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Total Efficient Domination in Fuzzy Graphs

XUE-GANG CHEN[1], MOO YOUNG SOHN[2], AND DE-XIANG MA[1]

[1]Department of Mathematics, North China Electric Power University, Beijing 102206, China
[2]Department of Mathematics, Changwon National University, Changwon 641-773, South Korea

Corresponding author: Moo Young Sohn (mysohn@changwon.ac.kr)

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ABSTRACT This study proposed total efficient domination in fuzzy graphs. The exact values on the total efficient domination number for several classes of fuzzy graphs are determined. A lower bound and an upper bound for the total efficient domination number in terms of maximum strong arc neighborhood degree and the order are obtained. In addition, a new relationship between total efficient domination number and total efficient domatic number is established. Finally, we design an algorithm to determine the minimum fuzzy cardinality of the total efficient dominating set of a fuzzy tree T or decide that T has no total efficient dominating set.

INDEX TERMS Fuzzy graph, fuzzy tree, total efficient dominating set.

I. INTRODUCTION

It is a challenge obtaining full details about real world problems, therefore, the vagueness and uncertainty in the description has led to the growth of fuzzy graph theory. A mathematical framework to describe uncertainty in real life situation was first suggested by Zadeh [2]. Rosenfeld [3] introduced fuzzy graph and several other fuzzy analogs of graph theoretic concepts such as paths, cycles and connectivity.

One of the most interesting graph theoretical concept is domination in graphs which was introduced by Ore in 1962 [8]. The concept of total domination was introduced by Cockayne and Hedetniemi [7]. Kulli and Patwari [1] introduced total efficient domination in graphs. A remarkable beginning in fuzzy graphs for the concept domination was made by Somasundaram and Somasundaram [4].

Revathi *et al.* [6] defined total perfect domination in fuzzy graphs using strong arcs. This present study discussed total efficient domination in fuzzy graphs using strong arcs.

This paper is organized as follows. Section 2 comprises of preliminaries and in section 3, the total efficient domination of a fuzzy graph is defined *(Definition 1)*. A complete fuzzy graph has no total efficient dominating set *(Theorem 1)*. For several classes of fuzzy graphs such as a path, a fuzzy cycle and a complete bipartite fuzzy graph, the exact values on the total efficient domination number were determined *(Theorem 2,3,4)*. In section 4, lower bounds and upper bounds for the total efficient domination number were obtained in terms of maximum strong arc neighborhood degree and the order *(Theorem 5,6)*. In addition, a new relationship between total efficient domination number and total efficient domatic number was established *(Theorem 7)*. In section 5, a vertex data structure was designed. Combining with a labeling method, an algorithm was designed to determine the minimum fuzzy cardinality of the total efficient dominating set of a fuzzy tree T or decide that T has no total efficient dominating set. Section 6 gives the conclusion of the study.

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II. PRELIMINARIES

A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of membership functions on fuzzy sets $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. We denote the underlying crisp graph by $G^* = (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout the paper, we assume that $\sigma^* = V$.

In a fuzzy graph $G = (\sigma, \mu)$, a *path* P of length n is a sequence of distinct vertices $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, \ldots, n$ and the degree of membership of the weakest arc is defined as its *strength*. If $u_0 = u_n$ and $n \geq 3$ then P is called a *cycle*, and a *fuzzy cycle* if it contains more than one weakest arc. For any two vertices x and y, let $d(x, y)$ denote the length of the shortest path between x and y. The *strength of connectivity* between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $Conn_G(x, y)$.

A fuzzy graph $G = (\sigma, \mu)$ is *connected* if for every $u, v \in \sigma^*$, $Conn_G(u, v) > 0$. An arc (u, v) is said to be a *strong arc* if $\mu(u, v) \geq Conn_G(u, v)$ and the vertex u is a strong neighbor to v.

The *order* p and *size* q of a fuzzy graph $G = (\sigma, \mu)$ are defined as $p = \sum_{v \in V} \sigma(v)$ and $q = \sum_{(u,v) \in E} \mu(u, v)$.

The *strong arc neighbourhood degree* of a vertex v is defined by sum of the membership values of the strong adjacent vertices of v and is denoted by $d_N(v)$. That is $d_N(v) = \sum_{u \in N_S(v)} \sigma(u)$, where $N_S(v) = \{u \in V : (u, v)$ is a strong arc $\}$. The *minimum strong arc neighbourhood degree* of the fuzzy graph G is defined by $\delta_N(G) = \min\{d_N(u) : u \in V\}$ and the *maximum strong arc neighbourhood degree* of the fuzzy graph G is defined by $\Delta_N(G) = \max\{d_N(u) : u \in V\}$.

A fuzzy graph $G = (\sigma, \mu)$ is said to be a *complete* if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. A fuzzy graph $G = (\sigma, \mu)$ is said to be a *bipartite* if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$, then G is called a *complete bipartite fuzzy graph.*

Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $u, v \in V$. The vertex u *dominates* the vertex v in G if (u, v) is a strong arc. A subset D of V is called a *perfect dominating set* of G if each vertex not in D is dominated by exactly one vertex of D.

A perfect dominating set D in a fuzzy graph G is said to be *total perfect dominating set* if every vertex in D is dominated by at least one vertex of D. The minimum fuzzy cardinality of the total perfect dominating set is the *total perfect domination number* and is denoted by $\gamma_{tpf}(G)$.

III. EXACT VALUES ON TOTAL EFFICIENT DOMINATION NUMBER

In the section, the new concept of total efficient dominating set in fuzzy graph is introduced. Furthermore, we determine the exact values on total efficient domination number for several classes of fuzzy graphs.

Definition 1: A perfect dominating set D in a fuzzy graph G is said to be a total efficient dominating set if every vertex in D is dominated by exactly one vertex of D. The minimum fuzzy cardinality of a total efficient dominating set is the total efficient domination number of G and it is denoted by $\gamma_{tef}(G)$.

A total efficient dominating set of G with minimum fuzzy cardinality is called a γ_{tef}-set of G. If G has no total efficient dominating set, we define $\gamma_{tef}(G) = 0$. Consider the fuzzy graph G in Fig. 1. Total efficient dominating set of G is $\{b, c, g, f\}$, $\{a, b, g, f\}$, $\{a, d, g, h\}$ and $\{c, d, g, h\}$. Total efficient domination number $\gamma_{tef}(G) = \sigma(b) + \sigma(c) + \sigma(g) + \sigma(f) = 1.9$. Let G be a complete fuzzy graph. Then all arcs in G are strong and each vertex dominates to all other vertices. We have the following.

Theorem 1: Let G be a complete fuzzy graph with $n \geq 3$ vertices. Then G has no total efficient dominating set.

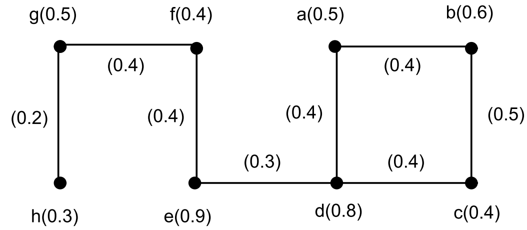

FIGURE 1. A fuzzy graph G.

Theorem 2: Let $P_n = v_1 v_2 \ldots v_n$ be a path with n vertices, then

$$\gamma_{tef}(P_n) = \begin{cases} \sum_{i=0}^{k-1}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})], & \text{if } r = 0 \\ 0, & \text{if } r = 1 \\ \sum_{i=0}^{k}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})], & \text{if } r = 2 \\ \min\{\sum_{i=0}^{k}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})], & \text{if } r = 3 \\ \sum_{i=0}^{k}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})]\}. & \end{cases}$$

where $n = 4k + r$, $0 \leq r < 4$.

Proof: Since P_n is a path, all arcs in P_n are strong. Let D be a total efficient dominating set of P_n. We will discuss it from the following cases.

Case 1: $n \equiv 0(mod\ 4)$. Assume that $n = 4k$, where $k \geq 1$. In order to dominate vertex v_1, it follows that $v_2 \in D$. It is obvious that $v_1 \notin D$. Then $v_3 \in D$. So $D = \{v_{4i+2}, v_{4i+3} : i = 0, 1, \ldots, k-1\}$ is the unique total efficient dominating set of P_n. Hence, $\gamma_{tef}(P_n) = \sum_{i=0}^{k-1}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})]$.

Case 2: $n \equiv 1(mod\ 4)$. It is obvious that P_n has no total efficient dominating set. Hence, $\gamma_{tef}(P_n) = 0$.

Case 3: $n \equiv 2(mod\ 4)$. Assume that $n = 4k + 2$, where $k \geq 0$. In order to dominate vertex v_1, it follows that $v_2 \in D$. It is obvious that $v_3 \notin D$. Then $v_1 \in D$. So $D = \{v_{4i+1}, v_{4i+2} : i = 0, 1, \ldots, k\}$ is the unique total efficient dominating set of P_n. Hence, $\gamma_{tef}(P_n) = \sum_{i=0}^{k}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})]$.

Case 4: $n \equiv 3(mod\ 4)$. Assume that $n = 4k + 3$, where $k \geq 0$. In order to dominate vertex v_1, it follows that $v_2 \in D$. If $v_1 \in D$, then $D = \{v_{4i+1}, v_{4i+2} : i = 0, 1, \ldots, k\}$ is a total efficient dominating set of P_n. If $v_1 \notin D$, then $v_3 \in D$ and $D = \{v_{4i+2}, v_{4i+3} : i = 0, 1, \ldots, k\}$ is a total efficient dominating set of P_n. So P_n has exactly two total efficient dominating sets. Hence, $\gamma_{tef}(P_n) = \min\{\sum_{i=0}^{k}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})], \sum_{i=0}^{k}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})]\}$. ■

Theorem 3: Let $C_n = v_1 v_2 \ldots v_n v_1$ be a fuzzy cycle with n vertices. Then

(1) If $n \equiv i(mod\ 4)$ for $i = 1, 2, 3$, then C_n has no total efficient dominating set.

(2) If $n \equiv 0(mod\ 4)$, then $\gamma_{tef}(C_n) = \min\{\sum_{i=0}^{k-1}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})], \sum_{i=0}^{k-1}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})], \sum_{i=0}^{k-1}[\sigma(v_{4i+3}) + \sigma(v_{4i+4})], \sum_{i=0}^{k-2}[\sigma(v_{4i+4}) + \sigma(v_{4i+5})] + \sigma(v_1) + \sigma(v_{4k})\}$, where $n = 4k$, $(k \geq 1)$.

Proof: It is obvious that if $n \equiv i(mod\ 4)$ for $i \in \{1, 2, 3\}$, then C_n has no total efficient dominating set. Suppose that $n \equiv 0(mod\ 4)$. Assume that $n = 4k$, where $k \geq 1$. Let $D_1 = \{v_{4i+1}, v_{4i+2} : i = 0, 1, \ldots, k-1\}$, $D_2 = \{v_{4i+2}, v_{4i+3} : i = 0, 1, \ldots, k-1\}$, $D_3 = \{v_{4i+3}, v_{4i+4} : i = 0, 1, \ldots, k-1\}$, $D_4 = \{v_{4i+4}, v_{4i+5} : i = 0, 1, \ldots, k-2\} \cup \{v_1, v_{4k}\}$. Then fuzzy cycle C_n has exactly four total efficient dominating sets D_i for $i = 1, 2, 3, 4$. Hence, $\gamma_{tef}(C_n) = \min\{\sum_{i=0}^{k-1}[\sigma(v_{4i+1}) + \sigma(v_{4i+2})], \sum_{i=0}^{k-1}[\sigma(v_{4i+2}) + \sigma(v_{4i+3})], \sum_{i=0}^{k-1}[\sigma(v_{4i+3}) + \sigma(v_{4i+4})], \sum_{i=0}^{k-2}[\sigma(v_{4i+4}) + \sigma(v_{4i+5})] + \sigma(v_1) + \sigma(v_{4k})\}$. ■

Theorem 4: Let G be a complete bipartite fuzzy graph with partition V_1 and V_2. Then $\gamma_{tef}(G) = \min\{\sigma(u) : u \in V_1\}) + \min\{\sigma(v) : v \in V_2\}$.

Proof: Since G is a complete bipartite fuzzy graph, all arcs are strong arcs. Each vertex in V_1 dominates all vertices in V_2 and each vertex in V_2 dominates all vertices in V_1. Hence in a complete bipartite fuzzy graph, the total efficient dominating sets are any set containing exactly two vertices, one in V_1 and the other in V_2. Hence, $\gamma_{tef}(G) = \min\{\sigma(u) : u \in V_1\}) + \min\{\sigma(v) : v \in V_2\}$. ■

IV. BOUNDS ON TOTAL EFFICIENT DOMINATION NUMBER

Since every total efficient dominating set of G is also a total perfect dominating set of G, we have the following.

Proposition 1: For any connected fuzzy graph G, if $\gamma_{tef}(G)$ exists, then $\gamma_{tpf}(G) \leq \gamma_{tef}(G)$.

It is obvious that $\gamma_{tpf}(G) = \gamma_{tef}(G)$ if and only if there exists a γ_{tpf}-set D of G such that D is a total efficient dominating set of G.

Let (u, v) be a strong arc of connected fuzzy graph G. Let $\sigma_1 = \min\{\sigma(v) : v \in V\}$ and $\sigma_2 = \max\{\sigma(v) : v \in V\}$. Define $\tau_{uv} = \sum_{w \in (N_S(u) \cup N_S(v)) \setminus \{u,v\}} \sigma(w)$. Let $\tau = \min\{\tau_{uv} : (u, v)$ is a strong arc of $G\}$.

Theorem 5: Let G be a connected fuzzy graph. If $\gamma_{tef}(G)$ exists, then

(1) $\frac{p\sigma_1}{\Delta_N(G)} \leq \gamma_{tef}(G) \leq \frac{2p\sigma_2}{\tau + 2\sigma_2}$.

(2) $\gamma_{tef}(G) = \frac{2p\sigma_2}{\tau+2\sigma_2}$ if and only if there exists a γ_{tef}-set D of G such that $\tau_{uv} = \tau$ for any arc $(u, v) \in E(G[D])$ and $\sigma(w) = \sigma_2$ for any $w \in D$.

(3) $\gamma_{tef}(G) = \frac{p\sigma_1}{\Delta_N(G)}$ if and only if there exists a γ_{tef}-set D of G such that $d_N(w) = \Delta_N(G)$ and $\sigma(w) = \sigma_1$ for any $w \in D$.

Proof: (1) Let D be a γ_{tef}-set of G. Then $\bigcup_{(u,v)\in E(G[D])}(N_S(u) \cup N_S(v)) \setminus \{u, v\}) = V - D$. It follows that $\sum_{(u,v)\in E(G[D])} \sum_{w\in(N_S(u)\cup N_S(v))\setminus\{u,v\}} \sigma(w) = \sum_{w\in V-D} \sigma(w) = p - \gamma_{tef}(G)$. That is $p - \gamma_{tef}(G) = \sum_{(u,v)\in E(G[D])} \tau_{uv} \geq \tau\frac{|D|}{2}$. Since $\gamma_{tef}(G) = \sum_{w\in D} \sigma(w) \leq \sigma_2|D|$, it follows that $|D| \geq \frac{\gamma_{tef}(G)}{\sigma_2}$. So, $p - \gamma_{tef}(G) \geq \tau\frac{|D|}{2} \geq \tau\frac{\gamma_{tef}(G)}{2\sigma_2}$. That is $\gamma_{tef}(G) \leq \frac{2p\sigma_2}{\tau+2\sigma_2}$.

Since D is a total efficient dominating set of G, $\bigcup_{u\in D} N_S(u) = V$. So, $p = \sum_{v\in V} \sigma(w) = \sum_{u\in D} \sum_{w\in N_S(u)} \sigma(w) \leq |D|\Delta_N(G)$. Since $\gamma_{tef}(G) = \sum_{v\in D} \sigma(v) \geq \sigma_1|D|$, $|D| \leq \frac{\gamma_{tef}(G)}{\sigma_1}$. Therefore, $p \leq |D|\Delta_N(G) \leq \frac{\gamma_{tef}(G)}{\sigma_1}\Delta_N(G)$. So, $\gamma_{tef}(G) \geq \frac{p\sigma_1}{\Delta_N(G)}$.

(2) Suppose that $\gamma_{tef}(G) = \frac{2p\sigma_2}{\tau+2\sigma_2}$. Let D be a γ_{tef}-set of G. Then all inequalities in the above proof must be equal. That is $\tau_{uv} = \tau$ for any arc $(u, v) \in E(G[D])$ and $\sigma(w) = \sigma_2$ for any $w \in D$. Conversely, suppose that there exists a γ_{tef}-set D of G such that $\tau_{uv} = \tau$ for any arc $(u, v) \in E(G[D])$ and $\sigma(w) = \sigma_2$ for any $w \in D$. Then $p - \gamma_{tef}(G) = \sum_{(u,v)\in E(G[D])} \tau_{uv} = \tau\frac{|D|}{2}$ and $\gamma_{tef}(G) = \sum_{w\in D} \sigma(w) = \sigma_2|D|$. So, $p - \gamma_{tef}(G) = \tau\frac{\gamma_{tef}(G)}{2\sigma_2}$. That is $\gamma_{tef}(G) = \frac{2p\sigma_2}{\tau+2\sigma_2}$.

(3) By a similar proof as that in (2), the result holds. ■

Theorem 6: For any fuzzy graph G, then $\gamma_{tef}(G) = p$ if and only if $G = mK_2$, where $m \geq 1$.

Proof: Suppose that $G = mK_2$. Obviously $\gamma_{tef}(G) = p$.

Conversely suppose $\gamma_{tef}(G) = p$. We now prove that $G = mK_2$. Assume that $G \neq mK_2$. Then there exists one vertex u such that it has at least two strong adjacent vertices in G. Let D be a γ_{tef}-set of G. Since $\gamma_{tef}(G) = p$, it implies that $V - D = \emptyset$. Hence $u \in D$. It implies that u has at least two strong adjacent vertices in D, which is a contradiction. So $G = mK_2$. ■

Definition 2: Let $G = (\sigma, \mu)$ be a fuzzy graph. The total efficient domatic number $d_{tef}(G)$ of G is the maximum order of a partition of the vertex set of G into total efficient dominating sets of G.

By the definition on the total efficient domatic number, we have the following.

Proposition 2: (1) For any fuzzy cycle C_{4k}, $k \geq 1$, $d_{tef}(C_{4k}) = 2$.

(2) For any complete bipartite fuzzy graph $K_{m,n}$, $1 \leq m \leq n$, $d_{tef}(K_{m,n}) = m$.

Theorem 7: Let $G = (\sigma, \mu)$ be a connected fuzzy graph. If $\gamma_{tef}(G)$ exists, then $d_{tef}(G) \leq \frac{p}{\gamma_{tef}(G)}$.

Proof: Assume that $d_{tef}(G) = d$. Let $D_1, D_2, \ldots, D_d$ be the partition of the vertex set of G such that D_i is a total efficient dominating set of G, where $1 \leq i \leq d$. It follows that $\gamma_{tef}(G) \leq \sum_{u\in D_i} \sigma(u)$ for $1 \leq i \leq d$. Since $V = \bigcup_{i=1}^{d} D_i$, it follows that $\sum_{i=1}^{d} \gamma_{tef}(G) \leq \sum_{i=1}^{d} \sum_{u\in D_i} \sigma(u) = \sum_{u\in V} \sigma(u) = p$. So, $d \leq \frac{p}{\gamma_{tef}(G)}$. That is $d_{tef}(G) \leq \frac{p}{\gamma_{tef}(G)}$. ■

V. ALGORITHM

It is well known that the total efficient domination problem is NP-hard in general. A polynomial-time algorithm was designed to determine the minimum fuzzy cardinality of the total efficient dominating set of a fuzzy tree T or decide that T has no total efficient dominating set. One vertex of P_2 is defined as the center. Let $K_{1,r}$ denote a star with r leaves, and let $S(r, s)$ denote a double star with two support vertices such that one support vertex is adjacent to r leaves and the other support vertex is adjacent to s leaves. Let $S(T)$ denote the support vertex set of T.

Proposition 3: Let $T = (\sigma, \mu)$ be a connected fuzzy tree. Let u and v be two support vertices of T.

(1) If $d(u, v) = 1$ and $\gamma_{tef}(G)$ exists, then u and v belong to every total efficient dominating set of T.

(2) If $d(u, v) = 2$, then T has no total efficient dominating set.

(3) Suppose that $d(u, v) = 3$ and $\gamma_{tef}(G)$ exists. Let $P = uwtv$ be the path in T between u and v. Then w and t do not belong to any total efficient dominating set of T.

In the following, three types of trees were defined. Let $\mathcal{T}_1$ be a tree obtained from a vertex x, a double star $S(r, s)$, and l vertex disjoint stars $K_{1,r_1}, \ldots, K_{1,r_l}$ by joining an edge from x to a support vertex of $S(r, s)$ and joining an edge from x to a leaf of each star K_{1,r_i}, where $l \geq 0$ and $r_i \geq 2$ for $i = 1, \ldots, l$.

Let $\mathcal{T}_2$ be a tree obtained from a vertex x, a star $K_{1,r}$ with $r \geq 1$, and l vertex disjoint stars $K_{1,r_1}, \ldots, K_{1,r_l}$ by joining an edge from x to a center of $K_{1,r}$ and joining an edge from x to a leaf of each star K_{1,r_i}, where $l \geq 1$ and $r_i \geq 2$ for $i = 1, \ldots, l$.

Let $\mathcal{T}_3$ be a tree obtained from a vertex x and l vertex disjoint stars $K_{1,r_1}, \ldots, K_{1,r_l}$ by joining an edge from x to a leaf of each star K_{1,r_i}, where $l \geq 1$ and $r_i \geq 2$ for $i = 1, \ldots, l$.

The difference among them is as follows: $\mathcal{T}_1 - x$ contains exactly one double star and l vertex disjoint stars such that vertex x is joined to a leaf of each star, where $l \geq 0$. $\mathcal{T}_2 - x$ contains exactly one star $K_{1,r}$ such that x is joined to its center and l vertex disjoint stars such that vertex x is joined to a leaf of each star, where $l \geq 1$. $\mathcal{T}_3 - x$ contains only l vertex disjoint stars such that vertex x is joined to a leaf of each star, where $l \geq 1$.

Given a tree T, if T is isomorphic to $\mathcal{T}_i$, then T is called a $\mathcal{T}_i$-type tree, where $i = 1, 2, 3$.

In Fig. 2, G_1 is $\mathcal{T}_1$-type tree with $l = 1$, G_2 is $\mathcal{T}_2$-type tree with $l = 2$ and G_3 is $\mathcal{T}_3$-type tree with $l = 2$.

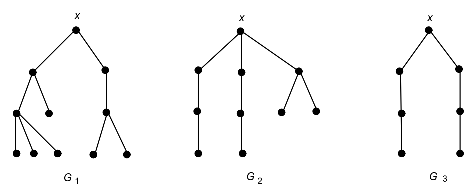

FIGURE 2. Three types of Tree.

Proposition 4: Let $T = (\sigma, \mu)$ be a connected fuzzy tree rooted at vertex r. Let v be a vertex with the longest distance from r. Let u, w, x be the parent of v, u, w, respectively. Let T_x denote the subtree of T induced by x and its descendants. If T has a total efficient dominating set, then T_x is a $\mathcal{T}_i$-type tree, where $i \in \{1, 2, 3\}$.

Proof: Since u is a support vertex and $d(u, x) = 2$, it follows that x is not a support vertex by Proposition 3. Let $C(x)$ denote the children set of vertex x. For any $w \in C(x)$, w is adjacent to at most one support vertex in T_x. Hence each component of $T_x - x$ is a double star or a star with at least two vertices. We will discuss it from the following cases.

Case 1: $T_x - \{x\}$ contains a double star $S(r, s)$. Without loss of generality, we can assume that both u and w are support vertices of $S(r, s)$. By Proposition 3, $T_x - \{x\}$ contains exactly one double star. Otherwise, T has two support vertices with distance two and T has no total efficient dominating set, which is a contradiction. Similarly, vertex x is not adjacent to a support vertex. Hence, any other component of $T_x - \{x\}$ is a star with at least three vertices and vertex x is joining to a leaf of the star. Therefore, T_x is a $\mathcal{T}_1$-type tree.

Case 2: $T_x - \{x\}$ does not contain a double star $S(r, s)$. Since $vuwx$ is a path in T_x, $N_S(w) = \{x, u\}$. Then $T_x - \{x\}$ contains at least a star with at least three vertices such that x is joining to one leaf of the star by an edge.

Suppose that $T_x - \{x\}$ contains a star $K_{1,r}$ with $r \geq 1$ such that x is joining to the center of $K_{1,r}$ by an edge. By Proposition 3, $T_x - \{x\}$ contains exactly one star such that x is joining to the center of $K_{1,r}$ by an edge. Otherwise, T has two support vertices with distance two and T has no total efficient dominating set, which is a contradiction. Hence, any other component of $T_x - \{x\}$ is a star with at least three vertices and vertex x is joining to a leaf of each star. Therefore, T_x is a $\mathcal{T}_2$-type tree. If $T_x - \{x\}$ does not contain a star $K_{1,r}$ with $r \geq 1$ such that x is joining to the center of $K_{1,r}$ by an edge, then it is obvious that T_x is a $\mathcal{T}_3$-type tree. ■

Suppose that T has a total efficient dominating set D. If T has $\mathcal{T}_1$-type subtree T_x, then every support vertex of T_x belongs to D. For each star component of $T_x - x$, its support vertex is dominated by exactly one leaf of T. If T has $\mathcal{T}_2$-type subtree T_x, then every support vertex of T_x belongs to D, and each support vertex of T_x is dominated by exactly one leaf of T.

Proposition 5: Suppose that T_x is a $\mathcal{T}_1$-type subtree of tree T. Let $T' = T - T_x$, where x is joining to a vertex y of T' by an edge. Then T has a γ_{tef}-set D if and only if T' has a γ_{tef}-set D' such that $y \notin D'$. Furthermore, $\gamma_{tef}(T) = \gamma_{tef}(T') + \gamma_{tef}(T_x)$.

Proof: Let D be a γ_{tef}-set of T. Since every support vertex in T_x belongs to D, it follows that $x, y \notin D$ by Proposition 3. Let $D' = D \cap V(T')$ and $D'' = D \cap V(T_x)$. Then D'' is a total efficient dominating set of T_x and D' is a total efficient dominating set of T' such that $y \notin D'$. Hence $\gamma_{tef}(T') + \gamma_{tef}(T_x) \leq \sum_{v \in D'} \sigma(v) + \sum_{v \in D''} \sigma(v) = \sum_{v \in D} \sigma(v) = \gamma_{tef}(T)$. Let D' be γ_{tef}-set of T' such that $y \notin D'$, and D'' be γ_{tef}-set of T_x. Then $D' \cup D''$ is a total efficient dominating set of T. Hence $\gamma_{tef}(T) \leq \sum_{v \in D' \cup D''} \sigma(v) \leq \gamma_{tef}(T') + \gamma_{tef}(T_x)$. So $\gamma_{tef}(T) = \gamma_{tef}(T') + \gamma_{tef}(T_x)$. Hence all inequalities above must be equal. So, T has a γ_{tef}-set D if and only if T' has a γ_{tef}-set D' such that $y \notin D'$. ■

Proposition 6: Suppose that T_x is a $\mathcal{T}_2$-type subtree of tree T. Let $T' = T - T_x$, where x is joining to a vertex y of T' by an edge. Then T has a γ_{tef}-set D if and only if T' has a γ_{tef}-set D' such that $y \notin D'$. Furthermore, $\gamma_{tef}(T) = \gamma_{tef}(T') + \gamma_{tef}(T_x)$.

Proof: Let D be a γ_{tef}-set of T. Since every support vertex in T_x belongs to D, it follows that $x, y \notin D$ by Proposition 3. Let $D' = D \cap V(T')$ and $D'' = D \cap V(T_x)$. Then D'' is a total efficient dominating set of T_x and D' is a total efficient dominating set of T' such that $y \notin D'$. Hence $\gamma_{tef}(T') + \gamma_{tef}(T_x) \leq \sum_{v \in D'} \sigma(v) + \sum_{v \in D''} \sigma(v) = \sum_{v \in D} \sigma(v) = \gamma_{tef}(T)$.

Let D' be γ_{tef}-set of T' such that $y \notin D'$, and D'' be γ_{tef}-set of T_x. It follows that $x \notin D''$ by Proposition 3.

Then $D' \cup D''$ is a total efficient dominating set of T. Hence $\gamma_{tef}(T) \leq \sum_{v \in D' \cup D''} \sigma(v) \leq \gamma_{tef}(T') + \gamma_{tef}(T_x)$. So $\gamma_{tef}(T) = \gamma_{tef}(T') + \gamma_{tef}(T_x)$. Hence all inequalities above must be equal. So, T has a γ_{tef}-set D if and only if T' has a γ_{tef}-set D' such that $y \notin D'$. ■

Proposition 7: Suppose that T_x is a $\mathcal{T}_3$-type subtree of tree T. Let $T' = T - T_x$, where x is joining to a vertex y of T' by an edge. Then T has a total efficient dominating set if and only if T' has a total efficient dominating set.

Proof: Suppose T has a total efficient dominating set D. It is obvious that every support vertex in T_x belongs to D. By Proposition 3, $x \notin D$. Then $D \cap V(T')$ is a total efficient dominating set of T'. So T' has a total efficient dominating set.

Suppose that T' has a total efficient dominating set D'. Let $S(T_x) = \{u_1, \ldots, u_l\}$ be the support vertex set of T_x. For each support vertex u_i, assume that u_i' and u_i'' are two vertices adjacent to u_i, where u_i' is a leaf and u_i'' is not a leaf. If $y \in D'$, let $D = D' \cup \{u_i, u_i' : i = 1, \ldots, l\}$. If $y \notin D'$, let $D = D' \cup \{u_1, u_1''\} \cup \{u_i, u_i' : i = 2, \ldots, l\}$. In any case, D is a total efficient dominating set of T. So T has a total efficient dominating set. ■

We use a labeling algorithm, which appears in [9] for the first time and is afterwards widely used in the literature for solving the domination-related problem in [10] and [11]. To obtain a polynomial time algorithm for obtaining the total efficient dominating set of a tree, a vertex data structure should be designed as follows.

Let r and v be two vertices with the longest path in T. Root the tree T at vertex r. The height of T is the maximum distance between r and all other vertices. Let h be the height of T . The i-th level A_i $(0 \leq i \leq h)$ is the set of vertices of T which are at distance i from the root. For such a rooted tree T with n vertices, we can number the vertices of T with $v_1, v_2, \ldots, v_n$ as follows. We go on every level starting from level h to level 1.

For each i $(1 \leq i \leq h)$, the vertices were traversed on level i in arbitrary order, from left to right.

Finally, the parents of all vertices of T were listed (the vertex v_n has no parent and is represented as $p(v_n) = 0$), and thus, T can be represented by a data structure called a vertex parent array. The vertex v_n is called the root of T. For any vertex x, let T_x denote the subtree of T induced by vertex x and its descendants. Let $C(x) = N_S(x) \cap V(T_x)$.

Suppose that T_x is a $\mathcal{T}_1$-type tree. Let u_0 and w_0 be two support vertices of $S(r, s)$. If $l \geq 1$, then let u_i be the center of the star K_{1,r_i}, for $1 \leq i \leq l$. Assume that $N_S(u_i) \cap N_S(x) = \{w_i\}$ for $0 \leq i \leq l$.

Suppose that T_x is a $\mathcal{T}_2$-type tree. Let u_0 be the center of $K_{1,r}$, and u_i be the center of the star K_{1,r_i} for $1 \leq i \leq l$. Assume that $N_S(u_i) \cap N_S(x) = \{w_i\}$ for $1 \leq i \leq l$.

Suppose that T_x is a $\mathcal{T}_3$-type tree. Let u_i be the center of the star K_{1,r_i} for $1 \leq i \leq l$. Assume that $N_S(u_i) \cap N_S(x) = \{w_i\}$ for $1 \leq i \leq l$.

In the following, we can assume that every $\mathcal{T}_i$-type tree can be relabeled as above.

Algorithm 1 Computes the Total Efficient Domination Number of a Fuzzy Tree T

Input: A rooted tree T represented by its vertex parent array $[v_1, v_2, \cdots, v_n]$.
A pair of membership functions σ and μ of a fuzzy tree T.
Output: Total efficient domination number of the fuzzy tree T.

$I \leftarrow \emptyset$;
for all $v \in V(T)$ **do**
$l(v) \leftarrow 0$; $\bar{l}(v) \leftarrow 0$;
end for
While there exists a vertex v such that $d(v, v_n) \geq 4$ **do**
Choose a vertex v such that $d(v, v_n)$ is maximum.
Let u, w, x, y be the parent of v, u, w, x, respectively.
Consider the subtree T_x of T.
If T_x is not $\mathcal{T}_i$-type tree, where $i \in \{1, 2, 3\}$ **then** return $\gamma_{tef}(T) = 0$.
If T_x is $\mathcal{T}_1$-type tree **then**
$I \leftarrow I \cup \{y\}$; $S \leftarrow S(T_x)$;
If there exists a vertex u_i such that $C(u_i) \subseteq I$ $(1 \leq i \leq l)$ or $S \cap I \neq \emptyset$ or y is a support vertex **then** return $\gamma_{tef}(T) = 0$.
else
For i=1 to l **do**
Choose a vertex $v_i \in C(u_i) \setminus I$ such that $\sigma(v_i) + L(v_i) + \sum_{t \in C(u_i) \setminus (I \cup \{v_i\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in C(u_i) \setminus (I \cup \{w\})} \overline{L}(t) : w \in C(u_i) \setminus I\}$;
$S \leftarrow S \cup \{v_i\}$;
end for
$T \leftarrow T - T_x$;
$\overline{L}(y) \leftarrow \overline{L}(y) + \sum_{w \in S}(\sigma(w) + L(w)) + \sum_{w \in V(T_x) \setminus S} \overline{L}(w)$;
end if
end if
If T_x is $\mathcal{T}_2$-type tree **then**
$I \leftarrow I \cup \{y\}$; $S \leftarrow S(T_x)$;
If there exists a vertex $u \in S$ such that $C(u) \subseteq I$ or $S \cap I \neq \emptyset$ or y is a support vertex **then** return $\gamma_{tef}(T) = 0$.
else
For i=0 to l **do**
Choose a vertex $v_i \in C(u_i) \setminus I$ such that $\sigma(v_i) + L(v_i) + \sum_{t \in C(u_i) \setminus (I \cup \{v_i\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in C(u_i) \setminus (I \cup \{w\})} \overline{L}(t) : w \in C(u_i) \setminus I\}$;
$S \leftarrow S \cup \{v_i\}$;
end for
$T \leftarrow T - T_x$;
$\overline{L}(y) \leftarrow \overline{L}(y) + \sum_{w \in S}(\sigma(w) + L(w)) + \sum_{w \in V(T_x) \setminus S} \overline{L}(w)$;
end if
end if
If T_x is $\mathcal{T}_3$-type tree **then**
$S \leftarrow S(T_x)$;
For i=1 to l **do**
If $C(u_i) \setminus I \neq \emptyset$ **then**
Choose a vertex $v_i \in C(u_i) \setminus I$ such that $\sigma(v_i) + L(v_i) + \sum_{t \in C(u_i) \setminus (I \cup \{v_i\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in C(u_i) \setminus (I \cup \{w\})} \overline{L}(t) : w \in C(u_i) \setminus I\}$;
$S \leftarrow S \cup \{v_i\}$;

Algorithm 1 *(Continued.)* Computes the Total Efficient Domination Number of a Fuzzy Tree T

end if
end for
If there exists w_i such that $w_i \in I$ **then**
If $C(u_i) \subseteq I$ or $u_i \in I$ **then** return $\gamma_{tef}(T) = 0$
else
$T \leftarrow T - T_{w_i}$; $I \leftarrow I \cup \{x\}$;
$\overline{L}(x) \leftarrow \overline{L}(x) + \sum_{w \in \{u_i, v_i\}}(\sigma(w) + L(w)) + \sum_{w \in V(T_{w_i}) \setminus \{u_i, v_i\}} \overline{L}(w)$;
end if
end if
If there exist u_i and u_j such that $C(u_i) \cup C(u_j) \subseteq I$ and $i \neq j$ **then** return $\gamma_{tef}(T) = 0$.
If there exists u_i such that $C(u_i) \subseteq I$ and $C(u_j) \nsubseteq I$ for $j \neq i$ **then**
$I \leftarrow I \cup \{y\}$; $T \leftarrow T - T_x$;
$\overline{L}(y) \leftarrow \overline{L}(y) + \sum_{w \in S \cup \{w_i\}}(\sigma(w) + L(w)) + \sum_{w \in V(T_x) \setminus (S \cup \{w_i\})} \overline{L}(w)$;
end if
If for any u_i such that $C(u_i) \nsubseteq I$ **then**
$T \leftarrow T - T_x$;
$L(y) \leftarrow L(y) + \sum_{w \in S}(\sigma(w) + L(w)) + \sum_{w \in V(T_x) \setminus S} \overline{L}(w)$;
$\overline{L}(y) \leftarrow \overline{L}(y) + \min\{\sum_{w \in (S \setminus \{v_i\}) \cup \{w_i\}}(\sigma(w) + L(w)) + \sum_{w \in V(T_x) \setminus ((S \setminus \{v_i\}) \cup \{w_i\})} \overline{L}(w) : i = 1, 2, \ldots, l\}$;
end if
end if
end while
If T is a P_1, **then** return $\gamma_{tef}(T) = 0$.
If T is a P_2, **then if** $V(P_2) \cap I \neq \emptyset$, **then** return $\gamma_{tef}(T) = 0$ **else** $\gamma_{tef}(T) = \sigma(u) + \sigma(v) + L(u) + L(v)$.
If T is a star with center vertex u, **then if** $u \in I$ or $N_S(u) \subseteq I$, **then** return $\gamma_{tef}(T) = 0$ **else** Choose $v \in N_S(u)$ such that $\sigma(v) + L(v) + \sum_{t \in N_S(u) \setminus (I \cup \{v\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in N_S(u) \setminus (I \cup \{w\})} \overline{L}(t) : w \in N_S(u) \setminus I\}$;
$\gamma_{tef}(T) = \sum_{w \in \{u,v\}}(\sigma(w) + L(w)) + \sum_{w \in N_S(u) \setminus \{v\}} \overline{L}(w)$.
If T is a double star, **then if** $S(T) \cap I \neq \emptyset$, **then** return $\gamma_{tef}(T) = 0$ **else** $\gamma_{tef}(T) = \sum_{w \in S(T)}(\sigma(w) + L(w)) + \sum_{w \in V(T) \setminus S(T)} \overline{L}(w)$.
return $\gamma_{tef}(T)$

Theorem 8: Algorithm 1 produces the minimum fuzzy cardinality of a total efficient dominating set of a tree T in $O(n^2)$ time.

Proof: We now discuss the running time of the Algorithm 1. At each iteration of the "*while*" loop of the algorithm, it take $O(|V(T_x)|)$ time to decide T_x is $\mathcal{T}_i$-type tree for $i = 1, 2, 3$. If T_x is $\mathcal{T}_1$-type tree or $\mathcal{T}_2$-type tree, we need $O(|V(T_x)|)$ time to give a label of the vertex y. If T_x is $\mathcal{T}_3$-type tree, we need at most $O(|V(T_x)|d_N(x))$ time to give a label of the vertex y. Since $d_N(x) \leq \Delta_N(T)$, we need $O(|V(T_x)|\Delta_N(T))$ time to give a label of the vertex y. Since Algorithm visits each T_x of T once and $\Delta_N(T) \leq n - 1$, it follows that the Algorithm 1 can be computed in $O(n^2)$ time.

For the correctness of the algorithm, it is sufficient to consider T with a vertex v such that $d(v, v_n) \geq 4$. Otherwise, the algorithm obviously produces the minimum fuzzy cardinality of a total efficient dominating set of T. Choose a vertex v such that $d(v, v_n)$ is maximum. Let u, w, x, y be the parent of v, u, w, x, respectively. Consider the subtree T_x of T. By Proposition 4, if T_x is not $\mathcal{T}_i$-type tree for $i \in \{1, 2, 3\}$, then T has no total efficient dominating set and $\gamma_{tef}(T) = 0$. Without loss of generality, we can assume T_x is $\mathcal{T}_i$-type tree for $i \in \{1, 2, 3\}$. By Propositions 5, 6, 7, it is sufficient to prove that we obtain a minimum fuzzy cardinality of a total efficient dominating set of T_x or T_w. So the proof of Theorem 8 is followed from the following three cases.

Case 1: T_x is $\mathcal{T}_1$-type tree. Let D be a γ_{tef}-set of T. Then $S(T_x) \subseteq D$ and $|D \cap C(u_i)| = 1$ for every $u_i \in S(T_x)$, where $1 \leq i \leq l$. Hence if there exists a u_i such that $C(u_i) \subseteq I$ $(1 \leq i \leq l)$ or $S(T_x) \cap I \neq \emptyset$ or y is a support vertex, then T has no total efficient dominating set of T. For each u_i, since we choose a vertex $v_i \in C(u_i) \setminus I$ such that $\sigma(v_i) + L(v_i) + \sum_{t \in C(u_i) \setminus (I \cup \{v_i\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in C(u_i) \setminus (I \cup \{w\})} \overline{L}(t) : w \in C(u_i) \setminus I\}$, it follows that we obtain a minimum fuzzy cardinality of a total efficient dominating set of T_x.

Case 2: T_x is $\mathcal{T}_2$-type tree. Let D be a γ_{tef}-set of T. Then $S(T_x) \subseteq D$ and $|D \cap C(u_i)| = 1$ for every $u_i \in S(T_x)$, where $0 \leq i \leq l$. Hence if there exists a u_i such that $C(u_i) \subseteq I$ $(0 \leq i \leq l)$ or $S(T_x) \cap I \neq \emptyset$ or y is a support vertex, then T has no total efficient dominating set of T. For each u_i, since we choose a vertex $v_i \in C(u_i) \setminus I$ such that $\sigma(v_i) + L(v_i) + \sum_{t \in C(u_i) \setminus (I \cup \{v_i\})} \overline{L}(t) = \min\{\sigma(w) + L(w) + \sum_{t \in C(u_i) \setminus (I \cup \{w\})} \overline{L}(t) : w \in C(u_i) \setminus I\}$, it follows that we obtain a minimum fuzzy cardinality of a total efficient dominating set of T_x.

Case 3: T_x is $\mathcal{T}_3$-type tree. Let D be a γ_{tef}-set of T. Then $S(T_x) \subseteq D$. If $y \in D$, then $|D \cap C(u_i)| = 1$ for every $u_i \in S(T_x)$, where $1 \leq i \leq l$. If $y \notin D$, then there exists i such that $w_i \in D$ and $|D \cap C(u_j)| = 1$ for every $u_j \in S(T_x)$, where $1 \leq j \leq l$ and $j \neq i$. Hence if there exists a u_i such that $C(u_i) \cup \{w_i\} \subseteq I$ $(1 \leq i \leq l)$ or $S(T_x) \cap I \neq \emptyset$ or $C(u_i) \cup C(u_j) \subseteq I$ $(i \neq j)$, then T has no total efficient dominating set of T. By a similar way as Case 2, in any cases, we obtain a minimum fuzzy cardinality of a total efficient dominating set of T_x or T_{w_i} for some $(1 \leq i \leq l)$.

At each iteration of the "*while*" loop of the algorithm, we find a minimum fuzzy cardinality of a total efficient dominating set of T_x or T_w. If $y \in I$, then it is saved to $L(y)$. If $y \notin I$, then it is saved to $\overline{L}(y)$. So by the parameters $L(y)$ and $\overline{L}(y)$, we know the minimum fuzzy cardinality of a total efficient dominating set in the deleted subtree of T_y. If $d(v, v_n) \leq 3$, then the tree is P_1, P_2, a star or a double star. It is easy to obtain its minimum fuzzy cardinality of a total efficient dominating set. Hence Algorithm 1 produces the minimum fuzzy cardinality of a total efficient dominating set of a tree T. ■

VI. CONCLUSION

This study proposed some new results on the total efficient dominating set of fuzzy graphs. The natural extension of this research work is to research on other types of fuzzy graphs. Furthermore, the fixed-parameter tractability of the total efficient domination problem of fuzzy graphs is our future research direction.

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XUE-GANG CHEN received the Ph.D. degree in applied mathematics from the Beijing Institute of Technology, in 2004. He is currently a Professor in mathematics with the North China Electric Power University of China. His research interests include graph theory and domination theory.

MOO YOUNG SOHN received the Ph.D. degree in mathematics from Kyungpook National University, in 1987. He is currently a Professor with the Changwon National University of Korea. His research interests include graph theory and domination theory.

DE-XIANG MA received the Ph.D. degree in applied mathematics from the Beijing Institute of Technology, in 2006. She is currently a Professor in mathematics with the North China Electric Power University of China. Her research interests include graph theory and fractional differential equations.