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Adaptive Robust Control Based on Moore-Penrose Generalized Inverse for Underactuated Mechanical Systems

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ABSTRACT To address the uncertainty existing in underactuated mechanical systems (UMSs) and their nonholonomic servo constraints, we propose a class of adaptive robust control based on the Moore-Penrose generalized inverse for UMSs in this paper. The uncertainty is considered as (possible fast) time-varying and bounded. However, the bound is unknown. To estimate the bound information, an adaptive law is designed, which combines leakage type and dead-zone type. This adaptive law can simultaneously regulate the control effort and computation speed. The proposed control can guarantee deterministic system performance, which is analyzed by using Lyapunov method. The effectiveness of proposed control is shown by an example of simplified two-wheeled self-balancing robot.

INDEX TERMS Underactuated mechanical systems, servo constraints, adaptive robust control, adaptive law, Moore-Penrose generalized inverse.

I. INTRODUCTION

Underactuated manipulators [1], [2], UAVs [3], surface vessels [4], spacecraft [5], [6], underwater vehicles [7] and so on have generally become active fields for the past few years, because they are all underactuated mechanical systems (UMSs) that have the advantages of low number of actuators, weight, cost and energy consumption. However, complex internal dynamics, lack of feedback linearization and nonholonomic behavior also accompany UMSs, which increase the difficulty of designing controller for such systems. Even so, many brilliant work for such systems have been achieved in control field, such as sliding-mode [8]–[10], LQR [11], partial feedback linearization [12], [13], LMI [14], [15], H_∞ [16] and so on. However, most of the control methods mentioned above are difficult to deal with nonholonomic servo constraints [17].

In 1992, Udwadia and Kalaba proposed a novel Udwadia-Kalaba equation [18] based on the Moore-Penrose generalized inverse, and through which can easily establish the motion equation of a mechanical system and solve the constraint forces without considering Lagrange multiplier. The Udwadia and Kalaba's work is a significant breakthrough in the field of dynamics. Based on the breakthrough, Udwadia proposed a novel servo control [19], which can easily handle nonholonomic servo constraints without relying on additional sophisticated mathematical tools in the control design of fully actuated mechanical systems. Furthermore, based on Udwadia's and Kalaba's work, in [20], Chen systematically proposed the concept of servo control for constrained mechanical systems and the design method of constraint forces. In [21], Liu et al. proposed a new servo control based on pioneer's work, which makes the advantages of Udwadia's control apply in controlling UMSs. However, Liu's work does not consider the uncertainty existing in UMSs. In real life, the uncertainty should be nonnegligible.

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In engineering applications, it is crucial to distinguish whether certain part of the uncertainty is known or not. Engineers and researchers always acquire the known portion of uncertainty though numerous observed data. As long as the bound information of uncertainty is clearly determined, deterministic control methods can be developed, such as [22]–[25]. However, some data of uncertainty is not easy to observe or is unlikely to repeat many times accurately, such as earthquake data. So the bound information of uncertainty is not always available.

It is where the rub is that simultaneously handle uncertainty existing in UMSs and their nonholonomic servo constraints. Therefore, we propose a novel adaptive robust control for UMSs along with Liu's work. The proposed control is designed based on the Moore-Penrose generalized inverse to deal with nonlinear time-varying underactuated system with uncertainty and nonholonomic servo constraint. We consider that the uncertainty is (possible fast) time-varying and bounded. However, the bound is unknown. So the control design is not based on pre-given bound information of uncertainty, which may otherwise be too conservative and render excessively large control effort. We design an adaptive law, which combines leakage type and dead-zone type, to estimate the bound of the uncertainty. The leakage term can adjust the value of the adaptive parameter according to the system performance, which further helps to regulate the control effort. The dead-zone term helps to simplify the adaptive law calculation, which further speeds up the algorithm practice.

The main contributions of this paper are threefold. First, the proposed adaptive robust control is able to deal with nonlinear time-varying UMSs and nonholonomic servo constraints. Second, the uncertainty is (possibly fast) time-varying and bounded. However, the bound is unknown. An adaptive law is designed to emulate a constant parameter vector which may be relevant to the bounding set. Therefore, the designed control is not based on a pre-given bound. Third, the adaptive law merged leakage type and dead-zone type is designed to estimate the bound of the uncertainty and regulate the control effort and speed up the algorithm practice.

The rest of this paper is organized as follows. In Section 2, the dynamic equation of constrained UMSs with (possible fast) time-varying uncertainty is described, and the forms of the servo constraints are also presented, which can be holonomic and nonholonomic. In Section 3, a class of adaptive robust control with adaptive law is proposed based on the Moore-Penrose generalized inverse to realize approximate constraint-following of UMSs. Furthermore, the control guaranteeing deterministic system performance (including uniform boundedness and uniform ultimate boundedness) is theoretically analyzed by using Lyapunov method. In Section 4, an example of simplified two-wheeled self-balancing robot is given to demonstrate the effectiveness of the control. Finally, the "Conclusion" section is given.

II. PRELIMINARY OF MOORE-PENROSE GENERALIZED INVERSE

For a given matrix $\varphi \in R^{m\times n}$ (i.e., $\varphi = A(\alpha,t)\bar{H}^{-1}(\alpha,t)B)$, the rank of φ is $R(\varphi) \geq 1$, the eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Assume the singular value decomposition (SVD) of φ is [26]

$$\varphi = U\sum V^T = \sum_{i=1}^{r}\lambda_i u_i v_i^T \tag{1}$$

where $\sum = diag[\lambda_i]_{r\times r}$, $U = [u_1, u_2, \cdots, u_r] \in R^{m\times r}$ and $V = [v_1, v_2, \cdots, v_r] \in R^{n\times r}$ are unitary matrixes. Furthermore, $[u_1, u_2, \cdots, u_r] \in R^{m\times r}$ and $[v_1, v_2, \cdots, v_r] \in R^{n\times r}$ are orthonormal sets over R^m and R^n. Then the Moore-Penrose generalized inverse [27]–[29] of φ is

$$\varphi^+ = V\sum^{-1}U^T \tag{2}$$

where the sign of the superscript "+" represents the Moore-Penrose generalized inverse.

Lemma 1 (See [26]): φ^+ satisfies the following characteristics

$$\begin{aligned}\varphi^+\varphi &= (\varphi^+\varphi)^T\\ \varphi\varphi^+ &= (\varphi\varphi^+)^T\\ \varphi^+\varphi\varphi^+ &= \varphi^+\\ \varphi\varphi^+\varphi &= \varphi\end{aligned} \tag{3}$$

φ^+ always exists for any given φ. When φ is full rank, in particular, φ is with linearly independent columns (i.e., $\varphi^T\varphi$ is invertible), then φ^+ can be found by

$$\varphi^+ = (\varphi^T\varphi)^{-1}\varphi^T \tag{4}$$

This special case is so-called left inverse, since here $\varphi^+\varphi = I$. In contrast, when the rows of φ are linearly independent and $\varphi\varphi^T$ is invertible, then we have

$$\varphi^+ = \varphi^T(\varphi\varphi^T)^{-1} \tag{5}$$

This is called right inverse for $\varphi\varphi^+ = I$.

Lemma 2 (See [30]): For any given matrix $\varphi \in R^{m\times n}$, there always exists a Moore-Penrose generalized inverse $\varphi^+ \in R^{n\times m}$, and the Moore-Penrose generalized inverse is unique.

Lemma 3 (See [26]): Consider a matrix $\varphi \in R^{m\times n}$ with $R(\varphi) = r \geq 1$, the following properties hold:

$$R(\varphi^T) = R(\varphi^+) = R(\varphi^+\varphi) \tag{6}$$

$$N(\varphi) = R(I - \varphi^+\varphi) \tag{7}$$

where $R(\cdot)$ denotes the range space of matrix, $N(\cdot)$ denotes the null space of matrix.

Remark 1: The Moore-Penrose generalized inverse was initially reported in [27], [28]. The common application of using Moore-Penrose generalized inverse is to find a "best fit" (least squares) solution to linear equations which lack unique solution. Furthermore, it can be used to compute the minimum norm solution to linear equations which have multiple solutions. Besides, the Moore-Penrose generalized inverse is extensively applied in inverse dynamics, such as [31]–[34].

III. DYNAMIC MODEL AND SERVO CONSTRAINTS

The dynamic model of constrained UMSs with uncertainty can be described as follows [35]

$$H(\alpha(t), \delta(t), t)\ddot{\alpha} = Q(\alpha(t), \dot{\alpha}(t), \delta(t), t) + B\tau \tag{8}$$

where $t \in R$ is time, $\alpha \in R^n$ is generalized coordinate, $\dot{\alpha} \in R^n$ represents the velocity, $\ddot{\alpha} \in R^n$ represents the acceleration, $H(\alpha, \delta, t) \in R^{n\times n}$ is the inertia matrix, $Q(\alpha, \dot{\alpha}, \delta, t) \in R^n$ represents the known force imposed on the system whose constraints are released, $\tau \in R^m$ is constraint force, $B \in R^{n\times m}$ is the matrix of control coefficients, $\delta \in \sum \subset R^p$ is the (possible fast time-varying) uncertain parameter. Furthermore, the set $\sum \subset R^p$, which stands for the possible bound of δ, is compact but unknown. Notice that the vector α can also chosen based on the specifies of the problem and the matrices/vectors $H(\alpha, \delta, t)$, $Q(\alpha, \dot{\alpha}, \delta, t)$ and B are of appropriate dimensions.

Assumption 1: The functions $H(\cdot)$ and $Q(\cdot)$ are continuous (this can be generalized to be Lebesgue measurable in t).

In Eq. (8), the $\tau \in R^m$ is provided by active servo controls under some legitimate pre-specified servo constraints. Now, we assume that the system is subjected to following servo constraints [18], [19], [36]

$$\sum_{i=1}^{n} A_{li}(\alpha, t)\dot{\alpha} = c_l(\alpha, t), \quad l = 1, 2, \cdots, m, \tag{9}$$

where $\dot{\alpha}_i$ is the ith component of $\dot{\alpha}$ and $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1 in α ant t, $1 \le m \le n$.

These constraints are the first-order form and may be nonintegrable and thus is nonholonomic in general. They can be put in the matrix form

$$A(\alpha, t)\dot{\alpha} = c(\alpha, t) \tag{10}$$

where $A = [A_{li}]_{m\times n}$ and $c = [c_1, c_2, \cdots, c_m]^T$,.

We assume that these constraint equations are differentiable. Differentiating the constraints Eq. (9) with respect to t yields

$$\sum_{i=1}^{n} A_{li}(\alpha, t)\ddot{\alpha} + \sum_{i=1}^{n}(\frac{d}{dt}A_{li}(\alpha, t))\dot{\alpha}_i = \frac{d}{dt}c_l(\alpha, t) \tag{11}$$

where

$$\frac{d}{dt}A_{li}(\alpha, t) = \sum_{k=1}^{n} \frac{\partial A_{li}(\alpha, t)}{\partial \alpha_k}\dot{\alpha}_k + \frac{\partial A_{li}(\alpha, t)}{\partial t}$$

$$\frac{d}{dt}c_l(\alpha, t) = \sum_{k=1}^{n} \frac{\partial c_l(\alpha, t)}{\partial \alpha_k}\dot{\alpha}_k + \frac{\partial c_l(\alpha, t)}{\partial t}$$

Then, the Eq. (11) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{n} A_{li}(\alpha, t)\ddot{\alpha}_i &= -\sum_{i=1}^{n}(\frac{d}{dt}A_{li}(\alpha, t))\dot{\alpha}_i + \frac{d}{dt}c_l(\alpha, t)\\ &=: b_l(\alpha, \dot{\alpha}, t)\end{aligned} \tag{12}$$

which, in matrix form, is

$$A(\alpha, t)\ddot{\alpha} = b(\alpha, \dot{\alpha}, t) \tag{13}$$

where $b = [b_1, b_2, \cdots, b_m]^T$, which is the second-order form [37].

Remark 2: For a given configuration, the forms of Eqs. (10) and (13) are interpreted as follows: the Eq. (10) governs the velocity $\dot{\alpha}$, and the Eq. (13) governs the acceleration $\ddot{\alpha}$. From [17], a large variety of control problems, such as optimality, trajectory following and stabilization, could apply the forms of Eqs. (10) and (13).

Remark 3: A constrained underactuated mechanical system (UMS) could be described by Eq. (8) with Newtonian or Lagrangian mechanics, which is a second-order equation of motion. In this paper, the focus is to let an UMS, equipped with servo controls follow a set of constraints (Eqs. (10) and (13),) by generating the required constraint forces through servo controls, which is a control problem in reality. For pre-specified servo constraints, which can be the pre-given trajectory and set-point, we combine holonomic and nonholonomic constraints in Eq. (10) and further convert them to second-order form [Eq. (13)]. The conversion can acquire the mathematical conformity between the system model and constraints, which is very suitable to further design the constraint forces in the next section.

IV. ADAPTIVE ROBUST CONTROL DESIGN

The uncertainty existing in UMSs should be considered in reality. Therefore, we decompose the matrices H, Q as follows [38]:

$$H(\alpha, \delta, t) = \bar{H}(\alpha, t) + \Delta H(\alpha, \delta, t) \tag{14}$$

$$Q(\alpha, \dot{\alpha}, \delta, t) = \bar{Q}(\alpha, \dot{\alpha}, t) + \Delta Q(\alpha, \dot{\alpha}, \delta, t) \tag{15}$$

where $\bar{H}$ and $\bar{Q}$ denote the "nominal" portions, while ΔH and ΔQ are the corresponding uncertain ones. We assume that $\bar{H} > 0$. Here, the functions $\bar{H}(\cdot)$, $\Delta H(\cdot)$, $\bar{Q}(\cdot)$, $\Delta Q(\cdot)$ are all continuous.

Let

$$E(\alpha, t) := \bar{H}^{-1}(\alpha, t) \tag{16}$$

$$\Delta E(\alpha, \delta, t) := H^{-1}(\alpha, \delta, t) - \bar{H}^{-1}(\alpha, t) \tag{17}$$

$$F(\alpha, \delta, t) := \bar{H}(\alpha, t)H^{-1}(\alpha, \delta, t) - I \tag{18}$$

Combining Eqs. (16) to (18), we get

$$H^{-1}(\alpha, \delta, t) = E(\alpha, t) + \Delta E(\alpha, \delta, t) \tag{19}$$

$$\Delta E(\alpha, \delta, t) = E(\alpha, t)F(\alpha, \delta, t) \tag{20}$$

Assumption 2: For each $(\alpha, t) \in R^n \times R$, rank $A(\alpha, t) \ge 1$.

Assumption 3: Based on the provision of Assumption. 2, for given $P \in R^{m\times m}$, $P > 0$, let

$$\Upsilon(\alpha, \delta, t) := PA(\alpha, t)E(\alpha, t)F(\alpha, \delta, t)B \times [A(\alpha, t)\bar{H}^{-1}(\alpha, t)B]^{+}P^{-1} \quad (21)$$

There exists a constant $\Gamma_E > -1$ such that for all $(\alpha, t) \in R^n \times R$,

$$\frac{1}{2}\min_{\delta\in\sum} \lambda_m(\Upsilon(\alpha, \delta, t) + \Upsilon^T(\alpha, \delta, t)) \geq \Gamma_E \quad (22)$$

Let

$$\bar{b}(\alpha, \dot{\alpha}, t) := b(\alpha, \dot{\alpha}, t) - A(\alpha, t)\bar{H}^{-1}(\alpha, t)\bar{Q}(\alpha, \dot{\alpha}, t) \quad (23)$$

Assumption 4: There is $(A(\alpha, t)\bar{H}^{-1}(\alpha, t)B) \in R^{m\times n} \times R^{n\times n} \times R^m$ for all $\bar{b}(\alpha, \dot{\alpha}, t)$ and P in Assumption. 3, we assume

$$\bar{b}(\alpha, \dot{\alpha}, t) = (A(\alpha, t)\bar{H}^{-1}(\alpha, t)B)(A(\alpha, t) \times \bar{H}^{-1}(\alpha, t)B)^{+}\bar{b}(\alpha, \dot{\alpha}, t) \quad (24)$$

and

$$P^{-1} = (A(\alpha, t)\bar{H}(\alpha, t)^{-1}B)(A(\alpha, t)\bar{H}^{-1}(\alpha, t)B)^{+}P^{-1} \quad (25)$$

Assumption 5:

1) There exists an unknown constant vector $\chi \in (0, \infty)^k$ and a known function $\Psi(\cdot) : (0, \infty)^k \times R^n \times R^n \times R \to R_+$ such that, for all $(\alpha, \dot{\alpha}, t) \in R^n \times R^n \times R, \delta \in \sum$,

$$\begin{aligned}(1 + \Gamma_E)^{-1} \max_{\delta\in\sum}[\|PA(\alpha, t)E(\alpha, t)\Delta Q(\alpha, \dot{\alpha}, \delta, t) \\ + PA(\alpha, t)\Delta E(\alpha, \delta, t)Q(\alpha, \dot{\alpha}, \delta, t) \\ + PA(\alpha, t)\Delta E(\alpha, \delta, t)B(p_1 + p_2)\|] \\ \leq \Psi(\chi, \alpha, \dot{\alpha}, t)\end{aligned} \quad (26)$$

2) For each $(\chi, \alpha, \dot{\alpha}, t)$, the function $\Psi(\chi, \alpha, \dot{\alpha}, t)$ can be linearly factorized with respect to χ; there exists a function $\tilde{\Psi}(\cdot) : R^n \times R^n \times R \to R_+$ such that

$$\Psi(\chi, \alpha, \dot{\alpha}, t) = \chi^T\tilde{\Psi}(\alpha, \dot{\alpha}, t) \quad (27)$$

Now, we propose the adaptive robust control

$$\tau(t) = p_1(\alpha, \dot{\alpha}, t) + p_2(\alpha, \dot{\alpha}, t) + p_3(\hat{\chi}, \alpha, \dot{\alpha}, t) \quad (28)$$

with

$$p_1(\alpha, \dot{\alpha}, t) = (A(\alpha, t)\bar{H}^{-1}(\alpha, t)B)^{+}[b(\alpha, \dot{\alpha}, t) - A(\alpha, t)\bar{H}^{-1}(\alpha, t)\bar{Q}(\alpha, \dot{\alpha}, t)] \quad (29)$$

$$p_2(\alpha, \dot{\alpha}, t) = -\kappa(A(\alpha, t)\bar{H}^{-1}(\alpha, t)B)^{+}P^{-1}e(\alpha, \dot{\alpha}, t) \quad (30)$$

$$p_3(\hat{\chi}, \alpha, \dot{\alpha}, t) = -(A(\alpha, t)\bar{H}^{-1}(\alpha, t)B)^{+}P^{-1} \times \gamma(\hat{\chi}, \alpha, \dot{\alpha}, t)\sigma(\hat{\chi}, \alpha, \dot{\alpha}, t)\Psi(\hat{\chi}, \alpha, \dot{\alpha}, t) \quad (31)$$

where

$$e(\alpha, \dot{\alpha}, t) = A(\alpha, t)\dot{\alpha} - c(\alpha, t) \quad (32)$$

$$\sigma(\hat{\chi}, \alpha, \dot{\alpha}, t) = e(\alpha, \dot{\alpha}, t)\Psi(\hat{\chi}, \alpha, \dot{\alpha}, t) \quad (33)$$

$$\gamma(n, \epsilon, \sigma) = \frac{\|\sigma\|^{n-2}}{\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1}} \quad (34)$$

Here $\kappa, \epsilon > 0$, $\kappa, \epsilon \in R$ and $n = 2, 3, \cdots$ are scalar constants.

The parameter $\hat{\chi}$ is governed by the following adaptive law:

$$\dot{\hat{\chi}} = \begin{cases} k_1\tilde{\Psi}(\alpha, \dot{\alpha}, t)\|e\| - k_2\hat{\chi} & \text{if } \tilde{\Psi}(\alpha, \dot{\alpha}, t)\|e\| > \varepsilon \\ -k_2\hat{\chi} & \text{if } \tilde{\Psi}(\alpha, \dot{\alpha}, t)\|e\| \leq \varepsilon \end{cases} \quad (35)$$

$\hat{\chi}_i(t_0) > 0$ (where $\hat{\chi}_i$ is the ith component of the vector $\hat{\chi}$, $i = 1, \ldots, k$), $k_1, k_2 > 0$, $k_1, k_2 \in R$.

Remark 4: Because of the uncertainty (that may be initial condition deviation or modeling error) existing in UMSs, the $e(\alpha, \dot{\alpha}, t) \neq 0$ is possible. Therefore, from the view of approximate constraint-following, the proposed adaptive robust control based on Moore-Penros generalized inverse is designed as three portions: p_1, p_2 and p_3. p_1 is designed as the nominal portion. p_2 is designed to suppress initial condition deviation. p_3 is designed to suppress uncertainty. Suppose the uncertainty is time-varying and bounded. However, the bound is unknown. Therefore, an adaptive law shown in Eq. (35), which combines leakage type and dead-zone type, is designed to estimate the bound of the uncertainty. The leakage term can adjust the value of the adaptive parameter according to the system performance, which further helps to regulate the control effort. The dead-zone term helps to simplify the adaptive law calculation, which further speeds up the algorithm practice. The resulting adaptive robust control guarantees deterministic system performance and regulates the control effort.

Remark 5: $\gamma(n, \epsilon, \sigma)$ is designed as the robust control gain, which is fractional and performance-based one. If the system performance gradually decreases, that is, $\|\sigma\|$ increases, then $\gamma(n, \epsilon, \sigma)$ will increase to augment p_3 to enhance system performance. On the contrary, if the system performance gradually gets better, that is, $\|\sigma\|$ decreases, then $\gamma(n, \epsilon, \sigma)$ will decrease to reduce p_3 and thus reduce the control cost.

Theorem 1 (See [39]): Let $\beta := [e^T\ (\hat{\chi} - \chi)^T]^T \in R^{m+k}$. Consider the underactuated mechanical system of Eq. (8), and suppose that Eq. Assumption. 2 to 5 are met. The adaptive robust control Eq. (28) renders the following performance:

1) *Uniform boundedness: For any $r > 0$, there is a $d(r) < \infty$, such that if $\|\beta(t_0)\| \leq r$, then $\|\beta(t)\| \leq d(r)$ for all $t \geq t_0$.*
2) *Uniform ultimate boundedness: For any $r > 0$ with $\|\beta(t_0)\| \leq r$, there exists a $\underline{d} > 0$, such that $\|\beta(t)\| \leq \bar{d}$ for any $\bar{d} > \underline{d}$ as $t \geq t_0 + T(\bar{d}, r)$, where $T(\bar{d}, r) < \infty$.*

Proof: A legitimate Lyapunov function candidate is given [40]:

$$V(e, \hat{\chi} - \chi) = e^TPe + k_1^{-1}(1 + \Gamma_E)(\hat{\chi} - \chi)^T(\hat{\chi} - \chi) \quad (36)$$

For a given uncertainty $\delta(\cdot)$ and the corresponding trajectory $\alpha(\cdot)$, $\dot{\alpha}(\cdot)$ and $\hat{\chi}(\cdot)$ of the control system. In the proof, arguments of functions are largely omitted except for a few critical ones. The derivative of V is given by

$$\dot{V} = 2e^T P\dot{e} + 2k_1^{-1}(1+\Gamma_E)(\hat{\chi}-\chi)^T\dot{\hat{\chi}} \quad (37)$$

We'll analyze each term of Eq. (37) separately. For the first term of the right hand side (RHS) of Eq. (37), we can obtain

$$\begin{aligned} 2e^T P\dot{e} &= 2e^T P[A\ddot{\alpha} - b] \\ &= 2e^T P[AH^{-1}Q + AH^{-1}B\tau - b] \\ &= 2e^T P[AH^{-1}Q + AH^{-1}B(p_1+p_2+p_3) - b] \end{aligned} \quad (38)$$

By Eqs. (14), (15) and (19), we have

$$\begin{aligned} &AH^{-1}Q + AH^{-1}B(p_1+p_2+p_3) - b \\ &= A(E+\Delta E)(\bar{Q}+\Delta Q) + A(E+\Delta E)B(p_1 + p_2 + p_3) - b \\ &= AE\bar{Q} + AE\Delta Q + A\Delta EQ + AEBp_1 + AEBp_2 \\ &\quad + A\Delta EB(p_1+p_2) + A(E+\Delta E)Bp_3 - b \end{aligned} \quad (39)$$

Based on Eq. (24) in Assumption. 4 and Eq. (29), we have

$$AE\bar{Q} + AEBp_1 - b = 0 \quad (40)$$

By Eq. (30) and Eq. (25) in Assumption. 4,

$$\begin{aligned} 2e^T PAEBp_2 &= 2e^T PAEB\{-\kappa(A\bar{H}^{-1}B)^+P^{-1}e\} \\ &= -2\kappa e^T e = -2\kappa\|e\|^2 \end{aligned} \quad (41)$$

Next, based on Assumption. 5 (1), we have

$$\begin{aligned} &2e^T P[AE\Delta Q + A\Delta EQ + A\Delta EB(p_1+p_2)] \\ &\le 2\|e\|\|PAE\Delta Q + PA\Delta EQ + PA\Delta EB(p_1+p_2)\| \\ &\le 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \end{aligned} \quad (42)$$

By Eqs. (20), (31) and (33) and Eq. (30) and Eq. (25) in Assumption. 4,

$$\begin{aligned} &2e^T PA(E+\Delta E)Bp_3 \\ &= 2e^T PAEB\{-(A\bar{H}^{-1}B)^+P^{-1}\gamma\sigma(\hat{\chi},\alpha,\dot{\alpha},t) \\ &\quad \times \Psi(\hat{\chi},\alpha,\dot{\alpha},t)\} + 2e^T PA\Delta EB\{-(A\bar{H}^{-1}B)^+ \\ &\quad \times P^{-1}\gamma\sigma(\hat{\chi},\alpha,\dot{\alpha},t)\Psi(\hat{\chi},\alpha,\dot{\alpha},t)\} \\ &= -2\gamma\|\sigma\|^2 + 2e^T PAEFB\{-(A\bar{H}^{-1}B)^+ \\ &\quad \times P^{-1}\gamma\sigma(\hat{\chi},\alpha,\dot{\alpha},t)\Psi(\hat{\chi},\alpha,\dot{\alpha},t)\} \end{aligned} \quad (43)$$

Based on Assumption. 3 and Rayleigh's principle [30], we have

$$\begin{aligned} &2e^T PAEFB\{-(A\bar{H}^{-1}B)^+P^{-1}\gamma\sigma\Psi(\hat{\chi},\alpha,\dot{\alpha},t)\} \\ &= -2\gamma\sigma^T\{PAEFB(A\bar{H}^{-1}B)^+P^{-1}\}\sigma \\ &= -2\gamma\sigma^T\frac{1}{2}\{PAEFB(A\bar{H}^{-1}B)^+P^{-1} \\ &\quad + [PAEFB(A\bar{H}^{-1}B)^+P^{-1}]^T\}\sigma \\ &= -2\gamma\sigma^T\frac{1}{2}(\Upsilon+\Upsilon^T)\sigma \\ &\le -2\gamma\frac{1}{2}\lambda_m(\Upsilon+\Upsilon^T)\|\sigma\|^2 \\ &\le -2\gamma\Gamma_E\|\sigma\|^2 \end{aligned} \quad (44)$$

Combining Eq. (43) and Eq. (44)

$$2e^T PA(E+\Delta E)Bp_3 \le -2\gamma(1+\Gamma_E)\|\sigma\|^2 \quad (45)$$

Combing Eqs. (34), (38), (40) to (42) and (45), we have

$$\begin{aligned} &2e^T P\dot{e} \\ &\le -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\gamma(1+\Gamma_E)\|\sigma\|^2 \\ &= -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\frac{\|\sigma\|^{n-2}}{\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1}} \\ &\quad \times (1+\Gamma_E)\|\sigma\|^2 \\ &= -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\frac{\|\sigma\|^{n}}{\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1}} \\ &\quad \times (1+\Gamma_E) \\ &\le -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\frac{\|\sigma\|^{n} - \epsilon^n}{\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1}} \\ &\quad \times (1+\Gamma_E) \\ &\le -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\frac{(\|\sigma\|-\epsilon)(\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1})}{\|\sigma\|^{n-1} + \|\sigma\|^{n-2}\epsilon + \cdots + \|\sigma\|\epsilon^{n-2} + \epsilon^{n-1}} \\ &\quad \times (1+\Gamma_E) \\ &= -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - 2\|\sigma\|(1+\Gamma_E) + 2\epsilon(1+\Gamma_E) \\ &= -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)[\Psi(\chi,\alpha,\dot{\alpha},t) \\ &\quad - \Psi(\hat{\chi},\alpha,\dot{\alpha},t)] + 2\epsilon(1+\Gamma_E) \\ &= -2\kappa\|e\|^2 + 2\|e\|(1+\Gamma_E)(\chi-\hat{\chi})^T\tilde{\Psi}(\alpha,\dot{\alpha},t) \\ &\quad + 2\epsilon(1+\Gamma_E) \end{aligned} \quad (46)$$

For the second term of the RHS of Eq. (37), by using adaptive law Eq. (35), if $\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\| > \varepsilon$, we can obtain

$$\begin{aligned} &2\,k_1^{-1}(1+\Gamma_E)(\hat{\chi}-\chi)^T\dot{\hat{\chi}} \\ &= 2\,k_1^{-1}(1+\Gamma_E)(\hat{\chi}-\chi)^T(k_1\tilde{\Psi}(\alpha,\dot{\alpha},t))\|e\| \\ &\quad - k_2\hat{\chi}) \\ &= 2(1+\Gamma_E)(\hat{\chi}-\chi)^T\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\| \\ &\quad - 2\,k_1^{-1}k_2(1+\Gamma_E)(\hat{\chi}-\chi)^T(\hat{\chi}-\chi) \\ &\quad - 2\,k_1^{-1}k_2(1+\Gamma_E)(\hat{\chi}-\chi)^T\chi \\ &\le 2(1+\Gamma_E)(\hat{\chi}-\chi)^T\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\| \\ &\quad - 2\,k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2 \\ &\quad + 2\,k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|\|\chi\| \end{aligned}$$

$$\begin{aligned}&\leq 2(1+\Gamma_E)(\hat{\chi}-\chi)^T\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\|\\&\quad -k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\end{aligned}\tag{47}$$

Combing Eqs. (37), (46) and (47) we get

$$\begin{aligned}\dot{V} \leq &-2\kappa\|e\|^2+2\epsilon(1+\Gamma_E)-k_1^{-1}k_2(1+\Gamma_E)\\&\times\|\hat{\chi}-\chi\|^2+k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\end{aligned}\tag{48}$$

If $\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\|\leq\varepsilon$, we can obtain

$$\begin{aligned}&2k_1^{-1}(1+\Gamma_E)(\hat{\chi}-\chi)^T\dot{\hat{\chi}}\\&= -2k_1^{-1}k_2(1+\Gamma_E)(\hat{\chi}-\chi)^T\hat{\chi}\\&= -2k_1^{-1}k_2(1+\Gamma_E)(\hat{\chi}-\chi)^T(\hat{\chi}-\chi)\\&\quad -2k_1^{-1}k_2(1+\Gamma_E)(\hat{\chi}-\chi)^T\chi\\&\leq -2k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +2k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|\|\chi\|\\&\leq -k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\end{aligned}\tag{49}$$

Combing Eqs. (37), (46) and (49), we have

$$\begin{aligned}\dot{V} &\leq -2\kappa\|e\|^2+2\|e\|(1+\Gamma_E)(\chi-\hat{\chi})^T\tilde{\Psi}(\alpha,\dot{\alpha},t)\\&\quad +2\epsilon(1+\Gamma_E)-k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\\&\leq -2\kappa\|e\|^2+2(1+\Gamma_E)\|\chi-\hat{\chi}\|\varepsilon\\&\quad +2\epsilon(1+\Gamma_E)-k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\end{aligned}\tag{50}$$

Based on Eqs. (48) and (50), we have that for all $\tilde{\Psi}(\alpha,\dot{\alpha},t)\|e\|$

$$\begin{aligned}\dot{V} &\leq -2\kappa\|e\|^2+2(1+\Gamma_E)\|\chi-\hat{\chi}\|\varepsilon\\&\quad +2\epsilon(1+\Gamma_E)-k_1^{-1}k_2(1+\Gamma_E)\|\hat{\chi}-\chi\|^2\\&\quad +k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2\\&\leq -\rho\|\beta\|^2+\Omega\end{aligned}\tag{51}$$

where $\rho:=\min\{2\kappa, k_1^{-1}k_2(1+\Gamma_E)\}$, $\|\beta\|^2:=\|e\|^2+\|\hat{\chi}-\chi\|^2$, $\Omega:=2(1+\Gamma_E)\|\chi-\hat{\chi}\|\varepsilon+2\epsilon(1+\Gamma_E)+k_1^{-1}k_2(1+\Gamma_E)\|\chi\|^2$

Upon invoking the standard arguments as in Chen [39] and Khalil [40], we conclude that the solution of the controlled UMSs satisfies uniform boundedness with

$$d(r)=\begin{cases}\sqrt{\dfrac{\Lambda_2}{\Lambda_1}}R & \text{if} \quad r\leq R\\ \sqrt{\dfrac{\Lambda_2}{\Lambda_1}}r & \text{if} \quad r>R\end{cases}\tag{52}$$

$$R=\sqrt{\frac{\Omega}{\rho}}\tag{53}$$

where $\Lambda_1 = \min\{\lambda_{min}(P), k_1^{-1}(1+\Gamma_E)\}$ and $\Lambda_2 = \max\{\lambda_{max}(P), k_1^{-1}(1+\Gamma_E)\}$.

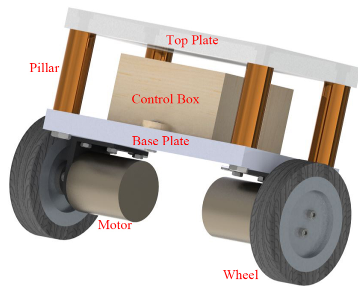

FIGURE 1. Three dimensions diagram of the TWSBR.

Furthermore, uniform ultimate boundedness also follows with

$$\underline{d}=\sqrt{\frac{\Lambda_2}{\Lambda_1}}R\tag{54}$$

$$T(\bar{d},r)=\begin{cases}0, & \text{if } r\leq\bar{d}\sqrt{\dfrac{\Lambda_1}{\Lambda_2}}\\ \dfrac{\Lambda_2r^2-(\Lambda_1^2/\Lambda_2)\bar{d}^2}{\rho\bar{d}^2(\Lambda_1/\Lambda_2)-\Omega}, & \text{otherwise.}\end{cases}\tag{55}$$

□

V. AN ILLUSTRATIVE EXAMPLE

We choose a typical UMS to verify the effectiveness of the proposed control, which is the two-wheeled self-balancing robot (TWSBR) shown in FIGURE. 1. This robot has been widely used as a test bed for nonlinear controls of UMSs, such as [11], [41]–[43]. In this paper, in order to highlight the characteristic of underactuation of the TWSBR and simplify the operation, we simplify the robot to a two-dimensional plane as shown in FIGURE. 2. The simplified TWSBR consists of a wheel and a pendulum, and it has two degrees of freedom but only one drive. The symbol definition of parameters and variables of the simplified TWSBR are detailedly listed in TABLE. 1. Choosing $\alpha=[\theta_w\ \theta_p]^T$ as the generalized coordinate of the simplified TWSBR, by Lagrangian mechanics, the dynamic model of the simplified TWSBR is given as follows:

$$\begin{cases}(\frac{3}{2}m_wr^2+m_pr^2)\ddot{\theta}_w+m_plr\sin(\theta_p)\ddot{\theta}_p\\ \quad +m_plr\dot{\theta}_p^2\cos(\theta_p)=\tau\\ \frac{4}{3}m_pl^2\ddot{\theta}_p+m_plr\sin(\theta_p)\ddot{\theta}_w+m_plr\dot{\theta}_w\dot{\theta}_p\cos(\theta_p)\\ \quad -m_plr\dot{\theta}_w\dot{\theta}_p^2\cos(\theta_p)+m_pgl\dot{\theta}_p\cos(\theta_p)=0\end{cases}\tag{56}$$

which can be expressed in Eq. (8) as

$$H(\alpha(t),\delta(t),t)\ddot{\alpha}=Q(\alpha(t),\dot{\alpha}(t),\delta(t),t)+B\tau\tag{57}$$

where

$$H=\begin{pmatrix}\frac{3}{2}m_wr^2+m_pr^2 & m_plr\sin(\theta_p)\\ m_plr\sin(\theta_p) & \frac{4}{3}m_pl^2\end{pmatrix},$$

$$Q=\begin{pmatrix}Q_1\\Q_2\end{pmatrix},\quad B=\begin{pmatrix}1\\0\end{pmatrix},$$

$$Q_1=-m_plr\dot{\theta}_p^2\cos(\theta_p)$$

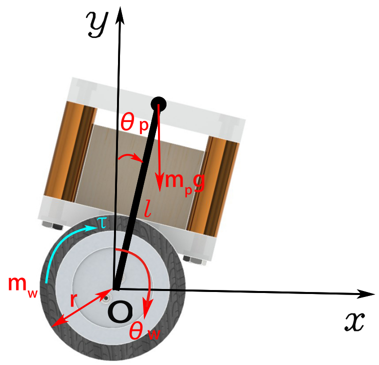

FIGURE 2. Two dimensions digram of the simplified TWSBR.

TABLE 1. The parameters and variables of the simplified TWSBR.

Parameters and Variables	Physical Significance	Unite
m_w	mass of the wheel	Kg
m_p	mass of the pendulum	Kg
l	length of pendulum	m
r	radius of wheel	m
θ_w	rotational angle of the wheel	rad
θ_p	rotational angle of the pendulum	rad
g	acceleration of gravity	m/s^2
τ	control torque actuating the pendulum arm	$N \cdot m$

$$Q_2 = -m_p lr\dot{\theta}_w\dot{\theta}_p \cos(\theta_p) + m_p lr\dot{\theta}_w\dot{\theta}_p^2 \cos(\theta_p) - m_p gl\dot{\theta}_p \cos(\theta_p)$$

To verify the robustness of the control for the system, m_w, m_p are considered as uncertain parameters (hence, $m_w = \bar{m}_w + \Delta m_w$, $m_p = \bar{m}_p + \Delta m_p$). We choose $\bar{m}_w = 1Kg$, $\Delta m_w = 0.1\cos(t)Kg$, $\bar{m}_p = 1Kg$, $\Delta m_p = 0.1\sin(t)Kg$, $r = 0.1m$, $l = 0.2\ m$, $g = 9.8\ m/s^2$ and $P = I_{1\times 1}$. Thus, Assumption. 2 to 4 can be easily verified. Assumption. 5 is met by choosing

$$\Psi(\chi, \alpha, \dot{\alpha}, t) = \chi_1\|\dot{\alpha}\|^2 + \chi_2\|\dot{\alpha}\| + \chi_3 = \begin{pmatrix}\chi_1 & \chi_2 & \chi_3\end{pmatrix}\begin{pmatrix}\|\dot{\alpha}\|^2 \\ \|\dot{\alpha}\| \\ 1\end{pmatrix} = \chi^T\tilde{\Psi}(\alpha, \dot{\alpha}, t) \tag{58}$$

where $\chi_{1,2,3}$ are unknown constant parameters. It is interesting to note that we can also satisfy Assumption. 5 by an alternative choice of $\tilde{\Psi}(\cdot)$ as

$$\chi_1\|\dot{\alpha}\|^2 + \chi_2\|\dot{\alpha}\| + \chi_3 \le \chi(\|\dot{\alpha}\| + 1)^2 =: \chi^T\tilde{\Psi}(\alpha, \dot{\alpha}, t) \tag{59}$$

where $\chi = max\{\chi_1, \frac{\chi_2}{2}, \chi_3\}$.

We assume that the simplified TWSBR is constrained by following nonholonomic servo constraint

$$\dot{\theta}_w - \dot{\theta}_p\cos(\theta_p) = \cos(t) \tag{60}$$

Differentiating Eq. (60) respect to time t once, we get

$$\ddot{\theta}_w - \ddot{\theta}_p\cos(\theta_2) + \dot{\theta}_p^2\sin(\theta_p) = -\sin(t) \tag{61}$$

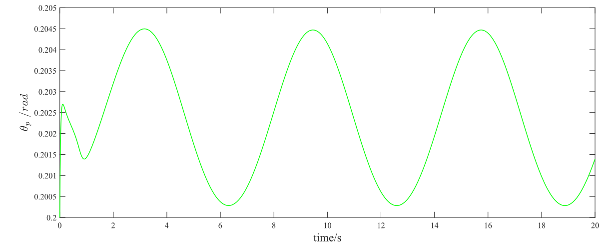

FIGURE 3. Time history of θ_p.

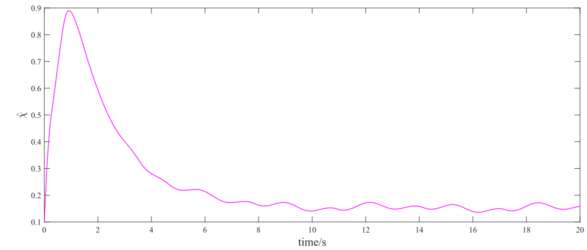

FIGURE 4. Time history of $\hat{\chi}$.

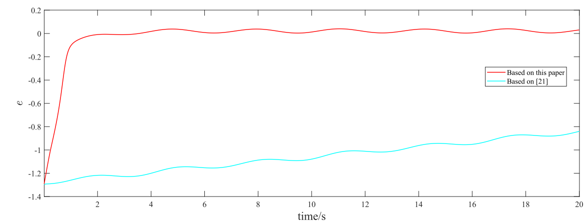

FIGURE 5. Time history of e.

Then, constraints Eqs. (60), and (61) can respectively be cast into the form of Eqs. (10) and (13) with

$$A_2 = \begin{pmatrix}1 & -\cos(\theta_p)\end{pmatrix}, \quad c_2 = \begin{pmatrix}\cos(t)\end{pmatrix}, \quad b_2 = \begin{pmatrix}-\dot{\theta}_p^2\sin(\theta_p) - \sin(t)\end{pmatrix}$$

FIGURE. 3 to 9 show the simulations by choosing $\kappa = 1$, $n = 5$, $\epsilon = 0.01$, $k_1 = 1$, $k_2 = 0.5$, $\varepsilon = 0.001$. The initial values of the simplified TWSBR are given as $\alpha(0) = [-0.1\ 0.2]^T$, $\dot{\alpha}(0) = [-0.2\ 0.1]^T$, $\hat{\chi}(0) = 0.1$. The simulation results are obtained through $ode15i$ algorithm in MATLAB. FIGURE. 3 shows the time history of the θ_p. FIGURE. 4 shows the time history of the adaptive parameter $\hat{\chi}$. Initially, the $\hat{\chi}$ quickly increases, because of initial condition deviation. After a while, the initial condition deviation is gradually suppressed and the system perform is met, resulting in the $\hat{\chi}$ decline and stability. The error between desired trajectory and actual that is e is shown in FIGURE. 5. As a comparison, by almost the same computational technique, the control in [23] is utilized, and whose time history of error is also shown in FIGURE. 5. This comparison denotes that the control we designed can get a good system performance under the influence of uncertainty. The time history of the control τ acted on the wheel is shown in FIGURE. 6. FIGURE. 3 to 6 verify that the proposed control can solve both uncertainties existing in simplified TWSBR

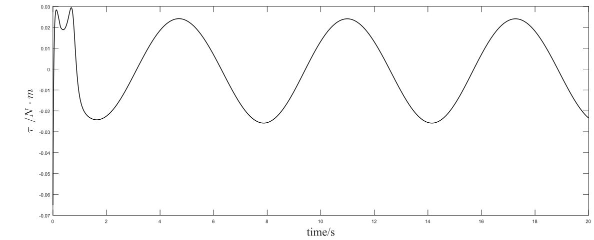

FIGURE 6. Time history of τ.

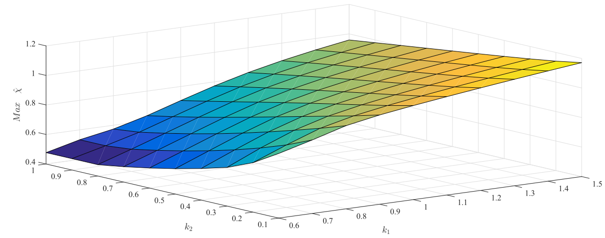

FIGURE 7. The max value of $\hat{\chi}$ VS k_1 and k_2.

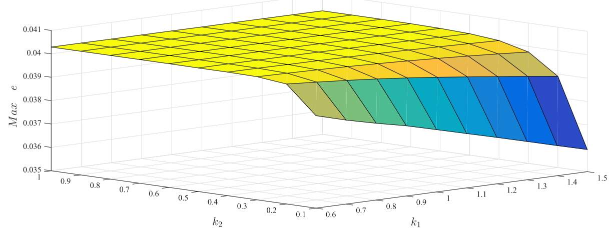

FIGURE 8. The max value of e VS k_1 and k_2.

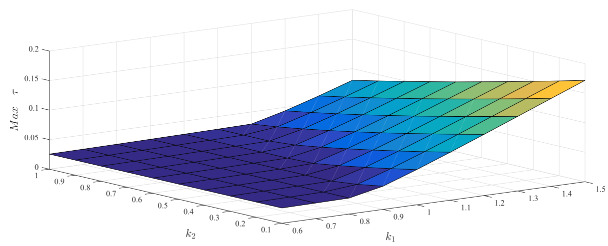

FIGURE 9. The max value of τ VS k_1 and k_2.

and its nonholonomic servo constraint. FIGURE. 7 shows the influence of the adaptive law parameters k_1 and k_2 on the max value of $\hat{\chi}$. FIGURE. 8 shows the influence of the adaptive law parameters k_1 and k_2 on the max value of e. FIGURE. 9 shows the influence of the adaptive law parameters k_1 and k_2 on the max value of τ.

VI. CONCLUSION

In this paper, we propose a novel adaptive robust control based on Moore-Penrose generalized inverse with adaptive law to address uncertainty existing in UMSs and its nonholonomic servo constraints, which is deterministic and can guarantee the deterministic performance. Theoretical analysis and numerical simulation demonstrate the effectiveness of the proposed control. Furthermore, for a practical engineering problem, based on this paper, we can easily acquire the constraint forces by following three steps: first step, establish the dynamic model of a constrained UMS with uncertainty by Newtonian or Lagrangian mechanics. Second step, give legitimate pre-specified servo constraints based on actual engineering requirement and then convert them to Eqs. (10) and (13). Third step, check the Assumption. 2 to 5, and then the constraint forces can be solved by Eq. (28), which makes the system subject to the servo constraints.

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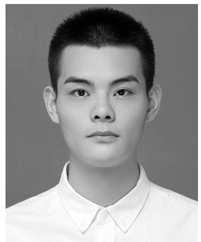

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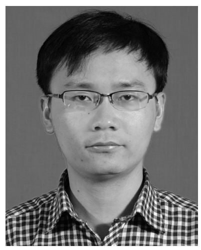

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