\hat{\rho}_{\mathcal{K}}$ decrease with the increasing ε, 3) $\hat{\rho}_{\mathcal{S}}$ outperforms $\hat{\rho}_{\mathcal{K}}$ for ρ small, and 4) $\hat{\rho}_{\mathcal{K}}$ outperforms $\hat{\rho}_{\mathcal{S}}$ for ρ large.

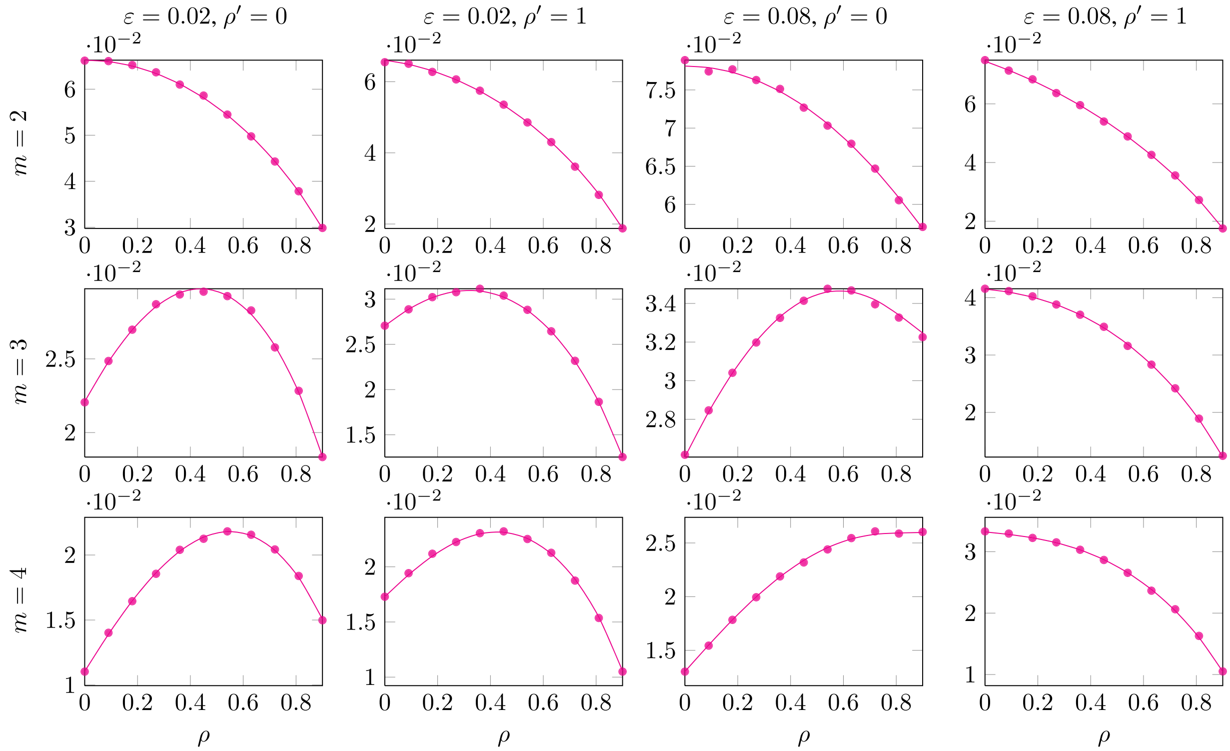


FIGURE 2. **Verification of (17) in Theorem 2 for $n = 10$, $\lambda_i = \lambda_j = 10^5$. From top to bottom, each row corresponds to a different number of channels $m \in \{2, 3, 4\}$, respectively; whereas from left to right, each column corresponds to theoretical and observed values of $\mathbb{V}(\omega_{\mathcal{K}})$ with different parameter settings. Good agreements are observed between the theoretical results (solid lines) and simulation counterparts (circles).**

TABLE 5. **RMSE for three estimators for $n = 100$, $\lambda_i = \lambda_j = 10^{10}$, $\varepsilon = \{0.02, 0.04, 0.06, 0.08\}$ and $\rho' = \{0, 0.9\}$.**

	$\varepsilon = 0.02$			$\varepsilon = 0.04$			$\varepsilon = 0.06$			$\varepsilon = 0.08$		
ρ	$\hat{\rho}_{\mathcal{S}}$	$\hat{\rho}_{\mathcal{P}}$	$\hat{\rho}_{\mathcal{K}}$	$\hat{\rho}_{\mathcal{S}}$	$\hat{\rho}_{\mathcal{P}}$	$\hat{\rho}_{\mathcal{K}}$	$\hat{\rho}_{\mathcal{S}}$	$\hat{\rho}_{\mathcal{P}}$	$\hat{\rho}_{\mathcal{K}}$	$\hat{\rho}_{\mathcal{S}}$	$\hat{\rho}_{\mathcal{P}}$	$\hat{\rho}_{\mathcal{K}}$
0.0	**0.0449**	0.2895	0.0462	**0.0459**	0.2286	0.0479	**0.0477**	0.1849	0.0503	**0.0487**	0.1547	0.0518
0.1	**0.0507**	0.3038	0.0517	**0.0536**	0.2489	0.0549	**0.0556**	0.2105	0.0567	**0.0576**	0.1836	0.0583
0.2	0.0568	0.3451	**0.0567**	0.0606	0.3036	**0.0593**	0.0666	0.2749	**0.0631**	0.0725	0.2528	**0.0669**
0.3	0.0616	0.4004	**0.0600**	0.0694	0.3772	**0.0645**	0.0802	0.3515	**0.0708**	0.0911	0.3388	**0.0776**
0.4	0.0646	0.4720	**0.0607**	0.0773	0.4576	**0.0673**	0.0933	0.4390	**0.0765**	0.1099	0.4290	**0.0869**
0.5	0.0671	0.5440	**0.0602**	0.0860	0.5463	**0.0700**	0.1079	0.5334	**0.0829**	0.1278	0.5206	**0.0945**
0.6	0.0682	0.6314	**0.0572**	0.0941	0.6367	**0.0704**	0.1205	0.6286	**0.0857**	0.1487	0.6191	**0.1031**
0.7	0.0684	0.7107	**0.0520**	0.1008	0.7270	**0.0683**	0.1360	0.7224	**0.0885**	0.1685	0.7172	**0.1086**
0.8	0.0682	0.8029	**0.0441**	0.1076	0.8268	**0.0633**	0.1472	0.8174	**0.0853**	0.1879	0.8131	**0.1094**
0.9	0.0685	0.8862	**0.0332**	0.1149	0.9216	**0.0538**	0.1618	0.9203	**0.0772**	0.2031	0.9131	**0.0996**
1.0	0.0671	0.9764	**0.0050**	0.1186	1.0174	**0.0132**	0.1692	1.0169	**0.0248**	0.2172	1.0139	**0.0390**
0.0	**0.0696**	0.8076	0.0710	**0.1062**	0.8676	0.1088	**0.1436**	0.8816	0.1473	**0.1816**	0.8871	0.1868
0.1	**0.0686**	0.7238	0.0718	**0.0986**	0.7742	0.1044	**0.1290**	0.7842	0.1373	**0.1611**	0.7884	0.1719
0.2	**0.0671**	0.6410	0.0711	**0.0886**	0.6789	0.0966	**0.1134**	0.6870	0.1249	**0.1383**	0.6888	0.1531
0.3	**0.0644**	0.5624	0.0685	**0.0786**	0.5872	0.0876	**0.0975**	0.5871	0.1111	**0.1176**	0.5908	0.1354
0.4	**0.0622**	0.4855	0.0660	**0.0709**	0.4928	0.0798	**0.0838**	0.4933	0.0983	**0.0970**	0.4921	0.1162
0.5	**0.0579**	0.4198	0.0604	**0.0616**	0.4080	0.0693	**0.0690**	0.3968	0.0823	**0.0789**	0.3932	0.0977
0.6	**0.0520**	0.3637	0.0531	**0.0544**	0.3269	0.0598	**0.0579**	0.3043	0.0682	**0.0620**	0.2980	0.0777
0.7	0.0459	0.3253	**0.0445**	**0.0474**	0.2594	0.0480	**0.0487**	0.2167	0.0525	**0.0506**	0.2058	0.0586
0.8	0.0371	0.3088	**0.0324**	0.0412	0.2102	**0.0346**	0.0447	0.1538	**0.0366**	0.0474	0.1299	**0.0388**
0.9	0.0299	0.3236	**0.0190**	0.0397	0.2114	**0.0210**	0.0477	0.1337	**0.0225**	0.0545	0.0933	**0.0236**
1.0	0.0257	0.3631	**0.0011**	0.0411	0.2567	**0.0023**	0.0545	0.1791	**0.0037**	0.0668	0.1424	**0.0054**

Upper panel: $\rho' = 0$; lower pannel: $\rho' = 0.9$. The minimum values are highlighted in bold face.

C. AN EXPAMPLE OF MULTI-CHANNEL SIGNAL DETECTION

As mentioned in Section I, the multi-channel signal detection is a frequently encountered problem in radar, sonar or communication, which can be mathematically modeled as

$$X_a^i = \theta s_a + N_a^i \tag{37}$$

where $1 \le i \le m$, $1 \le a \le n$, s_a is a Gaussian random signal of length n to be detected, $\{N_a^i\}_{a=1}^n$ is i.i.d. noise following

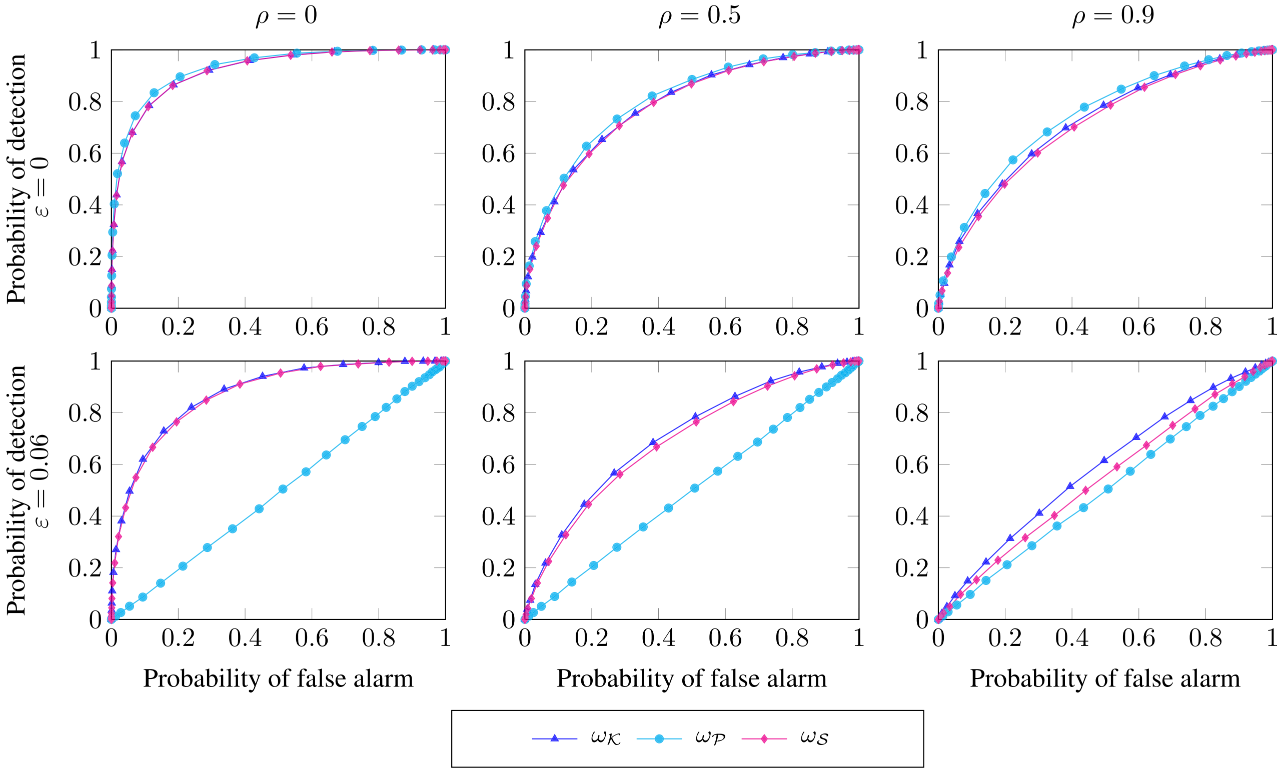


FIGURE 3. Comparison of ROC curves for $\varepsilon = 0$ and $\varepsilon = 0.06$. From top to bottom, three columns correspond to different $\rho \in \{0.0, 0.5, 0.9\}$, respectively; whereas from left to right, two rows correspond to $\varepsilon = 0$ (normal case) and $\varepsilon = 0.06$, respectively. Here the parameters $\lambda_i = \lambda_j = 100$, $m = 4$ and $n = 100$.

the MCGM. Our purpose is to determine the existence of the random signal s_a. Then, the signal detection problem in (37) can be cast into the following hypothesis test

$$\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta \neq 0 \end{cases} \tag{38}$$

base on APPMCC, KCC and AKT. We use the popular receiver operating characteristic (ROC) curve as a figure of merit to compare the performances with respect to APPMCC, KCC and AKT.

FIGURE 3 illustrates the ROC curves of AKT, APPMCC and KCC, corresponding to three scenarios with $\rho \in \{0, 0.5, 0.9\}$, respectively, $n = 100$, $m = 4$, $\theta = 0.4$, $\rho' = 0.5$, and $\lambda_i = \lambda_j = 100$. We can observed that, when $\epsilon = 0$ (top row),

- APPMCC, having the highest ROC curves, performs the best,
- AKT and KCC, having almost identical ROC curves (only slightly lower than those of APPMCC), perform comparably well;

when $\epsilon = 0.06$ (bottom row),

- AKT, whose ROC curves sit on the top, performs the best,
- KCC, whose ROC curves lie in between those of AKT and APPMCC, performs the second,
- APPMCC, whose ROC curves are nearly diagonal, performs the worst.

D. CONCLUSION

In this work, we investigated systematically the statistical properties of the average Kendall's tau (AKT) under the multivariate contaminated Gaussian model (Theorems 1 and 2 as well as Corollaries 1 to 3). To gain further insights on AKT, we also compared AKT with other two concordance coefficients, namely, the average Pearson's product moment correlation coefficient (APPMCC), and Kendall's concordance coefficient (KCC) in terms of the root mean squared error (RMSE). Moreover, we also revealed the robustness of AKT against impulsive noise by an example of multi-channel signal detection, in terms of the receiver operating characteristic (ROC) curve. Theoretical derivations and experimental results suggest that

- under a specific multivariate contaminated Gaussian model, analytic expressions of the mean and variance of AKT are available,
- when the noise is pure Gaussian, APPMCC is optimal, in the context of multi-channel signal detection, whereas AKT and KCC perform comparably well,
- under the MCGM that simulates impulsive noise, APPMCC performs far poorly than AKT and KCC,

- both AKT and KCC are robust against impulsive noise, in terms of RMSE,
- under the MCGM, AKT performs the best in terms of the ROC curve.

Possessing the above advantages, AKT might play a complementary role to the popular APPMCC and KCC in the field of multivariate analysis, including multi-channel signal detection in the presence of impulsive noise.

APPENDIX A
PROOF OF THEOREM 1

Proof: The numerator $\mathcal{T}$ of (3) can be simplified to [26]

$$\mathcal{T} = 4\sum_{a\neq b=1}^{n}\sum^{n} H(X_a^i - X_b^i)H(X_a^j - X_b^j) - n^{[2]} \quad (39)$$

Then, by the i.i.d. assumption, the above equation becomes

$$\mathbb{E}(\mathcal{T}) = 4n^{[2]}\mathbb{E}[H(X_1^i - X_2^i)H(X_1^j - X_2^j)] - n^{[2]} \quad (40)$$

Denote by $\varphi(x_1^i, y_1^j, x_2^i, y_2^j)$ the joint distribution of $(X_1^i, Y_1^j, X_2^i, Y_2^j)$, which can be written as

$$\begin{aligned}\varphi &= [(1-\varepsilon)\phi_1 + \varepsilon\psi_1][(1-\varepsilon)\phi_2 + \varepsilon\psi_2]\\ &= \underbrace{(1-\varepsilon)^2}_{\alpha_1}\underbrace{\phi_1\phi_2}_{\varphi_1}+\underbrace{\varepsilon(1-\varepsilon)}_{\alpha_2}\underbrace{\phi_1\psi_2}_{\varphi_2}+\underbrace{\varepsilon(1-\varepsilon)}_{\alpha_3}\underbrace{\phi_2\psi_1}_{\varphi_3}+\underbrace{\varepsilon^2}_{\alpha_4}\underbrace{\psi_1\psi_2}_{\varphi_4}\end{aligned} \quad (41)$$

where φ is abbreviated symbol of $\varphi(x_1^i, y_1^j, x_2^i, y_2^j)$. Write

$$U^i \triangleq \frac{X_1^i - X_2^i}{\sqrt{\mathbb{V}(X_1^i - X_2^i)}} \quad (42)$$

$$U^j \triangleq \frac{X_1^j - X_2^j}{\sqrt{\mathbb{V}(X_1^j - X_2^j)}} \quad (43)$$

Then, with regard to φ_1, φ_2, φ_3, φ_4, (U^i, U^j) obeys four standard bivariate normal distributions with correlations

$$\varrho_1 = \rho_{ij} \quad (44)$$

$$\varrho_2 = \frac{\rho_{ij} + \lambda_i\lambda_j\rho'_{ij}}{\sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}} \to \rho'_{ij} \quad as\ \lambda_i, \lambda_j \to \infty \quad (45)$$

$$\varrho_3 = \frac{\rho_{ij} + \lambda_i\lambda_j\rho'_{ij}}{\sqrt{1+\lambda_i^2}\sqrt{1+\lambda_j^2}} \to \rho'_{ij} \quad as\ \lambda_i, \lambda_j \to \infty \quad (46)$$

$$\varrho_4 = \rho'_{ij} \quad (47)$$

Substituting (10) into (40) along with (44)−(47) gives

$$\begin{aligned}\mathbb{E}(\mathcal{T}) &= 4n^{[2]}\sum_{i=1}^{4}\alpha_i\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\varrho_i\right) - n^{[2]}\\ &= \frac{2n^{[2]}}{\pi}[\alpha_1\sin^{-1}\rho_{ij} + 2\alpha_2\sin^{-1}\varrho_2 + \alpha_4\sin^{-1}\rho'_{ij}]\end{aligned} \quad (48)$$

which yields

$$\mathbb{E}(\omega_{\mathcal{K}}) = \frac{2\sum_{i\neq j=1}^{m}\sum^{m}[\alpha_1\sin^{-1}\varrho_1 + 2\alpha_2\sin^{-1}\varrho_2 + \alpha_4\sin^{-1}\varrho_4]}{m(m-1)\pi} \quad (49)$$

and hence the statement of (14) holds true. □

APPENDIX B
PROOF OF THEOREM 2

Proof: Write

$$H_{ij} \triangleq \sum_{a\neq b=1}^{n}\sum^{n} H(X_a^i - X_b^i)H(X_a^j - X_b^j) \quad (50)$$

Then, according to the definition in (6) and statement (39), we have

$$\omega_{\mathcal{K}} = \frac{4}{m(m-1)n(n-1)}\sum_{i\neq j=1}^{m}\sum^{m} H_{ij} - 1 \quad (51)$$

which yields

$$\mathbb{V}(\omega_{\mathcal{K}}) = \frac{16}{m^2(m-1)^2n^2(n-1)^2}\mathbb{V}[\sum_{i\neq j=1}^{m}\sum^{m} H_{ij}] \quad (52)$$

By the relationship of variance and expectation, we have

$$\mathbb{V}[\sum_{i\neq j=1}^{m}\sum^{m} H_{ij}] = \mathbb{E}[(\sum_{i\neq j=1}^{m}\sum^{m} H_{ij})^2] - \mathbb{E}^2[\sum_{i\neq j=1}^{m}\sum^{m} H_{ij}] \quad (53)$$

The second term on the right in the above equation can be determined by Theorem 1, which can be written as

$$\mathbb{E}[\sum_{i\neq j=1}^{m}\sum^{m} H_{ij}] = \frac{m(m-1)n(n-1)}{4}[\mathbb{E}(\omega_{\mathcal{K}}) + 1] \quad (54)$$

we only need to calculate the first term. Due to the symmetry in all the summation, it follows that

$$\begin{aligned}\mathbb{E}[(\sum_{i\neq j=1}^{m}\sum^{m} H_{ij})^2] &= \sum_{i\neq j=1}^{m}\sum^{m}\sum_{k\neq l=1}^{m}\sum^{m}\mathbb{E}[H_{ij}H_{kl}]\\ &= \sum_{i\neq j\neq k\neq l=1}^{m}\sum^{m}\sum^{m}\sum^{m}\mathbb{E}[H_{ij}H_{kl}]\\ &\quad + 4\sum_{i\neq j\neq k=1}^{m}\sum^{m}\sum^{m}\mathbb{E}[H_{ij}H_{ik}]\\ &\quad + 2\sum_{i\neq j=1}^{m}\sum^{m}\mathbb{E}[H_{ij}^2]\end{aligned} \quad (55)$$

where

$$\begin{aligned}H_{ij}H_{kl} = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n}\Big[&H(X_a^i - X_b^i)H(X_a^j - X_b^j)\\ &\times H(X_c^k - X_d^k)H(X_c^l - X_d^l)\Big]\end{aligned} \quad (56)$$

$$H_{ij}H_{ik} = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n}\Big[H(X_a^i - X_b^i)H(X_a^j - X_b^j) \times H(X_c^i - X_d^i)H(X_d^k - X_d^k)\Big] \quad (57)$$

$$H_{ij}^2 = \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n}\Big[H(X_a^i - X_b^i)H(X_a^j - X_b^j) \times H(X_c^i - X_d^i)H(X_c^j - X_d^j)\Big]. \quad (58)$$

The quadruple summation in (56) can be decomposed into

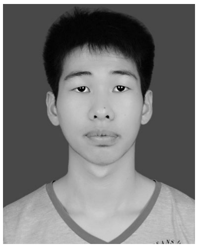
$$\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n} = \left[\varepsilon\sum_{a=1}^{n} + (1-\varepsilon)\sum_{a'=1}^{n}\right] \times \left[\varepsilon\sum_{b=1}^{n} + (1-\varepsilon)\sum_{b'=1}^{n}\right] \times \left[\varepsilon\sum_{c=1}^{n} + (1-\varepsilon)\sum_{c'=1}^{n}\right] \times \left[\varepsilon\sum_{d=1}^{n} + (1-\varepsilon)\sum_{d'=1}^{n}\right] \quad (59)$$

with the suffixes $(a\ b\ c\ d)$ being generated from the bivariate normal distribution ϕ, and suffixes $(a'\ b'\ c'\ d')$ from another bivariate normal distribution ψ.

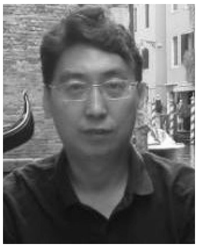
By expanding (59) according to different suffixes of $(a\ b\ c\ d)$ and $(a'\ b'\ c'\ d')$, we obtain 16 sub-quadruple summations, which can be further decomposed into 32 disjoint and exhaustive subsets. In other words, the statement (56) is a summation of 32 integrals of the form $\mathbb{E}(H(Z_1)H(Z_2)H(Z_3)H(Z_4))$, weighted by the corresponding constant factor β''_ℓ and subset cardinality α''_ℓ. Substituting the corresponding terms tabulated in TABLE 3 into (12), the expression of (56) can be easily obtained. In a similarly way, the expressions of (57) and (58) can also be obtained according to the results presented in TABLE 2 and TABLE 1, respectively. Combining the results (52)−(58) finally gives the statement (17). □

REFERENCES

[1] N. Kang, Z. Shang, and Q. Du, "Knowledge-aided structured covariance matrix estimator applied for radar sensor signal detection," *Sensors*, vol. 19, no. 3, p. 664, 2019.
[2] C. Chaux, L. Duval, A. Benazza-Benyahia, and J.-C. Pesquet, "A nonlinear stein-based estimator for multichannel image denoising," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3855–3870, Aug. 2007.
[3] M. M. Feraidooni, D. Gharavian, and S. Imani, "Signal-dependent interference reduction based on a new transmit covariance matrix and receive filter design for colocated MIMO radar," *Signal Image Video Process.*, vol. 13, no. 7, pp. 1275–1282, 2019.
[4] X. Xu, Z. Ye, and Y. Zhang, "DOA estimation for mixed signals in the presence of mutual coupling," *IEEE Trans. Signal Process.*, vol. 57, no. 9, pp. 3523–3532, Sep. 2009.
[5] J. D. Gibbons and S. Chakraborti, *Nonparametric Statistical Inference*, 5th ed. Boca Raton, FL, USA: CRC Press, 2010.
[6] D. D. Mari and S. Kotz, *Correlation and Dependence*. Singapore, World Scientific, 2001.
[7] W. Xu, Z. Chen, Y. Zhang, and L. Cheng, "Order statistics concordance coefficient with applications to multichannel Biosignal analysis," *IEEE J. Biomed. Health Inform.*, vol. 21, no. 5, pp. 1206–1215, Sep. 2016.
[8] S. Lim, H. Jung, and B. J. Jeong, "Efficient multi-channel signal detection algorithms for cognitive radio systems in TV white space," in *Proc. IEEE Int. Symp. World Wireless Multimedia Netw. (WoWMoM)*, Jun. 2012, pp. 1–5.
[9] M. Karimzadeh, A. M. Rabiei, and A. Olfat, "Soft-limited polarity-coincidence-array spectrum sensing in the presence of non-Gaussian noise," *IEEE Trans. Veh. Technol.*, vol. 66, no. 2, pp. 1418–1427, Feb. 2017.
[10] M. Y. Manesh and A. Olfat, "Sigmoid function detector in the Presence of heavy-tailed noise for multiple antenna cognitive radio networks," in *Proc. IEEE Int. Conf. Commun. (ICC)*, May 2017, pp. 1–5.
[11] R. A. Fisher, "Statistical methods, experimental design, and scientific inference," *Biometrics*, vol. 47, p. 1206, Sep. 1991.
[12] M. G. Kendall and B. B. Smith, "The problem of *m* rankings," *Ann. Math. Statist.*, vol. 10, no. 3, pp. 275–287 1939.
[13] J. Teles, "Concordance coefficients to measure the agreement among several sets of ranks," *J. Appl. Statist.*, vol. 39, no. 8, pp. 1749–1764, 2012.
[14] C. Genest, J. Nešlehová, and N. B. Ghorbal, "Estimators based on Kendall'S tau in multivariate copula models," *Austral. & New Zealand J. Statist.*, vol. 53, no. 2, pp. 157–177, 2011.
[15] H. El Maache and Y. Lepage, *Spearman's Rho and kendall's Tau for Multivariate Data Sets* (Lecture Notes-Monograph Series), vol. 42. 2003, pp. 113–130.
[16] K. C. Garwood and A. G. Strandberg, "Measuring agreement among ranks: Sustainability application," *Int. J. Bus. Intell. Res.*, vol. 7, pp. 45–62, Jan. 2016.
[17] M. Alvo and P. Cabilio, "A comparison of approximations to the distribution of average kendall tau," *Commun. Statist.—Theory Methods*, vol. 13, no. 25, pp. 3191–3216, 1984.
[18] W. Lee and J. Y. Ahn, "On the application of multivariate Kendall's tau and its interpretation," *Korean J. Appl. Statist.*, vol. 26, no. 3, pp. 495–509, 2013.
[19] W. L. Hays, "A note on average tau as a measure of concordance," *J. Amer. Stat. Assoc.*, vol. 55, no. 290, pp. 331–341, 1960.
[20] M. Raghavachari, "Measures of concordance for assessing agreement," in *Ratings and Rank Order Data*. Boston, MA, USA: Birkhäuser, 2005, pp. 245–263.
[21] J. Wen, X. Zhou, B. Liao, C. Guo, and S.-C. Chan, "Adaptive beamforming in an impulsive noise environment using matrix completion," *IEEE Commun. Lett.*, vol. 22, no. 4, pp. 768–771, Apr. 2018.
[22] V. C. Gogineni and S. Mula, "Proportionate adaptive filtering under Correntropy criterion in impulsive noise Environments," 2017, *arXiv:1707.00315*. [Online]. Available: https://arxiv.org/abs/1707.00315
[23] Z. Reznic, R. Zamir, and M. Feder, "Joint source-channel coding of a Gaussian mixture source over the Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 776–781, Mar. 2002.
[24] B. C. Y. Wong and I. F. Blake, "Detection in multivariate non-Gaussian noise," *IEEE Trans. Commun.*, vol. 42, no. 234, pp. 1672–1683, Feb. 1994.
[25] P. Willett and J. B. Thomas, "Mixture models for underwater burst noise and their relationship to a simple bivariate density representation," *IEEE J. Ocean. Eng.*, vol. OE-12, no. 1, pp. 29–37, Jan. 1987.
[26] W. Xu, Y. Hou, Y. S. Hung, and Y. Zou, "A comparative analysis of Spearman's rho and Kendall's tau in normal and contaminated normal models," *Signal Process.*, vol. 93, no. 1, pp. 261–276, 2013.
[27] R. Ma, W. Xu, Q. Wang, and W. Chen, "Robustness analysis of three classical correlation coefficients under contaminated Gaussian model," *Signal Process.*, vol. 104, pp. 51–58, Nov. 2014.
[28] W. Xu, C. Chen, J. Dai, Y. Zhou, and Y. Zhang, "Detection of known signals in additive impulsive noise based on Spearman's rho and Kendall's tau," *Signal Process.*, vol. 161, pp. 165–179, Aug. 2019.
[29] S. S. Wilks, "Order statistics," *Bull. Amer. Math. Soc.*, vol. 54, no. 1, pp. 6–50,1948.
[30] M. Kendall and J. D. Gibbons, *Rank Correlation Methods*, 5th ed. New York, NY, USA: Oxford Univ. Press, 1990.
[31] M. Kendall and A. Stuart, *The Advanced Theory of Statistics, Vol. 1: Distribution Theory*, 4th ed. London, U.K.: Griffin, 1977.
[32] S. S. Gupta, "Probability integrals of multivariate normal and multivariate t^1," *Ann. Math. Statist.*, vol. 34, no. 3, pp. 792–828, 1963.
[33] D. R. Childs, "Reduction of the multivariate normal integral to characteristic form," *Biometrika*, vol. 54, nos. 1–2, pp. 293–300, 1967.



HUADONG LAI received the B.Eng. degree in automation from the School of Automation, Guangdong University of Technology, Guangzhou, China, in 2018, where he is currently pursuing the M.S. degree. His research interests include statistical signal processing, machine learning, and wireless communications.



WEICHAO XU (M'06) received the B.Eng. and M.Eng. degrees in electrical engineering from the University of Science and Technology of China, Hefei, in 1993 and 1996, respectively, and the Ph.D. degree in biomedical engineering from The University of Hong Kong, Hong Kong, in 2002. He was a Research Associate with the Department of Electrical and Electronic Engineering, The University of Hong Kong, from 2002 to 2010. In 2011, he entered the 100-Talent Scheme with the Guangdong University of Technology, where he is currently a Professor. His research interests include mathematical statistics, computational statistics, information theory, machine learning, optimization theory, and digital signal processing and applications.