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The Extension of Bisimulation Quantified Modal Logic Based on Covariant-Contravariant Refinement

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ABSTRACT The notion of covariant-contravariant refinement (CC-refinement, for short) is a generalization of the notions of bisimulation and refinement. This paper interprets semantically a CC-refinement as bisimulation plus model restriction, that is, a CC-refinement model of a given model may be obtained from one bisimilar duplicate of this model by adding some transitions labelled by covariant actions and removing some transitions labelled by contravariant actions. By using certain proposition letter to witness a contravariant action, the standard bisimulation quantified modal logic is able to capture the characterization of this action, however, this fails for covariant actions. This paper, based on the notion of CC-refinement, introduces an extended bisimulation quantified modal logic with the universal modality $\blacksquare$ (EBQML$_\blacksquare$), describes syntactically CC-refinement quantification as the extended bisimulation quantification plus relativization, and establishes a translation from the language of CC-refinement modal μ-calculus to the language of EBQML$_\blacksquare$ such that every CC-refinement modal μ-formula is equivalent to its translation. The language of EBQML$_\blacksquare$ may be considered as a specification language for describing the properties of a system referring to reactive and generative actions, which are represented respectively by covariant and contravariant actions, and may be used to formalize some interesting problems in the field of formal methods.

INDEX TERMS Bisimulation quantification, modal logic, covariant-contravariant refinement modal μ-calculus, relativization.

I. INTRODUCTION

A number of different compatible relations between labelled transitions systems (LTSs) have been presented in the literature (see [1], [2]), which are adopted to capture the behaviour relations between processes. Among them, the notion of covariant-contravariant refinement (CC-refinement, for short), which generalizes the notions of bisimulation and refinement considered in [1], is often used to describe the refinement relations between systems referring to reactive (passive) and generative (active) actions (e.g., input/output (I/O) automata) [2]–[5]. The notion of CC-refinement partitions all actions into three sorts: covariant actions which capture the passive actions of a system; contravariant actions which represent the generative actions; and bivariant actions which are treated as in the usual notion of bisimulation. The transitions labelled with covariant actions in a given specification should be simulated by any correct implementation and the transitions for contravariant actions in an implementation must be allowed by its specification.

The notion of bisimulation is a basic one among the coinductively defined notions of behavior relations between systems. It is more easily understood and realized. Hence these behavior relations are often analyzed under the situation of bisimulation. In the notion of bisimulation, related states are required to satisfy the same propositional properties and have matching transition possibilities. Through weakening such propositional requirement, there is a natural way of approximating this notion. For example, given a subset P of the propositional properties, two systems are said to be P-restricted bisimilar whenever related states satisfy the same propositional properties except the ones in P and have matching transition possibilities. Bisimulation quantifier, a kind of non-standard propositional quantifier, is presented to quantify over all P-restricted bisimulations of a given system. This kind of quantifier is very interesting and useful. It was first introduced in [6] for intuitionistic propositional logic, and then investigated as a tool to prove uniform interpolation

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for modal logic and modal μ-calculus (see, e.g., [7]–[10]). Moreover, bisimulation quantifiers are useful in formal analysis, design and development of dynamic systems. For example, Kripke structures may be considered as the models of optimization problems in control theory and planning theory. A Kripke model is an LTS with its states labelled by propositional properties, that is, it consists of a set of states labelled by propositional properties and relations between these states. In this situation, the propositional properties in P are not as important as the ones not in P, and hence two P-restricted bisimilar Kripke models may be considered to be "equivalent".

In [11], Bozzano et al. presented and explored the first formal method to the design of Fault Detection and Identification (FDI) components for discrete event systems, which is available in autonomous critical systems, such as satellites and space rovers. This method uses LTSs as the models of systems. If the states in an LTS are labelled with all possible environmental properties affecting FDI, then FDI may work under different environments, that is, we could neglect those properties which are not important under some environment and try to search one bisimilar system except for those properties as desired.

Bisimulation quantified modal logic (BQML) is an extension of the modal system $\boldsymbol{K}$ with bisimulation quantifier $\exists_p$ where p is a proposition letter [12]. Given a set *Atom* of proposition letters, a formula $\exists_p\alpha$ is true in a pointed model M_u if there are a pointed model N_v satisfying α and a $\{p\}$-restricted bisimulation linking M_u and N_v. The modal μ-calculus augments the standard modal logic with the least and greatest fixed-point operators of monotone operators [13]. This gives a significant increase in expressive power, however μ-calculus formulas are hard to understand and it is also complex to construct their models. The modal μ-calculus is often used to describe some properties of a system in model checking. It is known that BQML is equivalent to the modal μ-calculus, which implies that BQML is able to express any monadic second-order property which is invariant under bisimulation [8]. More work on the expressivity and decidability of BQML may be found in [12], [14]. D'Agostino and Lenzi have given a sound and complete axiomatization for BQML via the modal μ-calculus in [15].

BQML focuses on reasoning and formalizing of the properties such as "there exists a bisimilar model, except for a proposition letter p, which satisfies φ". In such cases, a system is modeled using Kripke-structures and its properties will be expressed by the ones which are invariant under $\{p\}$-restricted bisimulation. Note that, here, we are interested in the system more than any model representing it. In other words, given a system S_1 presented as a Kripke model M, we are interested in whether there is some system S_2 differing from S_1 only in the propositional property p, which satisfies some property φ. Hence, in interpreting $\exists_p$, it is natural to consider all the interpretations of the propositional property p in all the models bisimilar to M.

The notion of simulation (refinement) is able to describe the refinement relations between reactive systems. Based on this notion, Laura Bozzelli et al. recently presented and explored refinement modal μ-calculus (RML$^\mu$) [1], [16], which contains a refinement operator (or, quantifier) $\exists_B$ where B is a set of actions, in addition to usual modal operators and fixed-point operators. The formula $\exists_B\psi$ intuitively expresses that we can refine the current model so as to realize ψ. In [1], Laura Bozzelli et al. semantically interpreted that a B-refinement of a given model can be obtained from a bisimilar duplicate of this model by deleting some transitions labelled by the actions in B. Further, concern proposition letters can be used to witness those desired transitions labelled by the actions in B. They have shown that refinement quantification can be seen as bisimulation quantification plus relativization, by defining an equivalent translation from the language of RML$^\mu$ to the language $\mathcal{L}_{bq\blacksquare}$ of BQML with the universal modality $\blacksquare$, that is, each refinement modal μ-formula is equivalent to its translation. The universal modality $\blacksquare$ and its duality $\blacklozenge$, also called master modality [12], quantify over all the accessible states from the actual state in a given model. This translation applies a bisimulation quantifier characterization of fixed-points by employing $\blacksquare$, which is given in [12, Lemma 2.43]. With the help of this translation, Laura Bozzelli et al. established the soundness of the presented axiom system for RML$^\mu$. The language $\mathcal{L}_{bq\blacksquare}$ may be considered as a specification language for describing the properties of reactive systems.

It is well known that the result of executing an epistemic action in a pointed model is a refinement of that model, and dually, for every refinement of a finite pointed model there is an epistemic action such that the result of its execution in that pointed model is a model bisimilar to the refinement [17]. In [18], it has been shown that a product update by an action model can decompose in copy and remove operations. This indeed corresponds to the semantical interpretation that "a B-refinement of a given model can be obtained from a bisimilar duplicate of this model by deleting some transitions labelled by the actions in B". It is easy to see that this kind of copy and remove operations are easily to realize by programming.

Following Laura Bozzelli et al's work, we considered CC-refinement modal logic (CCRML) in [19], which is obtained from the modal system $\boldsymbol{K}$ by adding CC-refinement operator $\exists_{(A_1,A_2)}$ where A_1 (A_2) is a set of all covariant (contravariant, resp.) actions. In this paper, we will investigate its extension with fixed-point operators: CC-refinement modal μ-calculus (CCRML$^\mu$). Intuitively, the formula $\exists_{(A_1,A_2)}\psi$ represents that we can refine the current model so that ψ is realized. Thus, given a specification expressed by a Kripke model M which involves passive and generative actions, the problem whether this specification has an implementation realizing some given property ψ may be formalized as the model checking problem: whether $\exists_{(A_1,A_2)}\psi$ holds in M.

From the above introduction, we know that the language $\mathcal{L}_{bq\blacksquare}$ can perfectly describe the characterizations of contravariant actions. However, unfortunately, this is impossible for covariant actions, which results that there is no translation from the language $\mathcal{L}^{\mu}_{(A_1,A_2)}$ of CCRML$^{\mu}$ to $\mathcal{L}_{bq\blacksquare}$ such that it is defined inductively and every $\mathcal{L}^{\mu}_{(A_1,A_2)}$-formula is equivalent to its translation whenever $A_1 \neq \emptyset$. In this paper, to remedy this, we present an extended bisimulation quantified modal logic with the universal modality (EBQML$_\blacksquare$), then define a relativization function in its language $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$, and by employing this relativization, establish an equivalent translation from $\mathcal{L}^{\mu}_{(A_1,A_2)}$ to $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$.

This paper is organized as follows. The next section presents CC-refinement modal μ-calculus and recalls the standard bisimulation quantified modal logic. Section 3 introduces the extended bisimulation quantified modal logic based on the notion of CC-refinement. Section 4 interprets semantically a CC-refinement as bisimulation plus model restriction. Section 5 establishes an equivalent translation from the language of refinement modal μ-calculus to the language of the extended bisimulation quantified modal logic with the universal modality. Finally Section 6 ends the paper with a brief discussion.

II. PRELIMINARY NOTIONS

In this section, we recall the notion of CC-refinement [2] and bisimulation quantified modal logic (BQML) [1], [12], where we refer to the notations used in [1], and present CC-refinement modal μ-calculus (CCRML$^{\mu}$).

Let A be a finite set of actions, and let $Atom$ be a countable set of proposition letters.

A. MODEL

Definition 1 (Kripke model): A Kripke model M is a triple $\langle S^M, R^M, V^M\rangle$ where

(1) *S^M is a non-empty set of states,*

(2) *$R^M : A \to 2^{S^M\times S^M}$ is an accessibility function which assigns a binary relation $R^M_b \subseteq S^M \times S^M$ to each action $b \in A$, and*

(3) *$V^M : Atom \to 2^{S^M}$ is a valuation function. For each $p \in Atom$, $V^M(p)$ is the set of states in M where p is true.*

A pair (M, u) with $u \in S^M$ is said to be a pointed Kripke model, often written as M_u.

In the following, we give a number of useful notations. For any binary relation R, set T and s, we define that:

$R(s) \triangleq \{v \mid sRv\}$,

$R(T) \triangleq \bigcup_{z\in T} R(z)$,

$\pi_1(R) \triangleq \{u \mid \exists w(uRw)\}$,

$\pi_2(R) \triangleq \{w \mid \exists u(uRw)\}$,

R^+ expresses the transitive closure of R, and

R^* expresses the reflexive and transitive closure of R.

For any model M,

$R^+_M \triangleq (\bigcup_{a\in A} R^M_a)^+$ and

$R^*_M \triangleq (\bigcup_{a\in A} R^M_a)^*$.

Given a model M, $a \in A$, $p \in Atom$, $R : A \to 2^{S^M\times S^M}$, $D \subseteq S^M \times S^M - R^M_a$ and $S \subseteq S^M$, the models $M \mid R$, $M + (a, D)$, $M \mid (a, S)$ and $M \mid (a, p)$ are defined as follows:

- $M \mid R \triangleq \langle S^M, R, V^M\rangle$
- $M + (a, D) \triangleq \langle S^M, R', V^M\rangle$, where $R'_b \triangleq R^M_b$ for all $b \in A - \{a\}$ and $R'_a \triangleq R^M_a \cup D$
- $M \mid (a, S) \triangleq \langle S^M, R', V^M\rangle$, where $R'_b \triangleq R^M_b$ for all $b \in A - \{a\}$ and $R'_a \triangleq R^M_a \cap (S^M \times S)$
- $M \mid (a, p) \triangleq M \mid (a, V^M(p))$.

As usual, we use the following notations:

- $\circ$ denotes the composition operator of relations,
- $\mathfrak{i}_{C,C'}$ with $C \subseteq C'$ indicates the graph of the inclusion function from C to C', that is, $\mathfrak{i}_{C,C'} \triangleq \{\langle b, b\rangle \mid b \in C \subseteq C'\}$, and
- $M \uplus N$ expresses the disjoint union of two models M and N such that $S^M \cap S^N = \emptyset$, which is defined by $S^{M\uplus N} \triangleq S^M \cup S^N$, $R^{M\uplus N}_b \triangleq R^M_b \cup R^N_b$ for each $b \in A$ and $V^{M\uplus N}(q) \triangleq V^M(q) \cup V^N(q)$ for each $q \in Atom$.

B. CC-REFINEMENT

Definition 2 (CC-refinement [2]): Let A_1, $A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Given two models M and N, a non-empty binary relation $\mathcal{Z} \subseteq S^M \times S^N$ is an (A_1, A_2)-refinement relation between M and N if, for every pair $\langle u, v\rangle$ in $\mathcal{Z}$, we have

(atoms) *$u \in V^M(q)$ iff $v \in V^N(q)$ for each $q \in Atom$;*

(forth) *for each $b \in A - A_2$ and $u' \in S^M$, $uR^M_b u'$ implies $v R^N_b v'$ and $u'\mathcal{Z}v'$ for some $v' \in S^N$;*

(back) *for each $b \in A - A_1$ and $v' \in S^N$, $v R^N_b v'$ implies $uR^M_b u'$ and $u'\mathcal{Z}v'$ for some $u' \in S^M$.*

*Here A_1 and A_2 are said to be **covariant** and **contravariant set** respectively. We say that N_v (A_1, A_2)-refines M_u (or, M_u (A_1, A_2)-simulates N_v), in symbols $M_u \succeq_{(A_1,A_2)} N_v$, if there exists an (A_1, A_2)-refinement relation between M and N linking u and v. We also write $\mathcal{Z} : M_u \succeq_{(A_1,A_2)} N_v$ to indicate that $\mathcal{Z}$ is an (A_1, A_2)-refinement relation such that $u\mathcal{Z}v$.*

The above notion generalizes the notions of ***bisimulation*** and ***refinement*** considered in [1]. Formally, a bisimulation relation is exactly an $(\emptyset, \emptyset)$-refinement, and a B-refinement relation an $(\emptyset, B)$-refinement. We write $\mathcal{Z} : M_u \underline{\leftrightarrow} N_v$ to represent that $\mathcal{Z}$ is a bisimulation witnessing that M_u is bisimilar to N_v. Given $P \subseteq Atom$, a binary relation $\mathcal{Z}$ is said to be a P-restricted bisimulation, in symbols $\mathcal{Z} : M \underline{\leftrightarrow}^P N$, if the bisimulation conditions (**forth**) and (**back**) are satisfied, and (**atoms**) holds whenever the set of proposition letters is reduced to $Atom - P$. If P is finite, say $P = \{p_1, \cdots, p_n\}$, we often write $M \underline{\leftrightarrow}^{p_1,\cdots,p_n} N$ instead of $M \underline{\leftrightarrow}^P N$.

Example 3: Consider two models M and N depicted in Figure 1, where $A_1 = \{a\}$, $A_2 = \{b\}$, and $V^M(q) = \emptyset$ and $V^N(q) = \emptyset$ for each $q \in Atom$. It is not difficult to see that the relation represented by the dash arrows is an (A_1, A_2)-refinement relation between M_s and N_{s_1}.

Proposition 4 [19]: Let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Then, for each A'_1, A''_1, A'_2 and A''_2 such that $A'_1 \cup A''_1 = A_1$ and

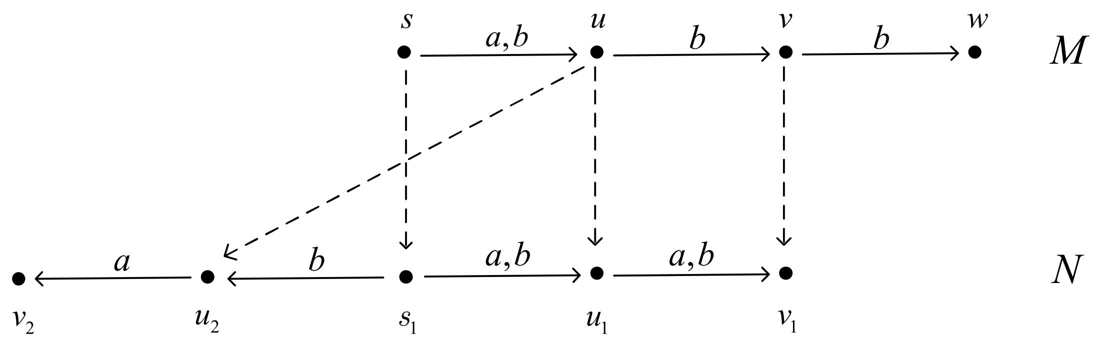

FIGURE 1. An ({*a*}, {*b*})-refinement between two given models *M* and *N*.

$A_2' \cup A_2'' = A_2$, *it holds that*

$$\succeq_{(A_1',A_2')} \circ \succeq_{(A_1'',A_2'')} = \succeq_{(A_1,A_2)} .$$

Convention: By Proposition 4, any CC-refinement may be captured by the CC-refinements with singleton covariant and contravariant sets (More information about this may be found in [19]). Hence, in the remainder of the paper, we focus on singleton covariant and contravariant sets.

C. CC-REFINEMENT MODAL μ-CALCULUS

In this subsection, we present CC-refinement modal μ-calculus (CCRML$^\mu$), which is obtained from the standard modal μ-calculus by adding CC-refinement quantifiers.

Definition 5 (Language $\mathcal{L}^{\mu}_{(A_1,A_2)}$): Let Var be a set of variables. The language $\mathcal{L}^{\mu}_{(A_1,A_2)}$ of CC-refinement modal μ-calculus is generated by the BNF grammar below, where $A_1,\ A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$, $b \in A$, $q \in Atom$ and $x \in Var$:

$$\varphi ::= x \mid q \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_b\varphi \mid \exists_{(A_1,A_2)}\varphi \mid \mu x.\varphi$$

The fixed-point variable x is bounded in $\mu x.\varphi$ and is required to occur positively in φ (namely occur only in the scope of even number of negations). The modal operator $\Diamond_b$ and propositional connectives $\bot$, $\top$, $\vee$, $\leftrightarrow$ and $\to$ are defined in the standard manner. Moreover, we write $\forall_{(A_1,A_2)}\varphi$ for $\neg\exists_{(A_1,A_2)}\neg\varphi$, and $\nu x.\varphi$ for $\neg\mu x.\neg\varphi[\neg x\backslash x]$.

In the above definition, since $\vee$-clause and $\wedge$-clause are dual, it is available to write any of them as a primary clause. In this paper, we choose the latter one as a primary clause for all BNF grammars, whose reason will be discussed in the proof of Proposition 11.

The fragment of $\mathcal{L}^{\mu}_{(A_1,A_2)}$ involving no fixed-point operator is indeed the language $\mathcal{L}_{(A_1,A_2)}$ of CC-refinement modal logic [19], and $\mathcal{L}^{\mu}_{(\emptyset,A_2)}$ is indeed the language of RML$^\mu$ [1]. If A_1 is singleton, say $A_1 = \{a_1\}$, we write $\exists_{(a_1,A_2)}\varphi$ (or $\forall_{(a_1,A_2)}\varphi$) instead of $\exists_{(A_1,A_2)}\varphi$ (resp., $\forall_{(A_1,A_2)}\varphi$), and similar if A_2 is singleton or both A_1 and A_2 are singleton.

Convention: To save the space, we shall write '**iff**' instead of 'if and only if'. Given the statements: $S_1, \cdots, S_n$, whenever S_1 if and only if S_2, and S_2 if and only if S_3, $\cdots$, and S_{n-1} if and only if S_n, we shall write 'S_1 iff S_2 iff $\cdots$ iff S_n' to ease the expression.

Given a model M, the notion of a formula $\psi \in \mathcal{L}^{\mu}_{(A_1,A_2)}$ being satisfied in M at a state u is defined inductively as follows:

$M_u \models q$	iff	$u \in V^M(q)$, where $q \in Atom$
$M_u \models \neg\varphi$	iff	$M_u \not\models \varphi$
$M_u \models \varphi_1 \wedge \varphi_2$	iff	$M_u \models \varphi_1$ and $M_u \models \varphi_2$
$M_u \models \Box_b\varphi$	iff	$M_v \models \varphi$ for all $v \in R^M_b(u)$
$M_u \models \exists_{(A_1,A_2)}\varphi$	iff	$M_u \succeq_{(A_1,A_2)} N_v$ and $N_v \models \varphi$ for some N_v
$M_u \models \mu x.\varphi$	iff	$u \in \bigcap\{T \subseteq S^M : \Vert\varphi(x)\Vert^M_{[x\mapsto T]} \subseteq T\}$
$M_u \models \nu x.\varphi$	iff	$u \in \bigcup\{T \subseteq S^M : \Vert\varphi(x)\Vert^M_{[x\mapsto T]} \supseteq T\}$

Here, $\Vert\varphi(x)\Vert^M_{[x\mapsto T]} \triangleq \{w \in S^M : M^{[x\mapsto T]}_w \models \varphi(x)\}$ and the model $M^{[x\mapsto T]}$ is obtained from M by setting

$$V^{M^{[x\mapsto T]}}(r) \triangleq \begin{cases} V^M(r) & \text{if } r \neq x \\ T & \text{if } r = x \end{cases}$$

The semantics of $\mu x.\varphi$ ($\nu x.\varphi$) clause captures exactly the fact that the least (greatest, resp.) fixed-point is the intersection (union, resp.) of all the prefixed (postfixed, resp.) points. We can see [20], [21] for more information about the modal μ-calculus.

As usual, a formula $\alpha \in \mathcal{L}^{\mu}_{(A_1,A_2)}$ is valid, denoted by $\models \alpha$, if $M_u \models \alpha$ for each pointed model M_u. It is easy to see that P-restricted bisimulation preserves the satisfiability of $\mathcal{L}^{\mu}_{(A_1,A_2)}$-formulas containing no proposition letter from P.

Proposition 6: Let $M_u \underline{\leftrightarrow}^P N_v$ and $\varphi \in \mathcal{L}^{\mu}_{(A_1,A_2)}$ such that $p \notin \varphi$ for all $p \in P$. Then

$$M_u \models \varphi \ \textit{iff}\ N_v \models \varphi.$$

Proof: By the induction on φ. □

D. BISIMULATION QUANTIFIED MODAL LOGIC

Now we recall the language $\mathcal{L}_{bq}$ of BQML which augments the standard modal language $\mathcal{L}_K$ by adding bisimulation quantifier, and its version with the universal modality $\mathcal{L}_{bq\blacksquare}$ [1] (also refer to [12, Section 2.3], in which universal modality is called master modality).

Definition 7 (Language $\mathcal{L}_{bq\blacksquare}$): The language $\mathcal{L}_{bq\blacksquare}$ of bisimulation quantified modal logic with the universal modality $\blacksquare$ is defined by the BNF grammar below, where $b \in A$ and $p, q \in Atom$:

$$\varphi ::= q \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \Box_b\varphi \mid \blacksquare\varphi \mid \widetilde{\exists}_p\varphi$$

The dual $\widetilde{\forall}_p\varphi$ is an abbreviation for $\neg\widetilde{\exists}_p\neg\varphi$, and $\blacklozenge\varphi$ for $\neg\blacksquare\neg\varphi$. The clause $\widetilde{\exists}_p\varphi$ and $\blacksquare\varphi$ are interpreted as:

$$M_u \models \widetilde{\exists}_p\varphi \text{ iff } M_u \underline{\leftrightarrow}^P N_v \text{ and } N_v \models \varphi \text{ for some } N_v$$
$$M_u \models \blacksquare\varphi \text{ iff } M_v \models \varphi \text{ for all } v \in R^*_M(u)$$

Due to the duality, it is available to choose $\blacksquare\varphi$ or $\blacklozenge\varphi$ as a primary clause in the above BNF. Here, we choose the former because the operator $\blacksquare$ is used more often in Section V. Moreover, we write $\widetilde{\exists}$ and $\widetilde{\forall}$ for the bisimulation quantifiers in order to distinguish them from the CC-refinement quantifiers $\exists$ and $\forall$, referring to the notations used in [1].

Also, a formula $\alpha \in \mathcal{L}_{bq\blacksquare}$ is valid, denoted by $\models \alpha$, if $M_u \models \alpha$ for each pointed model M_u. P-restricted bisimulation also preserves the satisfiability of $\mathcal{L}_{bq\blacksquare}$-formulas containing no proposition letter from P.

Proposition 8: Let $M_u \underline{\leftrightarrow}^P N_v$ and $\varphi \in \mathcal{L}_{bq\blacksquare}$ such that $p \notin \varphi$ for all $p \in P$. Then

$$M_u \models \varphi \text{ iff } N_v \models \varphi.$$

Proof: By the induction on φ. □

III. THE EXTENDED BISIMULATION QUANTIFIED MODAL LOGIC

This section presents the extended bisimulation quantified modal logic (EBQML) and introduces its useful properties.

A. PRESENTING MOTIVATION

In [1], a relativization $\bullet^{(a,p)} : \mathcal{L}_{bq} \to \mathcal{L}_{bq}$ to the proposition letter p for the action a was presented, with the help of which, a certain proposition letter may be used to witness a contravariant action, and then every refinement formula is translated into an equivalent $\mathcal{L}_{bq}$-formula. However, there is, unfortunately, no translation $t : \mathcal{L}_{(A_1,A_2)} \to \mathcal{L}_{bq}$ such that $\models t(\psi) \leftrightarrow \psi$ for each $\mathcal{L}_{(A_1,A_2)}$-formula ψ whenever $A_1 \neq \emptyset$. The key reason is that $\mathcal{L}_{bq}$ can not describe perfectly the characterizations of covariant actions. The detailed proof will be given below. Hence, we will intend to explore an extended version of $\mathcal{L}_{bq}$ to remedy this in this paper.

To prove the statement mentioned in the above paragraph, Proposition 10 and Proposition 11 are needed to simplify its proof. Firstly, we give an auxiliary notion, which is regarded as a syntactic entity that transforms formulas to formulas.

Definition 9 (Context): A context of $\mathcal{L}_{bq}$ is obtained by the following:

$$F ::= \varphi \mid \text{-} \mid \neg F \mid (F_1 \wedge F_2) \mid \Box_b F \mid \widetilde{\exists}_p F$$

where $b \in A$, $p \in Atom$ and $\varphi \in \mathcal{L}_{bq}$.

If F is a context and α is a $\mathcal{L}_{bq}$-formula, we write $F(\alpha)$ for the formula obtained by replacing the '-' in F with α.

Proposition 10: Given a context F(-) in $\mathcal{L}_{bq}$ and $p \in Atom$ such that $p \notin F$, we have that

$$M_s \models F(p) \text{ iff } M_s \models F(\bot), \quad \text{whenever } V^M(p) = \emptyset.$$

Proof: Let M_s be a pointed model with $V^M(p) = \emptyset$. We proceed by the induction on F.

For $F(\text{-}) \equiv \varphi$ with $\varphi \in \mathcal{L}_{bq}$, we have $F(p) = F(\bot) = \varphi$.

For $F(\text{-}) \equiv \text{-}$, $M_s \not\models p$ follows from $V^M(p) = \emptyset$.

For $F(\text{-}) \equiv \neg F'(\text{-})$, $M_s \models \neg F'(p)$ iff $M_s \not\models F'(p)$ iff $M_s \not\models F'(\bot)$ by the induction hypothesis iff $M_s \models \neg F'(\bot)$.

For $F(\text{-}) \equiv F_1(\text{-}) \wedge F_2(\text{-})$, we have $M_s \models F_1(p) \wedge F_2(p)$ iff $M_s \models F_1(p)$ and $M_s \models F_2(p)$ iff $M_s \models F_1(\bot)$ and $M_s \models F_2(\bot)$ by the induction hypothesis iff $M_s \models F_1(\bot) \wedge F_2(\bot)$.

For $F(\text{-}) \equiv \Box_b F'(\text{-})$, $M_s \models \Box_b F'(p)$ iff $M_u \models F'(p)$ for each $u \in R^M_b(s)$ iff $M_u \models F'(\bot)$ for each $u \in R^M_b(s)$ by the induction hypothesis iff $M_s \models \Box_b F'(\bot)$.

For $F(\text{-}) \equiv \widetilde{\exists}_r F'(\text{-})$, we have that $M_s \models \widetilde{\exists}_r F'(p)$ iff $M_s \underline{\leftrightarrow}^r N_w$ and $N_w \models F'(p)$ for some N_w. Let N'_w be the w-generated submodel of N. Then

$$M_s \underline{\leftrightarrow}^r N_w \underline{\leftrightarrow} N'_w$$

and next $N'_w \models F'(p)$ due to $N_w \models F'(p)$ and Proposition 8. Below, we check that $V^{N'}(p) = \emptyset$. Let $v \in S^{N'}$. Clearly,

$$\exists b_1, \cdots, b_n \in A(w \xrightarrow{b_1} v_1 \cdots \xrightarrow{b_n} v_n = v).$$

Because of $M_s \underline{\leftrightarrow}^r N'_w$,

$$\exists u_1, \cdots, u_n \in S^M(s \xrightarrow{b_1} u_1 \cdots \xrightarrow{b_n} u_n)$$

such that $M_{u_i} \underline{\leftrightarrow}^r N'_{v_i}$ $(1 \leq i \leq n)$. Further, since $r \neq p$ due to $p \notin F$, by the condition (**atoms**), $u_n \in V^M(p)$ if and only if $v \in V^{N'}(p)$. Hence, we get that $V^{N'}(p) = \emptyset$ due to $V^M(p) = \emptyset$. Then, by the induction hypothesis, from $N'_w \models F'(p)$, it follows that $M_s \underline{\leftrightarrow}^r N'_w \models F'(\bot)$, that is $M_s \models \widetilde{\exists}_r F'(\bot)$. The converse implication can be proved similarly. □

Proposition 11: For every context F(-) in $\mathcal{L}_{bq}$, we have that

$$\models F(p) \text{ for all } p \in Atom \text{ implies } \models F(\bot).$$

Proof: Suppose that F(-) is a context in $\mathcal{L}_{bq}$ and $\models F(p)$ for each $p \in Atom$. Let M_s be an arbitrary pointed model. We proceed by the induction on F.

For $F(\text{-}) \equiv \varphi$ with $\varphi \in \mathcal{L}_{bq}$, clearly $F(p) = F(\bot) = \varphi$.

For $F(\text{-}) \equiv \text{-}$, it holds trivially since $\not\models p$ for all $p \in Atom$.

For $F(\text{-}) \equiv \neg F'(\text{-})$, taking $q \in Atom$ such that $q \notin F$, we have $\models \neg F'(q)$. Let M_s be an arbitrary pointed model and then $M_s \models \neg F'(q)$. We consider two cases in the following. If $V^M(q) = \emptyset$, it is clear that $M_s \models \neg F'(\bot)$ by Proposition 10. Then we analyze the case with $V^M(q) \neq \emptyset$. Let M'_s be the model obtained from M_s by setting $V^{M'}(q) = \emptyset$. As $M'_s \models \neg F'(q)$ (due to $\models \neg F'(q)$) and $V^{M'}(q) = \emptyset$, we get $M'_s \models \neg F'(\bot)$ by Proposition 10. Further, since $q \notin F$, it is straightforward to check that

$$M'_s \models \neg F'(\bot) \text{ iff } M_s \models \neg F'(\bot)$$

by the induction on F'. Hence $M_s \models \neg F'(\bot)$ follows immediately.

For $F(\text{-}) \equiv F_1(\text{-}) \wedge F_2(\text{-})$, it follows from $\models F_1(p) \wedge F_2(p)$ for every $p \in Atom$ that $\models F_1(p)$ and $\models F_2(p)$ for every $p \in Atom$. By the induction hypothesis, it holds that $\models F_1(\bot)$ and $\models F_2(\bot)$, which implies $\models F_1(\bot) \wedge F_2(\bot)$. **Note that:** if $\vee$-clause was the primary clause, i.e., we need to check the case: $F(\text{-}) \equiv F_1(\text{-}) \vee F_2(\text{-})$, it is not difficult to see that the induction hypothesis could not work well in this case.

For $F(\text{-}) \equiv \Box_b F'(\text{-})$, since $\models \Box_b F'(p)$ for each $p \in Atom$, we easily get that $\models F'(p)$ for each $p \in Atom$ (its proof: Suppose that $N_w \not\models F'(q)$ for some $q \in Atom$. Let $N'_{w'}$ be the model obtained from N_w by adding a new state w' and a new transition $w' \xrightarrow{b} w$. Then $N'_w \not\models \Box_b F'(q)$, contradiction). By the induction hypothesis, we have that $\models F'(\bot)$. Next, it is clear that $\models \Box_b F'(\bot)$ (its proof: Assume that $N_w \not\models \Box_b F'(\bot)$. Then $N_v \not\models F'(\bot)$ for some $v \in R^N_b(w)$, contradiction).

For $F(\text{-}) \equiv \widetilde{\exists}_r F'(\text{-})$, this is analyzed by the strategy similar to the one for the case: $F(\text{-}) \equiv \neg F'(\text{-})$. □

Proposition 12: There is no translation $t : \mathcal{L}_{(A_1,A_2)} \to \mathcal{L}_{bq}$ with $A_1 \neq \emptyset$ which satisfies that there exists a context $F_{\exists_{(A_1,A_2)}}(\text{-})$ in $\mathcal{L}_{bq}$ such that

$$t(\exists_{(A_1,A_2)}\varphi) = F_{\exists_{(A_1,A_2)}}(t(\varphi))$$

and

$$\begin{aligned} t(p) &= p \\ t(\bot) &= \bot \\ t(\neg\varphi) &= \neg t(\varphi) \\ t(\varphi_1 \wedge \varphi_2) &= t(\varphi_1) \wedge t(\varphi_2) \\ t(\Diamond_b \varphi) &= \Diamond_b t(\varphi), \end{aligned}$$

and that for each $\mathcal{L}_{(A_1,A_2)}$-formula ψ, $\models t(\psi) \leftrightarrow \psi$.

Proof: By contradiction, we suppose that there is a translation $t : \mathcal{L}_{(A_1,A_2)} \to \mathcal{L}_{bq}$ with $A_1 \neq \emptyset$ such that, for some context $F_{\exists_{(a_1,a_2)}}(\text{-})$ in $\mathcal{L}_{bq}$, $t(\exists_{(a_1,a_2)}\varphi) = F_{\exists_{(a_1,a_2)}}(t(\varphi))$, $t(p) = p$, $t(\bot) = \bot$, $t(\neg\varphi) = \neg t(\varphi)$, $t(\varphi_1 \wedge \varphi_2) = t(\varphi_1) \wedge t(\varphi_2)$ and $t(\Diamond_b\varphi) = \Diamond_b t(\varphi)$, and such that $\models t(\psi) \leftrightarrow \psi$ for each $\mathcal{L}_{(A_1,A_2)}$-formula ψ. In $\mathcal{L}_{(A_1,A_2)}$, clearly, we have that:

(1) $\models \exists_{(a_1,a_2)}\Diamond_{a_1} q$ for all $q \in Atom$

(2) $\models \neg\exists_{(a_1,a_2)}\Diamond_{a_1}\bot$.

Then, as $\models t(\psi) \leftrightarrow \psi$, we get

(1′) $\models F_{\exists_{(a_1,a_2)}}\Diamond_{a_1} q \leftrightarrow \exists_{(a_1,a_2)}\Diamond_{a_1} q$ for all $q \in Atom$

(2′) $\models F_{\exists_{(a_1,a_2)}}\Diamond_{a_1}\bot \leftrightarrow \exists_{(a_1,a_2)}\Diamond_{a_1}\bot$.

Further, $\models F_{\exists_{(a_1,a_2)}}\Diamond_{a_1} q$ for all $q \in Atom$ follows from (1) and (1′), which implies that $\models F_{\exists_{(a_1,a_2)}}\Diamond_{a_1}\bot$ by Proposition 11. Hence, $\models \exists_{(a_1,a_2)}\Diamond_{a_1}\bot$ holds due to (2′), contradicting (2). □

B. LANGUAGE AND SEMANTICS

Now we give the language of the extended bisimulation quantified modal logic with universal modality (EBQML$_\blacksquare$) and its semantics.

Definition 13 (Language $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$): The language $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$ (for short, $\mathcal{L}_{ebq\blacksquare}$) of the extended bisimulation quantified modal logic with the universal modality $\blacksquare$ and $\emptyset \neq A_1, A_2 \subseteq A$ such that $A_1 \cap A_2 = \emptyset$, is generated by the BNF grammar below, where $b \in A$ and $p, q, p_1, p_2 \in Atom$:

$$\varphi ::= q \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \Box_b\varphi \mid \blacksquare\varphi \mid \widetilde{\exists}_{(p_1,p_2)}\varphi \mid \widehat{\exists}_p\varphi$$

Here, A_1 and A_2 correspond to covariant and contravariant set respectively. We write $\widehat{\forall}_p\varphi$ for $\neg\widehat{\exists}_p\neg\varphi$, and $\widetilde{\forall}_{(p_1,p_2)}\varphi$ for $\neg\widetilde{\exists}_{(p_1,p_2)}\neg\varphi$. The language $\mathcal{L}_{(\emptyset,A_2)-ebq\blacksquare}$ is indeed $\mathcal{L}_{bq\blacksquare}$ and the language $\mathcal{L}_{(A_1,\emptyset)-ebq\blacksquare}$ will be discussed in Section 6.

To define the notion of satisfiability of $\mathcal{L}_{ebq\blacksquare}$-formulas, we apply the notion of model structure instead of model.

Definition 14 (Model structure): A model structure is a triple (M, a, D) where M is a model, $a \in A$ and $D \subseteq S^M \times S^M - R^M_a$.

$(M, a, D)_s$ with $s \in S^M$ is said to be a pointed model structure. Moreover, the model structures $(M, a, D) \mid (b, S)$ and $(M, a, D) \mid (b, p)$, where $b \in A$, $p \in Atom$ and $S \subseteq S^M$, are defined as follows:

- $(M, a, D) \mid (b, S) \triangleq (M \mid (b, S), a, D)$, and
- $(M, a, D) \mid (b, p) \triangleq (M, a, D) \mid (b, V^M(p))$.

Definition 15 (Bisimilarity Between Model Structures): Given two pointed model structures $(M, a, D)_s$ and $(M', a, D')_{s'}$, a binary relation $\mathcal{Z} \subseteq S^M \times S^{M'}$ is a bisimulation relation between $(M, a, D)_s$ and $(M', a, D')_{s'}$, in symbols $\mathcal{Z} : (M, a, D)_s \underline{\leftrightarrow}_* (M', a, D')_{s'}$, if $s\mathcal{Z}s'$ and for each pair $\langle u, u'\rangle$ in $\mathcal{Z}$,

(atoms) *$u \in V^M(q)$ iff $u' \in V^{M'}(q)$ for each $q \in Atom$;*

(forth) *for each $b \in A$ and $v \in S^M$, $uR^M_b v$ implies $u'R^{M'}_b v'$ and $v\mathcal{Z}v'$ for some $v' \in S^{M'}$, and $\langle u, v\rangle \in D$ implies $\langle u', w'\rangle \in D'$ and $v\mathcal{Z}w'$ for some $w' \in S^{M'}$;*

(back) *for each $b \in A$ and $v' \in S^{M'}$, $u'R^{M'}_b v'$ implies $uR^M_b v$ and $v\mathcal{Z}v'$ for some $v \in S^M$, and $\langle u', v'\rangle \in D'$ implies $\langle u, w\rangle \in D$ and $w\mathcal{Z}v'$ for some $w \in S^M$.*

Analogous to the notion of P-restricted bisimulation between models, we define the one between model structures and use $\mathcal{Z} : (M, a, D)_s \underline{\leftrightarrow}^P_* (M', a, D')_{s'}$ to indicate that $\mathcal{Z}$ is a ***P*-restricted bisimulation relation** between $(M, a, D)_s$ and $(M', a, D')_{s'}$.

Let $\psi \in \mathcal{L}_{ebq\blacksquare}$. Given a model structure (M, a, D) with $a \in A_1$, the notion of the formula ψ being satisfied in (M, a, D) at a state $s \in S^M$ is defined inductively in Table 1.

From Table 1, it is easy to see that, actually, $M + (a, D)$ and (M, a, D) depict the same model and $(M, a, D) \mid (b, S)$ depicts the model obtained from M by adding the a-labelled transitions in D and preserving only the b-labelled transitions entering the states in S.

As usual, we say that two model structures $(M_1, a, D_1)_u$ and $(M_2, a, D_2)_v$ are equivalent if $(M_1, a, D_1)_u \models \psi$ if only if $(M_2, a, D_2)_v \models \psi$ for all $\psi \in \mathcal{L}_{ebq\blacksquare}$.

Convention: In the sequel, for such structure symbols: $M + (a, D)$, $M \mid (b, S)$, (M, a, D) and $(M, a, D) \mid (b, S)$, we always suppose $a \in A_1$ and $b \in A_2$ whenever referring

TABLE 1. The satisfiability of $\mathcal{L}_{ebq\blacksquare}$-formulas in a model structure (M, a, D) at a state $s \in S^M$.

$(M,a,D)_s \models q$	iff	$s \in V^M(q)$, where $q \in Atom$
$(M,a,D)_s \models \neg\varphi$	iff	$(M,a,D)_s \not\models \varphi$
$(M,a,D)_s \models \varphi_1 \wedge \varphi_2$	iff	$(M,a,D)_s \models \varphi_1$ and $(M,a,D)_s \models \varphi_2$
$(M,a,D)_s \models \Box_b\varphi$	iff	$(M,a,D)_w \models \varphi$ for all $w \in R_b^M(s)$, where $b \neq a$
$(M,a,D)_s \models \Box_a\varphi$	iff	$(M,a,D)_w \models \varphi$ for all $w \in (R_a^M \cup D)(s)$
$(M,a,D)_s \models \blacksquare\varphi$	iff	$(M,a,D)_w \models \varphi$ for all $w \in (D \cup \bigcup_{b\in A} R_b^M)^*(s)$
$(M,a,D)_s \models \widetilde{\exists}_{(p_1,p_2)}\varphi$	iff	for some $(N,a,D')_w$, $(M,a,D)_s \underline{\leftrightarrow}_*^{p_1,p_2} (N,a,D')_w \models \varphi$
$(M,a,D)_s \models \widehat{\exists}_p\varphi$	iff	for some $(N,a,D')_w$ and $B \subseteq (S^N)^2 - (R_a^N \cup D')$, $(M,a,D)_s \underline{\leftrightarrow}_*^p (N,a,D')_w$ and $(N,a,D' \cup B)_w \models \varphi$

to covariant and contravariant actions, and if no ambiguity, we often abbreviate these agent symbols.

C. USEFUL PROPERTIES

Model structures have some useful properties. We begin with the invariance of $\mathcal{L}_{ebq\blacksquare}$-satisfiability under P-restricted bisimulation:

Proposition 16: Let $(M,B)_s \underline{\leftrightarrow}_*^P (N,D)_w$ and $\psi \in \mathcal{L}_{ebq\blacksquare}$ such that $p \notin \psi$ for all $p \in P$. Then

$$(M,B)_s \models \psi \;\; \textit{iff} \;\; (N,D)_w \models \psi.$$

Proof: By the induction on ψ. □

Proposition 17: $(M,B)_s \underline{\leftrightarrow}_*^P (N,D)_w$ implies that $(M+B, \emptyset)_s \underline{\leftrightarrow}_*^P (N+D, \emptyset)_w$.

Proof: Straightforward. □

In the following, we intend to show that $(M+B, \emptyset)$ and (M,B) are equivalent. Moreover, in Section V, we will intend to show the equivalence of the given translation. To prove these results, Proposition 19 is needed to simplify their proofs.

Proposition 19 reveals that, given $\mathcal{Z} : ((M+B, \emptyset) \mid S)_s \underline{\leftrightarrow}_*^P (N, \emptyset)_w$ with $q \in Atom - P$, $S \subseteq V^M(q)$ and $D_1 \subseteq (S^N \times S^N) - R_{a_1}^N$, we can construct $(N^\diamond, D^\diamond)_{w'}$, $S^\diamond \subseteq V^{N^\diamond}(q)$ and $D_1^\diamond \subseteq (S^{N^\diamond} \times S^{N^\diamond}) - (R_{a_1}^{N^\diamond} \cup D^\diamond)$, which satisfy $(M,B)_s \underline{\leftrightarrow}_*^P (N^\diamond, D^\diamond)_{w'}$ and

$$((N^\diamond + (D^\diamond \cup D_1^\diamond)) \mid S^\diamond, \emptyset)_{w'} \underline{\leftrightarrow}_*^q (N + D_1, \emptyset)_w.$$

In the following, we will explain the idea behind the constructions. Without loss of generality, we assume that M and N' are disjoint. Since $\mathcal{Z} : ((M+B, \emptyset) \mid S)_s \underline{\leftrightarrow}_*^P (N, \emptyset)_w$, in order to realize $(M,B)_s \underline{\leftrightarrow}_*^P (N^\diamond, D^\diamond)_{w'}$, at **first** glance, $N^\diamond$ can be obtained from $M \uplus N$ by modifying the accessibility relation $R_{a_2}^{M \uplus N}$ so as to provide a matching transition for each transition $u \xrightarrow{a_2} z$ with $u \in \pi_1(\mathcal{Z})$ and $z \notin S$ in M. We will add the a_2-labelled transitions depicted by the dash arrows in Figure 2. **Next**, it is not difficult to see that, in this construction, the desired $D^\diamond$ has to come from N. But this is not always done successfully. For example,

Example 18: Consider the models M and N depicted in Figure 3. Here, $V^M(p) = V^N(p) = \emptyset$ for each $p \in Atom$ and the dash (thick dash) arrows represent the a_1-transitions in B (resp., D_1). We easily see $((M+B, \emptyset) \mid S)_s \underline{\leftrightarrow}_* (N, \emptyset)_w$ via the binary relation $\mathcal{Z} = \{\langle s, w\rangle, \langle u, z\rangle, \langle v, z\rangle\}$, where it is clear that $S = \emptyset$ since $S \subseteq V^M(q)$. According to the construction mentioned above, after adding the transition $z \xrightarrow{a_2} m$, it can be guaranteed that $(M, \emptyset)_s \underline{\leftrightarrow}_* (N^\diamond, \emptyset)_w$ via the binary relation $\{\langle s, w\rangle, \langle u, z\rangle, \langle m, m\rangle\}$. There is no doubt that it is the most convenient to obtain $D^\diamond$ from the corresponding transitions in N of the ones in B. However, disappointedly, we just get $D^\diamond = \emptyset$ by this due to $D^\diamond \cap R_{a_1}^{N^\diamond} = \emptyset$ so that $(M,B)_s \not\underline{\leftrightarrow}_* (N^\diamond, D^\diamond)_w$.

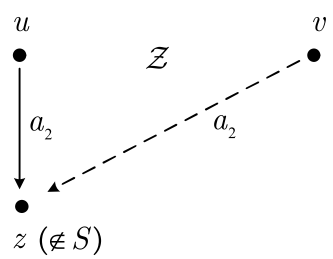

FIGURE 2. The added a_2-transitions in the construction of the model $N^\diamond$.

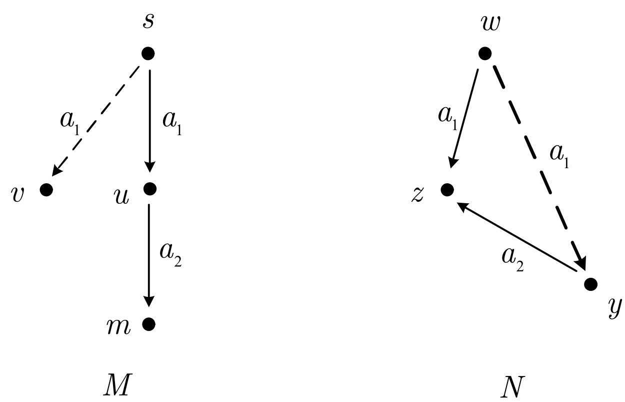

FIGURE 3. An example to depict how to construct the desired $D^\diamond$

To obtain a desired $D^\diamond$ of the above example, similar as in the proof for Proposition 24 (2.2) in [19], we intend to replace each $v' \in \pi_2(\mathcal{Z})$ by all the pairs of the form $\langle v', u'\rangle$ in $\mathcal{Z}^{-1}$. Moreover, the transitions from these new states $\langle v', u'\rangle$ in $N^\diamond$ are prescribed according to the ones related to v' in N. In particular, the transitions between two new states $\langle v_1, u_1\rangle$ and $\langle v_2, u_2\rangle$ are captured by the rule, for all $a \in A$:

$$\langle v_1, u_1\rangle R_a^{N^\diamond} \langle v_2, u_2\rangle \quad \text{iff} \quad u_1\, R_a^{M|S} u_2 \text{ and } v_1\, R_a^N v_2, \qquad (*)$$

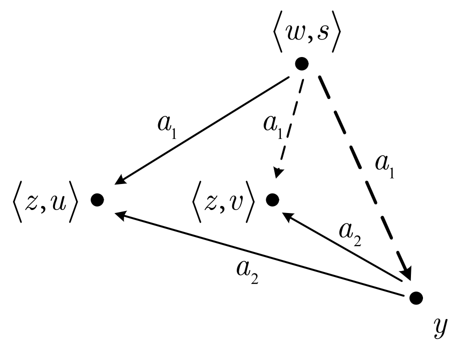

FIGURE 4. The part from N in the remedied $N^\diamond$.

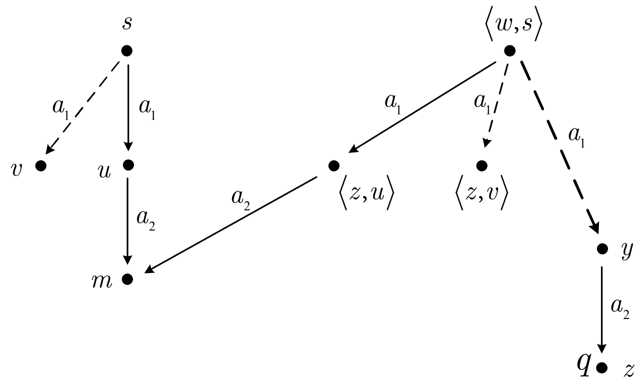

FIGURE 5. The final model $N^\diamond$ obtained by applying the remedied constructions.

and the a_2-labelled transitions between the new state $\langle v', u'\rangle$ and the state z' in $S^M - S$ are captured by the rule:

$$\langle v', u'\rangle R_{a_2}^{N^\diamond} z' \text{ iff } u' R_{a_2}^M z' \text{ and } z' \notin S. \qquad (**)$$

The part from N in the remedied $N^\diamond$ is depicted as in Figure 4, in which $D^\diamond$ consists of the dash arrow, corresponding to $\langle s, v\rangle$ (in B). In the proof of Proposition 19, $D^\diamond$ is defined as

$$D^\diamond \triangleq B \cup \{\langle\langle v_1, u_1\rangle, \langle v_2, u_2\rangle\rangle \mid u_1\, B u_2, \; v_1\, R_{a_1}^N v_2, u_1 \mathcal{Z} v_1 \text{ and } u_2 \mathcal{Z} v_2\}.$$

Clearly, it holds that $(M, B)_s \underline{\leftrightarrow}_* (N^\diamond, D^\diamond)_{\langle w,s\rangle}$.

Further, we will construct $D_1^\diamond$ and $S^\diamond$ of the above example. To realize

$$((N^\diamond + (D^\diamond \cup D_1^\diamond)) \mid S^\diamond, \emptyset)_{w'} \underline{\leftrightarrow}_*^q (N + D_1, \emptyset)_w,$$

the transitions $y \overset{a_2}{\to} \langle z, u\rangle$ and $y \overset{a_2}{\to} \langle z, v\rangle$ need to be kept. We may assign q to be true at the states $\langle z, u\rangle$ and $\langle z, v\rangle$ and set $S^\diamond \triangleq \{\langle z, u\rangle, \langle z, v\rangle\}$ so as to keep these transitions in the model $(N^\diamond + (D^\diamond \cup D_1^\diamond)) \mid S^\diamond$, as desired. Unfortunately, it is destroyed that $(M, B)_s \underline{\leftrightarrow}_* (N^\diamond, D^\diamond)_{\langle w,s\rangle}$ due to the condition (**atoms**). To remedy this flaw, we will preserve the states in $R_{N+D_1}^*(D_1(\pi_2(\mathcal{Z})))$ for $N^\diamond$, and assign q to be true at the a_2-accessible states in this part. In the proof of Proposition 19, the q-states in the part from N, together with S, form $S^\diamond$, and $D_1^\diamond$ is defined as

$$D_1^\diamond \triangleq (D_1 \cap (S^{N^\diamond})^2) \cup \{\langle\langle v', u'\rangle, z'\rangle \mid v' D_1\, z' \text{ and } u' \mathcal{Z} v'\}.$$

Actually, since the bisimilarity between two states depends on the bisimilarity between their generated submodels, it is enough to keep the states in $\mathcal{Z}^{-1} \cup R_{N+D_1}^*(D_1(\pi_2(\mathcal{Z})))$ in the part from N of $N^\diamond$. Thus the **final** remedied model $N^\diamond$ is described in Figure 5, in which the dash (thick dash) arrows represent the a_1-transitions in $D^\diamond$ (resp., $D_1^\diamond$).

From Figure 5, it is not difficult to observe that it is also available to, in the last step of the construction, add the matching transitions for each transition $u \overset{a_2}{\to} z$ with $u \in \pi_1(\mathcal{Z})$ and $z \notin S$ in M. That is, we firstly construct the part from N by the method mentioned in the preceding paragraphs, denoted as the model N', then obtain the desired model $N^\diamond$ from $M \uplus N'$ by adding the a_2-labelled transitions according to the rule $(**)$. We proceed by this strategy in the proof for Proposition 19.

Proposition 19: Given $\mathcal{L}_{ebq\blacksquare}$ and a model M, let $P \subseteq Atom$ and $q \in Atom - P$, and let $S \subseteq V^M(q)$ (or $S = \pi_2(R_{a_2}^M)$). If

$$((M + B, \emptyset) \mid S)_s \underline{\leftrightarrow}_*^P (N, \emptyset)_w$$

and $D_1 \subseteq (S^N \times S^N) - R_{a_1}^N$ then there exist $(N^\diamond, D^\diamond)_{w'}$, $D_1^\diamond \subseteq (S^{N^\diamond} \times S^{N^\diamond}) - (R_{a_1}^{N^\diamond} \cup D^\diamond)$ and $S^\diamond \subseteq V^{N^\diamond}(q)$ (resp., $S^\diamond = \pi_2(R_{a_2}^{N^\diamond})$) such that

$$(M, B)_s \underline{\leftrightarrow}_*^P (N^\diamond, D^\diamond)_{w'}$$

and

$$((N^\diamond + (D^\diamond \cup D_1^\diamond)) \mid S^\diamond, \emptyset)_{w'} \underline{\leftrightarrow}_*^q (N + D_1, \emptyset)_w.$$

In particular:

(1) *if $B = \emptyset$ then we may take $D^\diamond = \emptyset$,*
(2) *if $D_1 = \emptyset$ then take $D_1^\diamond = \emptyset$,*
(3) *if $S = V^M(q)$ then take $S^\diamond = V^{N^\diamond}(q)$, and*
(4) *if $S = \pi_2(R_{a_2}^M)$ then*

$$(N^\diamond + (D^\diamond \cup D_1^\diamond), \emptyset)_{w'} \underline{\leftrightarrow}_* (N + D_1, \emptyset)_w.$$

Proof: Suppose $S \subseteq V^M(q)$. Let $\mathcal{Z} : ((M + B, \emptyset) \mid S)_s \underline{\leftrightarrow}_*^P (N, \emptyset)_w$ and $D_1 \subseteq (S^N \times S^N) - R_{a_1}^N$, and let $M \mid S = \langle S^M, R, V^M\rangle$. We firstly construct the model structure (N', D) as follows, and then obtain $(N^\diamond, D^\diamond)$ from $M \uplus N'$.

(N_1') $S^{N'} \triangleq \mathcal{Z}^{-1} \cup R_{N+D_1}^*(D_1(\pi_2(\mathcal{Z})))$ (Here we assume that $S^N \cap \mathcal{Z}^{-1} = \emptyset$).

(N_2') For each $b \in A$, $R_b^{N'} \subseteq S^{N'} \times S^{N'}$ is obtained from R_b^N by preserving the transitions between the states in $S^N \cap S^{N'}$, and prescribing the behaviour of a new state $\langle v, u\rangle$ according to the rule $(*)$. Formally,

$$R_b^{N'} \triangleq (R_b^N \cap (S^{N'})^2) \cup \{\langle\langle v, u\rangle, \langle v', u'\rangle\rangle \mid uRu', v\, R_b^N v', u \mathcal{Z} v \text{ and } u' \mathcal{Z} v'\}.$$

(N_3') For each $r \in Atom$,

$$V^{N'}(r) \triangleq (V^N(r) \cap S^{N'}) \cup \{\langle v, u\rangle \mid u \mathcal{Z} v \text{ and } v \in V^N(r)\}.$$

Note that we do not modify the assignment of q here.

(N_4') $D \subseteq S^{N'} \times S^{N'}$ is defined as

$$D \triangleq \{\langle\langle v, u\rangle, \langle v', u'\rangle\rangle \mid uBu', v\, R^N_{a_1} v', u\mathcal{Z}v \text{ and } u'\mathcal{Z}v'\}.$$

(N_5') $D_1' \subseteq S^{N'} \times S^{N'}$ is defined as

$$D_1' \triangleq (D_1 \cap (S^{N'})^2) \cup \{\langle\langle v, u\rangle, z\rangle \mid vD_1\, z \text{ and } u\mathcal{Z}v\}.$$

Clearly, $D \subseteq S^{N'} \times S^{N'} - R^{N'}_{a_1}$ and $D_1' \subseteq S^{N'} \times S^{N'} - (R^{N'}_{a_1} \cup D)$.

We w.l.o.g. assume that M and N' are disjoint. Let the model $N^\diamond$ be obtained from $M \uplus N'$ by assigning q to be true at the a_2-accessible states in $R^*_{N+D_1}(D_1(\pi_2(\mathcal{Z})))$, and for every $\langle v, u\rangle \in \mathcal{Z}^{-1}$ and $z \in S^M$, imposing the clause:

$$\langle v, u\rangle R^{N^\diamond}_{a_2} z \;\text{ iff }\; uR^M_{a_2} z \text{ and } z \notin S.$$

Then define

$$D^\diamond \triangleq B \cup D \;\text{ and }\; D_1^\diamond \triangleq D_1'.$$

It is not difficult to see that $D^\diamond \subseteq S^{N^\diamond} \times S^{N^\diamond} - R^{N^\diamond}_{a_1}$ and $D_1^\diamond \subseteq S^{N^\diamond} \times S^{N^\diamond} - (R^{N^\diamond}_{a_1} \cup D^\diamond)$. Set

$$\mathcal{Z}' \triangleq \mathrm{i}_{S^M, S^{N^\diamond}} \cup \{\langle u, \langle v, u\rangle\rangle \mid u\mathcal{Z}v\}.$$

It is routine to show that

$$\mathcal{Z}' : (M, B)_s \underline{\leftrightarrow}^P_* (N^\diamond, D^\diamond)_{\langle w,s\rangle}.$$

On the other hand, put

$$S^\diamond \triangleq S \cup (V^{N^\diamond}(q) \cap S^{N'})$$

Clearly, $S^\diamond \subseteq V^{N^\diamond}(q))$. Thus, we immediately have

$$\mathcal{Z}'' : ((N^\diamond + (D^\diamond \cup D_1^\diamond)) \mid S^\diamond, \emptyset)_{\langle w,s\rangle} \underline{\leftrightarrow}^q_* (N + D_1, \emptyset)_w$$

where

$$\mathcal{Z}'' \triangleq \{\langle v, v\rangle \mid v \in S^N \cap S^{N^\diamond}\} \cup \{\langle\langle v, u\rangle, v\rangle \mid u\mathcal{Z}v \text{ and } v \in R^*_N(w)\}.$$

For $S = \pi_2(R^M_{a_2})$, we apply the same constructions as the ones in the above case except that $V^{N^\diamond} \triangleq V^M \cup V^{N'}$ and $S^\diamond \triangleq \pi_2(R^{N^\diamond}_{a_2})$. It is easy to observe that no a_2-translation will be removed and the assignment of q will not be modified, namely,

$$N^\diamond \mid S^\diamond = N^\diamond = M \uplus N'.$$

Therefore we get

$$\mathcal{Z}'' : (N^\diamond + (D^\diamond \cup D_1^\diamond), \emptyset)_{\langle w,s\rangle} \underline{\leftrightarrow}_* (N + D_1, \emptyset)_w.$$

Furthermore, if $B = \emptyset$ then $D^\diamond = \emptyset$, if $D_1 = \emptyset$ then $D_1^\diamond = \emptyset$, and if $S = V^M(q)$ then $S^\diamond = V^{N^\diamond}(q)$. □

Proposition 20: Let $\psi \in \mathcal{L}_{ebq\blacksquare}$. Then

$$(M, B)_s \models \psi \;\textit{ iff }\; (M + B, \emptyset)_s \models \psi.$$

Proof: Proceed by the induction on ψ. We analyze the clauses: $\psi \equiv \widehat{\exists}_p\varphi$ and $\psi \equiv \widetilde{\exists}_{(p_1,p_2)}\varphi$, and the analyses are routine for the other clauses.

(1) $\psi \equiv \widehat{\exists}_p\varphi$

Let $(M, B)_s \models \widehat{\exists}_p\varphi$. So $(M, B)_s \underline{\leftrightarrow}^p_* (N, D)_w$ and $(N, D \cup D_1)_w \models \varphi$ for some $(N, D)_w$ and D_1. By Proposition 17, $(M + B, \emptyset)_s \underline{\leftrightarrow}^p_* (N + D, \emptyset)_w$ follows from $(M, B)_s \underline{\leftrightarrow}^p_* (N, D)_w$. Due to $(N, D \cup D_1)_w \models \varphi$, by the induction hypothesis,

$$((N + D) + D_1, \emptyset)_w = (N + (D \cup D_1), \emptyset)_w \models \varphi$$

and next $(N + D, D_1)_w \models \varphi$ holds by applying the induction hypothesis again. Hence, from $(M + B, \emptyset)_s \underline{\leftrightarrow}^p_* (N + D, \emptyset)_w$ and $(N + D, D_1)_w \models \varphi$, it follows that $(M + B, \emptyset)_s \models \widehat{\exists}_p\varphi$.

Assume that $(M + B, \emptyset)_s \models \widehat{\exists}_p\varphi$. Then, for some N_w and D_1, $(M + B, \emptyset)_s \underline{\leftrightarrow}^p_* (N, \emptyset)_w$ and $(N, D_1)_w \models \varphi$. By the induction hypothesis, we get $(N + D_1, \emptyset)_w \models \varphi$ due to $(N, D_1)_w \models \varphi$. Further, as $(M + B, \emptyset)_s \underline{\leftrightarrow}^p_* (N, \emptyset)_w$, by Proposition 19 (4), there exist $(N', D)_{w'}$ and $D_1' \subseteq (S^{N'} \times S^{N'}) - (R^{N'}_{a_1} \cup D)$ such that $(M, B)_s \underline{\leftrightarrow}^p_* (N', D)_{w'}$ and $(N + D_1, \emptyset)_w \underline{\leftrightarrow}_* (N' + (D \cup D_1'), \emptyset)_{w'}$. Next, by Proposition 16, we get $(N' + (D \cup D_1'), \emptyset)_{w'} \models \varphi$ due to $(N + D_1, \emptyset)_w \models \varphi$, and then $(N', D \cup D_1')_{w'} \models \varphi$ by the induction hypothesis. Hence, $(M, B)_s \underline{\leftrightarrow}^p_* (N', D)_{w'}$ and $(N', D \cup D_1')_{w'} \models \varphi$ imply that $(M, B)_s \models \widehat{\exists}_p\varphi$.

(2) $\psi \equiv \widetilde{\exists}_{(p_1,p_2)}\varphi$

Let $(M, B)_s \models \widetilde{\exists}_{(p_1,p_2)}\varphi$. Then there exists $(N, D)_w$ such that $(M, B)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N, D)_w \models \varphi$, due to which we get $(M + B, \emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N + D, \emptyset)_w$ by Proposition 17 and $(N + D, \emptyset)_w \models \varphi$ by the induction hypothesis. Thus $(M + B, \emptyset)_s \models \widetilde{\exists}_{(p_1,p_2)}\varphi$ holds.

Suppose that $(M + B, \emptyset)_s \models \widetilde{\exists}_{(p_1,p_2)}\varphi$. Then, for some N_w, $(M + B, \emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N, \emptyset)_w$ and $(N, \emptyset)_w \models \varphi$. Because of $(M + B, \emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N, \emptyset)_w$, by Proposition 19 (2) and (4), we have that $(M, B)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N', D)_{w'}$ and $(N, \emptyset)_w \underline{\leftrightarrow}_* (N' + D, \emptyset)_{w'}$ for some $(N', D)_{w'}$. Then, due to $(N, \emptyset)_w \underline{\leftrightarrow}_* (N' + D, \emptyset)_{w'}$ and $(N, \emptyset)_w \models \varphi$, it holds that $(N' + D, \emptyset)_{w'} \models \varphi$ by Proposition 16, which implies that $(N', D)_{w'} \models \varphi$ by the induction hypothesis. Finally, $(M, B)_s \models \widetilde{\exists}_{(p_1,p_2)}\varphi$ follows from $(M, B)_s \underline{\leftrightarrow}^{p_1,p_2}_* (N', D)_{w'}$ and $(N', D)_{w'} \models \varphi$. □

Resorting to the above proposition, it is enough to consider the satisfiability of $\mathcal{L}_{ebq\blacksquare}$-formulas in such model structure $(M, \emptyset)$.

Proposition 21: Let $P \subseteq Atom$ and $q \in Atom - P$. Then $(M, \emptyset)_s \underline{\leftrightarrow}^P_* (N, \emptyset)_w$ implies that

$$((M, \emptyset) \mid (a, q))_s \underline{\leftrightarrow}^P_* ((N, \emptyset) \mid (a, q))_w.$$

Proof: The proof is routine. □

IV. CC-REFINEMENT AS BISIMULATION PLUS MODEL RESTRICTION

This section gives CC-refinement's semantical interpretation as bisimulation plus model restriction, based on which we will establish a relativization function in $\mathcal{L}_{ebq\blacksquare}$ in Section V.

In the following, we describe this semantical interpretation by demonstrating Lemma 25 and Lemma 26. Lemma 25 reveals intuitively that: an (a_1,a_2)-refinement model of a given model may be obtained from one bisimulation of this model by **adding** some a_1-labelled transitions and **removing**

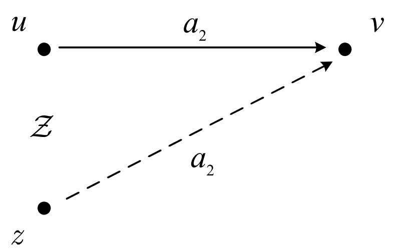

FIGURE 6. The *added* a_2-labelled transitions in the construction of the model $M +_{\mathcal{Z}} N$.

some a_2-labelled transitions. Given two disjoint models M and N such that $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_w$, Definition 22 gets such a bisimulation of M: the model $M +_{\mathcal{Z}} N$. In the following, we will explain the construction of $M +_{\mathcal{Z}} N$. Since $N_w \underline{\leftrightarrow} (M \uplus N)_w$ holds trivially and $M_s \succeq_{(a_1,a_2)} N_w$, based on the notion of (a_1, a_2)-refinement, it is not difficult to see that the construction can be realized by modifying the accessibility relations $R_{a_1}^{M \uplus N}$ and $R_{a_2}^{M \uplus N}$, namely, we focus on how to, **conversely**, ***delete*** some a_1-labelled transitions and ***add*** some a_2-labelled transitions.

Given a transition $u \xrightarrow{a_2} v$ in M, if the following conditions hold

(1) $u \in \pi_1(\mathcal{Z})$ and $v \in \pi_1(\mathcal{Z})$, and
(2) $\exists z(u\mathcal{Z}z$ and $\forall z'(v\mathcal{Z}z' \Rightarrow z \stackrel{a_2}{\nrightarrow} z'))$

then the transition $u \xrightarrow{a_2} v$ does not reflect in N. In Definition 22, the set of all these states v is denoted by $S_{a_2-}^{M,\mathcal{Z}}$, which is defined as

$$S_{a_2-}^{M,\mathcal{Z}} \triangleq \pi_1(\mathcal{Z}) \cap \pi_2(R_{a_2}^M \cap (\mathcal{Z} \circ \overline{R_{a_2}^N \circ \mathcal{Z}^{-1}})).$$

Here $\overline{R_{a_2}^N \circ \mathcal{Z}^{-1}}$ is the complementation of $R_{a_2}^N \circ \mathcal{Z}^{-1}$, namely, $\overline{R_{a_2}^N \circ \mathcal{Z}^{-1}} = S^N \times S^M - R_{a_2}^N \circ \mathcal{Z}^{-1}$. Clearly, the definition of $S_{a_2-}^{M,\mathcal{Z}}$ is induced by the conditions (1) and (2). Moreover, it is easy to see that if $u \in \pi_1(\mathcal{Z})$ and $v \in S^M - \pi_1(\mathcal{Z})$ then $u \xrightarrow{a_2} v$ also does not reflect in N. To afford a matching transition for $u \xrightarrow{a_2} v$ where $v \in (S^M - \pi_1(\mathcal{Z})) \cup S_{a_2-}^{M,\mathcal{Z}}$, we will ***add*** the a_2-labelled transition depicted by the dash arrow in Figure 6.

Similarly, given a transition $u \xrightarrow{a_1} v$ in N, through the above (1) and (2) with a_1-action instead of a_2-action, we describe the motivation behind introducing the set $R_{a_1+}^{N,\mathcal{Z}}$ in Definition 22, which is defined as

$$R_{a_1+}^{N,\mathcal{Z}} \triangleq (\mathcal{Z}^{-1} \circ \overline{R_{a_1}^M \circ \mathcal{Z}}\ ) \cap R_{a_1}^N.$$

Since M is given and fixed, to meet the requirement for (**back**), these a_1-labelled transitions will be obliged to be ***deleted***.

Definition 22: Given two disjoint models M and N such that $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_w$, the model $M +_{\mathcal{Z}} N$ is obtained from $M \uplus N$ by

(1) *deleting the a_1-labelled transitions in $R_{a_1+}^{N,\mathcal{Z}}$, and*
(2) *adding the a_2-labelled transitions in*

$$\{\langle z, v\rangle \mid v \in (S^M - \pi_1(\mathcal{Z})) \cup S_{a_2-}^{M,\mathcal{Z}} \text{ and } z(\mathcal{Z}^{-1} \circ R_{a_2}^M)v\}.$$

Here,
$R_{a_1+}^{N,\mathcal{Z}} \triangleq (\mathcal{Z}^{-1} \circ \overline{R_{a_1}^M \circ \mathcal{Z}}\ ) \cap R_{a_1}^N$*, and*
$S_{a_2-}^{M,\mathcal{Z}} \triangleq \pi_1(\mathcal{Z}) \cap \pi_2(R_{a_2}^M \cap (\mathcal{Z} \circ \overline{R_{a_2}^N \circ \mathcal{Z}^{-1}}\ ))$.

Example 23: For the models M and N in Figure 1 with the (a, b)-refinement relation $\mathcal{Z}$ between them depicted by the dash-arrows, the model $M +_{\mathcal{Z}} N$ is given in Figure 7. Here, the dash-arrows represent the ***added*** *new b-labelled transitions (neither in M nor in N), and the transitions $u_1 \xrightarrow{a} v_1$ and $u_2 \xrightarrow{a} v_2$ from N are* ***deleted***. *It is not difficult to see that $M_s \underline{\leftrightarrow} (M +_{\mathcal{Z}} N)_{s_1}$ via the relation $\mathcal{Z} \cup \{\langle v, v\rangle, \langle w, w\rangle\}$.*

Proposition 24 [19]: (1) $M_{s_1} \underline{\leftrightarrow} M'_{s_2} \succeq_{(A_1,A_2)} N'_{w_2} \underline{\leftrightarrow} N_{w_1}$ *implies* $M_{s_1} \succeq_{(A_1,A_2)} N_{w_1}$.

(2) *If $M_s \succeq_{(A_1,A_2)} N_w$ then there exist $M'_{s'}$, $N'_{w'}$ and $\mathcal{Z}$ such that $M_s \underline{\leftrightarrow} M'_{s'}$, $N_w \underline{\leftrightarrow} N'_{w'}$, and $\mathcal{Z} : M'_{s'} \succeq_{(A_1,A_2)} N'_{w'}$ that is an injective partial function from $S^{M'}$ to $S^{N'}$, namely, $\mathcal{Z}$ satisfies*
(2.1) $\forall z \in S^{M'}\ \forall v_1, v_2 \in S^{N'}(z\mathcal{Z}v_1$ *and* $z\mathcal{Z}v_2 \Rightarrow v_1 = v_2)$;
(2.2) $\forall v \in S^{N'}\ \forall z_1, z_2 \in S^{M'}(z_1\mathcal{Z}v$ *and* $z_2\mathcal{Z}v \Rightarrow z_1 = z_2)$.

Lemma 25: If $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_w$ then

$$M_s \underline{\leftrightarrow} (M +_{\mathcal{Z}} N)_w \text{ and } ((M +_{\mathcal{Z}} N) \mid R)_w \underline{\leftrightarrow} N_w$$

where R is obtained from $R^{M +_{\mathcal{Z}} N}$ by setting $R_b = R_b^N$ for $b = a_1, a_2$.

Proof: Let $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_w$. W.l.o.g., we may suppose that M and N are disjoint and $\mathcal{Z}$ is an injective partial function from S^M to S^N by Proposition 24. Below we check

$$\mathcal{Z} \cup \mathrm{i}_{S^M, S^{M +_{\mathcal{Z}} N}} : M_s \underline{\leftrightarrow} (M +_{\mathcal{Z}} N)_w.$$

Here we borrow the notations in Definition 22.

For $\langle u, u'\rangle \in \mathrm{i}_{S^M, S^{M +_{\mathcal{Z}} N}}$, the proof is straightforward. In the following, we consider another case where $u\mathcal{Z}u'$. The condition **(atoms)** holds trivially, and we next check **(forth)** and **(back)**. Let $b \in A$ and $S^\blacktriangle \triangleq (S^M - \pi_1(\mathcal{Z})) \cup S_{a_2-}^{M,\mathcal{Z}}$.

(forth) Let $uR_b^M v$. We consider three cases based on b.

Case 1 $b \neq a_1, a_2$
Because of $\mathcal{Z} : M_u \succeq_{(a_1,a_2)} N_{u'}$ and $uR_b^M v$, there exists $v' \in S^N$ such that $u'R_b^N v'$ and $v\mathcal{Z}v'$. Further, due to the definition of $R^{M +_{\mathcal{Z}} N}$, we have $u'R_b^{M +_{\mathcal{Z}} N} v'$.

Case 2 $b = a_1$
Since $\mathcal{Z} : M_u \succeq_{(a_1,a_2)} N_{u'}$ and $uR_b^M v$, $u'R_b^N v'$ and $v\mathcal{Z}v'$ for some $v' \in S^N$. Next we need to check $u'R_b^{M +_{\mathcal{Z}} N} v'$. Based on the fact that $\mathcal{Z}$ is an injective partial function from S^M to S^N, from $u\mathcal{Z}u'$, $uR_b^M v$ and $v\mathcal{Z}v'$, it follows that

$$\langle u', v'\rangle \notin \mathcal{Z}^{-1} \circ \overline{R_b^M \circ \mathcal{Z}}.$$

That is $\langle u', v'\rangle \notin R_{b+}^{N,\mathcal{Z}}$. Thus $\langle u', v'\rangle \in R_b^{M +_{\mathcal{Z}} N}$ by the definition of $R^{M +_{\mathcal{Z}} N}$.

Case 3 $b = a_2$
Then we have $v \notin S^\blacktriangle$ or $v \in S^\blacktriangle$. If $v \notin S^\blacktriangle$ then $v \in \pi_1(\mathcal{Z}) - S_{b-}^{M,\mathcal{Z}}$. So, by the definition of $S_{b-}^{M,\mathcal{Z}}$, due to $uR_b^M v$, we have $\langle u, v\rangle \notin \mathcal{Z} \circ \overline{R_b^N \circ \mathcal{Z}^{-1}}$. Next, from $u\mathcal{Z}u'$, it follows that $u'(R_b^N \circ \mathcal{Z}^{-1})v$, which implies that

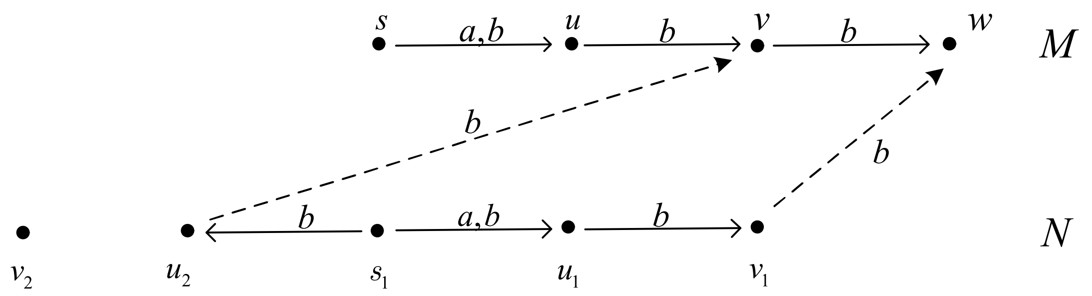

FIGURE 7. The model $M +_{\mathcal{Z}} N$ for the models M and N with the (a, b)-refinement relation $\mathcal{Z}$ between them in Figure 1.

$u'R_b^N v'$ (also, $u'R_b^{M+_{\mathcal{Z}}N}v'$) and $v\mathcal{Z}v'$ for some $v' \in S^N$ ($\subseteq S^{M+_{\mathcal{Z}}N}$). If $v \in S^\blacktriangle$ then together with $u\mathcal{Z}u'$ and $uR_b^M v$, we have $u'R_b^{M+_{\mathcal{Z}}N} v$ by the definition of $R^{M+_{\mathcal{Z}}N}$. Moreover, $\langle v, v\rangle \in \mathrm{i}_{S^M, S^{M+_{\mathcal{Z}}N}}$ as desired.

(back) Let $u'R_b^{M+_{\mathcal{Z}}N}v'$. It is clear that $u' \in S^N$ due to $\mathcal{Z} : M_u \succeq_{(a_1,a_2)} N_{u'}$. Then ($v' \in S^\blacktriangle$ and $b = a_2$) or $v' \in S^N$ by the definition of $R^{M+_{\mathcal{Z}}N}$.

If $v' \in S^\blacktriangle$ and $b = a_2$ then, due to $u'R_b^{M+_{\mathcal{Z}}N}v'$ and by the definition of $R^{M+_{\mathcal{Z}}N}$, we have that $u_0\mathcal{Z}u'$ and $u_0\, R_b^M v'$ for some $u_0 \in S^M$. Since $\mathcal{Z}$ is an injective partial function from S^M to S^N, $u = u_0$ immediately follows from $u\mathcal{Z}u'$ and $u_0\mathcal{Z}u'$. Hence $uR_b^M v'$ as desired. Also we have $\langle v', v'\rangle \in \mathrm{i}_{S^M, S^{M+_{\mathcal{Z}}N}}$. Next we consider another case where $v' \in S^N$. So $u'R_b^N v'$ by the definition of $R^{M+_{\mathcal{Z}}N}$. Here we analyze two cases: $b \neq a_1$ and $b = a_1$.

If $b \neq a_1$ then it follows from $\mathcal{Z} : M_u \succeq_{(a_1,a_2)} N_{u'}$ and $u'R_b^N v'$ that $uR_b^M v$ and $v\mathcal{Z}v'$ for some $v \in S^M$, as desired.

If $b = a_1$, $\langle u', v'\rangle \in R_b^N - \underline{R_{b+}^{N,\mathcal{Z}}}$ by the definition of $R^{M+_{\mathcal{Z}}N}$, i.e., $\langle u', v'\rangle \notin \mathcal{Z}^{-1} \circ R_b^M \circ \mathcal{Z}$ by the definition of $\underline{R_{b+}^{N,\mathcal{Z}}}$. Because of $u\mathcal{Z}u'$, $u(R_b^M \circ \mathcal{Z})v'$ immediately. Therefore, for some $v \in S^M$, we get $uR_b^M v$ and $v\mathcal{Z}v'$, as desired.

Further, it is easy to see that

$$(M +_{\mathcal{Z}} N) \mid R = N \uplus M$$

Then we get $((M +_{\mathcal{Z}} N) \mid R)_w = (N \uplus M)_w \underline{\leftrightarrow} N_w$ due to $w \in S^N$. □

Now we have known that an (a_1,a_2)-refinement of a given model may be obtained from one bisimulation of this model by **adding** some a_1-labelled transitions and **removing** some a_2-labelled transitions. Below, we use certain proposition letter to witness these **removed** a_2-labelled transitions.

Lemma 26: Let $p \in Atom$. If $M_s \succeq_{(a_1,a_2)} N_w$ then, for some $N'_{w'}$ and B,

$$M_s \underline{\leftrightarrow}^p N'_{w'} \quad \text{and} \quad ((N' + B) \mid p)_{w'} \underline{\leftrightarrow}^p N_w.$$

Proof: Let $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_w$. By Lemma 25, we have $M_s \underline{\leftrightarrow} (M +_{\mathcal{Z}} N)_w$. Here, we borrow the notations in Definition 22. Let the model N' be obtained from $M +_{\mathcal{Z}} N$ by setting $V^{N'}(p) = \pi_2(R_{a_2}^N)$. Clearly, $M_s \underline{\leftrightarrow}^p N'_w$ due to $M_s \underline{\leftrightarrow} (M +_{\mathcal{Z}} N)_w$. Put $B \triangleq \underline{R_{a_1+}^{N,\mathcal{Z}}}$. Then it is obvious that $B \subseteq S^{N'} \times S^{N'} - R_{a_1}^{N'}$. Let $(N' + B) \mid p = \langle S^{N'}, R, V^{N'}\rangle$. By Definition 22, it is not difficult to see that, if $R_{a_2} = R_{a_2}^N$ then

$$(N' + B) \mid p = \langle S^{N \uplus M}, R^{N \uplus M}, V^{N'}\rangle,$$

which implies that $((N' + B) \mid p)_w \underline{\leftrightarrow}^p N_w$ as desired. Thus, to complete the proof, it suffices to show $R_{a_2} = R_{a_2}^N$. Now we verify this below.

Let $uR_{a_2}^N u'$. Then $u' \in \pi_2(R_{a_2}^N) = V^{N'}(p)$. Next, due to $\langle u, u'\rangle \in R_{a_2}^N \subseteq R_{a_2}^{M+_{\mathcal{Z}}N} = R_{a_2}^{N'}$ and $u' \in V^{N'}(p)$, we have $uR_{a_2}u'$. Let $uR_{a_2}u'$. Then $u' \in V^{N'}(p)$ and $uR_{a_2}^{M+_{\mathcal{Z}}N}u'$. Thus $u' \in S^N$ because of $V^{N'}(p) = \pi_2(R_{a_2}^N)$. Further, $uR_{a_2}^N u'$ follows from $uR_{a_2}^{M+_{\mathcal{Z}}N}u'$ and $u' \in S^N$ by the definition of $M +_{\mathcal{Z}} N$. □

In the above constructed model $(N' + B) \mid p$, it is easy to observe that, only p-states are accessible for the contravariant action a_2, or, we say that p witnesses the contravariant action a_2, and the a_1-labelled transitions in B are new for N'. The converse of Lemma 26 also holds.

Lemma 27: If $M_s \underline{\leftrightarrow}^p N'_{w'}$ and $((N' + B) \mid p)_{w'} \models \varphi$ with $p \notin \varphi$ then $M_s \succeq_{(a_1,a_2)} N_w \models \varphi$ for some N_w.

Proof: Let $\mathcal{Z} : M_s \underline{\leftrightarrow}^p N'_{w'}$ and $((N' + B) \mid p)_{w'} \models \varphi$ with $p \notin \varphi$. The model N is obtained from $(N' + B) \mid p$ by setting $V^N(p) = \mathcal{Z}(V^M(p))$. Obviously

$$\mathrm{i}_{S^{N'}, S^N} : ((N' + B) \mid p)_{w'} \underline{\leftrightarrow}^p N_{w'}.$$

Then, $N_{w'} \models \varphi$ due to $((N' + B) \mid p)_{w'} \models \varphi$ and $p \notin \varphi$, by Proposition 6. Moreover, it is routine to check that $\mathcal{Z} : M_s \succeq_{(a_1,a_2)} N_{w'}$. ■ □

Laura Bozzelli et al. have obtained the same conclusions for a_2-refinement [1] as Lemma 25 and Lemma 26, which corresponds to the case where $A_1 = \emptyset$ and $A_2 = \{a_2\}$.

V. RELATIVIZATION

In Section IV, we interpret semantically a CC-refinement as one bisimulation followed by model restriction. In this section, we intend to describe syntactically CC-refinement quantification as the extended bisimulation quantification plus relativization. Further we propose an equivalent translation from $\mathcal{L}^{\mu}_{(A_1,A_2)}$ with $A_1, A_2 \neq \emptyset$ to $\mathcal{L}_{ebq\blacksquare}$.

Similar as in [1], a relativization in $\mathcal{L}_{ebq\blacksquare}$, to certain proposition letters for contravariant actions, devotes to select the desired transitions for these contravariant actions. Hence,

TABLE 2. The translation t from $\mathcal{L}^{\mu}_{(A_1,A_2)}$ with $A_1, A_2 \neq \emptyset$ to $\mathcal{L}_{ebq\blacksquare}$.

$t(q)$	$=$	q	
$t(\neg\varphi)$	$=$	$\neg t(\varphi)$	
$t(\varphi_1 \wedge \varphi_2)$	$=$	$t(\varphi_1) \wedge t(\varphi_2)$	
$t(\Box_b\varphi)$	$=$	$\Box_b t(\varphi)$	
$t(\exists_{(a_1,a_2)}\varphi)$	$=$	$\widehat{\exists}_p(t(\varphi))^{(a_2,p)}$	where $p \notin t(\varphi)$
$t(\nu x.\varphi)$	$=$	$\widetilde{\exists}_{(p_1,p_2)}((p_1 \vee p_2) \wedge \blacksquare((p_1 \vee p_2) \rightarrow t(\varphi[p_1 \vee p_2 \backslash x])))$	where $p_1, p_2 \notin \varphi$
$t(\mu x.\varphi)$	$=$	$\widetilde{\forall}_{(p_1,p_2)}(\blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \rightarrow (p_1 \vee p_2)) \rightarrow (p_1 \vee p_2))$	where $p_1, p_2 \notin \varphi$

such a relativization semantically also corresponds to an arrow-elimination relativization, but not a state-elimination relativization. We may refer to [22] for arrow-eliminating approach in the modal logic, [23] state-eliminating, and [24], [25] their differences.

Definition 28 (Relativization): We say that a function $\bullet^{(a,p)} : \mathcal{L}_{ebq\blacksquare} \rightarrow \mathcal{L}_{ebq\blacksquare}$ is a relativization to the proposition letter p for the action $a \in A_2$, whenever the following conditions hold,

$$
\begin{array}{lll}
q^{(a,p)} & = q & \\
(\neg\varphi)^{(a,p)} & = \neg\varphi^{(a,p)} & \\
(\varphi \wedge \psi)^{(a,p)} & = \varphi^{(a,p)} \wedge \psi^{(a,p)} & \\
(\Box_a\varphi)^{(a,p)} & = \Box_a(p \rightarrow \varphi^{(a,p)}) & \\
(\Box_b\varphi)^{(a,p)} & = \Box_b\varphi^{(a,p)} & \textit{if } b \neq a \\
(\blacksquare\varphi)^{(a,p)} & = \blacksquare\varphi^{(a,p)} & \\
(\widehat{\exists}_q\varphi)^{(a,p)} & = \widehat{\exists}_q\varphi^{(a,p)} & \textit{if } p \neq q \\
(\widehat{\exists}_p\varphi)^{(a,p)} & = \widehat{\exists}_r(\varphi[r\backslash p])^{(a,p)} & \textit{where } r \notin \varphi \\
(\widetilde{\exists}_{(p_1,p_2)}\varphi)^{(a,p)} & = \widetilde{\exists}_{(p_1,p_2)}\varphi^{(a,p)} & \textit{if } p \neq p_2 \\
(\widetilde{\exists}_{(p_1,p)}\varphi)^{(a,p)} & = \widetilde{\exists}_{(p_1,r)}(\varphi[r\backslash p])^{(a,p)} & \textit{where } r \notin \varphi
\end{array}
$$

Next, we translate $\mathcal{L}^{\mu}_{(A_1,A_2)}$-formulas with $A_1, A_2 \neq \emptyset$ into equivalent $\mathcal{L}_{ebq\blacksquare}$-formulas. In the remainder of this paper, we will assume $A = \{a_1, a_2\}$ with $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$ for the sake of simplicity.

Definition 29: A function $t : \mathcal{L}^{\mu}_{(A_1,A_2)} \rightarrow \mathcal{L}_{ebq\blacksquare}$ is said to be a translation if the conditions in Table 2 hold.

In Table 2, p_1 (p_2) is used to witness the agent a_1 (a_2, resp.). By employing the bisimulation quantification and universal modality, $t(\nu x.\varphi)$ ($t(\mu x.\varphi)$) equation captures the intuitive meaning of a greatest (least, resp.) fixed-point as the least upper (greatest lower, resp.) bound of the sets of states which are postfixed (prefixed, resp.) points of φ [12]. Moreover, since φ is finite, we easily observe that such a translation exists necessarily.

Below, we give a crucial conclusion to prove the equivalence between $\mathcal{L}^{\mu}_{(A_1,A_2)}$-formulas and their translations.

Lemma 30: Let $t : \mathcal{L}^{\mu}_{(A_1,A_2)} \rightarrow \mathcal{L}_{ebq\blacksquare}$ with $A_1, A_2 \neq \emptyset$ be a translation and $n < \omega$. Then

$$(M, \emptyset)_s \models (t(\psi))^{\Delta_n} \;\textit{ iff }\; ((M, \emptyset) \mid p_n)_s \models (t(\psi))^{\Delta_{n-1}},$$

where $\bullet^{\Delta_n} \triangleq (\cdots(\bullet^{(a_2,p_1)})\cdots)^{(a_2,p_n)}$, $\{p_i\}_{i<n}$ are pairwise different and $p_n \notin t(\psi)$.

Proof: Proceed by the induction on ψ. We check several typical cases and the others are routine to prove. We assume that $M \mid p_n = \langle S^M, R, V^M\rangle$.

Case 1 $\psi \equiv q$ where $q \in Atom$

$(M, \emptyset)_s \models (t(q))^{\Delta_n}$
iff $(M, \emptyset)_s \models q$
iff $s \in V^M(q)$
iff $((M, \emptyset) \mid p_n)_s \models q$
iff $((M, \emptyset) \mid p_n)_s \models (t(q))^{\Delta_{n-1}}$

Case 2 $\psi \equiv \Box_b\varphi$ where $b \neq a_2$

$(M, \emptyset)_s \models (t(\Box_b\varphi))^{\Delta_n}$
iff $(M, \emptyset)_s \models (\Box_b t(\varphi))^{\Delta_n}$
iff $(M, \emptyset)_s \models \Box_b(t(\varphi))^{\Delta_n}$
iff $(M, \emptyset)_w \models (t(\varphi))^{\Delta_n}$ for all $w \in R^M_b(s)$
iff $((M, \emptyset) \mid p_n)_w \models (t(\varphi))^{\Delta_{n-1}}$ for all $w \in R^M_b(s)$ (by I.H.)
iff $((M, \emptyset) \mid p_n)_w \models (t(\varphi))^{\Delta_{n-1}}$ for all $w \in R_b(s)$
iff $((M, \emptyset) \mid p_n)_s \models \Box_b(t(\varphi))^{\Delta_{n-1}}$
iff $((M, \emptyset) \mid p_n)_s \models (t(\Box_b\varphi))^{\Delta_{n-1}}$

Case 3 $\psi \equiv \Box_{a_2}\varphi$

$(M, \emptyset)_s \models (t(\Box_{a_2}\varphi))^{\Delta_n}$
iff $(M, \emptyset)_s \models (\Box_{a_2} t(\varphi))^{\Delta_n}$
iff $(M, \emptyset)_s \models \Box_{a_2}(p_n \rightarrow (\cdots \rightarrow (p_1 \rightarrow (t(\varphi))^{\Delta_n})\cdots))$
iff $(M, \emptyset)_w \models p_n \rightarrow (\cdots \rightarrow (p_1 \rightarrow (t(\varphi))^{\Delta_n})\cdots)$ for all $w \in R^M_{a_2}(s)$
iff $(M, \emptyset)_w \models p_n \Rightarrow (\cdots \Rightarrow ((M, \emptyset)_w \models p_1 \Rightarrow (M, \emptyset)_w \models (t(\varphi))^{\Delta_n})\cdots)$ for all $w \in R^M_{a_2}(s)$
iff $(M, \emptyset)_w \models p_n \Rightarrow (\cdots \Rightarrow ((M, \emptyset)_w \models p_1 \Rightarrow ((M, \emptyset) \mid p_n)_w \models (t(\varphi))^{\Delta_{n-1}})\cdots)$ for all $w \in R^M_{a_2}(s)$ (by I.H.)
iff $((M, \emptyset) \mid p_n)_w \models p_{n-1} \rightarrow (\cdots \rightarrow (p_1 \rightarrow (t(\varphi))^{\Delta_{n-1}})\cdots)$ for all $w \in R_{a_2}(s)$
iff $((M, \emptyset) \mid p_n)_s \models \Box_{a_2}(p_{n-1} \rightarrow (\cdots \rightarrow (p_1 \rightarrow (t(\varphi))^{\Delta_{n-1}})\cdots))$
iff $((M, \emptyset) \mid p_n)_s \models (\Box_{a_2} t(\varphi))^{\Delta_{n-1}}$
iff $((M, \emptyset) \mid p_n)_s \models (t(\Box_{a_2}\varphi))^{\Delta_{n-1}}$

Case 4 $\psi \equiv \exists_{(a_1,a_2)}\varphi$

Let $(M, \emptyset)_s \models (t(\exists_{(a_1,a_2)}\varphi))^{\Delta_n}$. Then, by Definition 29, for some $q \in Atom$ such that $q \notin t(\varphi)$,

$$(M, \emptyset)_s \models (\widehat{\exists}_q(t(\varphi))^{(a_2,q)})^{\Delta_n}.$$

By Definition 28, we may w.l.o.g. assume that q and $p_1, \cdots, p_n$ are different proposition letters. Then $(M, \emptyset)_s \models \widehat{\exists}_q((t(\varphi))^{(a_2,q)})^{\Delta_n}$. Next, $(M, \emptyset)_s \underline{\leftrightarrow}^q_*(N, \emptyset)_w$ and

$(N, D_1)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_n}$ for some N_w and D_1. Further, by Proposition 21, it follows from $(M,\emptyset)_s \underline{\leftrightarrow}^q_*(N,\emptyset)_w$ that

$$((M,\emptyset) \mid p_n)_s \underline{\leftrightarrow}^q_*((N,\emptyset) \mid p_n)_w,$$

i.e.,

$$(M \mid p_n, \emptyset)_s \underline{\leftrightarrow}^q_*(N \mid p_n, \emptyset)_w.$$

Moreover, due to $(N, D_1)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_n}$ and Proposition 20, it holds that

$$(N + D_1, \emptyset)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_n},$$

and so, by the induction hypothesis,

$$((N + D_1, \emptyset) \mid p_n)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}},$$

i.e.,

$$(N \mid p_n + D_1, \emptyset)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}.$$

Applying Proposition 20 again, we get

$$(N \mid p_n, D_1)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}.$$

This, together with $(M \mid p_n, \emptyset)_s \underline{\leftrightarrow}^q_*(N \mid p_n, \emptyset)_w$, implies that

$$(M \mid p_n, \emptyset)_s \models \widehat{\exists}_q((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}},$$

that is,

$$((M,\emptyset) \mid p_n)_s \models (t(\exists_{(a_1,a_2)}\varphi))^{\Delta_{n-1}}.$$

Let $((M,\emptyset) \mid p_n)_s \models (t(\exists_{(a_1,a_2)}\varphi))^{\Delta_{n-1}}$. Then

$$((M,\emptyset) \mid p_n)_s \models (\widehat{\exists}_q(t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}$$

where $q \notin t(\varphi)$. Similarly, we may suppose that q and $p_1, \cdots, p_n$ are different proposition letters, and then we get

$$((M,\emptyset) \mid p_n)_s \models \widehat{\exists}_q((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}.$$

So, for some N_w and D_1, we have $((M,\emptyset) \mid p_n)_s \underline{\leftrightarrow}^q_*(N,\emptyset)_w$ and $(N, D_1)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}$. Further, because of $((M,\emptyset) \mid p_n)_s \underline{\leftrightarrow}^q_*(N,\emptyset)_w$, by Proposition 19 (1) and (3), there exist $N'_{w'}$ and D'_1 such that $(M,\emptyset)_s \underline{\leftrightarrow}^q_*(N',\emptyset)_{w'}$ and

$$((N' + D'_1) \mid p_n, \emptyset)_{w'} \underline{\leftrightarrow}^{p_n}_*(N + D_1, \emptyset)_w.$$

Next, it follows from $(N, D_1)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}$ that $(N + D_1, \emptyset)_w \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}$ by Proposition 20. Thus, also as $((N' + D'_1) \mid p_n, \emptyset)_{w'} \underline{\leftrightarrow}^{p_n}_*(N + D_1, \emptyset)_w$ and $p_n \notin t(\psi)$, by Proposition 16,

$$((N' + D'_1) \mid p_n, \emptyset)_{w'} \models ((t(\varphi))^{(a_2,q)})^{\Delta_{n-1}}.$$

Hence $(N' + D'_1, \emptyset)_{w'} \models ((t(\varphi))^{(a_2,q)})^{\Delta_n}$ by the induction hypothesis and then $(N', D'_1)_{w'} \models ((t(\varphi))^{(a_2,q)})^{\Delta_n}$ by Proposition 20. Finally, from $(M,\emptyset)_s \underline{\leftrightarrow}^q_*(N',\emptyset)_{w'}$ and $(N', D'_1)_{w'} \models ((t(\varphi))^{(a_2,q)})^{\Delta_n}$, it follows that

$$(M,\emptyset)_s \models \widehat{\exists}_q((t(\varphi))^{(a_2,q)})^{\Delta_n}.$$

That is $(M,\emptyset)_s \models (t(\exists_{(a_1,a_2)}\varphi))^{\Delta_n}$.

Case 5 $\psi \equiv \nu x.\varphi$

By the definition of $t(\nu x.\varphi)$, we need to prove that

$$(M,\emptyset)_s \models (\widetilde{\exists}_{(q_1,q_2)}\alpha)^{\Delta_n}$$

if and only if

$$((M,\emptyset) \mid p_n)_s \models (\widetilde{\exists}_{(q_1,q_2)}\alpha)^{\Delta_{n-1}},$$

where $q_1, q_2 \notin \varphi$ and

$$\alpha \triangleq (q_1 \vee q_2) \wedge \blacksquare((q_1 \vee q_2) \to t(\varphi[q_1 \vee q_2 \backslash x])).$$

We apply the analysis similar as the one in Case 4. Here, note that we will rely on Proposition 19 (1), (2) and (3). □

Now we arrive at the equivalence between each $\mathcal{L}^{\mu}_{(A_1,A_2)}$-formula and its t-translation.

Proposition 31: Let $\psi \in \mathcal{L}^{\mu}_{(A_1,A_2)}$ with $A_1, A_2 \neq \emptyset$. Then

$$M_s \models \psi \textit{ iff } (M,\emptyset)_s \models t(\psi).$$

Proof: Proceed by the induction on ψ. We only deal with the non-trivial clauses: $\psi \equiv \mu x.\varphi$ and $\psi \equiv \exists_{(a_1,a_2)}\varphi$.

(1) $\psi \equiv \mu x.\varphi$

Let $M_s \models \mu x.\varphi$. We prove by contradiction and assume that $(M,\emptyset)_s \not\models t(\mu x.\varphi)$. By the definition of $t(\mu x.\varphi)$, there exists N_w such that $(M,\emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}(N,\emptyset)_w$ and $(N,\emptyset)_w \models \blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \to (p_1 \vee p_2)) \to (p_1 \vee p_2)$ where $p_1, p_2 \notin \varphi$. So $(N,\emptyset)_w \models \blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \to (p_1 \vee p_2)) \wedge \neg p_1 \wedge \neg p_2$, which implies $(N,\emptyset)_w \not\models p_1$, $(N,\emptyset)_w \not\models p_2$ and $(N,\emptyset)_w \models \blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \to (p_1 \vee p_2))$. By the semantic interpretation of $\blacksquare$ and the induction hypothesis, for all $u \in R^*_N(w)$, $N_u \models \varphi[p_1 \vee p_2 \backslash x] \to (p_1 \vee p_2)$. Then $N_w \models \varphi[p_1 \vee p_2 \backslash x] \to (p_1 \vee p_2)$. Further, since $N_w \not\models p_1$ and $N_w \not\models p_2$, $N_w \not\models \varphi[p_1 \vee p_2 \backslash x]$ follows. By the semantics definition of $\mu x.\varphi$, we get $N_w \not\models \mu x.\varphi$. However, because of $M_s \underline{\leftrightarrow}^{p_1,p_2} N_w$ due to $(M,\emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}(N,\emptyset)_w$, $N_w \models \mu x.\varphi$ follows from $M_s \models \mu x.\varphi$ and $p_1, p_2 \notin \varphi$, by Proposition 6. Contradict.

Assume $(M,\emptyset)_s \models t(\mu x.\varphi)$. Then, we have $(M,\emptyset)_s \models \widetilde{\forall}_{(p_1,p_2)}(\blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \to (p_1 \vee p_2)) \to (p_1 \vee p_2))$. For every $T \subseteq S^M$, it is clear that

$$(M,\emptyset)_s \underline{\leftrightarrow}^{p_1,p_2}_*(M^{[(p_1 \vee p_2 \backslash x) \mapsto T]},\emptyset)_s,$$

where the model $M^{[(p_1 \vee p_2 \backslash x) \mapsto T]}$ is obtained from M by assigning p_i to be true at the roots (the states with no entering transitions) and the states in T entered by a_i-labelled transitions where $i = 1, 2$. So

$$(M^{[(p_1 \vee p_2 \backslash x) \mapsto T]},\emptyset)_s \models \blacksquare(t(\varphi[p_1 \vee p_2 \backslash x]) \to (p_1 \vee p_2)) \to (p_1 \vee p_2).$$

By the semantics of $\blacksquare$ and the induction hypothesis, $M_u^{[(p_1 \vee p_2 \backslash x) \mapsto T]} \models \varphi[p_1 \vee p_2 \backslash x] \to (p_1 \vee p_2)$ for all $u \in R^*_M(s)$ implies that $M_s^{[(p_1 \vee p_2 \backslash x) \mapsto T]} \models p_1 \vee p_2$. Thus, for any $T \subseteq S^M$, if

$$\{u \in R^*_M(s) : (M^{[(p_1 \vee p_2 \backslash x) \mapsto T]})_u \models \varphi[p_1 \vee p_2 \backslash x]\} \subseteq T$$

then $s \in T$. Hence

$$s \in \bigcap\{T \subseteq S^M : \|\varphi\|^M_{[x\mapsto T]} \subseteq T\}.$$

That is $M_s \models \mu x.\varphi$ by the semantics of $\mu x.\varphi$.

(2) $\psi \equiv \exists_{(a_1,a_2)}\varphi$

(Assume $p \in Atom$ and $p \notin t(\varphi)$, easily prove $p \notin \varphi$)

$M_s \models \exists_{(a_1,a_2)}\varphi$

iff $M_s \succeq_{(a_1,a_2)} N_w$ and $N_w \models \varphi$ for some N_w

iff $M_s \underline{\leftrightarrow}^p N'_{w'}$ and $((N'+D) \mid p)_{w'} \models \varphi$ for some $N'_{w'}$ and D (by Lemma 26 and 27)

iff $(M,\emptyset)_s \underline{\leftrightarrow}^p_* (N',\emptyset)_{w'}$ and $((N'+D) \mid p, \emptyset)_{w'} \models t(\varphi)$ for some $N'_{w'}$ and D (by I.H.)

iff $(M,\emptyset)_s \underline{\leftrightarrow}^p_* (N',\emptyset)_{w'}$ and $(N'+D,\emptyset)_{w'} \models (t(\varphi))^{(a_2,p)}$ for some $N'_{w'}$ and D (by Lemma 30)

iff $(M,\emptyset)_s \underline{\leftrightarrow}^p_* (N',\emptyset)_{w'}$ and $(N',D)_{w'} \models (t(\varphi))^{(a_2,p)}$ for some $N'_{w'}$ and D (by Proposition 20)

iff $(M,\emptyset)_s \models \widehat{\exists}_p(t(\varphi))^{(a_2,p)}$

iff $(M,\emptyset)_s \models t(\exists_{(a_1,a_2)}\varphi)$ □

Laura Bozzelli et al. have also obtained an equivalent translation from $\mathcal{L}^{\mu}_{(\emptyset,A_2)}$ to $\mathcal{L}_{bq\blacksquare}$ by employing a relativization defined in $\mathcal{L}_{bq\blacksquare}$ [1].

VI. CONCLUSIONS AND DISCUSSION

The notion of CC-refinement generalizes the notions of bisimulation and refinement. An (a_1,a_2)-refinement model of a given model may be obtained from one bisimilar model of this model by **removing** some a_2-labelled transitions and **adding** some a_1-labelled transitions. This can be much easier to realize by programming. Based on the notion of CC-refinement, this paper considers the extended bisimulation quantified modal logic with the universal modality, and gives an equivalent translation from $\mathcal{L}^{\mu}_{(A_1,A_2)}$ to its language $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$, where $A_1, A_2 \neq \emptyset$. The language $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$ captures perfectly the characterizations of covariant and contravariant actions. Thus, $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$ may be considered as a specification language for describing the properties of a system which refers to covariant and contravariant actions and may formalize some interesting problems in the field of formal method.

As BQML, in some applications, e.g., planning optimization in artificial intelligence, a bisimulation of a given system, except for some inessential propositional properties, may be considered as its equivalent system. Furthermore, for example, given a specification presented as a Kripke model M which refers to the set A_1 (A_2) of passive (generative, resp.) actions, the problem whether this specification has an special implementation which satisfies a given property ψ may be boiled down to the model checking problem:

$$M \models t(\exists_{(a_1,a_2)}\psi),$$

that is,

$$M \models \widehat{\exists}_p(t(\psi))^{(a_2,p)}$$

where $p \notin t(\psi)$. Hence, based on the characterization of the bisimulation quantifiers in $\mathcal{L}_{(A_1,A_2)-ebq\blacksquare}$, which can be realized easily by programming, the problem whether there exists a desired implementation of a given specification involving passive and generative actions may be solved by using model checking technique.

The language $\mathcal{L}^{\mu}_{(\emptyset,A_2)}$ is indeed the one of refinement modal μ-calculus introduced in [1]. The language $\mathcal{L}_{(A_1,\emptyset)-ebq\blacksquare}$ involves different bisimulation quantifiers: $\widetilde{\exists}_p$ and $\widehat{\exists}$. The clauses $\widetilde{\exists}_p\varphi$ and $\widehat{\exists}\varphi$ are interpreted by:

$(M,a,D)_s \models \widetilde{\exists}_p\varphi$ iff $(M,a,D)_s \underline{\leftrightarrow}^p_* (N,a,D')_w \models \varphi$ for some $(N,a,D')_w$

$(M,a,D)_s \models \widehat{\exists}\varphi$ iff $(M,a,D)_s \underline{\leftrightarrow}_* (N,a,D')_w$ and $(N,a,D' \cup B)_w \models \varphi$ for some $(N,a,D')_w$ and $B \subseteq (S^N)^2 - (R^N_a \cup D')$

Also, we say that a function $t : \mathcal{L}^{\mu}_{(A_1,\emptyset)} \to \mathcal{L}_{(A_1,\emptyset)-ebq\blacksquare}$ is a translation if the following conditions hold,

$$\begin{aligned}
t(q) &= q\\
t(\neg\varphi) &= \neg t(\varphi)\\
t(\varphi_1 \wedge \varphi_2) &= t(\varphi_1) \wedge t(\varphi_2)\\
t(\Box_b\varphi) &= \Box_b t(\varphi)\\
t(\exists_{(a_1,\emptyset)}\varphi) &= \widehat{\exists}\, t(\varphi)\\
t(\nu x.\varphi) &= \widetilde{\exists}_p(p \wedge \blacksquare(p \to t(\varphi[p\backslash x]))) \quad \text{where } p \notin \varphi\\
t(\mu x.\varphi) &= \widetilde{\forall}_p(\blacksquare(t(\varphi[p\backslash x]) \to p) \to p) \text{ where } p \notin \varphi
\end{aligned}$$

Here, note that, we do not need a relativization because of $A_2 = \emptyset$. Moreover, similar as in this paper, the statements and their proofs with minor changes can still work for $\mathcal{L}_{(A_1,\emptyset)-ebq\blacksquare}$. That is, we still can obtain the result that every $\mathcal{L}^{\mu}_{(A_1,\emptyset)}$-formula is equivalent to its t-translation. We leave it to the reader to check this.

We have given the sound and complete axiomatization and decidability of CCRML$^{\mu}$ in another paper, which is waiting for the publication. We will further explore the axiomatization and decidability of EBQML, also referring to the axiomatization and decidability of BQML [12], [14], [15]. This investigation will be interesting and also complex.

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