IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received October 3, 2019, accepted October 21, 2019, date of publication October 28, 2019, date of current version November 11, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2949920

Underdetermined Direction-of-Arrival Estimation Using Difference Coarray in the Presence of Unknown Nonuniform Noise

GUOJUN JIANG[1,2], MIANZHI WANG[3], (Student Member, IEEE), XINGPENG MAO[1,2], (Member, IEEE), CHENG QIAN[4], YONGTAN LIU[1,2], (Senior Member, IEEE), AND ARYE NEHORAI[3], (Life Fellow, IEEE)

[1]School of Electronics and Information Engineering, Harbin Institute of Technology, Harbin 150001, China
[2]Key Laboratory of Marine Environmental Monitoring and Information Processing, Ministry of Industry and Information Technology, Harbin 150001, China
[3]Preston M. Green Department of Electrical and Systems Engineering, Washington University in St. Louis, St. Louis, MO 63130, USA
[4]Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904, USA

Corresponding author: Xingpeng Mao (mxp@hit.edu.cn)

This work was supported by the Key Program of National Natural Science Foundation of China under Grant 61831009. The work of M. Wang and A. Nehorai was supported by the ONR under Grant N00014-13-1-0050 and Grant N00014-17-1-2371.

ABSTRACT The coarray techniques, e.g., nested and coprime arrays, can significantly improve degrees of freedom (DOFs) via constructing a so-called difference coarray, which enables underdetermined direction-of-arrival (DOA) estimation within reach in the presence of unknown nonuniform noise. There are repeated lags in the difference coarray, which also contain useful statistical information. In this paper, the repeated lags are properly used for DOA estimation algorithm design in unknown nonuniform noise environments. Specifically, the number of repeated lags in the difference coarray is rigorously given. Then these repeated lags and unique lags are judiciously rearranged to form a pseudo data set, which is composed of linearly independent vectors. Based on the pseudo data set, we propose two algorithms for DOA estimation in the presence of unknown nonuniform noise. One is a searching algorithm without source number knowledge (SASNK), and the other is a multi-snapshot compressive sensing method (MSCS) with better DOA estimation performance. The MSCS also does not require source number information. Numerical results are included to showcase the effectiveness of the proposed algorithms.

INDEX TERMS Direction-of-arrival (DOA) estimation, nested array, coprime array, repeated lags, pseudo data set, unknown nonuniform noise.

I. INTRODUCTION

Underdetermined direction-of-arrival (DOA) estimation, i.e., estimating K DOAs from $N < K$ sensors, is a problem of significance in engineering and science [1]–[4]. One way to deal with this problem is to construct a virtual array that offers more degrees of freedom (DOFs) than the physical array. The coarray techniques [5]–[12], e.g., nested and coprime arrays, play such a role. It has been shown that given an N-sensor such array, one can obtain a difference coarray which offers $O(N^2)$ DOFs through simply vectorizing the sample covariance matrix (SCM) of the received data [5], [8]. Thus, up to $O\left(N^2\right)$ DOAs can be resolved from an N-sensor array.

The basic idea of coarray technique is to employ two or more uniform linear arrays (ULAs) with specifically selected number of sensors and inter-sensor spacings, such that the DOFs in the difference coarray can be significantly increased. For example, let us consider an N-sensor nested array consisting of two ULAs. Assume that the first ULA has N_1 sensors with inter-sensor spacing d, and the second ULA has N_2 sensors with inter-sensor spacing $(N_1 + 1)d$, where $N = N_1 + N_2$ and $N_1 = N_2$. Then $(N^2/2 + N - 1)$ unique lags in total can be found in the difference coarray [5], [13]. Thus the number of identifiable sources can be larger than the number of sensors, and many algorithms

The associate editor coordinating the review of this manuscript and approving it for publication was Huawei Chen.

have been designed for underdetermined DOA estimation, e.g., [14]–[17] and references therein.

In some practical cases, the noise power at each sensor is different, yielding nonuniform noise. Thus, the noise covariance matrix is a diagonal matrix with unequal diagonal elements [18]. In this case, the DOA estimation performance of many existing methods based on nested and coprime arrays suffer severe degradation. Some methods have been proposed to improve the DOA performance for such arrays in the presence of unknown nonunform noise. One simple compressive sensing (CS) based method is proposed in [19] for efficient DOA estimation. However, its DOA estimation performance is not very good since it directly eleminates the diagonal elements of the covariance matrix. Recently, the authors in [20] propose a covariance matrix reconstruction based CS method (CMRCS) for robust estimation performance by averaging the diagonal elements. However, the DOA performance still needs to be improved in small sample or low signal-to-noise ratio (SNR) cases.

It is seen in the above nested array example that there are actually N^2 lags in the difference coarray. However, only $(N^2/2 + N - 1)$ of them are unique, which implies that there are redundant/repeated entries in the difference coarray. In the ideal case, there is no extra signal or noise variance in these redundant lags, and the unique lags have sufficient information to estimate DOAs. However, in practice, only the SCM is available. All the entries in the estimated difference coarray have auto- or cross-correlations of the signal or noise covariance estimates. In small sample or low SNR cases, the signal or noise variance corruption is much more severe than that in the high SNR or large sample scenarios. The repeated entries contain such correlation information, which can be used for alleviating the noise and enhancing the robustness of DOA estimation algorithms. However, some existing algorithms, such as [5] and [7], ignore most repeated entries, which means that only partial statistical information about the received data is employed. As a result, many of them do not perform satisfactorily in low SNR or small sample scenarios. The method in [21] averages all the covariance values corresponding to each repeated lag to estimate DOAs, but it needs source number knowledge a priori. The methods in [10]–[12] also use all the covariance values for DOA estimation without source number knowledge under the assumption that the noise is uniform, i.e., noise power at each sensor is the same. However, they do not consider nonuniform noise. The CS methods in [13], [14] estimate DOAs in nonuniform noise environments, but they need to estimate the noise power. In the presence of unknown nonuniform noise, estimating all the noise power increases the number of estimation parameters and thus the DOA estimation performance degrades.

To fully utilize the correlations of the received data and overcome the aforementioned difficulties, we propose two efficient algorithms in this paper. Specifically, we first study the number of repeated lags appearing more than once and twice in the difference coarray. Then we construct a pseudo data set by properly rearranging these repeated and unique entries, such that the useful correlation information about the received data is contained in this pseudo data set. Finally, based on the pseudo data set composed of multiple snapshots, two algorithms are developed for DOA estimation. It is worth highlighting that the proposed methods do not require source number information while the MUSIC methods [21], [22] do require such information a priori. The proposed algorithms are valid when the noise is uniform or nonuniform. In the case of nonuniform noise, the proposed algorithms have better estimation performance improvement, because the proposed pseudo data set chooses small noise power to form vectors, and removes the big noise power. Numerical results are included to showcase the effectiveness of the proposed algorithms.

The rest of the paper is organized as follows. Section II presents the data model and briefly summarizes the existing coarray configurations. In Section III, the number of repeated lags in the difference coarray is studied. Section IV presents the pseudo data set and the proposed algorithms. Numerical examples are included in Section V. Finally, Section VI draws the conclusion of this paper.

Throughout the paper, scalars are denoted by lowercase letters, e.g., a. Vectors are denoted by boldface lowercase letters, e.g., $\mathbf{a}$. Matrices are denoted by boldface uppercase letters, e.g., $\mathbf{A}$. We list some notational conventions which will be used in the paper.

- $\lfloor a \rfloor$: a number rounded to the nearest integer a and $\lfloor a \rfloor \leq a$
- $\lceil a \rceil$: a number rounded to the nearest integer a and $\lceil a \rceil \geq a$
- diag($\mathbf{a}$): a diagonal matrix whose diagonal elements are given by $\mathbf{a}$
- $||\mathbf{a}||_1$: the l_1 norm for the vector $\mathbf{a}$
- $||\mathbf{a}||_2$: the Euclidean norm for the vector $\mathbf{a}$
- $\|\mathbf{A}\|_F$: the Frobenius norm of $\mathbf{A}$
- $\mathbf{A}^*$: complex conjugate of $\mathbf{A}$
- $\mathbf{A}^T$: transpose of $\mathbf{A}$
- $\mathbf{A}^H$: conjugate transpose of $\mathbf{A}$
- $\mathbf{A}^{-1}$: inverse of $\mathbf{A}$
- $\mathbf{A}^\dagger$: pseudo-inverse of $\mathbf{A}$
- $\mathbb{E}[\mathbf{A}]$: mathematical expectation of $\mathbf{A}$
- vec($\mathbf{A}$): vectorizing matrix $\mathbf{A}$
- max eig$\{\mathbf{A}\}$: the maximum eigenvalue of $\mathbf{A}$
- $\mathbf{A} \odot \mathbf{B}$: Khatri-Rao product of $\mathbf{A}$ and $\mathbf{B}$

II. PRELIMINARIES

A. DATA MODEL

Assume that there is an N-sensor linear array receiving K narrowband signals from directions $\{\theta_1, \ldots, \theta_K\}$. The received data is given as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{s}(t)$ is the signal vector, $\mathbf{n}(t)$ is the noise vector, and $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ is the array manifold with its (i, k)th element being $[\mathbf{A}]_{i,k} = e^{j2\pi d_i \Delta \sin\theta_k/\lambda}$, in which d_i is the sensor position and $\Delta = \lambda/2$ is the inter-sensor spacing with

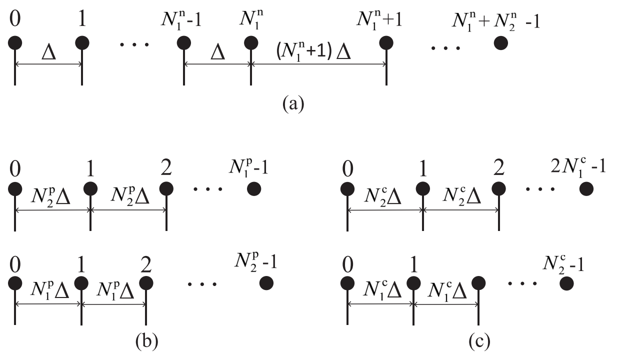


FIGURE 1. (a) A nested array with $N_1^n + N_2^n$ sensors. (b) A PCA with $N_1^p + N_2^p - 1$ sensors. (c) A coprime array with $2N_1^c + N_2^c - 1$ sensors.

λ being the wavelength. We assume a nonuniform Gaussian noise environment, i.e., the noise power at each sensor is different and unknown. Then the noise covariance matrix is

$$\mathbf{R_n} = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2\}, \tag{2}$$

where σ_n^2 is the noise power of the nth sensor.

Suppose that the sources are independent and uncorrelated with the noise. Then the covariance matrix of $\mathbf{x}(t)$ is

$$\mathbf{R} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)] = \mathbf{A}\mathbf{R_s}\mathbf{A}^H + \mathbf{R_n}, \tag{3}$$

where $\mathbf{R_s} = \mathbb{E}[\mathbf{s}(t)\mathbf{s}^H(t)]$ is the source covariance matrix. The vectorization of the covariance matrix $\mathbf{R}$ is expressed as

$$\mathbf{v} = \mathrm{vec}(\mathbf{R}) = \left(\mathbf{A}^* \odot \mathbf{A}\right)\boldsymbol{\alpha} + \mathbf{e}, \tag{4}$$

where $\boldsymbol{\alpha} = [\alpha_1^2, \ldots, \alpha_K^2]^T$, with α_k^2 being the power of the kth source, and $\mathbf{e} = \mathrm{vec}(\mathbf{R_n}) = [\mathbf{e}_1^T \mathbf{e}_2^T \ldots \mathbf{e}_N^T]^T$ with $\mathbf{e}_n$ being a vector of all zeros expect σ_n^2 at the nth position. We suppose that all signals are of equal power.

B. REVIEW OF COARRAYS AND IMPORTANT NOTATIONS

To make the paper self-contained, we now briefly review the existing coarray configurations, i.e., nested array [5], prototype coprime array (PCA) [6] and coprime array [13]. Note that, we use the superscripts $(\cdot)^{\mathrm{n}}$, $(\cdot)^{\mathrm{p}}$ and $(\cdot)^{\mathrm{c}}$ stand for nested array, PCA, and coprime array, respectively. We also define $\mathcal{L}$ as the set that contains all the consecutive lags and $\mathcal{D}$ as the set containing all the lags in the difference coarray, while $\mathcal{L}_{s,1}$, $\mathcal{L}_{s,2}$, and $\mathcal{L}_c$ represent the self-difference set of the first ULA, the self-difference set of the second ULA, and cross-difference set between the first and second ULAs, respectively.

1) NESTED ARRAY

Assume a nested array with $N^{\mathrm{n}} = N_1^{\mathrm{n}} + N_2^{\mathrm{n}}$ sensors, including two concatenated ULAs, where the first ULA has N_1^{n} sensors with spacing Δ and the second ULA has N_2^{n} sensors with spacing $(N_1^{\mathrm{n}} + 1)\Delta$. The nested array is shown in Fig. 1 (a). Let $\mathbf{d}^{\mathrm{n}} = [d_1^{\mathrm{n}}, \ldots, d_{N_1^{\mathrm{n}}+N_2^{\mathrm{n}}}^{\mathrm{n}}]^T$ be the sensor position vector of the nested array. It is easy to check that the (i, j)th entry in $\mathbf{R}$ matches lag $(d_i^{\mathrm{n}} - d_j^{\mathrm{n}})$. Thus, by varying i and j from 1 to $(N_1^{\mathrm{n}} + N_2^{\mathrm{n}})$, we can construct the following set that contains all the lags:

$$\mathcal{D}^{\mathrm{n}} = \left\{d_i^{\mathrm{n}} - d_j^{\mathrm{n}}, 1 \leqslant i, j \leqslant N^{\mathrm{n}}\right\}, \tag{5}$$

where there are $(N^{2\mathrm{n}} - 2)/2 + N^{\mathrm{n}}$ (N^{n} is even and $N_1^{\mathrm{n}} = N_2^{\mathrm{n}}$) or $(N^{2\mathrm{n}} - 1)/2 + N^{\mathrm{n}}$ (N^{n} is odd and $N_2^{\mathrm{n}} = N_1^{\mathrm{n}} + 1$) consecutive lags within the range $\mathcal{L}^{\mathrm{n}}$. For the nested array, all the lags are in the consecutive range. So the number of unique lags equals to that of consecutive lags.

2) PROTOTYPE COPRIME ARRAY (PCA)

Consider a pair of coprime integers N_1^{p} and N_2^{p}, where $2 < N_1^{\mathrm{p}} < N_2^{\mathrm{p}}$. A PCA is composed of two ULAs, where the first ULA has N_1^{p} sensors with spacing $N_2^{\mathrm{p}}\Delta$ and the second ULA has N_2^{p} sensors with spacing $N_1^{\mathrm{p}}\Delta$, as shown in Fig. 1 (b). Let $\mathbf{d}^{\mathrm{p}} = [d_1^{\mathrm{p}}, \ldots, d_{N_1^{\mathrm{p}}+N_2^{\mathrm{p}}-1}^{\mathrm{p}}]^T$ be the sensor position vector of the PCA. The difference coarray of the PCA is determined by

$$\mathcal{D}^{\mathrm{p}} = \left\{d_i^{\mathrm{p}} - d_j^{\mathrm{p}}, 1 \leqslant i, j \leqslant N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 1\right\}, \tag{6}$$

where there are $(N_1^{\mathrm{p}}N_2^{\mathrm{p}} + N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 2)$ unique lags. There are $(2N_1^{\mathrm{p}} + 2N_2^{\mathrm{p}} - 1)$ consecutive lags within the range $\mathcal{L}^{\mathrm{p}} = \left\{l^{\mathrm{p}} \left| -N_1^{\mathrm{p}} - N_2^{\mathrm{p}} + 1 \le l^{\mathrm{p}} \le N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 1\right.\right\}$.

The difference sets of the PCA are defined as follows:

$$\mathcal{L}_{s,1}^{\mathrm{p}} = \left\{l_{s,1}^{\mathrm{p}} \left| l_{s,1}^{\mathrm{p}} = \pm(n_1^{\mathrm{p}}N_2^{\mathrm{p}})\right.\right\}, \tag{7}$$

$$\mathcal{L}_{s,2}^{\mathrm{p}} = \left\{l_{s,2}^{\mathrm{p}} \left| l_{s,2}^{\mathrm{p}} = \pm(n_2^{\mathrm{p}}N_1^{\mathrm{p}})\right.\right\}, \tag{8}$$

$$\mathcal{L}_{c}^{\mathrm{p}} = \left\{l_{c}^{\mathrm{p}} \left| l_{c}^{\mathrm{p}} = \pm(n_2^{\mathrm{p}}N_1^{\mathrm{p}} - n_1^{\mathrm{p}}N_2^{\mathrm{p}})\right.\right\}, \tag{9}$$

where $0 \le n_1^{\mathrm{p}} \le N_1^{\mathrm{p}} - 1$ and $0 \le n_2^{\mathrm{p}} \le N_2^{\mathrm{p}} - 1$.

3) COPRIME ARRAY

A coprime array contains two ULAs, where the first ULA has $2N_1^{\mathrm{c}}$ sensors with spacing $N_2^{\mathrm{c}}\Delta$, and the second ULA has N_2^{c} sensors with spacing $N_1^{\mathrm{c}}\Delta$, as shown in Fig. 1 (c). Note that $2N_1^{\mathrm{c}}$ and N_2^{c} are coprime integers. Let $\mathbf{d}^{\mathrm{c}} = [d_1^{\mathrm{c}}, \cdots, d_{2N_1^{\mathrm{c}}+N_2^{\mathrm{c}}-1}^{\mathrm{c}}]^T$ be the sensor position vector of the coprime array. The difference coarray is determined by the sensor position set

$$\mathcal{D}^{c} = \left\{d_i^{\mathrm{c}} - d_j^{\mathrm{c}}, 1 \leqslant i, j \leqslant 2N_1^{\mathrm{c}} + N_2^{\mathrm{c}} - 1\right\}, \tag{10}$$

where there are $(3N_1^{\mathrm{c}}N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - N_2^{\mathrm{c}})$ unique lags. There are $(2N_1^{\mathrm{c}}N_2^{\mathrm{c}} + 2N_1^{\mathrm{c}} - 1)$ consecutive lags within the range $\mathcal{L}^{\mathrm{c}} = \left\{l^{\mathrm{c}} \left| -N_1^{\mathrm{c}}N_2^{\mathrm{c}} - N_1^{\mathrm{c}} + 1 \le l^{\mathrm{c}} \le N_1^{\mathrm{c}}N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1\right.\right\}$.

Similarly, we can write the difference sets of the coprime array as

$$\mathcal{L}_{s,1}^{\mathrm{c}} = \left\{l_{s,1}^{\mathrm{c}} \left| l_{s,1}^{\mathrm{c}} = \pm(n_1^{\mathrm{c}}N_2^{\mathrm{c}})\right.\right\}, \tag{11}$$

$$\mathcal{L}_{s,2}^{\mathrm{c}} = \left\{l_{s,2}^{\mathrm{c}} \left| l_{s,2}^{\mathrm{c}} = \pm(n_2^{\mathrm{c}}N_1^{\mathrm{c}})\right.\right\}, \tag{12}$$

$$\mathcal{L}_{c}^{\mathrm{c}} = \left\{l_{c}^{\mathrm{c}} \left| l_{c}^{\mathrm{c}} = \pm(n_2^{\mathrm{c}}N_1^{\mathrm{c}} - n_1^{\mathrm{c}}N_2^{\mathrm{c}})\right.\right\}, \tag{13}$$

where $0 \le n_1^{\mathrm{c}} \le 2N_1^{\mathrm{c}} - 1$ and $0 \le n_2^{\mathrm{c}} \le N_2^{\mathrm{c}} - 1$.

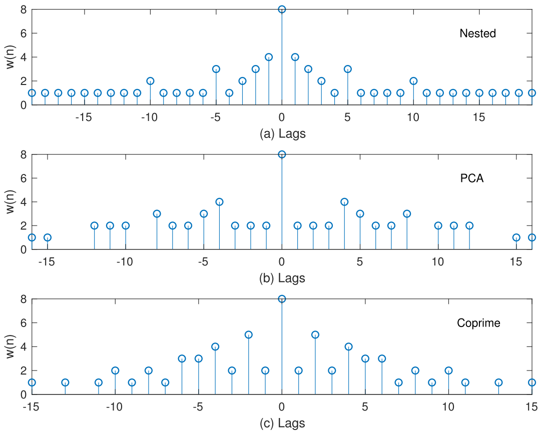


FIGURE 2. **Weight function value of lags for (a) nested array, (b) PCA, and (c) coprime array.**

III. REPEATED LAGS IN DIFFERENCE COARRAY

In this section, we will derive the number of repeated lags in the difference coarrays of nested array, the PCA, and the coprime array, respectively. Here, we consider the lags which appear more than once and twice, since they will be used to form a pseudo data set composed of linearly independent vectors in the next section. Before proceeding to the derivations, let us define $w(n)$ as the *weight function value (WFV)* which records the number of repetitions of the lag n in the difference coarray [5]. Specifically, we note that if lag n in the difference coarray appears only once, then WFV $w(n) = 1$, while if lag n appears more than once, $w(n) \geq 2$.

Fig. 2 (a), (b), and (c) show the WFVs of lags in the difference coarray of nested array, PCA, and coprime array, respectively. In Fig. 2(a), a nested array with $N_1^{\mathrm{n}} = 4$ and $N_2^{\mathrm{n}} = 4$ is considered. It can be seen that there are 39 unique lags, including 11 lags appearing more than once and 7 lags appearing more than twice. The numbers of lags appearing more than once and twice are the same in the sets $\mathcal{D}^{\mathrm{n}}$ and $\mathcal{L}^{\mathrm{n}}$. Fig. 2(b) plots the WFVs of lags of a PCA with $N_1^{\mathrm{p}} = 4$ and $N_2^{\mathrm{p}} = 5$, where there are 27 unique lags in $\mathcal{D}^{\mathrm{p}}$, including 23 lags appearing more than once and 7 lags appearing more than twice. There are 17 unique lags in $\mathcal{L}^{\mathrm{p}}$, including 17 lags with WFV greater than one and 7 lags with WFV greater than two. Fig. 2(c) plots the WFVs of lags of a coprime array with $N_1^{\mathrm{c}} = 2$ and $N_2^{\mathrm{c}} = 5$, where there are 27 unique lags in $\mathcal{D}^{\mathrm{c}}$, including 17 lags with WFV greater than one and 9 lags with WFV greater than two. In $\mathcal{L}^{\mathrm{c}}$, 17 lags appear more than once, and 9 lags appear more than twice.

Motivated by the results in Fig. 2, we have the following propositions which show the numbers of lags with WFV greater than one and two in nested array, PCA, and coprime array, respectively.

Proposition 1: Given a nested array with N^{n} sensors,

(a) There are $(2N^{\mathrm{n}} - 5)$ lags with WFV greater than one.

(b) There are $(2N^{\mathrm{n}} - 9)$ lags with WFV greater than two.

Proof: See Appendix A. ■

For the nested array, the numbers of lags appearing more than once and twice in $\mathcal{L}^{\mathrm{n}}$ are the same with those in $\mathcal{D}^{\mathrm{n}}$.

Proposition 2: Given a PCA with $(N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 1)$ sensors where $2 < N_1^{\mathrm{p}} < N_2^{\mathrm{p}}$,

(a)There are $(2N_1^{\mathrm{p}} + 2N_2^{\mathrm{p}} - 1)$ lags with WFV greater than one in $\mathcal{L}^{\mathrm{p}}$.

(b) There are $\left(2\left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 5\right)$ $(N_1^{\mathrm{p}} > 3)$ or $\left(2\left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 3\right)$ $(N_1^{\mathrm{p}} = 3)$ lags with WFV greater than two in $\mathcal{L}^{\mathrm{p}}$.

Proof: See Appendix B. ■

For the PCA, from Proposition 1 we know that all the lags in $\mathcal{L}^{\mathrm{p}}$ appear more than once in the difference coarray. Moreover, there are at least $(4N_1^{\mathrm{p}} + 4N_2^{\mathrm{p}} - 2\left\lceil \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rceil - 11)$ lags with WFV greater than one in $\mathcal{D}^{\mathrm{p}}$.

Proposition 3: Given a coprime array with $(2N_1^{\mathrm{c}} + N_2^{\mathrm{c}} - 1)$ sensors,

(a) There are at least $(N_1^{\mathrm{c}} N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1)$ lags with WFV greater than one in $\mathcal{L}^{\mathrm{c}}$.

(b) There are $(2N_1^{\mathrm{c}} + 2N_2^{\mathrm{c}} - 3)$ $(N_2^{\mathrm{c}} > 2)$ or $(2N_1^{\mathrm{c}} + 2N_2^{\mathrm{c}} - 5)$ $(N_2^{\mathrm{c}} = 2)$ lags with WFV greater than two in $\mathcal{L}^{\mathrm{c}}$.

Proof: See Appendix C. ■

For the coprime array, as the lags appearing more than once and twice in $(\mathcal{D}^{\mathrm{c}} - \mathcal{L}^{\mathrm{c}})$ are relatively few compared with those in $\mathcal{L}^{\mathrm{c}}$. So we do not consider additional calculations for them.

Remark 1: Unfortunately, there is no closed-form solution for some calculations of the number of lags with WFV greater than one or two in PCA and coprime array. However, in practical applications, since we know the number of sensors of the given PCA and coprime array, it is viable to pre-calculate the number of lags appearing more than once and twice and store them in the system.

IV. PROPOSED ALGORITHMS

We have given the number of lags appearing more than once and twice of the nested array, PCA, and coprime array in the above section. In the following, we will show how to make use of these repeated lags for DOA estimation by constructing a pseudo data set.

A. PROBLEM FORMULATION

In the ideal case with white Gaussian noise, the noise covariance matrix is diagonal with equal diagonal elements, so the covariance values obtained from $\mathbf{R}$ corresponding to one repeated lag are equal. Many conventional coarray based DOA estimators vectorize $\mathbf{R}$ to get a measurement vector on which they operate. Note that only one covariance value for one repeated lag is used to form the vector, which is enough to utilize all the source and noise information.

However, the noise power at each sensor may be different in many cases, e.g., nonuniformity of sensor noise or array imperfection. In such nonuniform noise case, the difference between noise power is considerable. The ideal covariance matrix $\mathbf{R}$ is not available in practice. We can only approach it, e.g., by using the SCM which takes the form of

$$\hat{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}^H(t), \tag{14}$$

where T is the number of snapshots. In this case, all the entries in the SCM contain auto- or cross-correlation terms of signal or noise. So the covariance values obtained from $\hat{\mathbf{R}}$ corresponding to one repeated lag are different. Rationally utilizing the information of repeated lags can alleviate the effect of noise and maintain the source information.

B. PSEUDO DATA SET

In this subsection, we utilize the different covariance values to construct a pseudo data set composed of linearly independent vectors. Without loss of generality, we take the nested array with N^{n} sensors (N^{n} is even) as an example to show how to form the pseudo data set $\{\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_{2N^{\mathrm{n}}-6}\}$ by using $(2N^{\mathrm{n}} - 5)$ lags with WFV greater than one and $(2N^{\mathrm{n}} - 9)$ lags with WFV greater than two.

Let N_s be the number of single lags (appearing only once) in the difference coarray. Assume the corresponding covariance values of all the single lags form a set $\mathbf{a}_s = \{a_1, a_2, \ldots, a_{N_s}\}$. First, we consider the $(2N^{\mathrm{n}} - 5)$ lags appearing more than once, including $(N^{\mathrm{n}} - 3)$ negative lags, $(N^{\mathrm{n}} - 3)$ positive lags, and lag 0. We arrange the negative and positive lags and let their corresponding covariance values in $\hat{\mathbf{R}}$ be

$$\begin{aligned} \mathbf{b}^{-(N^{\mathrm{n}}-3)} &= \{b_1^{-(N^{\mathrm{n}}-3)}, \cdots, b_{m_{N^{\mathrm{n}}-3}}^{-(N^{\mathrm{n}}-3)}\}, \\ &\cdots \\ \mathbf{b}^{-1} &= \{b_1^{-1}, b_2^{-1}, \cdots, b_{m_1}^{-1}\}, \\ \mathbf{b}^{1} &= \{b_1^{1}, b_2^{1}, \cdots, b_{m_1}^{1}\}, \\ &\cdots \\ \mathbf{b}^{N^{\mathrm{n}}-3} &= \{b_1^{N^{\mathrm{n}}-3}, \cdots, b_{m_{N^{\mathrm{n}}-3}}^{N^{\mathrm{n}}-3}\}, \end{aligned} \tag{15}$$

where m_i is the WFV of the ith positive repeated lag, and $\mathbf{b}^i$ contains its corresponding covariance values. In (15), $\mathbf{b}^{-(N^{\mathrm{n}}-3)}, \ldots, \mathbf{b}^{-1}$ correspond to the negative repeated lags in order from small to big, and $\mathbf{b}^{1}, \ldots, \mathbf{b}^{N^{\mathrm{n}}-3}$ correspond to the positive repeated lags in order from small to big. The number of values in $\mathbf{b}^{-i}$ is the same with that in $\mathbf{b}^{i}$ since $w(-n) = w(n)$. It should be noted that the covariance values in $\mathbf{b}^{i}$ is arranged in ascending order of magnitude of the covariance values corresponding to the ith repeated lag.

We choose b_1^1 from $\mathbf{b}^1$ and its corresponding conjugate value b_1^{-1} from $\mathbf{b}^{-1}$. Then, we respectively choose one value $b_1^2, b_1^3, \ldots, b_1^{N^{\mathrm{n}}-3}$ from $\mathbf{b}^2, \ldots, \mathbf{b}^{N^{\mathrm{n}}-3}$, and their corresponding conjugate values $b_1^{-2}, b_1^{-3}, \ldots, b_1^{-(N^{\mathrm{n}}-3)}$ from $\mathbf{b}^{-2}, \ldots, \mathbf{b}^{-(N^{\mathrm{n}}-3)}$. For lag 0, there are N^n covariance values, and the gap between different noise power is big. We first choose the minimum value in order to alleviate the noise. Then we let the above chosen values and the values in $\mathbf{a}_s$ form a vector $\hat{\mathbf{y}}_1$ in order of lags. After replacing only two items in $\hat{\mathbf{y}}_1$, which means we only replace b_1^1 and b_1^{-1} with b_2^1 and b_2^{-1} respectively, we get $\hat{\mathbf{y}}_2$. Similarly, after replacing only the values b_1^2 and b_1^{-2} with b_2^2 and b_2^{-2}, we get $\hat{\mathbf{y}}_3$. Correspondingly, $\hat{\mathbf{y}}_{N^{\mathrm{n}}-2}$ means that only the value of $b_1^{N^{\mathrm{n}}-3}$ and $b_1^{-(N^{\mathrm{n}}-3)}$ are replaced by $b_2^{N^{\mathrm{n}}-3}$ and $b_2^{(-N^{\mathrm{n}}-3)}$. Thus we can get $(N^{\mathrm{n}} - 2)$ vectors: $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_{N^{\mathrm{n}}-2}$.

In the above paragraph, we use the lags appearing more than once to form vectors. Next, we form more vectors by using the lags appearing more than twice. Since there are $(2N^{\mathrm{n}} - 9)$ lags with WFV greater than two, including $(N^{\mathrm{n}} - 5)$ negative lags, $(N^{\mathrm{n}} - 5)$ positive lags, and lag 0, we can form $(N^{\mathrm{n}} - 5)$ new vectors by using the same way as forming $\hat{\mathbf{y}}_i$ $(1 \leq i \leq N^{\mathrm{n}} - 2)$. For example, if $m_1 > 2$, we form $\hat{\mathbf{y}}_{N^n-1}$ by replacing b_1^1 and b_1^{-1} in $\hat{y}_1$ with b_3^1 and b_3^{-1}. At last, we only replace the value in $\hat{\mathbf{y}}_1$, which corresponds to lag 0, with the second minimun value in the N^{n} covariance values to form $\hat{\mathbf{y}}_{2N^{\mathrm{n}}-6}$. Thus we get the pseudo data set $\{\hat{\mathbf{y}}_1, \ldots, \hat{\mathbf{y}}_{N^{\mathrm{n}}-2}, \ldots, \hat{\mathbf{y}}_{2N^{\mathrm{n}}-6}\}$.

Note that, the covariance values are complex expect those corresponding to lag 0. According to the linearly independent property of complex, we know that $\{\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_{2N^{\mathrm{n}}-6}\}$ in the pseudo data set are linearly independent. Any new formed vector will be linearly dependent with them, i.e., any new formed vector can be expressed by a linear combination of the vectors in the pseudo data set. For example, if $m_1 > 3$, we can form a new vector $\hat{\mathbf{y}}_{2N^{\mathrm{n}}-5}$ by replacing b_1^1 and b_1^{-1} with b_4^1 and b_4^{-1}, respectively. Then $\hat{\mathbf{y}}_{2N^{\mathrm{n}}-5} = \hat{\mathbf{y}}_1 + q_1(\hat{\mathbf{y}}_2 - \hat{\mathbf{y}}_1) + q_2(\hat{\mathbf{y}}_{N^{\mathrm{n}}-1} - \hat{\mathbf{y}}_1) + 0(\hat{\mathbf{y}}_3 - \hat{\mathbf{y}}_1) + \ldots + 0(\hat{\mathbf{y}}_{2N^{\mathrm{n}}-6} - \hat{\mathbf{y}}_1)$, where q_1 and q_2 are not equivalent to zero.

For the i-th diagonal element of covariance matrix, its value is positively correlated with the noise power at the i-th sensor. Among the N^n covariance values corresponding to lag 0, the minimum two are chosen to form vectors, which means the minimum two noise power are chosen. In this way, the effect of nonuniform noise can be mitigated. Moreover, for each repeated lag, we can obtain a set, e.g., b^i. Two elements with two minimun amplitudes in every set are chosen to form vectors.

Similarly, when N^{n} is odd, we can get the pseudo data set with $(2N^{\mathrm{n}} - 6)$ linearly independent vectors. For the PCA and coprime array, we have similar properties. Specifically,

- Given a PCA with $(N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 1)$ sensors where $2 < N_1^{\mathrm{p}} < N_2^{\mathrm{p}}$, we can obtain a pseudo data set with $\left(N_1^{\mathrm{p}} + N_2^{\mathrm{p}} + \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 3\right)$ $(N_1^{\mathrm{p}} > 3)$ or $\left(N_1^{\mathrm{p}} + N_2^{\mathrm{p}} + \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 2\right)$ $(N_1^{\mathrm{p}} = 3)$ linearly independent vectors in $\mathcal{L}^{\mathrm{p}}$. In addition, it is easy to infer that we can obtain at least $(2N_1^{\mathrm{p}} + 2N_2^{\mathrm{p}} - 3)$ $(N_1^{\mathrm{p}} > 3)$ or $(2N_1^{\mathrm{p}} + 2N_2^{\mathrm{p}} - 4)$ $(N_1^{\mathrm{p}} = 3)$ linearly independent vectors in $\mathcal{D}^{\mathrm{p}}$.
- Given a coprime array with $(2N_1^{\mathrm{c}} + N_2^{\mathrm{c}} - 1)$ sensors, we can obtain a pseudo data set with at least $(N_1^{\mathrm{c}}N_2^{\mathrm{c}}/2 + 3N_1^{\mathrm{c}}/2 + N_2^{\mathrm{c}} - 1)$ $(N_2^{\mathrm{c}} > 2)$ or $(N_1^{\mathrm{c}}N_2^{\mathrm{c}}/2 + 3N_1^{\mathrm{c}}/2 + N_2^{\mathrm{c}} - 2)$ $(N_2^{\mathrm{c}} = 2)$ linearly independent vectors in both $\mathcal{L}^{\mathrm{c}}$ and $\mathcal{D}^{\mathrm{c}}$.

C. DOA ESTIMATION

In the above section, we have formed a pseudo data set. Note that, each vector in the data set can be considered as one snapshot of the virtual array. The proposed data set can provide multiple snapshots, while the existing methods in [14], [20], [21] only obtain one snapshot for the virtual array. Thus the proposed data set can also be applied to the DOA estimation methods which are invalid or suffer performance loss/down in the condition of single snapshot for the virtual array, such as the proposed searching algorithm without source number knowledge (SASNK) in the following. We now propose two algorithms based on the pseudo data set composed of multiple snapshots. We assume T' is the number of snapshots in the pseudo data set.

1) SEARCHING ALGORITHM WITHOUT SOURCE NUMBER KNOWLEDGE (SASNK)

Assume the number of consecutive lags in the difference coarray is $(2M+1)$. Then each $\hat{\mathbf{y}}_i$ is one snapshot of the $(2M+1)$-sensor array. Thus, after spatial smoothing [5] or the technique in [21] used for $\hat{\mathbf{y}}_i$, a covariance matrix can be obtained as

$$\hat{\mathbf{R}}_{\mathrm{ss},i} = \mathbf{A}_{\mathrm{ss}}\mathbf{R}_{\mathrm{s}}\mathbf{A}_{\mathrm{ss}}^H + \hat{\mathbf{R}}_{\mathbf{nn}}, \tag{16}$$

where $\mathbf{A}_{\mathrm{ss}} = [\mathbf{a}_{\mathrm{ss}}(\theta_1), \ldots, \mathbf{a}_{\mathrm{ss}}(\theta_K)]$ is the steering matrix of a virtual $(M+1)$-sensor ULA. It is obvious that one vector, $\hat{\mathbf{y}}_i$, forms one covariance matrix. Therefore, the pseudo data sets of the nested array, the PCA, and the coprime array can respectively form the corresponding numbers of covariance matrices.

In [5], a MUSIC algorithm has been proposed for DOA estimation. The drawback of this method is that it requires knowledge of the source number. However, we may not know the source number in practice. Although there are many detection methods, such as [23]–[25], that can be applied to find the source number, in the low SNR regime, the probability of detection of these methods is still not optimistic.

Unlike MUSIC, a searching algorithm which does not require source number knowledge is proposed in [26]. However, this method can not be directly applied to the difference coarray because there is only one snapshot in the virtual array. Moreover, the performance of the method in [26] suffers serious degradation when SNR is low. It is because that only one formed covariance matrix is conjugate symmetric while the others are not in the presence of noise. We have formed a pseudo data containing a number of snapshots, and each snapshot can be uesd to form a covariance matrix, e.g., $\hat{\mathbf{R}}_{\mathrm{ss},i} \in \mathbb{C}^{(M+1)\times(M+1)}$ as in (16). In our paper, the matrix $\hat{\mathbf{R}}_{\mathrm{ss},i}$ is always conjugate symmetric since we choose one covariance value of a negative repeated lag and simultaneously choose the conjugate covariance value corresponding to the positive lag to form each $\hat{\mathbf{y}}_i$. In the following we use the matrices to estimate DOAs without source number knowledge.

For the kth source, there always exists a vector $\mathbf{c}_k \in \mathbb{C}^{(M+1)\times 1}$ which is orthogonal to the space spanned by the other $(K-1)$ steering vectors, and thus we have

$$\sum_{m=1}^{K} \mathbf{a}_{\mathrm{ss}}^H(\theta_m)\mathbf{c}_k = \mathbf{a}_{\mathrm{ss}}^H(\theta_k)\mathbf{c}_k. \tag{17}$$

Algorithm 1 SASNK

1: Obtain $\mathbf{x}(t)$, $t = 1, \ldots, T$;
2: Calculate the sample covariance matrix $\hat{\mathbf{R}}$ as in (14);
3: Form the pseudo data set $\{\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \cdots, \hat{\mathbf{y}}_{T'}\}$, and form the corresponding covariance matrices;
4: Construct $\mathbf{F}$ and $\mathbf{G}$ by using the covariance matrices;
5: Estimate DOAs by finding the peaks of $P(\theta)$ as in (21);
6: **return** the estimated DOAs $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_K$.

If there is no noise, we have the following result

$$\hat{\mathbf{R}}_{\mathrm{ss},i}\mathbf{c}_k = \sum_{m=1}^{K} \alpha_k^2 \mathbf{a}_{\mathrm{ss}}(\theta_m)\mathbf{a}_{\mathrm{ss}}^H(\theta_m)\mathbf{c}_k = g_i \mathbf{a}_{\mathrm{ss}}(\theta_k). \tag{18}$$

From (18), we know that if θ is a true DOA, there always exists a scalar g_i which makes $\hat{\mathbf{R}}_{\mathrm{ss},i}\mathbf{c} = g_i\mathbf{a}_{\mathrm{ss}}(\theta)$. In the presence of noise, we have the following optimization problem:

$$\begin{aligned} &\min_{\theta,\mathbf{g},\mathbf{c}} \mathbf{J}(\theta, \mathbf{g}, \mathbf{c}) = \sum_{i=1}^{M+1} \|\hat{\mathbf{R}}_{\mathrm{ss},i}\mathbf{c} - g_i\mathbf{a}_{\mathrm{ss}}(\theta)\|^2 \\ &s.t.\ \|\mathbf{g}\|^2 = 1, \end{aligned} \tag{19}$$

where $\mathbf{g} = [g_1, \ldots, g_{M+1}]^T \in \mathbb{C}^{(M+1)\times 1}$. Since $\mathbf{c}$ and $\mathbf{g}$ are unknown parameters, we first expand the cost function in (19), and then utilize $\mathbf{a}_{\mathrm{ss}}^H\mathbf{a}_{\mathrm{ss}} = M+1$ and let the first derivative of the expanded cost function with respect to $\mathbf{c}$ be zero. Thus (19) is reduced to

$$\begin{aligned} &\min_{\theta,\mathbf{g}} \mathbf{J}(\theta, \mathbf{g}) = M+1 - \mathbf{g}^H\mathbf{G}^H(\theta)\mathbf{F}^{\dagger}\mathbf{G}(\theta)\mathbf{g} \\ &s.t.\ \|\mathbf{g}\|^2 = 1, \end{aligned} \tag{20}$$

where $\mathbf{F} = \sum_{i=1}^{M+1} \hat{\mathbf{R}}_{\mathrm{ss},i}^H\hat{\mathbf{R}}_{\mathrm{ss},i}$ and $\mathbf{G} = [\hat{\mathbf{R}}_{\mathrm{ss},1}^H\mathbf{a}_{\mathrm{ss}}(\theta), \ldots, \hat{\mathbf{R}}_{\mathrm{ss},M+1}^H\mathbf{a}_{\mathrm{ss}}(\theta)]$. Only if $\mathbf{g}$ is the eigenvector of $\mathbf{G}^H(\theta)\mathbf{F}^{\dagger}\mathbf{G}(\theta)$ corresponding to the maximum eigenvalue, we can get the minimum of $\mathbf{J}$. Thus we obtain the pseudo output power spectrum for the virtual array as

$$P(\theta) = \frac{1}{M+1 - \max \mathrm{eig}\{\mathbf{G}^H(\theta)\mathbf{F}^{\dagger}\mathbf{G}(\theta)\}}. \tag{21}$$

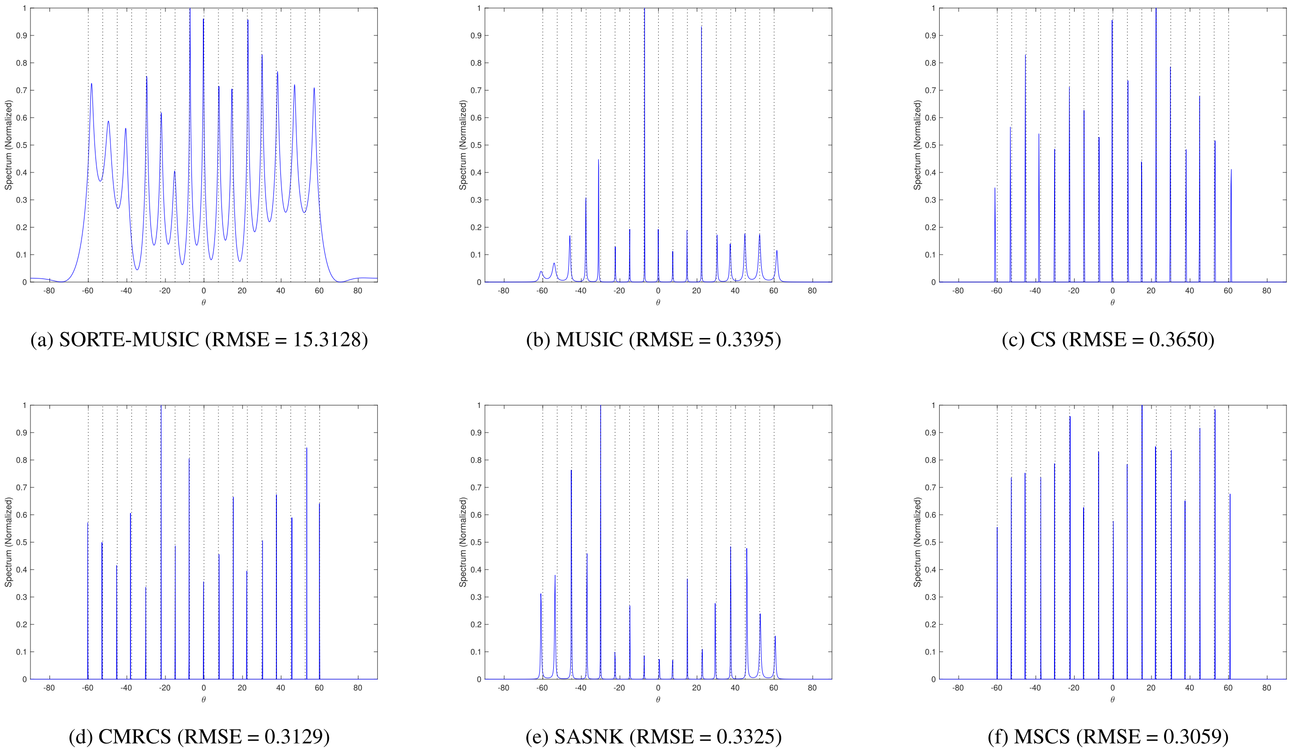
We call this searching algorithm without source number knowledge SASNK. The implementation of SASNK is summarized in Algorithm 1.

2) MULTI-SNAPSHOT COMPRESSIVE SENSING METHOD (MSCS)

We have formed the pseudo data set with linearly independent snapshots. Here we use these snapshots in set $\mathcal{D}$ to estimate DOAs in the presence of unknown nonuniform noise. In this way, all the DOFs and all the useful information of the received data can be utilized. Now we use the pseudo data to solve the following problem:

$$\min_{\mathbf{S}'} \frac{1}{2}\left\|\mathbf{Y} - \mathbf{B}\mathbf{S}'\right\|_F^2 + \eta\left\|\mathbf{s}'\right\|_1, \tag{22}$$

where $\mathbf{Y} = [\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \cdots, \hat{\mathbf{y}}_{T'}]$, $\mathbf{B}$ is a sensing matrix composed of the searching steering vectors and is defined over a finite



(a) SORTE-MUSIC (RMSE = 15.3128) (b) MUSIC (RMSE = 0.3395) (c) CS (RMSE = 0.3650)

(d) CMRCS (RMSE = 0.3129) (e) SASNK (RMSE = 0.3325) (f) MSCS (RMSE = 0.3059)

FIGURE 3. Spatial spectra ($T = 400$, SNR = 0dB).

grid $\theta_1, \ldots, \theta_G$, $\mathbf{s}'(n) = \left\|[\mathbf{S}'(n,1), \mathbf{S}'(n,2), \ldots \mathbf{S}'(n,T')]\right\|_2$, and η is a parameter which controls the tradeoff between the sparsity of the spectrum and the residual terms. To reduce the computational complexity [27], the above problem can be solved as

$$\begin{aligned} &\min\ \mu + \eta\gamma \\ &s.t.\ \|[\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{T'}]\|_2^2 \le \mu, \\ &\qquad \|\mathbf{S}'(j,:)\|_2 \le \mathbf{r}_j, \quad \|\mathbf{r}\|_1 \le \gamma, \end{aligned} \tag{23}$$

where $\mathbf{z}_i^T = \mathbf{Y}(:,i) - \mathbf{BS}'(:,i)$ with $i = 1, 2, \ldots, T'$, and $\mathbf{r}$ represents the sparse entries in the search grids to be estimated. The corresponding spatial spectrum is

$$P'(\theta_i) = \|\mathbf{S}'(i,:)\|_2, \quad i = 1, 2, \ldots, G. \tag{24}$$

We call this improved method MSCS. Essentially, we improve the performance of DOA estimation by increasing the number of snapshots in the pseudo data set. Since the number of linearly independent vectros in the pseudo data set is not big, the computational complexity is acceptable.

The computational complexities of the proposed algorithms are shown in Table 1. The complexities of MUSIC, CS and CMRCS are also given.

Remark 2: SASNK uses all the useful consecutive lags in set $\mathcal{L}$ for DOA estimation, while MSCS uses all the useful lags in set $\mathcal{D}$ to estimate DOAs. For nested array, there is no difference since $\mathcal{L}^{\mathrm{n}} = \mathcal{D}^{\mathrm{n}}$. But for PCA and coprime array, they are different since there are holes in their difference coarrays. Note that, for PCA and coprime array, we can first use matrix completion technique [20], [28] to fill holes and then use all the DOFs to estimate DOAs.

TABLE 1. Computational complexity.

SASNK	$O\left(N^2T + (M+1)^4G\right)$
MSCS	$O\left(N^2T + T'^3G^3\right)$
MUSIC	$O\left(N^2T + T'^3G^3\right)$
CS or CMRCS	$O\left(N^2T + N^4(G+N)\right)$

V. NUMERICAL EXAMPLES

In this section, we evaluate the performance of the SASNK and MSCS by comparing them with the SORTE-MUSIC, MUSIC [21], CS [14], and CMRCS [20] methods. The MUSIC-SORTE method means that the number of sources is estimated by the SORTE method [29]. We assume that the MUSIC method knows the number of sources a priori. For the CS, CMRCS and MSCS, a grid interval of 0.2° is used to form the dictionary. The SNR is defined as

$$\mathrm{SNR} = 10\log_{10}\frac{\mathrm{trace}(\mathbf{R}_s)}{(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2)}. \tag{25}$$

Without loss of generality, we consider an 8-sensor nested array with $N_1^{\mathrm{n}} = 4$ and $N_2^{\mathrm{n}} = 4$ as an example to showcase the performance of the proposed algorithms. According to Proposition 1, there are 11 lags with WFV greater than one and 7 lags with WFV greater than two. In this case, the pseudo data set has 10 linearly independent vectors, which means we can obtain 10 snapshots for the virtual array. Throughout of the simulations, the covariance matrix of the nonuniform noise is

$$\mathbf{R_n} = \mathrm{diag}\{30, 20, 5, 13, 6, 7, 1, 0.1, 9, 0.01\}, \tag{26}$$

and the root mean square error (RMSE) is defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{M_c K}\sum_{i=1}^{K}\sum_{j=1}^{M_c}\left(\hat{\theta}_{i,j} - \theta_i\right)^2}, \tag{27}$$

where M_c is the number of Monte Carlo runs.

A. SPATIAL SPECTRA

We first consider 17 uncorrelated narrowband sources uniformly distributed between $-60°$ and $60°$. The number of snapshots is set to be 400 and the SNR is set as 0 dB. The normalized spatial spectra of SORTE-MUSIC, MUSIC, CS, CMRCS, SASNK, and MSCS are shown in Figs. 3(a), (b), (c), (d), (e) and (f), respectively, where the vertical dot lines show the true DOAs. It is observed that the SORTE-MUSIC method fails to identify 17 sources accurately, while MSCS has the best estimation performance. For MUSIC, CS, CMRCS, and SASNK, 17 peaks can be seen, but they have more estimation error. The MUSIC and SASNK has similar RMSE, while MUSIC needs source number knowledge.

B. RMSE AND PR

We now examine the RMSE and probability of resolution (PR) performance of the MUSIC-SORTE, MUSIC, SASNK, CS, CMRCS and MSCS. The PR is defined as the event when all DOAs are estimated within 1.5° of their corresponding true values, which means that the difference between the true value of every DOA and its estimated value is less than 1.5°.

In the following examples, the number of independent Monte Carlo runs is $M_c = 200$. Assume that there are 13 uncorrelated sources which are uniformly selected from $[-60°, 60°]$. The Cramér-Rao Bound (CRB) [20], [30] is also included.

Fig. 4 shows RMSE performance as a function of SNR. The number of snapshots is 400. It can be observed that the MUSIC-SORTE achieves the worst estimation accuracy, mainly because it sometimes fails to estimate the correct source number. The performance of SASNK approaches that of MUSIC when SNR $\geqslant -5$ dB. Compared with SASNK, the slightly better performance of MUSIC is because that it knows the exact number of sources while SASNK does not. Therefore, MUSIC can get exact information of noise subspace for DOA estimation. The spectrum obtained by SASNK does not rely on the source number as well as any signal/noise subspace. Note that, it is hard to obtain the number source knowledge in reality. Thus the SASNK is more applicable in practice. The MSCS algorithm has the smallest RMSE, followed by CMRCS. Compared to CS and CMRCS, the better performance of the MSCS is due to the proposed pseudo data set which mitigates the nonuniform noise effect by specifically selecting the covariance values to form the linearly independent vectors. Fig. 5 shows the corresponding

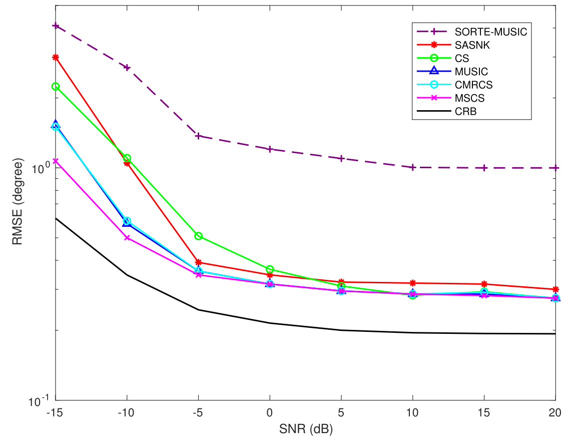


FIGURE 4. RMSE versus SNR ($T = 400$).

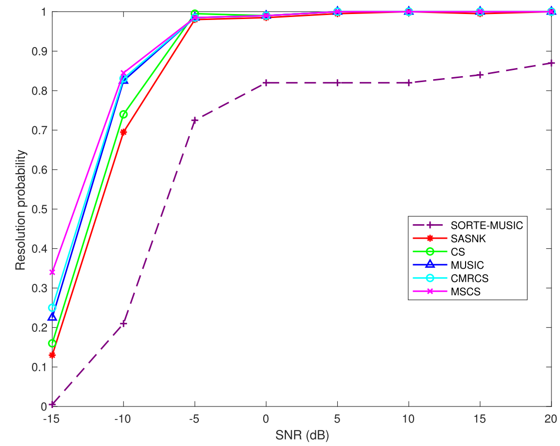


FIGURE 5. PR versus SNR ($T = 400$).

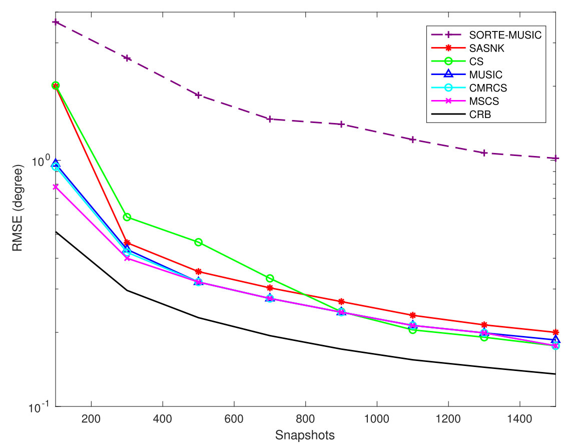


FIGURE 6. RMSE versus number of snapshots (SNR = -5 dB).

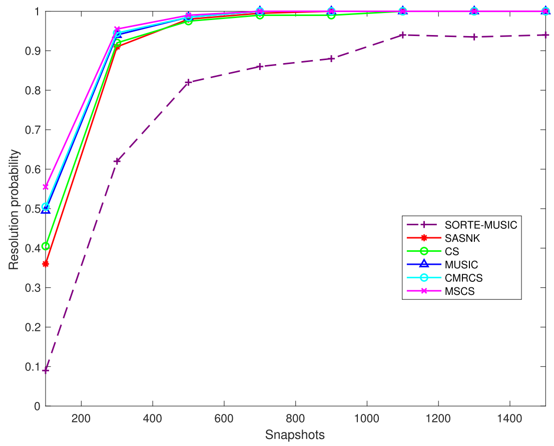


FIGURE 7. PR versus number of snapshots (SNR = −5 dB).

PR performance versus SNR. It is seen that MSCS has the highest PR. When the SNR approaches 5 dB, the PRs of MSCS, CMRCS, MUSIC, CS, and SASNK reach 100%, except for the MUSIC-SORTE method.

Similar results can also be found in Figs. 6 and 7, where the RMSE and PR performances are plotted versus the number of snapshots. In the simulation, the SNR is -5 dB and the remaining parameters keep the same as Fig. 4. It can be seen from Figs. 6 and 7 that SORTE-MUSIC has the worst performance. The MSCS algorithm has the best performance, followed by CMRCS, MUSIC, and SASNK.

VI. CONCLUSION

In this paper, we derived the number of repeated lags in the difference coarrays of the nested array, PCA, and coprime array in the presence of unknown nonuniform noise. To perform DOA estimation, we employed the repeated lags to form a pseudo data set that consists of multiple virtual linearly independent measurement vectors which contain all the useful information of the SCM. Two algorithms, named "SASNK" and "MSCS", are included for DOA estimation based on the multiple virtual snapshots from the pseudo data set. Numerical examples showed the good performance of SASNK and MSCS. In the future, we plan to extend the results to more sparse linear arrays and consider the case when there is mutual coupling.

APPENDIX A
PROOF OF PROPOSITION 1

A. PROOF OF (A)

We define that an integer valued function $\eta(i)$ denotes the number of single lags in the difference coarray formed by the ith sensor in the nested array.

1)

When N^{n} is even, it is easy to find that a positive lag in the difference coarray is single only if the lag belongs to the set $\mathcal{D}_1^{\mathrm{n}} = \left\{d_i^{\mathrm{n}} - d_j^{\mathrm{n}}, N_1^{\mathrm{n}}+1 \leqslant i \leqslant N_2^{\mathrm{n}}, 1 \leqslant j \leqslant N_1^{\mathrm{n}}\right\}$. Then for the positive lags, the following properties are true for the function $\eta(i)$ [1]:

(1) $\eta(i) = 0$ for $i \leqslant N_1^{\mathrm{n}}$.
(2) $\eta(i) = 1$ for $i = N_1^{\mathrm{n}}+1$.
(3) $\eta(i) = N_1^{\mathrm{n}}$ for $N_1^{\mathrm{n}}+1 < i < N_1^{\mathrm{n}}+N_2^{\mathrm{n}}$.
(4) $\eta(i) = N_1^{\mathrm{n}}+1$ for $i = N_1^{\mathrm{n}}+N_2^{\mathrm{n}}$.

The corresponding negative lags in the set $-\mathcal{D}_1^{\mathrm{n}}$ have the same results. According to properties (1), (2), (3), and (4), the number of single lags in the difference co-array is

$$\mathbb{N}_{\mathrm{ue}} = 2\left[1 + \left(N_2^{\mathrm{n}} - 2\right)N_1^{\mathrm{n}} + N_1^{\mathrm{n}} + 1\right] = \frac{N^{2\mathrm{n}}}{2} - N^{\mathrm{n}} + 4. \tag{28}$$

From [5], we know that the number of consecutive lags in the difference coarray is $\frac{N^{2\mathrm{n}}-2}{2} + N^{\mathrm{n}}$. Then the number of lags which appear more than once is

$$\mathbb{N}_{\mathrm{me}} = \frac{N^{2\mathrm{n}}-2}{2} + N^{\mathrm{n}} - (\frac{N^{2\mathrm{n}}}{2} - N^{\mathrm{n}} + 4) = 2N^{\mathrm{n}} - 5. \tag{29}$$

2)

When N^{n} is odd, we have $N_1^{\mathrm{n}} + N_2^{\mathrm{n}} = N^{\mathrm{n}}, N_1^{\mathrm{n}} + 1 = N_2^{\mathrm{n}}$. After a similar process as the even case, the number of single lags is

$$\mathbb{N}_{\mathrm{uo}} = 2N_1^{\mathrm{n}}N_2^{\mathrm{n}} - 2N_1^{\mathrm{n}} + 4. \tag{30}$$

Then the number of lags whose WFVs are greater than 1 is

$$\mathbb{N}_{\mathrm{mo}} = \frac{N^{2\mathrm{n}}-1}{2} + N^{\mathrm{n}} - 2N_1^{\mathrm{n}}N_2^{\mathrm{n}} + 2N_1^{\mathrm{n}} - 4 = 2N^{\mathrm{n}} - 5. \tag{31}$$

From (29) and (31), we can see that for both even and old numbers of sensors in a nested array, the number of lags whose WFV is greater than one is $(2N^{\mathrm{n}} - 5)$.

B. PROOF OF (B)

1)

When N^{n} is even, $N^{\mathrm{n}} = 2N_1^{\mathrm{n}}$.

For positive lags, $w(n) = N_1^{\mathrm{n}}+1-n$ when $1 \leq n \leq N_1^{\mathrm{n}}$. It is easy to know that there are $(N_1^{\mathrm{n}} - 2)$ lags with WFV greater than two when $n \in [1, N_1^{\mathrm{n}}]$. In the rest of the positive range $(N_1^{\mathrm{n}}+1, N_1^{2\mathrm{n}} + N_1^{\mathrm{n}} - 1]$, there are $(N_1^{\mathrm{n}} - 3)$ lags with WFV greater than two located at $\{N_1^{\mathrm{n}}+1, 2(N_1^{\mathrm{n}}+1), \ldots, (N_1^{\mathrm{n}} - 3)(N_1^{\mathrm{n}}+1)\}$.

The negative lags have the same results.

In conclusion, the number of lags appearing more than twice is $4N_1^{\mathrm{n}} - 9 = 2N^{\mathrm{n}} - 9$.

2)

When N^{n} is odd, $N_2^{\mathrm{n}} = N_1^{\mathrm{n}} + 1$.

For positive lags, $w(n) = N_1^{\mathrm{n}} + 1 - n$ when $1 \leq n \leq N_1^{\mathrm{n}}$. Thus there are $(N_1^{\mathrm{n}}-2)$ lags with WFV greater than two when $1 \leq n \leq N_1^{\mathrm{n}}$. In the rest of the positive range $(N_1^{\mathrm{n}}+1, N_1^{2\mathrm{n}} + 2N_1^{\mathrm{n}}]$, there are $(N_1^{\mathrm{n}} - 2)$ lags with WFV greater than two located in $\{N_1^{\mathrm{n}}+1, 2(N_1^{\mathrm{n}}+1), \ldots, (N_1^{\mathrm{n}} - 2)(N_1^{\mathrm{n}}+1)\}$.

The negative lags have the same results.

In conclusion, the number of lags appearing more than twice is $4N_1^{\mathrm{n}} - 7 = 2N^{\mathrm{n}} - 9$.

From the above derivation, we know that for both even and old numbers of sensors in a nested array, the number of lags whose WFV is greater than two is $(2N^{\mathrm{n}} - 9)$.

APPENDIX B
PROOF OF PROPOSITION 2

(1) The lag 0 is located in the range $\mathcal{L}^{\mathrm{p}}$. It is easy to find that $w(0) = N_1^{\mathrm{p}} + N_2^{\mathrm{p}} - 1 > 2$.

(2) Only one positive lag in the set $\mathcal{L}_{s,1}^{\mathrm{p}}$, i.e., N_2^{p}, is located in the range $\mathcal{L}^{\mathrm{p}}$. For the lag N_2^{p}, $w(N_2^{\mathrm{p}}) = N_1^{\mathrm{p}} - 1$, because there exist

$$\begin{aligned} N_2^{\mathrm{p}} - 0 \cdot N_2^{\mathrm{p}} &= N_2^{\mathrm{p}}, \\ 2N_2^{\mathrm{p}} - N_2^{\mathrm{p}} &= N_2^{\mathrm{p}}, \\ &\cdots \\ (N_1^{\mathrm{p}} - 1)N_2^{\mathrm{p}} - (N_1^{\mathrm{p}} - 2)N_2^{\mathrm{p}} &= N_2^{\mathrm{p}}. \end{aligned} \tag{32}$$

So if $N_1^{\mathrm{p}} = 3$, then $w(N_2^{\mathrm{p}}) = 2$, and if $N_1^{\mathrm{p}} > 3$, then $w(N_2^{\mathrm{p}}) > 2$. Similarly, the negative lag $-N_2^{\mathrm{p}}$ in the set $\mathcal{L}_{s,1}^{\mathrm{p}}$ has the same result.

(3) The following positive lags in the set $\mathcal{L}_{s,2}^{\mathrm{p}}$, i.e.,

$$\mathcal{L}_{s,2,p}^{\mathrm{p}} = \left\{ N_1^{\mathrm{p}}, 2N_1^{\mathrm{p}}, \ldots, \left(1 + \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor\right) N_1^{\mathrm{p}} \right\}, \tag{33}$$

are located in the range $\mathcal{L}^{\mathrm{p}}$. From the above paragraph, it is easy to induce that

$$\begin{aligned} w(N_1^{\mathrm{p}}) &= N_2^{\mathrm{p}} - 1, \\ &\cdots \\ w\left(\left(1 + \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor\right) N_1^{\mathrm{p}}\right) &= N_2^{\mathrm{p}} - 1 - \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor. \end{aligned} \tag{34}$$

Note that only when $N_1^{\mathrm{p}} = 3$ and $N_2^{\mathrm{p}} = 4$, $N_2^{\mathrm{p}} - 1 - \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor = 2$, otherwise $N_2^{\mathrm{p}} - 1 - \left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor > 2$.

So the WFVs of $N_1^{\mathrm{p}}, 2N_1^{\mathrm{p}}, \ldots, \left(\left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor\right) N_1^{\mathrm{p}}$ are greater than two.

Also, the set $-\mathcal{L}_{s,2,p}^{\mathrm{p}}$ in the set $\mathcal{L}_{s,2}^{\mathrm{p}}$ has the same result.

(4) Given an arbitrary lag l_i in the set $\mathcal{L}_c^{\mathrm{p}}$. Let

$$l_i = n_1 N_1^{\mathrm{p}} - m_1 N_2^{\mathrm{p}} = -n_2 N_1^{\mathrm{p}} + m_2 N_2^{\mathrm{p}}, \tag{35}$$

where $n_1, m_1, n_2, m_2 \neq 0$ are integers satisfying $1 \leq n_1, n_2 \leq N_2^{\mathrm{p}} - 1$ and $1 \leq m_1, m_2 \leq N_1^{\mathrm{p}} - 1$. Then we have

$$(n_1 + n_2) N_1^{\mathrm{p}} = (m_1 + m_2) N_2^{\mathrm{p}}. \tag{36}$$

There exist

$$\begin{cases} n_1 + n_2 = N_2^{\mathrm{p}} \\ m_1 + m_2 = N_1^{\mathrm{p}}, \end{cases} \tag{37}$$

so the WFV of l_i is at least two. Assume another two integers $n_3, m_3 \neq 0$ satisfying

$$l_i = n_1 N_1^{\mathrm{p}} - m_1 N_2^{\mathrm{p}} = -n_3 N_1^{\mathrm{p}} + m_3 N_2^{\mathrm{p}}, \tag{38}$$

then we have

$$\begin{cases} n_1 + n_3 = N_2^{\mathrm{p}} \\ m_1 + m_3 = N_1^{\mathrm{p}}. \end{cases} \tag{39}$$

According to (37) and (39), we have $n_2 = n_3$ and $m_2 = m_3$. Thus the lag l_i appears only twice. So $w(l_i) = 2$.

Since $-\mathcal{L}_{s,2,p}^{\mathrm{p}} \cup \{-N_1^{\mathrm{p}}\} \cup \{0\} \cup \{N_1^{\mathrm{p}}\} \cup \mathcal{L}_{s,2,p}^{\mathrm{p}} \cup \mathcal{L}_c^{\mathrm{p}} = \mathcal{L}^{\mathrm{p}}$, the WFVs of all the lags in $\mathcal{L}^{\mathrm{p}}$ are greater than one. So the number of lags appearing more than once is $(2N_1^P + 2N_2^P - 1)$. And according to the above derivation, the number of lags appearing more than twice is $\left(2\left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 5\right)$ $(N_1^{\mathrm{p}} > 3)$ or $\left(2\left\lfloor \frac{N_2^{\mathrm{p}}}{N_1^{\mathrm{p}}} \right\rfloor + 3\right)$ $(N_1^{\mathrm{p}} = 3)$.

APPENDIX C
PROOF OF PROPOSITION 3
A. PROOF OF (A)

We first consider the positive lags in $\mathcal{L}^{\mathrm{c}}$. It has been shown in [13] that for a coprime array, the single lags in $\mathcal{L}^{\mathrm{c}}$ are located at $(a_1 N_1^{\mathrm{c}} + a_2 N_2^{\mathrm{c}})$, where $a_1 > 0$ and $a_2 > 0$ are integers. The maximum value of a_1 can be given as follows:

$$\begin{aligned} a_1 N_1^{\mathrm{c}} + N_2^{\mathrm{c}} &\leq N_1^{\mathrm{c}} N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1 \\ a_1 &\leq N_2^{\mathrm{c}} + 1 - \frac{N_2^{\mathrm{c}} + 1}{N_1^{\mathrm{c}}} \\ a_1 &< N_2^{\mathrm{c}} + 1. \end{aligned} \tag{40}$$

Similarly, the maximum value of a_2 is given by

$$\begin{aligned} N_1^{\mathrm{c}} + a_2 N_2^{\mathrm{c}} &\leq N_1^{\mathrm{c}} N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1 \\ a_2 &\leq N_1^{\mathrm{c}} - \frac{1}{N_2^{\mathrm{c}}} \\ a_2 &< N_1^{\mathrm{c}}. \end{aligned} \tag{41}$$

Therefore, $a_1 \in [1, N_2^{\mathrm{c}} + 1)$ and $a_2 \in [1, N_1^{\mathrm{c}})$. The distribution of a_1 and a_2 are shown in Fig. 8. The boundary and interior of part $R1$ represent all the integer combinations of a_1 and a_2 satisfying $a_1 N_1^{\mathrm{c}} + a_2 N_2^{\mathrm{c}} \leq N_1^{\mathrm{c}} N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1$. It is easy to see that the number of integers in part $R2$ is more than that in part $R1$. We calculate the number of integers in $R2$, which is $(N_2^{\mathrm{c}} + 1)N_1^{\mathrm{c}}/2$. Thus the number of integers in part $R1$ is smaller than $(N_2^{\mathrm{c}} + 1)N_1^{\mathrm{c}}/2$, which means that the number of positive single lags is smaller than $(N_2^{\mathrm{c}} + 1)N_1^{\mathrm{c}}/2$. Similarly, the number of negative single lags has the same result. In total, the number of single lags in $\mathcal{L}^{\mathrm{c}}$ is smaller than $(N_2^{\mathrm{c}} + 1)N_1^{\mathrm{c}}$.

In this case, the number of lags whose WFVs are greater than one is at least

$$2N_1^{\mathrm{c}} N_2^{\mathrm{c}} + 2N_1^{\mathrm{c}} - 1 - (N_2^{\mathrm{c}} + 1)N_1^{\mathrm{c}} = N_1^{\mathrm{c}} N_2^{\mathrm{c}} + N_1^{\mathrm{c}} - 1. \tag{42}$$

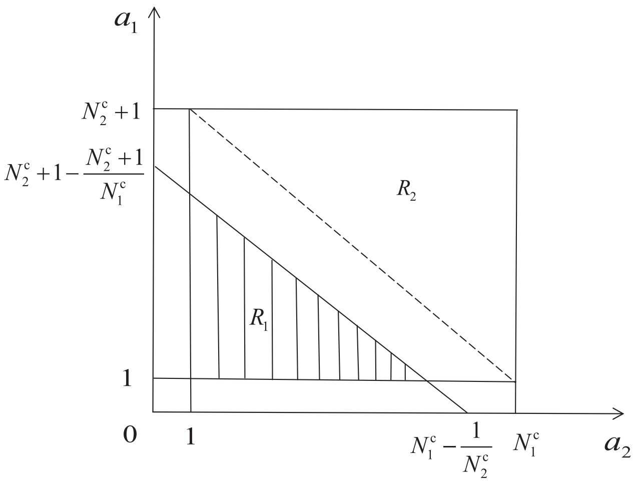


FIGURE 8. The geometry of N_1^c and N_2^c.

B. PROOF OF (B)

We first consider the positive lags in $\mathcal{L}^c$.

(1) The positive lags in the set $\mathcal{L}_{s,1}^c$, i.e., $N_2^c, \ldots, N_1^c N_2^c$, are located in the range $\mathcal{L}^c$, and $w(N_2^c) = 2N_1^c - 1, \ldots, w(N_1^c N_2^c) = N_1^c$. Since $w(N_2^c) > w(2N_2^c) > \cdots > w(N_1^c N_2^c) = N_1^c$, the number of positive lags in $\mathcal{L}_{s,1}^c$ appearing more than twice is N_1^c ($N_2^c > 2$) or $N_1^c - 1$ ($N_2^c = 2$).

(2) The positive lags in the set $\mathcal{L}_{s,2}^c$, i.e., $N_1^c, \ldots, N_2^c N_1^c$, are located in the range $\mathcal{L}^c$. As $w(N_2^c N_1^c)$ has been known in the above paragraph, here we consider the other $N_2^c - 1$ lags, and $w(N_1^c) = N_2^c, \ldots, w\left((N_2^c - 1)N_1^c\right) = 2$. So the number of positive lags in $\mathcal{L}_{s,2}^c$ appearing more than twice is $N_2^c - 2$.

(3) According to APPENDIX B, the lags in $\mathcal{L}_c^c$ appears no more than twice.

(4) $w(0) = 2N_1^c + N_2^c - 1 > 2$.

In conclusion, the number of lags with WFV greater than two is $2N_1^c + 2(N_2^c - 2) + 1 = 2N_1^c + 2N_2^c - 3$ ($N_2^c > 2$) or $2(N_1^c - 1) + 2(N_2^c - 2) + 1 = 2N_1^c + 2N_2^c - 5$ ($N_2^c = 2$).

ACKNOWLEDGMENT

Part of this article was presented at the 2018 IEEE Radar Conference.

REFERENCES

[1] G. Jiang, X. Mao, M. Wang, Y. Liu, and A. Nehorai, "Underdetermined DOA estimation with unknown source number in nonuniform noise," in *Proc. IEEE Radar Conf. (RadarConf)*, Apr. 2018, pp. 1422–1426.

[2] C. Zhang, Y. Wang, and F. Jing, "Underdetermined blind source separation of synchronous orthogonal frequency-hopping signals based on tensor decomposition method," *IEEE Access*, vol. 6, pp. 69407–69414, 2018.

[3] S. Zhao, T. Saluev, and D. L. Jones, "Underdetermined direction of arrival estimation using acoustic vector sensor," *Signal Process.*, vol. 100, pp. 160–168, Jul. 2014.

[4] Q. Shen, W. Liu, W. Cui, and S. Wu, "Underdetermined DOA estimation under the compressive sensing framework: A review," *IEEE Access*, vol. 4, pp. 8865–8878, 2016.

[5] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[6] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[7] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the music algorithm," in *Proc. Digit. Signal Process. Signal Process. Educ. Meeting (DSP/SPE)*, Sedona, AZ, USA, Jan. 2011, pp. 289–294.

[8] K. Han, P. Yang, and A. Nehorai, "Calibrating nested sensor arrays with model errors," *IEEE Trans. Signal Process.*, vol. 63, no. 11, pp. 4739–4748, Nov. 2015.

[9] Z. Tan and A. Nehorai, "Sparse direction of arrival estimation using co-prime arrays with off-grid targets," *IEEE Signal Process Lett.*, vol. 21, no. 1, pp. 26–29, Jan. 2014.

[10] C. Zhou, Y. Gu, X. Fan, Z. Shi, G. Mao, and Y. D. Zhang, "Direction-of-arrival estimation for coprime array via virtual array interpolation," *IEEE Trans. Signal Process.*, vol. 66, no. 22, pp. 5956–5971, Nov. 2018.

[11] C. Zhou, Y. Gu, Y. D. Zhang, Z. Shi, T. Jin, and X. Wu, "Compressive sensing-based coprime array direction-of-arrival estimation," *IET Commun.*, vol. 11, no. 11, pp. 1719–1724, 2017.

[12] X. Wu, W.-P. Zhu, and J. Yan, "A Toeplitz covariance matrix reconstruction approach for direction-of-arrival estimation," *IEEE Trans. Veh. Technol.*, vol. 66, no. 9, pp. 8223–8237, Sep. 2017.

[13] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1377–1390, Mar. 2015.

[14] Y. D. Zhang, M. G. Amin, and B. Himed, "Sparsity-based DOA estimation using co-prime arrays," in *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.*, Vancouver, BC, Canada, May 2013, pp. 3967–3971.

[15] Z. Shi, C. Zhou, Y. Gu, N. A. Goodman, and F. Qu, "Source estimation using Coprime array: A sparse reconstruction perspective," *IEEE Sensors J.*, vol. 17, no. 3, pp. 755–765, Feb. 2017.

[16] Z. Tan, Y. C. Eldar, and A. Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *IEEE Trans. Signal Process.*, vol. 62, no. 21, pp. 5565–5576, Nov. 2014.

[17] P. Pal and P. P. Vaidyanathan, "A grid-less approach to underdetermined direction of arrival estimation via low rank matrix denoising," *IEEE Signal Process. Lett.*, vol. 21, no. 6, pp. 737–741, Jun. 2014.

[18] C. E. Chen, F. Lorenzelli, R. E. Hudson, and K. Yao, "Stochastic maximum-likelihood DOA estimation in the presence of unknown nonuniform noise," *IEEE Trans. Signal Process.*, vol. 56, no. 7, pp. 3038–3044, Jul. 2008.

[19] Z.-Q. He, Z.-P. Shi, and L. Huang, "Covariance sparsity-aware DOA estimation for nonuniform noise," *Digit. Signal Process.*, vol. 28, pp. 75–81, May 2014.

[20] K. Liu and Y. D. Zhang, "Coprime array-based DOA estimation in unknown nonuniform noise environment," *Digit. Signal Process.*, vol. 79, pp. 66–74, Aug. 2018.

[21] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *IEEE Signal Process. Lett.*, vol. 22, no. 9, pp. 1438–1442, Sep. 2015.

[22] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[23] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 33, no. 2, pp. 387–392, Apr. 1985.

[24] L. Huang, C. Qian, H. C. So, and J. Fang, "Source enumeration for large array using shrinkage-based detectors with small samples," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 51, no. 1, pp. 344–357, Jan. 2015.

[25] K. Han and A. Nehorai, "Improved source number detection and direction estimation with nested arrays and ULAs using jackknifing," *IEEE Trans. Signal Process.*, vol. 61, no. 23, pp. 6118–6128, Nov. 2013.

[26] C. Qian, L. Huang, W. J. Zeng, and H. C. So, "Direction-of-arrival estimation for coherent signals without knowledge of source number," *IEEE Sensors J.*, vol. 14, no. 9, pp. 3267–3273, Sep. 2014.

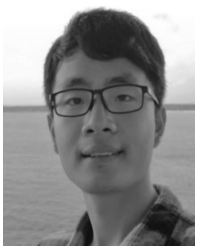
[27] D. Malioutov, M. Cetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3010–3022, Aug. 2005.

[28] C.-L. Liu, P. P. Vaidyanathan, and P. Pal, "Coprime coarray interpolation for DOA estimation via nuclear norm minimization," in *Proc. IEEE Int. Symp. Circuits Syst.*, May 2016, pp. 2639–2642.

[29] Z. He, A. Cichocki, S. Xie, and K. Choi, "Detecting the number of clusters in n-way probabilistic clustering," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 32, no. 11, pp. 2006–2021, Nov. 2010.

[30] M. Wang and A. Nehorai, "Coarrays, Music, and the Cramér-Rao bound," *IEEE Trans. Signal Process.*, vol. 65, no. 4, pp. 933–946, Feb. 2017.
[31] C.-L. Liu and P. P. Vaidyanathan, "Cramér-Rao bounds for coprime and other sparse arrays, which find more sources than sensors," *Digit. Signal Process.*, vol. 61, pp. 43–61, Feb. 2017.

GUOJUN JIANG received the B.S. and M.S. degrees from the Harbin Institute of Technology, Harbin, China, in 2012 and 2014, respectively, where she is currently pursuing the Ph.D. degree with the School of Electronics and Information Engineering. From 2017 to 2018, she was a Research Scholar with the Department of Electrical and Systems Engineering, Washington University in St. Louis, St. Louis, MO, USA. Her current research interests include array processing and graph signal processing.



MIANZHI WANG (S'15) received the B.S. degree from Fudan University, Shanghai, China, in 2013, and the M.S. and Ph.D. degrees from Washington University in St. Louis, St. Louis, MO, USA, in 2018, under the supervision of Dr. A. Nehorai. His current research interests include statistical signal processing for sensor arrays, optimization, and machine learning.

XINGPENG MAO (M'06) was born in Liaoning, China, in 1972. He received the B.S. degree in radio electronics from Northeast Normal University, Changchun, China, in 1993, and the M.S. and Ph.D. degrees from the Harbin Institute of Technology, China, in 1999 and 2004, respectively, all in electronics engineering. He joined the Harbin Institute of Technology as a Lecturer, in 1993, involved in the design and development of radar and other electronical system. From 2005 to 2006, he was a Postdoctoral Fellow with the University of Waterloo, Waterloo, ON, Canada. He is currently a Professor with the Department of Electronic and Information Engineering, Harbin Institute of Technology. His current research interest includes signal processing in wireless communication and radar. He is a Senior Member of the Chinese Institute of Electronics.

CHENG QIAN was born in Deqing, Zhejiang, China. He received the B.S. degree in communication engineering from Hangzhou Dianzi University, Hangzhou, China, in 2011, the M.S. and Ph.D. degrees in information and communication engineering from the Harbin Institute of Technology, China, in 2013 and 2017, respectively. He was a Visiting Ph.D. Student with the Department of ECE, University of Minnesota Twin Cities, Minneapolis, MN, USA, from 2014 to 2016. In the beginning of 2017, he returned back to the group in UMN as a Postdoctoral Researcher for half a year. After that, he joined the University of Virginia, where he is currently a Postdoctoral Associate with the Department of ECE. His current research interests include tensor decomposition, optimization with applications in signal processing, wireless communications, and bioinformatics.

YONGTAN LIU was born in 1936. He received the B.S. degree from Tsinghua University, in 1958. He is currently a Professor with the School of Electronics and Information Engineering, Harbin Institute of Technology. His current research interests include radar system and signal processing of high-frequency OTH radar, continuous wave radar, and microwave imaging radar. He is a member of the Chinese Academy of Sciences and the Chinese Academy of Engineering.

ARYE NEHORAI (S'80–M'83–SM'90–F'94–LF'17) received the B.S. and M.S. degrees from Technion, Haifa, Israel, and the Ph.D. degree from Stanford University, Stanford, CA, USA.

He was a Faculty Member with Yale University and the University of Illinois at Chicago. He was the Chair of the Preston M. Green Department of Electrical and Systems Engineering (ESE), Washington University in St. Louis (WUSTL), St. Louis, MO, USA, from 2006 to 2016, where he is currently the Eugene and Martha Lohman Professor of electrical engineering. Under the leadership, the undergraduate enrollment has more than tripled and the master's enrollment has grown seven-fold. He is also a Professor with the Division of Biology and Biomedical Sciences (DBBS), Division of Biostatistics, Department of Biomedical Engineering, and the Department of Computer Science and Engineering, and the Director of the Center for Sensor Signal and Information Processing, WUSTL.

Dr. Nehorai was a member of the Executive Committee of the IEEE Signal Processing Society (SPS), the Chair of the Publications Board, and the Vice President (Publications) of the IEEE SPS, from 2003 to 2005. He was a recipient of the 2006 IEEE SPS Technical Achievement Award and the 2010 IEEE SPS Meritorious Service Award. He was a recipient of several best paper awards in IEEE journals and conferences. He was an Editor-in-Chief of the IEEE Transactions on Signal Processing, from 2000 to 2002. In 2001, he was named as a University Scholar of the University of Illinois. He was the Founding Editor of the special columns on Leadership Reflections in the *IEEE Signal Processing Magazine*, from 2003 to 2006. He was an elected Distinguished Lecturer of the IEEE SPS for a term lasting, from 2004 to 2005. He was the Principal Investigator of the Multidisciplinary University Research Initiative (MURI) project titled Adaptive Waveform Diversity for Full Spectral Dominance, from 2005 to 2010. He has been a Fellow of the Royal Statistical Society, since 1996, and the AAAS, since 2012.