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H_∞ Consensus of Linear Multi-Agent Systems With Semi-Markov Switching Network Topologies and Measurement Noises

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ABSTRACT This paper investigates the H_∞ consensus of linear multi-agent systems with semi-Markov switching network topologies and measurement noises. The information that each agent measures its neighbors's has multiplicative noises. The switching of the network topologies are modeled by a semi-Markov process. Taking the external disturbance into account, H_∞ consensus of multi-agent systems with multiplicative noises is achieved under semi-Markov switching network topologies by using semi-Markov jump theory, stochastic theory and algebraic graph theory. Finally, simulation example is provided to illustrate the effectiveness of the theoretical results.

INDEX TERMS Multi-agent systems, H_∞ consensus, semi-Markov switching topologies, multiplicative noises.

I. INTRODUCTION

More and more attention has been paid to the distributed coordination of multi-agent systems in recent years. As an important and fundamental problem of the distributed coordination of multi-agent systems, the consensus problem has made considerable progress owing to its wide applications in robotics [1], flocking and swarms [2], [3], and sensor networks [4]. The consensus of multi-agent systems means to design a distributed control protocol using local neighboring information to achieve a global goal for all multi-agent states.

Recently, there has been an increasing interest in the research for semi-Markov jump systems [16]–[22], which can be considered as a generalisation of Markov jump systems. In multi-agent systems, the communication topologies among the agents may not be fixed, even randomly changing due to uncertain factors such as random failures and the change of environments. This class of time-varying random topologies can be considered to be switching and the switching signals are described by a Markov process. There are rich literatures on Markov switching topologies [10], [23]–[26]. However, the Markov switching topologies have many limitations in applications, since the sojourn time (the interval between two consecutive switching) obeys unique exponential distribution and the transition rates are constant due to the memoryless property of the exponential distribution. Unlike Markov switching topologies, the sojourn time of semi-Markov switching topologies is permitted to obey more general distribution, which causes the transition rates of semi-Markov switching topologies to be time-varying and depend on the sojourn time. Therefore, semi-Markov switching topologies are more general and have broader application prospect than Markov switching topologies. However, relatively few efforts are devoted to semi-Markov switching topologies. In [27], a leader-following consensus of nonlinear multi-agent systems with semi-Markovian switching topologies and communication time-delay was considered, where each agent has general linear dynamics. Dai *et al* [28] investigated the event-triggered leader-following consensus for multi-agent systems with semi-Markov switching topologies. The distributed consensus problem of multi-agent systems with semi-Markov switching topologies was solved by proposing a consensus protocol with sampled-data information in [29]. Containment control of stochastic multiagent systems with semi-Markovian switching topologies was considered in [34].

Meanwhile, due to the use of of sensors, quantization and wireless fading channels in the network, the information that

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each agent receives from its neighbours are often corrupted by various uncertain factors, such as random link failures, transmission noises and quantization errors. There are a lot of results about the stochastic consensus problem with additive measurement noises [5]–[12], where a time-descending gain is designed to reduced the effect of noises. There are also some research on the consensus problem with multiplicative noises, whose intensity depends on the states of agents. Ni *et al.* [13] studied the mean square consensus and strong consensus of continuous-time systems with multiplicative noises under fixed and switching network topologies by selecting properly consensus gains. In [14], distributed consensus of high-dimensional first order agents with relative-state-dependent measurement noises under an undirected graph was investigated and several small consensus gain theorems were given to ensure mean square and almost square consensus. By giving the stability criteria of stochastic differential delay equations, the necessary and sufficient conditions for consentability of linear multi agent systems with time delays and noises were revealed in [15]. To the best of our knowledge, consensus problem of multi-agent systems with measurement noises under semi-markov switching topologies still remains open and challenging, which is the first motivation of our study.

On the other hand, in practical applications, multi-agent systems are often affected by various disturbances such as measurement or calculation errors and channel fading, which may cause undesirable performance of the closed-loop multi-agent systems. To restrain the effects of disturbances, there have been many useful results on H_∞ control problems [30]–[33]. Based on the aforementioned works and the need to fill the gaps, in this paper, we discuss the H_∞ consensus of linear multi-agent systems with semi-markov switching network topologies and measurement noises by using semi-markov jump theory, stochastic theory and algebraic graph theory. The main contributions of this paper are summarized as follows: (1) compared with the existing works with Markov switching topologies, communication topologies in our paper are semi-Markov switching topologies, which is more general and challenging due to that the transition rates of semi-Markov switching are time-varying. (2) in practical application, networked systems are often in uncertain environments and are inevitably affected by measurement noises , therefore our paper considers the effects of measurement noises, which is different from the existing literatures with semi-Markov switching topologies and make the analysis more difficult since the traditional definitions and methods are not applicable to our problems due to the existence of randomness. (3) taking the effects of external disturbances into consideration, H_∞ consensus is investigated in our paper.

The paper is organized as follows. In section II, some useful preliminary results are introduced and the problem formulation is presented. In section III, the main results are investigated. In section IV, simulation result is presented to verify the theoretical analysis. In section V, the conclusion is given.

Notations. The following notations will be used in this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; A^T stands for the transpose of the real matrix A; $\mathbf{1}_N$ denotes the N-dimensional column vector with all ones; I_N represents an $N \times N$ identity matrix; $*$ denotes the term of matrices generated by symmetry; for real symmetric matrix P, $P > 0 (P \geq 0)$ means that matrix P is positive (semi-) definite; $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote its largest and smallest eigenvalues, respectively. $A \otimes B$ denotes the Kronecker product of matrices A and B. $\|\cdot\|$ indicates the Euclidean norm. For a given random variable or vector x, $E(x)$ represents its mathematical expectation.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. GRAPH THEORY

Let $\mathscr{G} = \mathscr{G}(\mathscr{V}, \mathscr{E}, \mathscr{A})$ be an undirected graph, where $\mathscr{V} = \{1, \ldots, N\}$ is the set of nodes and node i represents the ith agent. $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of edges. An edge $(i,j) \in \mathscr{E}$ means that there is an edge from node i to node j, i.e., agent i and j can receive information from each other. A path is a sequence of connected edges in a graph. If there is a path between any two nodes of a graph $\mathscr{G}$, then $\mathscr{G}$ is said to be connected, otherwise disconnected. $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is called the adjacency matrix of graph $\mathscr{G}$ with $a_{ij} = 1$ if $(i,j) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. The set of the node i's neighbours is denoted by $\mathscr{N}_i$, that is, for $j \in \mathscr{N}_i$, $a_{ij} = 1$. The degree matrix is defined as $\Delta = diag(\Delta_1, \ldots, \Delta_N)$, where $\Delta_i = \sum\limits_{j\in\mathscr{N}_i} a_{ij}$. The Laplacian of graph $\mathscr{G}$ is defined as $L = \Delta - \mathscr{A}$, which is symmetric.

B. SEMI-MARKOV SWITCHING TOPOLOGY

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a complete probability space and $\gamma(t)$ be a semi-Markov process. Denote the nth jump point of the process $\gamma(t)$ by t_n and $\gamma(t) = \gamma_n$, $t \in [t_n, t_{n+1})$. The sojourn-time of mode γ_n is denoted by $\tau_n = t_{n+1} - t_n$ and $\tau(t) := t - \sup\{t_n : t_n \leq t\}$. we give the following definition.

Definition 1: ([17]) A stochastic process $\gamma(t)$ is called a semi-Markov process on the probability space if the following conditions are satisfied for every $i, j \in \{1, \ldots, m\}$.

(i) $\mathrm{P}(\gamma_{n+1} = j, \tau_{n+1} \leq \tau|\gamma_n, t_n, \ldots, \gamma_0, t_0) = \mathrm{P}(\gamma_{n+1} = j, \tau_{n+1} \leq \tau|\gamma_n)$.

(iii) The probability $\mathrm{P}(\gamma_{n+1} = j, \tau_{n+1} \leq \tau|\gamma_n = i)$ is independent of n.

In this paper, $\mathscr{G}(\mathscr{V}, \mathscr{E}(\gamma(t)), \mathscr{A}(\gamma(t)))$ is used to describe the semi-markov switching topologies, and $\gamma(t)$ denotes the semi-Markov switching signal with the following probability transitions:

$$\mathrm{P}\{\gamma(t+h)=q|\gamma(t) = k\} = \begin{cases} \lambda_{kq}(\tau)h + o(h), & k \neq q \\ 1 + \lambda_{kq}(\tau)h + o(h), & k = q \end{cases} \tag{1}$$

where $\tau \geq 0$ is the sojourn time that indicates the time duration between two successive mode transitions,

$\lim_{h\to 0}\frac{o(h)}{h} = 0$, $\lambda_{kq}(\tau)$ is the transition rate from mode k at time t to mode $q(\neq k)$ at time $t+h$, and $\lambda_{kk}(\tau) = -\sum_{q=1,q\neq k}^{m}\lambda_{kq}(\tau)$. In practice, $\lambda_{kq}(\tau)$ is not easily obtained, but the parameter $\lambda_{kq}(\tau)$ belongs to a bounded interval, that is, $\lambda_{kq}(\tau)\in[\underline{\lambda}_{kq}, \overline{\lambda}_{kq}]$. In this paper, we assume every graph is connected.

Remark 1: Since the sojourn time τ follows a more general distribution, the transition rates $\lambda_{kq}(\tau)$ for the semi-Markov process are time-varying and depend on τ. When $\lambda_{kq}(\tau) = \lambda_{kq}$, the semi-Markov process becomes a Markov process. Therefore, Markov process is a special case of semi-Markov process.

C. PROBLEM FORMULATION

Consider a multi-agent system consisting of N agents. The dynamics of the N agents are described by the following systems:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + C\omega_i(t) \quad i = 1, 2, \ldots, N, \tag{2}$$

where $x_i(t)\in\mathbb{R}^n$ is the agent i's state, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times s}$, $C\in\mathbb{R}^{n\times q}$, $u_i(t)\in\mathbb{R}^s$ is agent i's input, and $\omega_i(t)\in\mathbb{L}_2^q[0,\infty)$ is the external disturbance.

The agent i can receive the information from its neighbors:

$$y_{ji} = x_j(t) + \mu_{ji}(x_j(t) - x_i(t))\eta_{ji}(t), j \in N_i \tag{3}$$

where $y_{ji}(t)$ denotes the measurement of $x_j(t)$ by agent i, and $\eta_{ji}(t)\in\mathbb{R}$ denotes the measurement noise, $\mu_{ji}\geq 0$. Denote $\mu = \max_{i,j=1}^{N}\mu_{ji}$. The noises satisfy the following assumption.

Assumption 1: The noise processes $\eta_{ji}(t), i, j = 1, \ldots, N$ satisfy $\int_0^t \eta_{ji}(s)ds = \varpi_{ji}(t), t \geq 0$, where $\{\varpi_{ji}(t), i, j = 1, \ldots, N\}$ are independent Brownian motions.

Assumption 2: Brownian motions $\varpi_{ji}(t), i, j = 1, \ldots, N$ and the semi-Markov process $\gamma(t)$ are independent.

We use the distributed protocol in the following form:

$$u_i(t) = K(\gamma(t))\sum_{j\in N_i} a_{ij}(\gamma(t))(y_{ji} - x_i(t)) \tag{4}$$

where $K(\gamma(t))\in\mathbb{R}^{s\times n}$ are the mode-dependent feedback gain matrices to be designed. let $e(t) = [(I_N - J_N)\otimes I_n]x(t)$, where $J_N = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$, $x(t) = [x_1^T(t), \ldots, x_N^T(t)]^T$, and $\omega(t) = [\omega_1^T(t), \ldots, \omega_N^T(t)]^T$. Substituting the protocol (4) into the system (2), and using Assumption 1, we get the following matrix form:

$$\begin{aligned} dx(t) = (I_N\otimes A)x(t)dt - (L(\gamma(t))\otimes BK(\gamma(t)))x(t)dt \\ + (I_N\otimes C)\omega(t)dt + W_1(t) \end{aligned} \tag{5}$$

where

$$W_1(t) = \sum_{i,j=1}^{N} a_{ij}(\gamma(t))\mu_{ji}(H_{i,j}(\gamma(t))\otimes BK(\gamma(t)))e(t)d\varpi_{ji}(t)$$

and as defined in [15], $H_{i,j} = [h_{kl}]_{N\times N}$ is an $N\times N$ matrix with $h_{ii} = -a_{ij}$, $h_{ij} = a_{ij}$ and all other elements being zero, $i, j = 1, \ldots, N$.

According to the definition of $e(t)$, for each fixed $\gamma(t) = k$, we have

$$\begin{aligned} de(t) = (I_N\otimes A)e(t)dt - (L(k)\otimes BK(k))e(t)dt \\ + \big((I_N - J_N)\otimes C\big)\omega(t)dt + W_2(t) \end{aligned} \tag{6}$$

where

$$W_2(t) = \sum_{i,j=1}^{N} a_{ij}(k)\mu_{ji}((I_N - J_N)H_{i,j}(k)\otimes BK(k))e(t)d\varpi_{ji}(t)$$

The controlled output function is defined as $z(t) = e(t)$.

Definition 2: Given $\delta > 0$, the H_∞ consensus of the linear multi-agent systems (2) is said to be achieved if there is a protocol (4) such that the states of the agents satisfy the following two requirements:

(i) When $\omega(t) = 0$, the linear multi-agent systems (2) with semi-Markov switching network topologies can reach mean square consensus, that is

$$\lim_{t\to\infty} E\|x_i(t) - x_j(t)\|^2 = 0, \quad i, j = 1, 2, \ldots, N. \tag{7}$$

(ii) When $\omega(t) \neq 0$, the performance random variable $z(t)$ satisfies

$$E\int_0^{+\infty}\|z(t)\|^2dt < \delta^2 E\int_0^{+\infty}\|\omega(t)\|^2dt \tag{8}$$

The control purpose of this paper is to find appropriate feedback control gains $K(k), k = 1, \ldots, m$ such that the H_∞ consensus of the multi-agent systems (2) and (4) can be achieved.

III. MAIN RESULTS

Theorem 1: Suppose that Assumption 1 and 2 holds. For a given index $\delta > 0$, the H_∞ consensus of linear multi-agent systems (2) and (4) with semi-Markov switching network topologies can be solved if there exist matrices $P(k)$ to guarantee the following matrix inequalities:

$$\Xi(k) = \begin{bmatrix} \Xi_{11}(k) & (I_N - J_N)\otimes P(k)C \\ (I_N - J_N)\otimes C^TP(k) & -\delta^2(I_N\otimes I_q) \end{bmatrix} < 0 \tag{9}$$

where

$$\begin{aligned} \Xi_{11}(k) &= I_N\otimes(A^TP(k) + P(k)A) \\ &\quad - L(k)\otimes\big(K^T(k)B^TP(k) + P(k)BK(k)\big) \\ &\quad + I_N\otimes I_n + \sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q)) \\ &\quad + \mu^2\frac{2(N-1)}{N}\lambda_{\max}(L(k))\big(I_N\otimes K^T(k)B^TP(k)BK(k)\big) \end{aligned} \tag{10}$$

Proof: Consider the following Lyapunov functional candidate:

$$V(e(t), \gamma(t)) = e^T(t)(I_N\otimes P(\gamma(t))e(t) \tag{11}$$

According the definition of derivative of $EV(e(t), \gamma(t))$, we have

$$\frac{dEV(e(t),\gamma(t))}{dt} = \lim_{h\to 0^+}\frac{1}{h}\{E\{V(e(t+h),\gamma(t+h))|e(t),\gamma(t)\} - EV(e(t),\gamma(t))\} \quad (12)$$

For $\gamma(t) = k$,

$$\begin{aligned}
&\frac{dEV(e(t),\gamma(t))}{dt}\\
&= \lim_{h\to 0^+}\frac{1}{h}\Big[\sum_{q=1,q\neq k}^{m} \mathrm{P}\{\gamma(t+h)=q|\\
&\quad \gamma(t)=k\}EV(e(t+h),q) + \mathrm{P}\{\gamma(t+h)=k|\gamma(t)=k\}\\
&\quad \times EV(e(t+h),k) - EV(e(t),k)\Big]\\
&= \lim_{h\to 0^+}\frac{1}{h}\Big[\sum_{q=1,q\neq k}^{m} \mathrm{P}\{\gamma_{n+1}=q,\tau_{n+1}\le\tau+h|\gamma_n=k,\\
&\quad \tau_{n+1}>\tau\}EV(e(t+h),q)\\
&\quad + \mathrm{P}\{\gamma_{n+1}=k,\tau_{n+1}>\tau+h|\gamma(t)=k,\\
&\quad \tau_{n+1}>\tau\}EV(e(t+h),k) - EV(e(t),k)\Big]\\
&= \lim_{h\to 0^+}\frac{1}{h}\Big[\sum_{q=1,q\neq k}^{m}\frac{p_{kq}(D_k(\tau+h)-D_k(\tau))}{1-D_k(\tau)}EV(e(t+h),q)\\
&\quad + \frac{1-D_k(\tau+h)}{1-D_k(\tau)}EV(e(t+h),k) - EV(e(t),k)\Big] \qquad (13)
\end{aligned}$$

where $p_{kq} = \mathrm{P}\{\gamma_{n+1} = q|\gamma_n = k\}$ is the probability of the process from mode k to mode q and $D_k(t) = \mathrm{P}(\tau_{n+1} \le t|\gamma_n = k)$ is the cumulative distribution function of the sojourn time when the topology stays in mode k. With a small h, the first-order approximation of $EV(e(t+h), q)$ is

$$\begin{aligned}
EV(e(t+h),q) &= EV(e(t),q) + E\Big(\frac{\partial V(e(t),q)}{\partial e}f\\
&\quad + \frac{1}{2}tr\big[g^T\frac{\partial^2 V(e(t),q)}{\partial e^2}g\big]\Big)h + o(h) \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
f(e,q) &= (I_N\otimes A)e(t) - (L(q)\otimes BK(q))e(t)\\
&\quad + \big((I_N-J_N)\otimes C\big)\omega(t),\\
g(e,q) &= (g_{11},\ldots,g_{1N},\ldots,g_{NN}),\\
g_{ij} &= a_{ij}(q)\mu_{ji}((I_N-J_N)H_{i,j}(q)\otimes BK(q))e(t)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{dEV(e(t),\gamma(t))}{dt}\\
&= \lim_{h\to 0^+}\frac{1}{h}\Big[\sum_{q=1,q\neq k}^{m}\frac{p_{kq}(D_k(\tau+h)-D_k(\tau))}{1-D_k(\tau)}\Big[EV(e(t),q)\\
&\quad + E\big(\frac{\partial V(e(t),q)}{\partial e}f(e,q)\\
&\quad + \frac{1}{2}tr[g^T(e,q)\frac{\partial^2 V(e(t),q)}{\partial e^2}g(e,q)]\big)h\Big]\\
&\quad + \frac{1-D_k(\tau+h)}{1-D_k(\tau)}\Big[EV(e(t),k) + E\big(\frac{\partial V(e(t),k)}{\partial e}f(e,k)\\
&\quad + \frac{1}{2}tr[g^T(e,k)\frac{\partial^2 V(e(t),k)}{\partial e^2}g(e,k)]\big)h\Big] - EV(e(t),k)\Big]
\end{aligned} \qquad (15)$$

According to $\lim_{h\to 0}\frac{D_k(\tau+h)-D_k(\tau)}{(1-D_k(\tau))h} = \lambda_k(\tau)$, $\lim_{h\to 0}\frac{D_k(\tau+h)-D_k(\tau)}{1-D_k(\tau)} = 0$, $\lim_{h\to 0}\frac{1-D_k(\tau+h)}{1-D_k(\tau)} = 1$, we have

$$\begin{aligned}
\frac{dEV(e(t),\gamma(t))}{dt} &= E\Big\{\sum_{q=1,q\neq k}^{m} p_{kq}\lambda_k(\tau)V(e(t),q)\\
&\quad + \frac{\partial V(e(t),k)}{\partial e}f + \frac{1}{2}tr[g^T\frac{\partial^2 V(e(t),k)}{\partial e^2}g]\\
&\quad - \lambda_k(\tau)V(e(t),k)\Big\}
\end{aligned} \qquad (16)$$

Define $\lambda_{kq}(\tau) = p_{kq}\lambda_k(\tau)$, $k \neq q$, and $\lambda_{kk}(\tau) = -\sum_{q=1,q\neq k}^{m}\lambda_{kq}(\tau)$, then it follows

$$\begin{aligned}
&\frac{dEV(e(t),\gamma(t))}{dt}\\
&= E\Big\{\sum_{q=1}^{m}\lambda_{kq}(\tau)V(e(t),q) + \frac{\partial V(e(t),k)}{\partial e}f\\
&\quad + \frac{1}{2}tr[g^T\frac{\partial^2 V(e(t),k)}{\partial e^2}g]\Big\}\\
&= E\Big\{e^T(t)\big[I_N\otimes(A^TP(k)+P(k)A) - (L(k)\otimes K^T(k)B^TP(k))\\
&\quad - (L(k)\otimes P(k)BK(k)) + \sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q))e(t)\big]\\
&\quad + 2\omega^T(t)\big((I_N-J_N)\otimes C^TP(k)\big)e(t)\\
&\quad + \sum_{i,j=1}^{N} a_{ij}(k)\mu_{ji}^2 e^T(t)\big(H_{i,j}^T(k)(I_N-J_N)^2H_{i,j}(k)\\
&\quad \otimes K^T(k)B^TP(k)BK(k)\big)e(t)\Big\}
\end{aligned} \qquad (17)$$

Note that $(I_N - J_N)^2 = I_N - J_N$, and $\sum_{i,j=1}^{N} a_{ij}(k)H_{i,j}^T(k)(I_N - J_N)H_{i,j}(k) = 2\frac{N-1}{N}L(k)$, hence we have

$$\begin{aligned}
&\frac{dEV(e(t),\gamma(t))}{dt}\\
&\le E\Big\{e^T(t)\big[I_N\otimes(A^TP(k)+P(k)A) - L(k)\\
&\quad \otimes\big(K^T(k)B^TP(k)+P(k)BK(k)\big) + \sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q))\\
&\quad + \mu^2\frac{2(N-1)}{N}\lambda_{\max}(L(k))\big(I_N\otimes K^T(k)B^TP(k)BK(k)\big)\big]e(t)\\
&\quad + 2\omega^T(t)\big((I_N-J_N)\otimes C^TP(k)\big)e(t)\Big\}
\end{aligned} \qquad (18)$$

When $\omega(t)=0$, denote

$$X(k)=I_N\otimes(A^TP(k)+P(k)A) - L(k)\otimes\big(K^T(k)B^TP(k)+P(k)BK(k)\big) + \sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q)) + \mu^2\frac{2(N-1)}{N}\lambda_{\max}(L(k))\big(I_N\otimes K^T(k)B^TP(k)BK(k)\big) \tag{19}$$

We get

$$\frac{dEV(e(t),\gamma(t))}{dt}\le\lambda_{\min}(X(k))E\|e(t)\|^2 \le\frac{\lambda_{\min}(X(k))}{\lambda_{\min}(P(k))}EV(e(t),\gamma(t)). \tag{20}$$

According to comparison theorem, we have

$$EV(e(t),\gamma(t))\le V(0)\exp\left\{\frac{\lambda_{\min}(X(k))}{\lambda_{\min}(P(k))}t\right\} \tag{21}$$

By Schur complement, the inequality (9) implies $X(k)<0$. Thus, we obtain

$$\lim_{t\to\infty}E\|e(t)\|^2=0 \tag{22}$$

When $\omega(t)\neq0$, using (18), we obtain

$$\frac{dEV(e(t),\gamma(t))}{dt}+E(z^T(t)z(t)-\delta^2\omega^T(t)\omega(t)) \le E(\eta^T(t)\Xi(k)\eta(t)) \tag{23}$$

where $\eta(t)=(e^T(t),\omega^T(t))^T$. Consequently, based on (9), we get

$$E\int_0^t(\|z(t)\|^2-\delta^2\|\omega(t)\|^2)dt\le-EV(e(t),r(t))\ \ (V(0)=0) \tag{24}$$

Thus $E\int_0^t\|z(t)\|^2dt<\delta^2E\int_0^t\|\omega(t)\|^2dt$. Let $t\to\infty$, we have

$$E\int_0^{+\infty}\|z(t)\|^2dt<\delta^2E\int_0^{+\infty}\|\omega(t)\|^2dt$$

Hence, the H_∞ consensus is obtained. The proof is completed. □

The sufficient condition for mean square consensus of multi-agent systems with semi-markov switching network topologies and measurement noises is given in theorem 1. Based on theorem 1, we give the design of controller gain below.

Theorem 2: Suppose that Assumption 1 and 2 holds. For a given index $\delta>0$, the H_∞ consensus of linear multi-agent systems (2) and (4) with semi-Markov switching network topologies can be solved if there exist matrices $\bar{P}(k)$, $\bar{K}(k)$ to guarantee the following linear matrix inequalities:

$$\begin{bmatrix}\Phi_{11}(k)&\Phi_{12}(k)&\Phi_{13}(k)&\Phi_{14}(k)&\Phi_{15}(k)\\ *&\Phi_{22}(k)&0&0&0\\ *&*&\Phi_{33}(k)&0&0\\ *&*&*&\Phi_{44}(k)&0\\ *&*&*&*&\Phi_{55}(k)\end{bmatrix}<0, \tag{25}$$

$k=1,\ldots,m$, where

$$\begin{aligned}
\Phi_{11}(k)&=I_N\otimes(\bar{P}(k)A^T+A\bar{P}(k)) - L(k)\otimes(\bar{K}^T(k)B^T+B\bar{K}(k)) + \lambda_{kk}(\tau)(I_N\otimes\bar{P}(k))\\
\Phi_{12}(k)&=I_N\otimes\bar{K}^T(k)B^T\\
\Phi_{13}(k)&=I_N\otimes\bar{P}(k)\\
\Phi_{14}(k)&=(I_N-J_N)\otimes C\\
\Phi_{15}(k)&=\Big[\sqrt{\lambda_{k1}(\tau)},\ldots,\sqrt{\lambda_{k,k-1}(\tau)},\sqrt{\lambda_{k,k+1}(\tau)},\ldots,\sqrt{\lambda_{k,m}(\tau)}\Big]H(k)\\
H(k)&=diag\,\underbrace{\{I_N\otimes\bar{P}(k),\ldots,I_N\otimes\bar{P}(k)\}}_{m-1}\\
\Phi_{22}(k)&=-\mu^{-2}\frac{N}{2(N-1)}\lambda_{\max}^{-1}(L(k))(I_N\otimes\bar{P}(k))\\
\Phi_{33}(k)&=-(I_N\otimes I_n)\\
\Phi_{44}(k)&=-\delta^2(I_N\otimes I_q)\\
\Phi_{55}(k)&=-diag\{I_N\otimes\bar{P}(1),\ldots,I_N\otimes\bar{P}(k-1), I_N\otimes\bar{P}(k+1),\ldots,I_N\otimes\bar{P}(m)\}
\end{aligned}$$

and the controller gain matrices can be designed by $K(k)=\bar{K}(k)\bar{P}^{-1}(k)$, $k=1\ldots,m$.

Proof: According to Schur complement, $\Xi(k)<0$ if and only if the following inequality hold:

$$-\delta^2(I_N\otimes I_n)<0 \tag{26}$$

and

$$I_N\otimes(A^TP(k)+P(k)A)-L(k)\otimes\big(K^T(k)B^TP(k) + P(k)BK(k)\big)+I_N\otimes I_n+\sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q)) + \mu^2\frac{2(N-1)}{N}\lambda_{\max}(L(k))\big(I_N\otimes K^T(k)B^TP(k)BK(k)\big) + \delta^{-2}(I_N-J_N)^2\otimes P(k)CC^TP(k)<0 \tag{27}$$

Obviously, (26) is always right. Denote $\bar{P}(k)=P^{-1}(k)$, $\bar{K}(k)=K(k)\bar{P}(k)$. Pre-multiplying and post-multiplying the inequality (27) by $I_N\otimes P^{-1}(k)$, we get

$$I_N\otimes(A\bar{P}(k)+\bar{P}(k)A^T)-L(k)\otimes\big(\bar{K}^T(k)B^T+B\bar{K}(k)\big) + \bar{P}(k)\Big[\sum_{q=1}^{m}\lambda_{kq}(\tau)(I_N\otimes P(q))\Big]\bar{P}(k) + I_N\otimes\bar{P}^2(k)+\mu^2\frac{2(N-1)}{N}\lambda_{\max}(L(k))(I_N \otimes\bar{K}^T(k)B^TP(k)B\bar{K}(k)\big)+\delta^{-2}(I_N-J_N)^2\otimes CC^T<0 \tag{28}$$

Based on Schur complement, the inequality (25) leads to (28). This completes the proof. □

In theorem 1, the term $\lambda_{kj}(\tau)$ is time-varying, so it is difficult to solve infinite number of linear matrix inequalities due to the fact that different τ produces different inequality. As in the works [16] and [27], the following theorem is given to get solvable conditions by using the the upper and lower bounds of the transition rate.

Theorem 3: Suppose that Assumption 1 and 2 holds. For a given index $\delta > 0$, the H_∞ consensus of linear multi-agent systems (2) and (4) with semi-Markovian switching network topologies can be solved if there exist matrices $\bar{P}(k)$, $\bar{K}(k)$ to guarantee the following linear matrix inequalities:

$$\begin{bmatrix} \underline{\Phi}_{11}(k) & \Phi_{12}(k) & \Phi_{13}(k) & \Phi_{14}(k) & \underline{\Phi}_{15}(k) \\ * & \Phi_{22}(k) & 0 & 0 & 0 \\ * & * & \Phi_{33}(k) & 0 & 0 \\ * & * & * & \Phi_{44}(k) & 0 \\ * & * & * & * & \Phi_{55}(k) \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \overline{\Phi}_{11}(k) & \Phi_{12}(k) & \Phi_{13}(k) & \Phi_{14}(k) & \overline{\Phi}_{15}(k) \\ * & \Phi_{22}(k) & 0 & 0 & 0 \\ * & * & \Phi_{33}(k) & 0 & 0 \\ * & * & * & \Phi_{44}(k) & 0 \\ * & * & * & * & \Phi_{55}(k) \end{bmatrix} < 0, \tag{30}$$

$k = 1, \ldots, m$, where

$$\begin{aligned} \underline{\Phi}_{11}(k) &= I_N \otimes (\bar{P}(k)A^T + A\bar{P}(k)) - L(k) \otimes (\bar{K}^T(k)B^T \\ &\quad + B\bar{K}(k)) + \underline{\lambda}_{kk}(I_N \otimes \bar{P}(k)) \\ \overline{\Phi}_{11}(k) &= I_N \otimes (\bar{P}(k)A^T + A\bar{P}(k)) - L(k) \otimes (\bar{K}^T(k)B^T \\ &\quad + B\bar{K}(k)) + \overline{\lambda}_{kk}(I_N \otimes \bar{P}(k)) \\ \underline{\Phi}_{15}(k) &= \left[\sqrt{\underline{\lambda}_{k1}}, \ldots, \sqrt{\underline{\lambda}_{k,k-1}}, \sqrt{\underline{\lambda}_{k,k+1}}, \ldots, \sqrt{\underline{\lambda}_{k,m}}\right] H(k) \\ \overline{\Phi}_{15}(k) &= \left[\sqrt{\overline{\lambda}_{k1}}, \ldots, \sqrt{\overline{\lambda}_{k,k-1}}, \sqrt{\overline{\lambda}_{k,k+1}}, \ldots, \sqrt{\overline{\lambda}_{k,m}}\right] H(k) \\ H(k) &= diag\, \underbrace{\{I_N \otimes \bar{P}(k), \ldots, I_N \otimes \bar{P}(k)\}}_{m-1} \end{aligned}$$

Moreover, the controller gain matrices can be designed by $K(k) = \bar{K}(k)\bar{P}^{-1}(k)$, $k = 1, \ldots, m$.

Proof: The proof is similar to Theorem 2 in [16] and is omitted.

Remark 2: The conditions in theorem 3 are relatively conservative. In order to decrease the conservativeness, a method that partitions the sojourn-time into S sections was proposed in [16].

Denote $\underline{\lambda}_{kj,s}$ and $\overline{\lambda}_{kj,s}$ as the lower and upper bounds of the transition rates during the sth section ($s = 1, 2 \ldots, S$), then we get the following less conservative corollary.

Corollary 1: Suppose that Assumption 1 and 2 holds. For a given index $\delta > 0$, the H_∞ consensus of linear multi-agent systems (2) and (4) with semi-Markov switching network topologies can be solved if there exist matrices $\bar{P}(k, s)$, $\bar{K}(k, s)$ to guarantee the following linear matrix inequalities:

$$\begin{bmatrix} \underline{\Phi}_{11}(k,s) & \Phi_{12}(k,s) & \Phi_{13}(k,s) & \Phi_{14}(k) & \underline{\Phi}_{15}(k,s) \\ * & \Phi_{22}(k,s) & 0 & 0 & 0 \\ * & * & \Phi_{33}(k) & 0 & 0 \\ * & * & * & \Phi_{44}(k) & 0 \\ * & * & * & * & \Phi_{55}(k,s) \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} \overline{\Phi}_{11}(k,s) & \Phi_{12}(k,s) & \Phi_{13}(k,s) & \Phi_{14}(k) & \overline{\Phi}_{15}(k,s) \\ * & \Phi_{22}(k,s) & 0 & 0 & 0 \\ * & * & \Phi_{33}(k) & 0 & 0 \\ * & * & * & \Phi_{44}(k) & 0 \\ * & * & * & * & \Phi_{55}(k,s) \end{bmatrix} < 0, \tag{32}$$

where $\Phi_{ij}(i, j = 1, \ldots, 5)$ are defined similarly as in theorem 3, except that $\bar{P}(k)$, $\bar{K}(k)$ are replaced by $\bar{P}(k, s)$, $\bar{K}(k, s)$. Moreover, the controller gain matrices can be designed by $K(k, s) = \bar{K}(k, s)\bar{P}^{-1}(k, s)$, $k = 1, \ldots, m$, $s = 1, 2 \ldots, S$.

IV. SIMULATION RESULT

In this section, we use an example to illustrate our theoretical results.

Example: Consider a linear multi-agent system with four agents where

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The communication topologies are semi-Makov switching with two modes and the Laplacian matrices are gained

$$L(1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$
$$L(2) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The transition rates are assumed to be $\lambda_{11}(k) \in [-4, -2]$, $\lambda_{12}(k) \in [0.5, 4]$, $\lambda_{21}(k) \in [1.2, 3.5]$, $\lambda_{22}(k) \in [-3, -1.5]$. By solving the LMIs in the theorem 3 with $\delta = 0.4$, $\mu = 0.15$, we get

$$\bar{P}(1) = \begin{bmatrix} 1.7183 & 0.0540 \\ 0.0540 & 0.1891 \end{bmatrix}, \quad \bar{P}(2) = \begin{bmatrix} 1.7219 & 0.0553 \\ 0.0553 & 0.1929 \end{bmatrix},$$
$$\bar{K}(1) = \begin{bmatrix} 0.2266 & 0.4743 \end{bmatrix}, \quad \bar{K}(2) = \begin{bmatrix} 0.2480 & 0.4938 \end{bmatrix},$$

By simple calculation, we have

$$K(1) = \begin{bmatrix} 0.05352.4929 \end{bmatrix}, \quad K(2) = \begin{bmatrix} 0.06242.5420 \end{bmatrix},$$

The semi-Markov switching signal is shown in Fig.1. The external disturbance is given by $\omega(t) = e^{-2t}$ and the initial states of the agents are chosen as $x_1(0) = (5, -2.5)^T$, $x_2(0) = (-3, 2)^T$, $x_3(0) = (2, -6)^T$, $x_4(0) = (-5, 6)^T$. Based on the above parameters, we get the simulation results in Fig.2. From Fig.2, it can be seen that the state trajectories of errors with $\omega(t)$ can be convergent to zero, which implies that the agents can achieve H_∞ consensus. The simulation results demonstrate the control output possesses robustness against the external disturbance and the uncertainty induced by the semi-Markov switching graphs and the measurement noises.

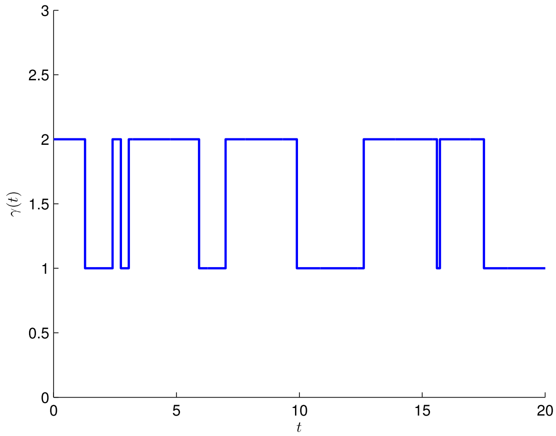

FIGURE 1. Semi-markov switching signal.

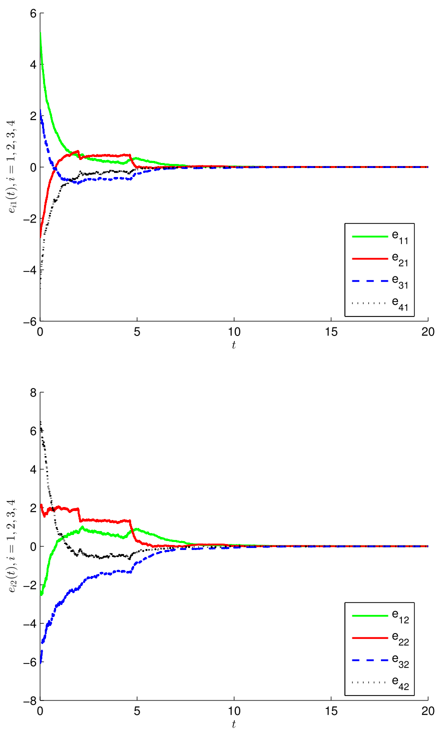

FIGURE 2. The error variable with $\omega(t)$.

V. CONCLUSION

In this paper, we have investigated the H_∞ consensus of linear multi-agent systems with semi-Markov switching network topologies and measurement noises. Each agent can measure or receive the information of its neighbours with multiplicative noises. By using semi-Markov jump theory, algebraic graph theory and stochastic theory, some sufficient conditions are given to ensure mean square consensus to be achieved when the external disturbance is absent. Meanwhile, H_∞ consensus of multi-agent systems is achieved when the external disturbance exists.

For further research, we have not minimized the performance index δ when solving the proposed LMIs. Moreover, it is interesting to consider the case with time-delay and nonlinear dynamics, and containment control.

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