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An Insight into Stability Conditions of Discrete-Time Systems With Time-Varying Delay

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ABSTRACT This note studies an interesting phenomenon for stability conditions of discrete-time systems with time-varying delay. The underlying reason behind this phenomenon is revealed, and thereafter some conclusions are drawn: (i) Stability conditions of discrete-time systems with time-varying delay are generally divided into two types: those obtained by summation inequalities with free-matrix variables and those obtained by the combination of summation inequalities without free-matrix variables and the reciprocally convex lemma; (ii) The conservatism between the two types of stability conditions can not be theoretically compared. To clearly demonstrate this interesting phenomenon and meanwhile, to further verify these conclusions, several bounded real lemmas are obtained via different bounding-inequality methods and applied to a numerical example.

INDEX TERMS Discrete-time system, Lyapunov functional, stability, summation inequality, time-varying delay.

I. INTRODUCTION

Since time delay is often encountered in the real world such as network and mechanical engineering [1], [2], the stability analysis for discrete-time systems with time-varying delay has attracted considerable attention. Until now, many remarkable results have been reported in the literature [1]–[39] such as the free-matrix-weighting technique [3], the delay-partitioning method [4], the bounding-equality method [1], [5]–[7]. Compared with other techniques, the bounding-inequality method, owing to its effectiveness and straightforwardness, has been widely used in the Lyapunov–Krasovskii (L–K) functional method.

When the L–K functional is constructed, the double summation term $\sum_{i=-h_2}^{-h_1-1}\sum_{j=i}^{-1} y_k^T(j)Ry_k(j)$ is commonly contained (see (9) below) since it could lead to a delay-dependent stability condition. Then, to estimate the summation term

$$\delta(k) := \sum_{i=-h_2}^{-h_1-1} y_k^T(i)Ry_k(i) \tag{1}$$

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arising in the forward difference of the L–K functional becomes an essential point in achieving less conservative stability conditions in terms of linear matrix inequalities (LMIs). When bounding $\delta(k)$, it is usually split into two parts: $\delta_1(k) := \sum_{i=-h(k)}^{-h_1-1} y_k^T(i)Ry_k(i)$ and $\delta_2(k) := \sum_{i=-h_2}^{-h(k)-1} y_k^T(i)Ry_k(i)$ to take full advantage of structural information.

As far as the authors know, three bounding-inequality methods have been established to estimate the summation term $\delta(k)$. Owing to the conservatism of Jensen inequality [1], the Wirtinger-based summation inequality (WBSI) was proposed [8]–[10]. Combined with the reciprocally convex lemma (RCL) [7], a more relaxed stability condition was obtained in [9]. This method is called WBSI + RCL method. Thereafter, a free-matrix-based summation inequality (FMBSI) was proposed by introducing some free matrix variables [11]. Based on FMBSI, a new stability condition was obtained in [11]. This method is called FMBSI method.

Recently, an improved summation inequality was proposed in [12] by considering both $\delta_1(k)$ and $\delta_2(k)$ together. In fact, this inequality can be directly obtained by combining WBSI and the improved RCL (see (5) below). For simplicity, this inequality is called CWBSI and the method is called WBSI + IRCL method. It is seen that WBSI + RCL and WBSI + IRCL methods belong to the same type: the combination of WBSI and RCL. As shown in [12], the maximum allowable upper bounds (MAUBs) obtained by the WBSI + IRCL method-based stability condition are larger than or equal to those obtained by the FMBSI method-based stability condition. However, it should be pointed out FMBSI provides a tighter bound of $\delta(k)$ than what CWBSI does. In this case, an interesting phenomenon arises: based on the same L–K functional, MAUBs obtained by FMBSI method are not larger than but even smaller than those obtained by WBSI + IRCL method while the inequality used in FMBSI method is more accurate than the one used in WBSI + IRCL method. This phenomenon is contrary to what we usually think. Why does this happen? To find the underlying reason can help us eliminate confusions about this phenomenon and deeply understand the formations of LMI-based stability conditions, which is the first motivation of this study.

With the help of more orthogonal polynomials, various kinds of accurate summation inequalities were recently developed, such as the auxiliary function-based inequalities [13], [14]. In general, existing summation inequalities reported in the literature could be classified into two types [6], [15]: those with free matrix variables and those without free matrix variables. Jensen inequality and WBSI belong to the former and FMBSI belongs to the latter. Since summation inequalities with free matrix variables transform the summation interval from the denominator to the numerator, RCL is no longer required when bounding $\delta(k)$. The relationship between these two types of summation inequalities was studied in [5]. It was pointed out that the two types of corresponding summation inequalities are actually equivalent in conservatism, that is, they produce the same tight bound of $\delta(k)$. However, what is the relationship between stability conditions, respectively, obtained by applying the two types of summation inequalities? To the best of authors' knowledge, this problem has not been studied in the literature. To answer this problem can help us better understand the roles of the two kinds of bounding-inequality methods in reducing the conservatism of stability conditions, which is the second motivation of this study.

In this note, we first present the interesting phenomenon via three numerical examples. Second, the underlying reason is revealed through careful study. Based on the phenomenon and the underlying reason, some conclusions are drawn: (i) according to the method applied to bound $\delta(k)$, stability conditions of discrete-time systems with time-varying can be divided into two types: RCL-based stability conditions and FM-based stability conditions; (ii) the conservatism between these two types can not be compared in theory. To further verify these conclusions, we apply several bounded real lemmas obtained by different bound-inequality methods to a numerical example.

Notations. Throughout this note, $sym\{X\}$ denotes $X + X^T$ for any square real matrix X. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stands for the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The notation $\mathbb{S}_+^n$ represents the set of symmetric positive definite matrices of $\mathbb{R}^{n\times n}$.

II. THE MAIN RESULT

In this section, three typical stability conditions are firstly recalled, which are, respectively, obtained by WBSI + RCL, WBSI + IRCL and FMBSI methods. Secondly, the interesting phenomenon is presented by applying the three stability conditions to numerical examples. Finally, the underlying reason is revealed and some conclusions are drawn.

A. WBSI, FMBSI AND RCLS

The following notations are defined for simplicity of presentation:

$$\begin{aligned} M &:= \begin{bmatrix} M_0 & M_1 \end{bmatrix}, \quad N := \begin{bmatrix} N_0 & N_1 \end{bmatrix}, \\ \Pi_0 &:= \begin{bmatrix} I & -I & 0 \end{bmatrix}, \quad \Pi_1 := \begin{bmatrix} I & I & -2I \end{bmatrix}, \\ \Pi &:= col\{\Pi_0, \Pi_1\}, \quad b_a := b - a, \\ \tilde{R} &:= diag\{R, 3R\}, \quad \tilde{R}_1 := diag\{R_1, 3R_1\}. \end{aligned}$$

Lemma 1 (WBSI [9], FMBSI [11]): For integers a and b satisfying $a < b$, matrices $R \in \mathbb{S}_+^n$, $M_0, M_1 \in \mathbb{R}^{3n\times n}$ and a vector function $\{x(i) \in \mathbb{R}^n | i \in [a, b]\}$, the following inequalities

$$\sum_{i=a}^{b-1} y^T(i)Ry(i) \geq \frac{1}{b_a}\vartheta^T \Pi^T \tilde{R}\Pi\vartheta \tag{2}$$

$$\sum_{i=a}^{b-1} y^T(i)Ry(i) \geq -\vartheta^T \Upsilon(b_a, M)\vartheta \tag{3}$$

hold, where

$$\begin{aligned} \vartheta &:= col\left\{x(b), x(a), \frac{1}{b_a+1}\sum_{i=a}^{b} x(i)\right\}, \\ y(i) &:= x(i+1) - x(i), \\ \Upsilon(b_a, M) &:= b_a M\tilde{R}^{-1}M^T + sym\{M\Pi\}. \end{aligned}$$

Remark 1: Ineq. (3) is a simplified version of FMBSI proposed in [11] by setting $Z_{ij} = M_i R^{-1} M_j^T$, $i \leq j \in \{0, 1\}$. In this case, the constraint inequality (9) in [11] obviously holds from the Schur complement. By setting $M_i = -\frac{2i+1}{b_a}\Pi_i^T R$, $i \in \{0, 1\}$, FMBSI (3) is reduced to WBSI (2). Actually, WBSI (2) and FMBSI (3) are equivalent, i.e., the two inequalities produce the same tight bounds [15]. However, compared with WBSI (2), FMBSI (3) moves the summation interval b_a from the denominator to the numerator, which makes RCL no longer required.

Lemma 2 (RCLs [7], [16], [17]): For matrices $R \in \mathbb{S}_+^n$ and $X, X_1, X_2 \in \mathbb{R}^{n\times n}$, the following matrix inequalities

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R & X \\ X^T & R \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R+(1-\alpha)T_1 & X \\ X^T & R+\alpha T_2 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R+(1-\alpha)Y_1 & Y(\alpha) \\ Y(\alpha)^T & R+\alpha Y_2 \end{bmatrix} \quad (6)$$

hold for $\forall \alpha \in (0, 1)$, where (4) is constrained by $\begin{bmatrix} R & X \\ X^T & R \end{bmatrix} \geq 0$, and

$$\begin{aligned} T_1 &:= R - XR^{-1}X^T, \quad T_2 := R - X^T R^{-1} X, \\ Y_1 &:= R - X_1 R^{-1} X_1^T, \quad Y_2 := R - X_2^T R^{-1} X_2, \\ Y(\alpha) &:= \alpha X_1 + (1-\alpha) X_2. \end{aligned}$$

Remark 2: The matrix inequality (4) is the matrix version of RCL [7]. The improved RCLs (5) and (6) are no longer constrained by $\begin{bmatrix} R & X \\ X^T & R \end{bmatrix} \geq 0$. Obviously, if the constraint is added, RCL (5) is more accurate than RCL (4). Additionally, if letting $X_1 = X_2 = X$, RCL (6) is reduced to RCL (5). Therefore, RCL (6) is more accurate than RCL (5). Notice that a novel RCL was proposed in [18] which is equivalent to RCL (6) in conservatism.

B. THREE TYPICAL STABILITY CONDITIONS

Consider the following linear discrete-time system with a time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-h(k)), & k \geq 0 \\ x(k) = \phi(k), & -h_2 \leq k \leq 0 \end{cases} \quad (7)$$

where $x(k) \in \mathbb{R}^n$ is the state vector; A and A_d are constant system matrices; $\phi(k)$ is the initial condition; with given integers h_1 and h_2, $h(k)$ is the time-varying delay satisfying

$$1 \leq h_1 \leq h(k) \leq h_2, \quad k \geq 0. \quad (8)$$

Before proceeding, the following notations are defined for simplicity:

$$\begin{aligned} h_k &:= h(k), \quad h_{k1} := h_k - h_1, \quad h_{2k} := h_2 - h_k, \\ h_{21} &:= h_2 - h_1, \quad x_k(i) := x(k+i), \\ y_k(i) &:= x_k(i+1) - x_k(i), \\ \eta(k) &:= col\left\{ x_k(0), \sum_{i=-h_1}^{-1} x_k(i), \sum_{i=-h_2}^{-h_1-1} x_k(i) \right\}. \end{aligned}$$

Based on the following L–K functional [9]:

$$\begin{aligned} V(k) = \eta^T(k)P\eta(k) &+ \sum_{i=-h_1}^{-1} x_k^T(i) Q_1 x_k(i) \\ &+ \sum_{i=-h_2}^{-h_1-1} x_k^T(i) Q_2 x_k(i) \\ &+ h_1 \sum_{i=-h_1}^{-1} \sum_{j=i}^{-1} y_k^T(j) R_1 y_k(j) \\ &+ h_{21} \sum_{i=-h_2}^{-h_1-1} \sum_{j=i}^{-1} y_k^T(j) R y_k(j), \end{aligned} \quad (9)$$

where matrices $P, Q_1, Q_2, R_1, R \in \mathbb{S}_+^n$ are to be determined, three stability conditions were, respectively, obtained in [9], [11] and [12].

Lemma 3: For given integers h_1 and h_2 satisfying (8), system (7) is asymptotically stable if there exist matrices $P \in \mathbb{S}_+^{3n}$, $Q_1, Q_2, R_1, R \in \mathbb{S}_+^n$, $S \in \mathbb{R}^{2n\times 2n}$ and $M_0, M_1, N_0, N_1 \in \mathbb{R}^{3n\times n}$ such that one of the following conditions holds:

Condition 1 (C1) (WBSI + RCL [9]):

$$\tilde{R}_S := \begin{bmatrix} \tilde{R} & S \\ S^T & \tilde{R} \end{bmatrix} \geq 0, \quad (10)$$

$$\Psi(h_k) - \mathcal{E}(\tilde{R}_S) < 0, \quad h_k \in \{h_1, h_2\}, \quad (11)$$

Condition 2 (C2) (CWBSI [12]):

$$\begin{bmatrix} \Psi(h_1) - \Upsilon_{5,1} & E_1^T S \\ * & -\tilde{R} \end{bmatrix} < 0, \quad (12)$$

$$\begin{bmatrix} \Psi(h_2) - \Upsilon_{5,2} & E_2^T S^T \\ * & -\tilde{R} \end{bmatrix} < 0, \quad (13)$$

Condition 3 (C3) (FMBSI [11]):

$$\begin{bmatrix} \Psi(h_1) + h_{21}\Xi & h_{21} F_2^T N \\ * & -\tilde{R} \end{bmatrix} 1 < 0, \quad (14)$$

$$\begin{bmatrix} \Psi(h_2) + h_{21}\Xi & h_{21} F_1^T M \\ * & -\tilde{R} \end{bmatrix} < 0, \quad (15)$$

where

$$\begin{aligned} \Psi(h_k) &= \Upsilon_1(h_k) + \Upsilon_2 + \Upsilon_3 - \Upsilon_4, \quad (16) \\ \Upsilon_1(h_k) &= sym\{\Gamma^T(h_k) P \Gamma_{12}\} + \Gamma_1^T P \Gamma_1 - \Gamma_2^T P \Gamma_2, \\ \Upsilon_2 &= e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 - e_4^T Q_2 e_4, \\ \Upsilon_3 &= e_s^T (h_1^2 R_1 + h_{21}^2 R) e_s, \quad \Upsilon_4 = E_0^T \tilde{R}_1 E_0, \\ \Gamma(h_k) &= col\{e_1, (h_1+1)e_5, (h_{k1}+1)e_6 + (h_{2k}+1)e_7\}, \\ \Gamma_1 &= col\{e_s, -e_2, -e_3 - e_4\}, \quad \Gamma_{12} = \Gamma_1 - \Gamma_2, \\ \Gamma_2 &= col\{e_0, -e_1, -e_2 - e_3\}, \\ E_0 &= col\{e_1 - e_2, e_1 + e_2 - 2e_5\}, \\ E_1 &= col\{e_2 - e_3, e_2 + e_3 - 2e_6\}, \\ E_2 &= col\{e_3 - e_4, e_3 + e_4 - 2e_7\}, \\ E &= col\{E_1, E_2\}, \\ e_0 &= 0_{n\times 7n}, \quad e_s = (A - I)e_1 + A_d e_3 \\ e_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_{n\times n} & 0_{n\times(7-i)n} \end{bmatrix}, \quad i \in \{1, \ldots, 7\}, \\ \Xi &= F_1^T sym\{M\Pi\} F_1 + F_2^T sym\{N\Pi\} F_2 \\ F_1 &= col\{e_2, e_3, e_6\}, \quad F_2 = col\{e_3, e_4, e_7\}, \\ \Upsilon_{5,1} &= \mathcal{E}\left(\begin{bmatrix} 2\tilde{R} & S \\ S^T & \tilde{R} \end{bmatrix}\right), \quad \Upsilon_{5,2} = \mathcal{E}\left(\begin{bmatrix} \tilde{R} & S \\ S^T & 2\tilde{R} \end{bmatrix}\right), \end{aligned}$$

and where $\mathcal{E}(W)$ means $E^T W E$ for an appropriate dimension matrix W.

Proof: Along the trajectory of system (7), the forward difference of $V(k)$ defined in (9) is computed:

$$\Delta V(k) \leq \zeta^T(k) \Psi(h_k) \zeta(k) - h_{21}\delta(k), \quad (17)$$

where $\Psi(h_k)$ is defined in (16), $\delta(k)$ is defined in (1) and

$$\zeta(k) := col\Big\{x_k(0), x_k(-h_1), x_k(-h_k), x_k(-h_2), \sum_{i=-h_1}^{0}\frac{x_k(i)}{h_1+1}, \sum_{i=-h_k}^{-h_1}\frac{x_k(i)}{h_{k1}+1}, \sum_{i=-h_2}^{-h_k}\frac{x_k(i)}{h_{2k}+1}\Big\}.$$

Applying FMBSI (3) to $\delta(k)$ yields:

$$\delta(k) = \delta_1(k) + \delta_2(k) \geq \zeta^T(k)\delta_{FMB}(h_k)\zeta(k), \quad (18)$$

where

$$\delta_{FMB}(h_k) = -F_1^T\Upsilon(h_{k1}, M)F_1 - F_2^T\Upsilon(h_{2k}, N)F_2.$$

Then, combining (17) with (18) leads to $C3$.

Letting $M_i = -\frac{2i+1}{h_{k1}}\Pi_i^T R$ and $N_i = -\frac{2i+1}{h_{2k}}\Pi_i^T R$, $i \in \{0, 1\}$ and applying RCL (5) to (18) leads to:

$$\begin{aligned}\delta_{FMB}(h_k) &= \frac{1}{h_{k1}}E_1^T\tilde{R}E_1 + \frac{1}{h_{2k}}E_2^T\tilde{R}E_2 \\ &= \frac{1}{h_{21}}\mathcal{E}\left(\begin{bmatrix}\frac{1}{\alpha}\tilde{R} & 0\\ 0 & \frac{1}{1-\alpha}\tilde{R}\end{bmatrix}\right) \\ &\geq \frac{1}{h_{21}}\mathcal{E}\left(\begin{bmatrix}\tilde{R}+(1-\alpha)T_1 & S\\ S^T & \tilde{R}+\alpha T_2\end{bmatrix}\right) \\ &:= \delta_{R2}(h_k) \end{aligned} \quad (19)$$

where $\alpha = h_{k1}/h_{21}$, $T_1 = \tilde{R} - S\tilde{R}^{-1}S^T$ and $T_2 = \tilde{R} - S^T\tilde{R}^{-1}S$. Then, combining (17) with Ineq. $\delta(k) \geq \zeta^T(k)\delta_{R2}(h_k)\zeta(k)$ leads to $C2$.

With $\tilde{R}_S \geq 0$, it follows from Remark 2 that

$$\delta_{R2}(h_k) \geq \frac{1}{h_{21}}\mathcal{E}(\tilde{R}_S) := \delta_{R1}. \quad (20)$$

Then, combining (17) with Ineq. $\delta(k) \geq \zeta^T(k)\delta_{R1}\zeta(k)$ leads to $C1$. This completes the proof. □

Remark 3: Based on the same L–K functional (9), three conditions are derived in Lemma 3 by employing WBSI + RCL, WBSI + IRCL and FMBSI methods, respectively. It is seen from the process that the bounding-inequality method plays an important role in deriving stability conditions for time-delay systems. Since a more accurate inequality usually leads to a more relaxed stability, developing more accurate summation inequalities is a hot topic in the research field of time-delay systems. Generally speaking, there are two main ways to reduce the conservatism of summation inequalities. One way is to take more summation information of the state into account when developing a new summation inequality. The other way is to introduce more free matrix variables so that more freedom is achieved to estimate $\delta(k)$.

C. AN INTERESTING PHENOMENON

From the proof of Lemma 3, we have

$$\begin{aligned}\delta(k) &\geq \zeta^T(k)\delta_{FMB}(h_k)\zeta(k) \\ &\geq \zeta^T(k)\delta_{R2}(h_k)\zeta(k) \geq \zeta^T(k)\delta_{R1}\zeta(k),\end{aligned}$$

in which $\delta(k) \geq \zeta^T(k)\delta_{R2}(h_k)\zeta(k)$ is just the improved summation inequality (CWBSI) proposed in [12]. Obviously, Ineq. $\delta(k) \geq \zeta^T(k)\delta_{FMB}(h_k)\zeta(k)$ is more accurate than CWBSI. Therefore, it seems that $C3$ should be less conservative than $C2$. In other words, the MAUBs obtained by $C3$ should be larger than, or at least equal to those obtained by $C2$. But that is not the case. This is clearly shown in the following three examples.

Example 1: Consider system (7) with

$$A = \begin{bmatrix}0.6480 & 0.0400\\ 0.1200 & 0.6540\end{bmatrix}, \quad A_d = \begin{bmatrix}-0.1512 & -0.0518\\ 0.0259 & -0.1091\end{bmatrix}. \quad (21)$$

MAUBs obtained by $C1$, $C2$ and $C3$ are, respectively, listed in Table 1. It is seen from Table 1 that MAUBs obtained by $C2$ are larger than or equal to those obtained by $C1$. This finding verifies some conclusions made before. However, it is also found that MAUBs obtained by $C3$ are not always larger than those obtained by $C2$. Especially, when h_1 takes large values such as 13, 20 or 25, MAUBs obtained by $C3$ are even smaller.

TABLE 1. MAUBs h_2 for different h_1 in Example 1.

h_1	1	3	5	7	11	13	20	25
$C1$	16	18	20	22	25	27	34	39
$C2$	17	19	21	22	26	28	35	40
$C3$	18	19	21	22	26	27	34	39

Example 2: Consider system (7) with

$$A = \begin{bmatrix}0.8 & 0\\ 0.05 & 0.9\end{bmatrix}, \quad A_d = \begin{bmatrix}-0.1 & 0\\ -0.2 & -0.1\end{bmatrix}. \quad (22)$$

MAUBs obtained by $C1$, $C2$ and $C3$ are listed in Table 2. It is easily seen that the MAUBs obtained by $C2$ are larger than or at least equal to those obtained by $C1$ while MAUBs obtained by $C3$ are not always larger than but even smaller than those obtained by $C2$ when h_1 takes values such as 3 or 11.

TABLE 2. MAUBs h_2 for different h_1 in Example 2.

h_1	1	3	5	7	9	11
$C1$	20	21	21	21	22	23
$C2$	21	22	22	22	23	24
$C3$	21	21	22	22	23	23

Example 3: Consider system (7) with

$$A = \begin{bmatrix}1 & 0.01\\ -0.1 & 0.99\end{bmatrix}, \quad A_d = \begin{bmatrix}0.003 & 0.001\\ 0.01 & 0.005\end{bmatrix}. \quad (23)$$

MAUBs obtained by $C1$, $C2$ and $C3$ are listed in Table 3. As expected, MAUBs obtained by $C2$ are larger than or equal to those obtained by $C1$. However, except the case of $h_1 = 1$, MAUBs obtained by $C3$ are all larger than those obtained by $C1$ or $C2$ when h_1 takes other values.

Combining Examples 1, 2 and 3, we find an unusual but interesting phenomenon. That is, MAUBs obtained by $C3$ may be larger than those obtained by $C2$ for one system while smaller for another system. Even for one same system such as shown in Example 1, MAUBs obtained by $C3$ might be

TABLE 3. MAUBs h_2 for different h_1 in Example 3.

h_1	1	3	5	7	9	11	13
$C1$	19	27	38	48	56	64	70
$C2$	19	28	40	50	58	66	72
$C3$	19	29	43	54	63	70	76

larger than those obtained by $C2$ when h_1 takes some values while become smaller when h_1 takes other values. It is worth noting that, for the three conditions, the same L–K functional is employed and meanwhile, the summation inequality employed in $C3$ is more general than those employed in $C1$ and $C2$. This interesting phenomenon attracts our attention and motivates us to find the underlying reason.

D. THE UNDERLYING REASON

Let us consider the three terms:

$$\delta_{FMB}(h_k) = -F_1^T \Upsilon(h_{k1}, M)F_1 - F_2^T \Upsilon(h_{2k}, N)F_2,$$
$$\delta_{R2}(h_k) = \frac{1}{h_{21}}\mathcal{E}\left(\begin{bmatrix} \tilde{R}+(1-\alpha)T_1 & S \\ S^T & \tilde{R}+\alpha T_2 \end{bmatrix}\right),$$
$$\delta_{R1} = \frac{1}{h_{21}}\mathcal{E}\left(\tilde{R}_S\right).$$

It is known that RCLs (4), (5) and (6) hold only for $\alpha \in (0, 1)$, exclusive of $\alpha = 0$ and $\alpha = 1$. Therefore, the inequality (19)

$$\delta_{FMB}(h_k) \geq \delta_{R2}(h_k)$$

holds only for $h_k \in \{h_1+1, \ldots, h_2-1\}$, exclusive of $h_k = h_1$ and $h_k = h_2$. So any of the inequalities

$$\delta_{FMB}(h_1) \geq \delta_{R2}(h_1)$$
$$\delta_{FMB}(h_2) \geq \delta_{R2}(h_2)$$

does not definitely hold. In other words, there is no relationship between the sizes of $\delta_{FMB}(h_1)$ and $\delta_{R2}(h_1)$ or between the sizes of $\delta_{FMB}(h_2)$ and $\delta_{R2}(h_2)$ in theory. As a result, the conservatism between Ineq. (12) and Ineq. (14) or between Ineq. (13) and Ineq. (15) can not be compared in theory. The end-point problem of RCL results in this interesting phenomenon. This is the underlying reason.

Additionally, if the constraint $\tilde{R}_S \geq 0$ is considered, the inequality

$$\delta_{R2}(h_k) \geq \delta_{R1}$$

obviously holds for any $h_k \in [h_1, h_2]$. In other words, the two inequalities $\delta_{R2}(h_1) \geq \delta_{R1}$ and $\delta_{R2}(h_2) \geq \delta_{R1}$ do hold. Therefore, $C2$ is always less conservative than $C1$, which is verified by previous numerical examples.

The above discussions are made on stability conditions that are derived from the L–K functional (9) and WBSI (2) or FMBSI (3). In fact, for stability conditions derived from the auxiliary function-based summation inequality [14] and its corresponding free-matrix-based summation inequality [13], the interesting phenomenon can also be observed and similar conclusions can also be made. In this case, the corresponding more complex L–K functional should be employed [8], [13].

According to the above discussions, we have the following general conclusions:

(i) Based on the methods employed to bound $\delta(k)$, stability conditions obtained for discrete-time systems with a time-varying delay can be divided into two types: FM-based stability conditions which are obtained by applying summation inequalities with free matrix variables and RCL-based stability conditions which are obtained by applying the combination of summation inequalities without free matrix variables and RCL.

(ii) The conservatism between these two types of stability conditions can not be compared theoretically even though corresponding summation inequalities are equivalent in conservatism and even though the same L–K functional is used.

(iii) As for RCL-based stability conditions with the same L–K functional considered, the more accurate RCL is employed, the more relaxed stability condition is obtained.

Remark 4: When checking the stability of time-delay systems, $C3$ is usually the first choice since MAUBs obtained by $C3$ are larger than or equal to those obtained by $C1$ or $C2$ for most cases. However, aside from the conservatism, the computation complexity is another important factor when choosing stability conditions. It is noted that the computation complexity is mainly reflected by the number of decision variables involved in stability conditions. Through computations, it is found that the numbers of decision variables involved in $C1$ and $C2$ are equal while the number involved in $C3$ is the largest. To be specific, the three numbers are, respectively, $10.5n^2+3.5n$, $10.5n^2+3.5n$ and $18.5n^2+3.5n$. Therefore, if the computation complexity is more concerned, $C2$ is the most suitable.

III. FURTHER VERIFICATION WITH BRLS

To clearly illustrate the interesting phenomenon and to further verify the above-drawn conclusions, several bounded real lemmas (BRLs) are obtained via different bounding-inequality methods in this section. The optimal H_∞ performance indexes (OHPIs) are calculated by applying BRLs to a numerical example, which could reflect the conservatism of BRLs. Smaller OHPI means less conservatism.

A. BRLS OBTAINED VIA DIFFERENT BOUNDING-INEQUALITY METHODS

Consider the following linear discrete-time system with a time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-h(k)) \\ \qquad\qquad + B\omega(k), & k \geq 0 \\ x(k) = \phi(k), & -h_2 \leq k \leq 0 \\ z(k) = Cx(k) \end{cases} \tag{24}$$

where $x(k)$, $z(k)$, $\omega(k)$ and $\phi(k)$ are the system state, the controlled output, the disturbance and the initial condition, respectively; A, A_d, B, C are system matrices; $h(k)$ is the time-varying delay satisfying (8). Based on H_∞ performance analysis of system (24), we further investigate the interesting phenomenon. The definition of H_∞ performance index is firstly presented. Secondly, with similar lines in [19], we have RCLs by applying WBSI + RCL, WBSI + IRCL and FMBI methods.

Definition 1 ([19]): For a given $\gamma > 0$, system (24) has H_∞ performance index γ if two conditions are satisfied: i) system (24) is asymptotically stable for $\omega(k) = 0$; and ii) the controlled output $z(k)$ satisfies $\|z(k)\| < \gamma\|\omega(k)\|$ for zero initial condition.

Lemma 4: For given integers h_1 and h_2 satisfying (8), system (24) has H_∞ performance index γ if there exist matrices $P \in \mathbb{S}_+^{3n}$, $Q_1, Q_2, R_1, R \in \mathbb{S}_+^n$, $S, X_1, X_2 \in \mathbb{R}^{2n\times 2n}$ and $M_0, M_1, N_0, N_1 \in \mathbb{R}^{3n\times n}$ such that one of the following conditions holds:

BRL 1 (WBSI (2) + RCL (4)): Ineq. (10) and (11) hold;
BRL 2 (WBSI (2) + RCL (5)): Ineq. (12) and (13) hold;
BRL 3 (WBSI (2) + RCL (6)):

$$\begin{bmatrix} \Psi(h_1) - \Upsilon_{6,1} & E_1^T X_1 \\ * & -\tilde{R} \end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} \Psi(h_2) - \Upsilon_{6,2} & E_2^T X_2^T \\ * & -\tilde{R} \end{bmatrix} < 0 \quad (26)$$

BRL 4 (FMBSI (3)): Ineq. (14) and (15) hold;
where

$$\Psi(h_k) = \Upsilon_1(h_k) + \Upsilon_2 + \Upsilon_3 - \Upsilon_4 + \Upsilon_5,$$
$$\Upsilon_5 = e_1^T C^T C e_1 - \gamma^2 e_8^T e_8,$$
$$e_0 = 0_{n\times 8n}, \quad e_s = (A-I)e_1 + A_d e_3 + B e_8,$$
$$e_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_{n\times n} & 0_{n\times(8-i)n} \end{bmatrix}, \quad i \in \{1,\ldots,8\},$$
$$\Upsilon_{6,1} = \mathcal{E}\left(\begin{bmatrix} 2\tilde{R} & X_2 \\ X_2^T & \tilde{R} \end{bmatrix}\right),$$
$$\Upsilon_{6,2} = \mathcal{E}\left(\begin{bmatrix} \tilde{R} & X_1 \\ X_1^T & 2\tilde{R} \end{bmatrix}\right),$$

and other terms such as $\Upsilon_1(h_k)$, Υ_2, Υ_3 and Υ_4 are defined in Lemma 3.

B. A NUMERICAL EXAMPLE

Example 4: Consider system (24) with

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and A and A_d are shown in (21). OHPIs obtained by BRLs 1, 2, 3 and 4 are listed in Table 4 with $h_1 = 1$ and different values of h_2 and in Table 5 with $h_1 = 3$ and different values of h_2. From Tables 4 and 5, it is seen that OHPIs obtained by BRL 3 are always less than or equal to those obtained by BRL 2 and that OHPIs obtained by BRL 2 are always less than or equal to those obtained by BRL 1. This further verifies Conclusion (iii). Meanwhile, it is also seen that OHPIs obtained by BRL 4 are larger than those obtained by BRLs 1, 2 and 3 when h_2 takes smaller values (e.g., 5 and 7) but smaller than those obtained by BRLs 1, 2 and 3 when h_2 takes bigger values (e.g., 13 and 15). This further verifies Conclusion (ii).

TABLE 4. OHPIs for system (24) with $h_1 = 1$.

h_2	5	7	9	11	13	15
BRL 1	3.552	4.623	6.097	8.362	12.598	24.380
BRL 2	3.541	4.588	5.871	7.660	10.618	16.941
BRL 3	3.540	4.587	5.859	7.607	10.404	15.949
BRL 4	3.592	4.722	5.997	7.596	10.031	14.701

TABLE 5. OHPIs for system (24) with $h_1 = 3$.

h_2	5	7	9	11	13	15
BRL 1	3.538	4.586	5.857	7.567	10.179	15.084
BRL 2	3.535	4.575	5.763	7.247	9.341	12.827
BRL 3	3.535	4.575	5.759	7.230	9.276	12.580
BRL 4	3.543	4.625	5.835	7.270	9.202	12.255

IV. CONCLUSION

This note has presented an interesting phenomenon for stability conditions of discrete-time systems with a time-varying delay. By closely studying this phenomenon, the underlying reason has been revealed and several useful conclusions have been drawn, which can help us deeply understand the construction of LMI-based stability conditions and the roles of different bounding-inequality methods in reducing the conservatism of stability conditions. Finally, several bounded real lemmas have been obtained to further verify some conclusions.

The conclusions made in this note are general, which can be applied to other research fields, such as T-S time-delay systems and neural time-delay networks, as long as the bounding-inequality methods are used when deriving stability conditions.

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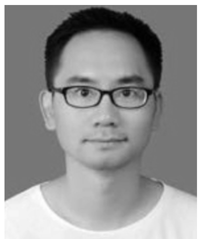

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