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# Geometric Algebra in Signal and Image Processing: A Survey

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**ABSTRACT** Recently, Geometric Algebra (GA) has attracted more and more attention in the field of signal and image processing. GA can treat multi-dimensional signals in a holistic way to keep the correlations among multiple dimensions and avoid information loss. So when traditional signal and image processing algorithms are redefined in GA space, they will be more powerful and achieve better performance in multi-dimensional signal processing. In this paper, we provide a comprehensive survey covering various GA-based algorithms. In particular, we first review the mathematic theories of GA and Reduced Geometric Algebra (RGA). Then, advanced GA-based algorithms are elaborately analyzed and compared, including GA-based Sparse representation model, GA-based Dictionary Learning method, Clifford Support Vector Machine, GA-based Feature extraction algorithms, GA-based adaptive filtering algorithms, GA-based Fourier-type transform, and GA-based edge detection algorithm. Finally, we discuss several open issues and challenges of GA, and point out possible research directions in the future.

**INDEX TERMS** Geometric algebra (GA), multivectors, multi-dimensional signals, image processing.

## I. INTRODUCTION

Geometric algebra (GA) has been considered as one of the most powerful tools in mathematics and has witnessed great success in a wide range of applications, such as physics, quantum computing, electromagnetism, satellite navigation, neural computing, camera geometry, image processing, robotics and computer vision, etc. The above-mentioned application fields are reviewed in 2013 [1]. Since it is impossible to make a complete overview for the enormous range of applications developed in the past decades, particularly, we try to give an overview on theory and applications of GA mainly in signal and image processing.

For multi-channel signals, traditional methods usually treat each channel as a different vector and process them independently, which may fail to exploit the correlations among multiple channels and lead to information loss. Fortunately, GA can transform multi-dimensional signals into multivectors and handle them in a holistic manner in a new multi-dimensional GA space. By this way, GA is able to keep the correlations among multiple dimensions and avoid information loss. So when traditional signal and image processing algorithms are redefined in GA space, they will be more powerful and achieve better performance in multi-dimensional signal processing.

In this paper, we provide a comprehensive and up-to-date survey covering various GA-based algorithms. We hope it may be helpful for researchers who are interested in the signal and image processing field of GA. For more details, please refer to the related references.

The rests of the paper are organized as follows. Section II briefly reviews the mathematic theories of GA and Reduced Geometric Algebra (RGA). In Section III, advanced GA-based algorithms for signal and image processing are elaborately analyzed and compared, including GA-based Sparse representation model, GA-based Dictionary Learning method K-GASVD, Clifford Fuzzy SVM(CFSVM), GA-based Feature extraction algorithms (GA-SIFT, GA-SURF, GA-ORB, GA-STIP), GA-based adaptive filtering algorithms (GA Least-Mean-Squares (GA-LMS), GA Least-Mean Kurtosis (GA-LMK)), sparse Fast Clifford Fourier transform(SFCFT), and GA-based edge detection algorithm. Section IV discusses several open issues and challenges of GA, their advantages and shortcomings are

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also analyzed. Section V forecasts the prospects of GA in signal and image processing. Finally, Section VI summarizes the paper and points out possible research directions in the future.

## II. RELATED WORK

### A. THE BASIC OF GEOMETRIC ALGEBRA

Geometric algebra (GA) [2], [3] is created by W.K. Clifford, also called Clifford algebra. GA provides a mathematical framework, which is ideal to constitute an extension of real, complex and quaternion algebras to complete associative algebras of subspaces of vector spaces in the general framework of vector. Geometric relationships and algorithms can be described by GA, compactly and geometrically. Thus, GA can completely make use of the information associated with different geometric data and improve the computational efficiency. In recent years, GA has become one of the research hotspots in the area of information science owing to the advantages mentioned above, it has been widely and successfully applied to object detection and face recognition, image processing and analysis [4]–[6].

Mathematically, suppose $G_n$ is denoted as a $2^n$-dimensional vector space, the basis of $G_n$ is denoted as

$$\left\{e_A = e_{a_1} \cdots e_{a_r} \,|A = (a_1, \ldots a_r) \in B\right\} \tag{1}$$

where $G_n$ can be denoted as $G_{p,q}$, and $n = p + q$.

In general, GA is non-commutative, the multiplication of GA will follow the next rules.

$$\begin{cases} e_{a_i}^2 = 1, & a_i = 1, \ldots, p \\ e_{a_i}^2 = -1, & a_i = p+1, \ldots, p+q \\ e_{a_i}e_{a_j} = -e_{a_j}e_{a_i}, & a_i \neq a_j \end{cases} \tag{2}$$

The power set $\Gamma$ of $\{1, \cdots, n\}$ can turn the basis into an ordered one with the index set $B$.

$$\{B = (a_1, \cdots a_r) \in \Gamma, 1 \leq a_1 \cdots a_r \leq n\} \tag{3}$$

For example, the basis in $2^3$ vector space can be described as

$$\left\{e_\varnothing, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\right\} \tag{4}$$

For convenience, in the rest of the paper, $A = a_1 \cdots a_r$ will be denoted as $A = a_{1\ldots r}$, and $e_\varnothing = 1$.

The set $G_n^t$ is denoted as following, representing $t$-vector part of $G_n$.

$$G_n^t = \{e_A \,|e_A \in B, |A| = t\} \tag{5}$$

An arbitrary element of GA is given as

$$x = \sum_{t=0}^{n} \langle x \rangle_t = \sum_A x_A e_A \tag{6}$$

where $x_A \in R$. For example, an element in $2^3$ vector space can be represented as

$$\begin{aligned} x &= \langle x \rangle_0 + \langle x \rangle_1 + \langle x \rangle_2 + \langle x \rangle_3 \\ &= x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_{12} + x_{13}e_{13} \\ &\quad + x_{23}e_{23} + x_{123}e_{123} \end{aligned} \tag{7}$$

The addition of GA can be defined as

$$x + y = \sum_A (x_A + y_A)e_A \tag{8}$$

The geometric product of GA can be written in the following form.

$$xy = x \cdot y + x \wedge y \tag{9}$$

where $x \cdot y$ and $x \wedge y$ represent inner product and outer product, respectively.

For performing more complex multivectors operations, several rules are defined as following.

a: The reversion of GA is

$$x^\dagger = \langle x \rangle_t^\dagger = (-1)^{\frac{t(t-1)}{2}} \langle x \rangle_t \tag{10}$$

b: The conjugation of GA is

$$x^* = \langle x \rangle_t^* = (-1)^{\frac{t(t+1)}{2}} \langle x \rangle_t \tag{11}$$

c: The module of GA is

$$|\langle x \rangle| = \sqrt{\sum_{k=0}^{n} \left|\langle x \rangle_t\right|^2} \tag{12}$$

$$\left|\langle x \rangle_t\right| = \sqrt{\langle x \rangle_t \cdot \langle x \rangle_t} \tag{13}$$

### B. THE BASIC OF REDUCED GEOMETRIC ALGEBRA

It is noted that the multiplication of GA above is not commutative, leading to the high algorithm complexity. For improvement, Shen *et al.* [7] propose a novel theory of reduced geometric algebra (RGA), including commutative multiplication rules and the geometric operations. The definition of the reduced geometric algebra (RGA) is given as follows:

$$\varepsilon_i = \frac{1}{2}(1 + e_ie_{n+i}) \in G_n^R, \quad i = 1, 2, \cdots, n \tag{14}$$

According to (14), the geometric product of is described as:

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \quad i \neq j \tag{15}$$

Then, we define:

$$\varepsilon_i^2 = \varepsilon_i\varepsilon_j = \begin{cases} \varepsilon_{i+1}, & i = 1, 2, \cdots, n-1 \\ \varepsilon_1, & i = n \end{cases} \tag{16}$$

For example, take $n = 3$, $\varepsilon_1^2 = \varepsilon_2, \varepsilon_2^2 = \varepsilon_3, \varepsilon_3^2 = \varepsilon_1$. According to (15) and (16), it is clearly that the multiplication of $\varepsilon_i$ is commutative. Specifically, RGA is denoted as $G_n^R$ and can be seen as the space that is generated by the collection of $\left\{\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}, \left\{\varepsilon_{ij} = \varepsilon_i\varepsilon_j, 1 \leqslant i \neq j \leqslant n\right\}\right\}$. The element $k$ in $G_2^R$ has the following form:

$$k = a^1\varepsilon_1 + a^2\varepsilon_2 + a^3\varepsilon_{12}, a^1, a^2, a^3 \in \mathbb{R} \tag{17}$$

The addition and subtraction operations in $G_2^R$ are almost the same as GA, here we present the multiplication operation only. $\forall k, l \in G_2^R$, suppose $k = a^1\varepsilon_1 + a^2\varepsilon_2 + a^3\varepsilon_{12}$ and

$l = b^1\varepsilon_1 + b^2\varepsilon_2 + b^3\varepsilon_{12}$, the multiplication in $G_2^R$ are given as:

$$\begin{aligned} kl &= \left(a^1\varepsilon_1 + a^2\varepsilon_2 + a^3\varepsilon_{12}\right)\left(b^1\varepsilon_1 + b^2\varepsilon_2 + b^3\varepsilon_{12}\right) \\ &= \left(a^1b^3 + a^2b^2 + a^3b^1\right)\varepsilon_1 + \left(a^1b^1 + a^2b^3 + a^3b^2\right)\varepsilon_2 \\ &\quad + \left(a^1b^2 + a^2b^1 + a^3b^3\right)\varepsilon_{12} \end{aligned} \tag{18}$$

where $\varepsilon_{12}\varepsilon_1 = \varepsilon_1\varepsilon_2\varepsilon_1 = \varepsilon_1\varepsilon_1\varepsilon_2 = \varepsilon_1^2\varepsilon_2 = \varepsilon_2\varepsilon_2 = \varepsilon_1$, similarly, $\varepsilon_{12}\varepsilon_2 = \varepsilon_2$. Moreover, $\varepsilon_{12}\varepsilon_{12} = \varepsilon_1\varepsilon_2\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_1\varepsilon_2\varepsilon_2 = \varepsilon_1^2\varepsilon_2^2 = \varepsilon_2\varepsilon_1 = \varepsilon_{12}$

As shown in (18), no superfluous components are produced in the results of the multiplication of $k$ and $l$. That is, the results only contain $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_{12}$ components, which successfully overcomes the disadvantages of quaternions with the data redundancy exited in the operations for color images. The norm of the element $k$ in $G_2^R$ is defined as

$$\|k\| = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_{12} = |a+b+c| = \sqrt{a^2+b^2+c^2} \tag{19}$$

The conjugate of $k$ is defined as

$$k^* = a'\varepsilon_1 + b'\varepsilon_2 + c'\varepsilon_{12} \tag{20}$$

Then

$$\begin{aligned} kk^* &= \left(a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_{12}\right)\left(a'\varepsilon_1 + b'\varepsilon_2 + c'\varepsilon_{12}\right) \\ &= \left(ca' + bb' + ac'\right)\varepsilon_1 + \left(aa' + cb' + bc'\right)\varepsilon_2 \\ &\quad + \left(ba' + ab' + cc'\right)\varepsilon_{12} = \|k\|^2 \end{aligned} \tag{21}$$

According to (21), the following equations are given:

$$\begin{cases} ca' + bb' + ac' = 0 \\ aa' + cb' + bc' = 0 \\ ba' + ab' + cc' = \|k\|^2 \end{cases} \tag{22}$$

When solving the equations in (22), the values of the individual components in (20) are correspondingly yielded, but not all the elements of $G_2^R$ are conjugate.

Thus, the inverse of element $k$ in $G_2^R$ can be defined as

$$k^{-1} = \frac{k^*}{\|k\|^2} \tag{23}$$

In RGA, multivectors, which are the extension of vectors to higher dimensions, are the basic units. Each multivector $K \in G_2^R$ is so described by

$$K = K^1\varepsilon_1 + K^2\varepsilon_2 + K^3\varepsilon_{12} \tag{24}$$

where $K^1, K^2, K^3 \in R$.

## III. APPLICATIONS TO SIGNAL AND IMAGE PROCESSING

### A. APPLICATIONS TO SPARSE REPRESENTATION MODELS FOR IMAGE PROCESSING

For image processing, the sparse representation model based on dictionary learning plays an important role. Taking advantages of the sparse representation, a color image can be well-represented as a sparse linear combination of elements from an appropriately chosen over-complete dictionary, rather than being separated into independent components. The singular value decomposition (SVD), as the most popular method for sparse representation, have recently attracted intensive interest and achieved great success. Since pure virtual quaternion can be perfectly embedded the spectral data of RGB channel of color image, the spectral information and spatial information of color image can be fully explored by quaternion. As a result, Xu *et al.* [8] propose a vector sparse representation model for color images based on quaternion matrix analysis, namely QSVD, which represents the color image as a quaternion matrix on account of the compatibility between quaternion matrix and color image to avoid losing the correlation information of three-color channels. In particular, a quaternion-based dictionary learning algorithm is proposed with the K-quaternion singular value decomposition, which generalized K-means clustering for QSVD. It is significant for color images that QSVD is able to uniformly transform the channel images to an orthogonal color space, preserving the inherent color structures completely during vector reconstruction.

Octonions provides an elegant mathematical tool to deal with octonion signals and the closest in its mathematical properties gives a formidable strategy for multi-dimensional signals which are no more than octonion signals. With original visual features, spectral normalization is performed under the framework of octonion algebra framework, Gao and Lam [9] further propose a new computational model for detecting salient regions in color images, which can accommodate more feature channels than quaternion. A singular value decomposition algorithm for octonion signal, namely OSVD, which is proposed by Shen and Wang [10]. Any $m \times n$ octonion matrix $F$, its SVD is represented as

$$F = U\Sigma V^H = U\begin{bmatrix} \Sigma' & 0 \\ 0 & 0 \end{bmatrix} V^H \tag{25}$$

where $UU^H = \mathbb{I}^{M\times M}$, $VV^H = \mathbb{I}^{N\times N}$, $\mathbb{I}$ is an identity matrix.

Wang *et al.* [11], [12] apply a new algorithm corresponding dictionary learning algorithm, namely K-GASVD, the details of K-GASVD algorithm are shown in Table 1. GA is used to map the color image into a high dimensional space for color image analysis and shows the superior performance compared with competing methods on color images. However, compared with traditional K-SVD algorithm [13], the proposed K-GASVD algorithm has higher computational complexity, because of noncommutativity of multiplication of GA.

Aiming at solving the shortage of K-GASVD which suffer from high computational complexity, taking advantage of the RGA theory, a novel multivector sparse representation model based on RGA is designed by Shen *et al.* [7], using K-RGA-based singular value decomposition (K-RGASVD) which not only considers the spatial and spectral information in multispectral images, but also achieves the removal of the redundancy among color channels and reduces low computational complexity.

**TABLE 1.** **The details of GA-based dictionary learning using K-GASVD method.**

| |
|---|
| Input: multispectral image patches: $F = \{f_i, 1 \leqslant i \leqslant K\} \in (\mathbb{G}_n)^{N\times K}$; GA-based dictionary: $D = \{d_i, 1 \leqslant i \leqslant M\} \in (\mathbb{G}_n)^{N\times M}$; the number of iterations: $J$. |
| Output: sparse coefficient: $A = \{a_i, 1 \leqslant i \leqslant K\} \in (\mathbb{G}_n)^{M\times K}$; dictionary after training: $D \in (\mathbb{G}_n)^{N\times M}$. |
| 1. Initialization: $j = 0$, initialize the GA-based dictionary $D \in (\mathbb{G}_n)^{N\times M}$ as random $M$ samples from $F \in (\mathbb{G}_n)^{N\times K}$. |
| 2. When $j < J$, repeat |
| (1) Sparse coding: compute sparse representation coefficient $a \in (\mathbb{G}_n)^M$ of $F \in (\mathbb{G}_n)^{N\times K}$ over $D \in (\mathbb{G}_n)^{N\times M}$ using GAOMP algorithm. |
| (2) Dictionary updating: update each dictionary atom $d_k \in (\mathbb{G}_n)^N$ in $D^{J-1}$. |
| (i) Define the set of patches using the $k$-th dictionary atom (i) $d_k \in (\mathbb{G}_n)^N$ as $\Omega_k = \{i \mid 1 \leqslant i \leqslant k, A(k,i) \neq 0\}$. |
| (ii) Calculate the sparse error matrix $E_k$ without using $d_k$. |
| $E_k = F - \sum_{i\neq k} D_i a^i$ |
| (iii) Select columns of $E_k$ corresponding to $\Omega_k$ to form $E_k^{\Omega}$. |
| $E_k^{\Omega} = E_k(:,i)\|_{i=\Omega_k}$ |
| (iv) Perform the SVD of $E_k^{\Omega}$ using GASVD: $E_k^{\Omega} = U\Sigma V^H$ and update $d_k$ as $U(:,1)$, set $a_{\Omega}^k = \Omega_k^T a^k$ to be $V^H(:,1)\Sigma(1,1)$. |

Therefore, the sparse representation model has been extended to multi-dimensional space for image processing, and the results have been demonstrated the sparse representation model based on GA achieve the state-of-the-art performance. The GA-based models treat color image as a multivector in the GA form and preserve the relationships of the multiple channels to avoid the loss of structures information among different channels.

### *B. APPLICTIONS TO SUPPORT VECTOR MACHINES*

Support Vector Machines (SVM) [14] is originally designed for binary classification. The main idea of the SVM algorithm is to separate the samples from different classes with a surface that maximize the boundary between classes. Each training point is processed equally and assigned to one class and only one class when using the traditional SVM to solve two classification problems.

However, some training points are damaged by noise in many practical applications and some points in the training data may be misclassified. Thus, Lin and Wang [15] propose Fuzzy Support Vector Machine (FSVM) to solve such problem. For FSVM, each sample has a fuzzy membership which represents the attitude of the corresponding point toward one class. Membership denotes how important is samples to the decision surface. Therefore, different input points have different contributions to the learning of decision surface. However, the performance of existing SVM algorithms are not very satisfying for multiple classes.

Therefore, extending SVM algorithms to GA space can solve above problem better. Bayro-Corrochano *et al.* [16] design the kernel function of nonlinear SVM by using Clifford algebra framework. [16] presents a design method for SVM for classification, which will be called Clifford SVM (CSVM). In [16], the real valued SVM is generalized to CSVM. The kernels involve Clifford algebra and geometric product can be used for nonlinear classification.

This method can also design recurrent Clifford SVM. Thus, Bayro-Corrochano *et al.* [17] introduce the recurrent Clifford Support Vector Machines (RCSVM). This method uses Clifford or geometric product to design the kernels so that nonlinear mappings can be used for nonlinear classification and recursive CSVM. By using CSVM with only one kernel can greatly reduce computational complexity. The mainly reason is that a Gramm matrix compact can be defined according to the formulation in terms of multivectors which needs less computations for multiple classes than for real-valued SVM.

Taking advantages of Clifford algebra theory, the multiple classes can be represented according to the dimension of Clifford algebra. Bayro-Corrochano and Arana-Daniel [18] propose the CSVM method which defines the optimization variables as a Clifford algebra multivector, which can be applied to classification, regression and recurrence. This method accepts multiple multivector inputs and multivector outputs, which similar to the Multiple-Input Multiple-Output (MIMO) architecture, so it can be applied to multiple classes. The complex, quaternion, and hyper-complex SVM algorithms can be recovered from CSVM. The proposed CSVM method appears wide range of applications, especially in image processing, pattern recognition, geometric computing and their applications, such as graphics, augmented reality, robot vision, etc.

Although CSVM learns the decision surface from multi distinct classes of the multiple input points, in many applications, each multiple input point may not be fully assigned to one of these multi-classes. Wang *et al.* [19] present the Clifford Fuzzy SVM (CFSVM), which applies a fuzzy membership to each multiple input point, and CSVM for multiple classes is reconstructed, so that different input points have different contributions to the learning of decision surface.

This method improves the CSVM in reducing the effect of outliers and noise in data points. CFSVM is suitable for applications where data points have unmodeled characteristics. The CFSVM extends the multiclass classification application range of [20], which can solve various kinds of tasks by setting different types of fuzzy membership.

### *C. FEATURE EXTRACTION ALGORITHMS*

Image interest point extraction algorithm as an important branch of image feature extraction, plays an important role in image analysis and is regarded as an indispensable basis for complex visual tasks, including image segmentation, image registration, image mosaic and motion recognition. Image interest point extraction is the process of extracting image interest points which are one type of local invariant feature for image. Local invariant feature can resist changes such as luminance reduction, rotation, image blur and scale variations.

Common feature extraction algorithms such as SURF [21] and SIFT [22] are widely used in various fields of image processing due to their high performance. However, most of these feature extraction algorithms are aimed at processing grayscale images. While those algorithms are used to extract interest points in multispectral images, multi-channel images usually are converted into grayscale images and then processed. This method ignores the correlation of each channel of the image, which is harm to the performance of feature extraction. To solve this issue, Li *et al.* [23] propose the GA-SIFT method, which is a novel algorithmic framework based on the SIFT for multispectral images. The GA-SIFT incorporates the GA theory into the traditional SIFT framework. Taking advantage of the GA theory, a multispectral image can be regarded as an embedding of a manifold in a higher dimensional spectral-spatial space. Both of the spectral and spatial information of the multispectral image can be retained. Based on the GA representation of multispectral images, GA-SIFT can make most use of spectral and spatial information and detect the feature points not only in the spatial space but also in the spectral space, that leads to a great performance for multispectral images.

However, GA-SIFT still has some shortcomings. Due to SIFT method has high complexity, GA-SIFT as the improved SIFT method is time-consuming in processing multispectral images. To solve this problem, Wang *et al.* [24] propose GA-SURF which incorporates the GA theory into the traditional SURF framework. Compared with SIFT, SURF uses a box filter to approximate the high complexity of the second Laplace derivative, which is faster to extract image interest points. Therefore, the improved method GA-SURF has higher calculation efficiency. The GA-SURF proposes a new Hessian matrix in GA space as following:

$$H(x,y,\sigma)=\begin{bmatrix}\varphi\left(L_{xx}(x,y,\sigma)\right) & \varphi\left(L_{xy}(x,y,\sigma)\right)\\ \varphi\left(L_{xy}(x,y,\sigma)\right) & \varphi\left(L_{yy}(x,y,\sigma)\right)\end{bmatrix} \tag{26}$$

where $L_{xx}(x,y,\sigma)$, $L_{yy}(x,y,\sigma)$ and $L_{xy}(x,y,\sigma)$ are the second-order derivatives of $L(x,y)$, the function $\varphi$ is used to retain the chromaticity image, more details are given in [24].

$$\begin{aligned}L(x,y)&=G_{n+2}\otimes f\otimes G_{n+2}\\&=\frac{1}{M^2N^2}\sum_{l=0}^{M-1}\sum_{s=0}^{N-1}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}g_if_ig_j'e_j\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}g_if_jg_k'\left(e_i\wedge e_j\wedge e_k\right)\Big)\end{aligned} \tag{27}$$

The new Hessian matrix can fully retain spectral information and spatial structure information of multispectral images. And the GA-SURF experiments demonstrate GA-SURF in comparison to the state-of-the-art algorithms is faster to compute, while not sacrificing performance, and show its potential as a homogeneous and efficient tool in various applications of multispectral image analysis.

In order to real-time multispectral image processing, Wang *et al.* [25] propose a novel feature extraction method, geometric algebra based oriented fast and rotated brief (GA-ORB), for multispectral images based on the theory GA. GA-ORB can extract multi-spectral image interest points in near real time, and improve performance by using spectral information and spatial information. The GA-ORB outperforms some previous algorithms with respect to distinctiveness and robustness in extracting and matching interest points, and it can be computed much faster.

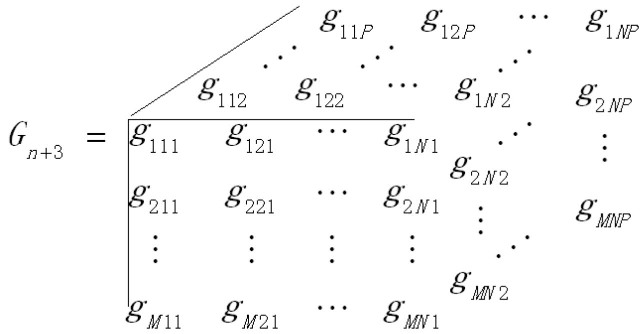

**FIGURE 1.** The distribution of each element in space.

GA can not only extract image interest points, but also be extended to three-dimensional space to extract spatio-temporal interest points (STIPs). Spatio-temporal interest point (STIP) is one type of local invariant feature for video and can be detected directly from video to describe moving objects, without the need for background modeling and foreground segmentation. Extracting STIPs often is an important step in action recognition. Wang *et al.* [26] incorporate the GA theory into the traditional Harris 3D algorithm and Gaussian pyramid framework and propose GA-STIP method which can obtain the scale space of multi-channel videos by GA representation. GA-STIP proposes a novel 3-dimensional Gaussian function in GA space as following.

$$G_{n+3}(x,y,t;\sigma,\tau)=\frac{1}{(2\pi)^{\frac{3}{2}}\sigma^2\tau}\exp\left(-\frac{x^2+y^2}{2\sigma^2}-\frac{t^2}{2\tau^2}\right) \tag{28}$$

In $G_{n+3}$, the Gaussian convolution kernel from (28) is described as FIGURE 1, where $g_{ijk}=\sum_{l=1}^{n}g_{ijkl}e_l$ and $g_{ijkl}\in R$, and $i,j,k$ denote the 3-dimension coordinate of spatial and temporal information, l denotes the corresponding channel. GA-STIP can extracted more robust spatio-temporal interest points by using spectral information in GA space. GA-STIP has improved performance in extracting features from a multi-channel video and can recognize complicated human activities.

## *D. APPLICATIONS TO FILTERING ALGORITHMS*

With the development of adaptive filters, many adaptive filtering algorithms are applied to signal processing domain in the past few years. However, the performance of existing filtering algorithm will decrease in the non-gaussian noise environment. Thus, adaptive filtering algorithms based on higher-order statistics is proposed in signal processing domain, such as the Least-Mean-Kurtosis (LMK) algorithm, which is first proposed by Tanrikulu and Constantinides [27]. It can be seen from the weight updating rule that the computational complexity of LMK is higher than Least-Mean-Square (LMS) algorithms, which is the reason why less attention is paid to the LMK algorithm. When dealing with the 3D and 4D signal, LMS and LMK algorithms process each dimension of signals separately, which may lead to performance loss due to neglecting the correlation between different components. Fortunately, representing 3D and 4D signal as a quaternion variable can solve such problem efficiently.

Took and Mandic [28] propose the quaternion LMS (QLMS) algorithm and augment QLMS algorithm (AQLMS) for adaptive filtering of 3D and 4D signal, such as wind, vector fields modeling. Actually, QLMS operates inherently based on the "augmented" statistics, i.e. both the covariance $E\left\{xx^T\right\}$ and pseudo-covariance $E\left\{xx^H\right\}$ of the tap input vector x are considered. However, the cost function of quaternion variables is a non-analytic function, thus the derivative of the cost function cannot be calculated directly in the quaternion domain. [28] uses pseudo-derivatives to address this problem which define the real-valued cost function by four real-valued components of quaternion variables, and then take the real-valued derivatives separately. The updating rule of QLMS algorithm obtained by pseudo-derivatives are described as follows:

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mu e(n)x^*(n) \tag{29}$$

This approach is a real-valued analytic mapping of cost function between quaternion and real-number domain. However, reformulating the cost function in real-valued domain is not conductive to the calculation of their gradients. Therefore, Mandic *et al.* [29] introduce the LMS algorithm based on the Hamilton-real (HR) calculus, which use the HR calculus to calculate derivatives of cost function directly in quaternion domain. According to HR calculus, the maximum change direction of the gradient is the conjugate gradient, which is consistent with the corresponding solution in the complex domain. Thus, HR calculus can be seen as an extension of the complex-real calculus (CR calculus) used in the complex domain to quaternion domain. The HR calculus in quaternion domain is given respectively as:

$$\begin{aligned}\frac{\partial f(q,q^i,q^j,q^k)}{\partial q} &= \frac{1}{4}[\frac{\partial f}{\partial q_a} - i\frac{\partial f}{\partial q_b} - j\frac{\partial f}{\partial q_c} - k\frac{\partial f}{\partial q_d}]\\ \frac{\partial f(q^*,q^{i*},q^{j*},q^{k*})}{\partial q} &= \frac{1}{4}[\frac{\partial f}{\partial q_a} + i\frac{\partial f}{\partial q_b} + j\frac{\partial f}{\partial q_c} + k\frac{\partial f}{\partial q_d}]\end{aligned} \tag{30}$$

where $f(\cdot)$ is a quaternion-valued function and $q$ is a quaternion variable, $q^i$,$q^j$,$q^k$ are the involutions of $q$ and $q^*$ is the quaternion conjugate.

The form of the weight updating rule of the QLMS algorithm based on the HR calculus is the same as that of [28]. This approach reduces the computational complexity with respect to the QLMS algorithm proposed in [28]. Nevertheless, the disadvantage of this method is that the traditional product and chain rules cannot be used due to the non-commutativity of quaternion variables. Besides, Took *et al.* [30] aim to address the uniqueness of the solutions to the stochastic gradient optimization problems, and provide an unified framework for the derivation and analysis of QLMS algorithm. [30] defines a new quaternion gradient ($i$-gradient) based on involutions and uses it to derive the updating rule of QLMS algorithm. $I$-gradient can be expressed as

$$\nabla_{\mathbf{w}^\eta} f(q,q^i,q^j,q^k) = \frac{\partial f}{\partial q^*} + \frac{1}{2}\frac{\partial f}{\partial q^a} \tag{31}$$

The additional term $\frac{\partial f}{\partial q^a}$ makes the convergence rate of the QLMS algorithm based on $I$ gradient faster than that of [28], [29]. This approach uses the HR calculus to calculate the gradient, the traditional product and chain rules also cannot be used. Thus, Xu *et al.* [31] use the generalized HR (GHR) calculus to derive the updating rule in quaternion domain and propose a new QLMS algorithm which is based on the GHR calculus. The GHR calculus adds the product and chain rules to HR calculus, which is advantage to quaternion analysis. Actually, GHR derivatives have the left GHR derivatives and the right GHR derivatives, [31] focuses on the left GHR derivatives. Experiment results show that the performance of the proposed QLMS algorithm based on the GHR calculus outperforms other existing QLMS algorithm. Took and Mandic [32] propose a widely linear QLMS (WL-QLMS) method to process Q-proper and Q-improper signals, which improves accuracies compared to the QLMS class of algorithms. Due to QLMS originates from the LMS algorithm, which may result in performance degradation in the non-Gaussian environment. Thus, LMK algorithm is extended to quaternion domain because of its excellent performance in non-Gaussian environment in order to enhance the performance of adaptive filters.

Chen *et al.* [33] propose QLMK to process 3D and 4D signals. The cost function of QLMK algorithm is defined by the negative kurtosis of the error signal in order to adapt the non-Gaussian data. The analysis shows that QLMK provides a new way that is responsive to dynamically changing environments. QLMK has a faster convergence rate and a smaller steady-state error compared with QLMS algorithm because its cost function is defined by high-order statistics. The weight updating rule of the QLMK algorithm is represented as follows:

$$\begin{aligned}\mathbf{w}(n+1) &= \mathbf{w}(n) + \mu\frac{\partial J_{QLMK}}{\partial \mathbf{w}}\\ &= \mathbf{w}(n) + 8\mu(|e(n)|^2 - 3E(|e(n)|^2))\,|e(n)|\,x^*(n)\end{aligned} \tag{32}$$

where $E(|e(n)|^2) = \beta E(|e(n-1)|^2) + |e(n)|^2$ , $0 < \beta < 1$

However, the computational complexity of this approach is high because it is also needed to map the cost function from quaternion to real-number domain and the cost function of QLMK algorithm is complex. Thus, in [34], they propose a novel QLMK adaptive filtering algorithm for 3D and 4D signal processes by using the GHR calculus. The weight updating rule are as follows:

$$\begin{aligned}\mathbf{w}(k+1) &= \mathbf{w}_k - \mu_s(\nabla_{\mathbf{w}} * J_k) \\ &= \mathbf{w}_k + \mu_s(3\hat{p}_k - e_k e_k^*)e_k x_k^* \end{aligned} \tag{33}$$

where the step size $\mu_s$ controls the convergence rate and the steady-state error of the QLMK algorithm and $\hat{p}_k = \lambda\hat{p}_{k-1} + e_k e_k^*$, $0 < \lambda < 1$, $\lambda$ is the forgetting factor.

The proposed QLMK algorithm minimizes the negated kurtosis of the error signal as a cost function in the quaternion domain, thus it solves the trade-off problem between the convergence rate and steady-state error of QLMS-type algorithm. Moreover, the proposed QLMK algorithm has a robust behavior for a wide range of noise signals due to its kurtosis-based cost function. The GHR calculus can reduce the computational complexity when compared with [33]. Due to the multi-dimensional signal cannot be represented by quaternion variables, some scholars propose to extend the adaptive filtering algorithm to GA space.

Lopes *et al.* [35] exploit GA theory to design a new adaptive filtering strategy and apply it to rotation estimation problem. For the Least-Squares cost function, the gradient is calculated according to geometric calculus (GC), which is the extension of GA to handle differential calculus.

The novel GA Least-Mean-Squares (GA-LMS) adaptive filtering algorithm, which retains characteristics of the standard adaptive filters and GA, and is developed to recursively estimate multivector. The multivector is a hypercomplex quantity to describe rotations in any dimension.

Besides, Al-Nuaimi *et al.* [36] exploit GA-LMS to recover the 6-degree-of-freedom alignment of two-point clouds related by a set of point correspondences. However, both [35] and [36] use GA-LMS algorithm for rotation estimation problem. Thus, Lopes and Lopes [35] present a new class of adaptive filters, namely GA adaptive filters (GAAFs). They are generated by formulating the minimization problem (a deterministic cost function) from the perspective of GA. The instantaneous cost function $J(i)$ is defined as follows:

$$J(w_{i-1}) = |D(i) - u_i * w_{i-1}|^2 = |E(i)|^2 \tag{34}$$

where $u_i$ is the input array, $D(i)$ is the desired signal, the scalar product between two multivectors is $u_i * w_{i-1} = \langle u_i w_{i-1}\rangle$, it is the scalar part (0-grade) of the geometric multiplication between $u_i$ and $w_{i-1}$.

According to GA theory and GC, the new GA-LMS updating rule is:

$$w_i = w_{i-1} + \mu u_i e(i) \tag{35}$$

The obtained updating rule is shown how to recover the following LMS adaptive filter variants: real-entries LMS, complex LMS, and quaternions LMS. The development of GAAFs is an attempt to unify different adaptive filtering methods under the same mathematical language. The GA-LMS algorithm can be used to estimate any kind of multivector, and the shape of its updating rule is invariant. However, GA-LMS also originates from the LMS algorithm, and its performance will also degradation in non-Gaussian environment. Thus, the LMK algorithm is extend to GA space in order to enhance the performance of adaptive filters for multi-dimensional signals processing.

Wang *et al.* [37] propose a novel Least-Mean-Kurtosis adaptive filtering algorithm based on geometric algebra (GA-LMK) which represents the multi-dimensional signal as a GA multivector. The cost function of GA-LMK algorithm is:

$$J(w_{i-1}) = 3\mathbb{E}^2|E(i)|^2 - \mathbb{E}|E(i)|^4 \tag{36}$$

where $E(i)$ is the error signal and $\mathbb{E}$ is the expectation operation.

The GA-LMK updating rule is given by:

$$\begin{aligned} w_i &= w_{i-1} - \mu[\partial_w J(w_{i-1})] \\ &= w_{i-1} + \mu h_i(3\hat{P}_i E(i) - E(i) * \widetilde{E}(i) * E(i)) \end{aligned} \tag{37}$$

where$\hat{P}_i = \mathbb{E}\left\{|E(i)|^2\right\}$ and it can be estimated by $\hat{P}_i = \beta\hat{P}_{i-1} + E(i) * \tilde{E}(i)$, $0 < \beta < 1$, $\beta$ is the forgetting factor.

The proposed GA-LMK algorithm minimizes the cost function based on the negated kurtosis of the error signal in GA space, and solves the trade-off problem between convergence rate and steady-state error better. Meanwhile, the misadjustment conditions of the GA-LMK algorithm under Gaussian noises is acquired, so as to better understanding the GA-LMK algorithm. The results show that the proposed GA-LMK algorithm outperform existing adaptive filtering algorithms in terms of multi-dimensional signal. GA-LMK algorithm has a faster convergence rate and a smaller steady-state error due to its cost function is defined by high-order statistics when compared with GA-LMS algorithm. Therefore, the performance of adaptive filter is greatly improved when processing multi-dimensional signals.

### *E. GEOMETRIC ALGEBRA FOURIER TRANSFORMATIONS*

The Fourier transform is an indispensable tool for many fields of physics, mathematics and computer science. In particular, image processing and solution and analysis of differential equations or signal cannot be imagined without Fourier transform any more. Based on GA, each multivector has a natural geometric interpretation. Therefore, it is very reasonable for multi-dimensional signals to generalize the Fourier transform to multivector valued functions in GA.

Different definitions of Fourier transform in GA have been developed. In the field of signal processing, a strategy is the usage of Clifford Fourier transform (CFT) for detecting and filtering particular components of signals, which is proposed by Jancewicz [38], a color image is considered as a function $f \in L^2\left(\mathbb{R}^2; \mathbb{R}^1_{4,0}\right)$ described by:

$$f(x, y) = r(x, y)e_1 + g(x, y)e_2 + b(x, y)e_3 + 0e_4 \tag{38}$$

where $r$, $g$ and $b$ correspond to the red, green and blue levels.

Let $f$ be a color image and B be an element of $\mathbb{S}^2_{3,0}$. The Clifford Fourier transform of $f$ with respect to $B$ is the valued function is described as:

$$\hat{f}_b(u,v) = \int_{\mathbb{R}^2} f(x,y) \perp \tilde{\phi}_{u,v,B,0,0,C}(-x,-y)dxdy \quad (39)$$

The concept of CFT provides a powerful framework used in image processing and analysis fields and can transforms vectors into general GA multivectors. The readers can find a detailed overview of the modern development of CFT in [39].

Sommer [2] consider study harmonic transforms, QFT and a type of CFT; B*ü*low and Sommer [2] consider study harmonic transforms, the quaternionic Fourier transform (QFT), and a type of CFT.

Actually, several attempts have been made to extend the classical approach of image processing with the Fourier transform to multi-dimensional signals. Batard and Berthier [40] introduce a spinor representation for images processing, focusing on segmentation and Clifford Fourier analysis. The grayscale image is processed by using the geometric information given by the Gauss map variations of the image. One main contribution is that the Riemannian geometry of the image surface is considered by this new transform, involving the bivector field coding the tangent plane and the spinor field that parametrizes the normal. Furthermore, a harmonic decomposition of the parametrization and applications to filtering are introduced. Mennesson *et al.* [41] propose a new set of Fourier-Mellin descriptors using CFT for color images, namely parallel-orthogonal Fourier-Mellin Descriptors (poFMD), Color Fourier-Mellin Descriptors (CFMD) and Color Fourier-Mellin Moment invariant (CFMDi), which are different extensions of the FourierMellin moments [42] computed from the CFT for grayscale images. These are invariant an dare avoided in direct similarity transformations (translation, rotation, scaling) and marginal treatment of color images, as a result, Their retrieval rates of proposed method is favourably compared to standard feature descriptors, testing with the purpose of object recognition on well-known color image databases.

Batard and Berthier [43] propose a new spinor Fourier transform for both gray-level and color image processing, which treats all the colorimetric information in a really non marginal way for color image processing. The construction involves group actions via spin characters, which are being parametrized by bivectors of the GA. Meanwhile, applications to low-pass filtering interpreted as diffusion process with heat equation are proposed. Yuan *et al.* [44] propose a template matching method which is constructed on the foundation of CFT, this method is applied to similar domain to extract the template forced spatial distribution pattern. Through the experiments test with ENSO forced global ocean surface wind segmentation, the results suggest that this algorithm is able to extract more attractive information which cannot be measured directly from the original data.

However, CFT has high computational complexity, especially for big data. To solve the problem with big data, Hassanieh *et al.* [45] develop a new approach of sparse fast Fourier transform (sFFT), especially sparse data. But when dealing with big data problems, this approach uses only a small subset of the input data to calculate a compressed Fourier transform. Therefore, a new algorithm called sparse fast CFT (SFCFT) is created by Wang *et al.* [46], combining sFFT and CFT to operate on multi-dimensional signals, which has advantages of the computing performance in scalar and vector fields. This method can not only can choose ''large'' coefficients for calculation with sFFT, but also can reduce useless data.

### *F. IMAGE EDGE DETECTION*

In this section, edge detection of color images and multispectral images are reviewed. Edge detection of color images is usually performed by applying the traditional techniques for grayscale images to the three color channels separately. An overview of edge detection techniques for color images is provided in [47].

Several edge detection methods treat RGB color triples as vectors and color images as vector fields, for instance, Schlemmer *et al.* [49] propose a novel approach based on Sobel operator and use vector value filter masks to detect color edges. Color value triples are converted in luminance and chrominance components. The chrominance part is detected by Clifford convolution with proper vector-value filters and a standard grayscale edge detection method is applied to the luminance component.

Then, an enhanced algorithm has been proposed in the Clifford algebra framework to extend the traditional concepts of convolution and Fourier transform to vector fields [49], which uses Canny method for edge detection on the grayscale component. Moreover, a hysteresis thresholding is exploited for both grayscale and color components. Compared to component-wise Canny color edge detector [50], the proposed Clifford color edge detector algorithm can not only achieves a comparable detection performance, but also reduces computational times. To achieve further speedup against the state-of-the-art color edge detection schemes. Franchini *et al.* [52] propose a hardware implementation of an edge detection method for color images, which define geometric product of vectors under the Clifford algebra framework to extend the convolution operator and the Fourier transform to vector fields. A prototype implementation of the specialized hardware structure on a field programmable gate array (FPGA) board has been introduced. And the proposed hardware architecture has been successfully applied for edge detection of multispectral magnetic resonance images.

Multispectral images such as medical images or remote sensing images, usually contain hundreds of spectral channels of the same scene, which provide a great deal of information. Of particular importance is the computationally efficient dealing with multispectral images formed by multiple spectral bands. In many cases, the applications of the multispectral image are based on edge detection to extract information, which is the most difficult process in image processing.

Therefore, the quality of the application depends on the accuracy of edge detection. Generally, most multispectral image edge detection algorithms are based on vector algebra [52], [53].

Because of the shortages of vector algebra which cannot completely express the correlation among the different layers of the multispectral image. As a consequence, such a strategy of edge detection algorithms cannot completely fully utilize the information associated with different spectrum layers. Xu *et al.* [55], Hui *et al.* [56] present a novel algorithm of multispectral image edge detection, which defines the pixel as a GA value rather than putting a pixel as an n-dimensional vector. Suppose multispectral image data is a matrix $IMG[n, w, h]$ where $n$ is the number of spectra, $w$ is the width and $h$ is the height. They define the spectral gradient $m$Clifford$(X)$ of a pixel $X$ as the following form:

$$\begin{aligned}\partial_X f(X) &= E^i \partial_{x^i} f(X) = E^i f_i(X) \\ &= e^1 f_1(X) + e^2 f_2(X) + \cdots \\ &\quad + e^n f_n(X) + e^1 \wedge e^2 f_{1,2}(X) \\ &\quad + \cdots + e^1 \wedge e^2 \cdots \wedge e^n f_{1,2,\ldots,n}(X) \end{aligned} \tag{40}$$

The derivative of $f(X)$ in the "direction" of a multivector $A \in G_n$ with $A := a^i E_i$, is given by

$$(A * \partial_X) f(X) = A * (\partial_X f(X)) = A * (E^i f_i(X)) = a^i f_i(X) \tag{41}$$

where $A * \partial_X$ represents the directed multivector derivative operator.

Compared with the maximal entropy edge detection algorithm, the proposed edge detection algorithm based on GA shows excellent performance at retaining and identifying edge information of a multispectral image and exhibits the possible application to help doctors make effective diagnose.

## IV. SUMMARY AND CHALLENGES

### A. SUMMARY

As an important tool for signal and image processing, GA is suitable for representation and computation of different geometric data, by either constructing geometric structure in multi-dimensional space, or extending a set of existing algorithms for multi-dimensional signals. Here, we place the focus on GA and review a wide variety of applications based on GA in signal and image processing including, the sparse representation models, Support Vector Machines (SVM), feature extraction algorithm, filtering algorithms, Fourier-type transform and image edge detection.

We begin this survey by introducing the basis of GA: the definition and properties of GA. These approaches can be very effective in solving the multi-dimensional signal problem with different geometric frameworks in a unified way. Next, the applications of GA are presented in detail:

(1) The sparse representation models based on GA are introduced, which attract wide interests and achieve great success in a wide range of image processing fields. We show that GA improve the effectiveness of SVD.

(2) Furthermore, SVM is introduced by using GA framework for various tasks. GA-based SVM models are suitable for processing multiclass SVM, using geometric product and redefining the optimization variables as multivectors. It has shown that GA-based SVM models for classification have better performance compared with traditional SVM methods.

(3) The feature extraction algorithms are discussed, GA-based feature extraction algorithms are powerful for signal and image processing, including color images, multispectral images and video. GA-based feature extraction algorithms use GA theory to detect and describe the local interest points in multiple channels. The results show that GA-based feature extraction algorithms show potential performance compared with state-of-the-art algorithms while not sacrificing performance.

(4) The adaptive filtering algorithms are represented, extending from quaternion to higher dimension. GA-based filtering algorithms represent a multi-dimensional signal as a GA multivector, the results show that proposed GA-based filtering algorithms can outperform significantly existing state-of-the art algorithms in terms of convergence rate and steady-state error.

(5) It is important for multi-dimensional signals to generalize the Fourier transform to multivector valued functions in GA, which is an extension of Fourier transform to vector fields based on Clifford product of multivectors. Meanwhile, the GA-based Fourier transform algorithms have a great influence on signal and image processing.

(6) Finally, GA is applied to image edge detection, which exhibits an excellent performance on image processing. GA-based image edge detection algorithms are better at preserving and identifying edge information of images.

### B. CHALLENGES

Future research will include:

(1) The construction of appropriate GA space aims at specific multi-dimensional signals and image processing. In different cases, the corresponding GA models are built for processing different inputs. However, recently, the architecture of the models based on GA are designed relatively simple, which cannot satisfy the various inputs in real world.

(2) The cost of time and space of the GA-based algorithms are more than the traditional algorithms, such as GA-based feature extraction algorithm, GA-based image edge detection, etc. The reason is that GA-based algorithms have some multi-cycles with non-commutative multiplications.

(3) Investigate various applications of signal and image processing based on GA. As expected, GA is potential in multi-dimensional domain, many applications of GA still need to be explored.

(4) How to derive algorithms based on analytical demands using the basic GA operators. The generalization of traditional operators into GA space is a complicated work, which needs to select proper function or derive specific algorithms.

## V. PROSPECTS

In the field of image and signal processing, GA has attracted many researchers' attention. However, there are many challenges mentioned above needed to be solved. Therefore, GA will still be a hot research topic in the future, and there are some research prospects as follows.

(1) Most of the existing algorithms in signal and image processing, split the correlation between the spatial domain and the temporal domain, ignoring the important structural correlation. It has shown that the GA can achieve better performance, preserving more correlation information among multi-dimensional signals and images. Therefore, in order to deal with various signals and images, it is significant to design a framework and application platform combining the theoretical GA framework, expression and analysis of multi-dimensional signals, and multi-dimensional image models.

(2) Signal and image processing based on GA has been wildly applied in the multi-dimensional space. These applications are required to reduce computational complexity, so the efficiency of GA-based algorithms must be improved to achieve better performance with less time consumption and less space complexity. The future research can combine with parallel computing to improve the efficiency of computation for multi-dimensional signals and image processing.

(3) Future works should pay more attention on enhancing the adaptability of GA-based algorithms and considering more extensive applications for signal and image processing.

(4) Currently, how to derive GA-based algorithms depend on analytical demands using basic GA operators. In the future, different GA operators should be unified in one framework, unifying existing GA variables and operations.

## VI. CONCLUSION

In conclusion, this paper presents a comprehensive review on GA in signal and image processing. The application of GA has been summarized, with the analysis of their advantages and shortcomings. Also, the challenges and prospects of various applications proposed by many researchers have been given. With continuous developments in computer vision field and the improvement of computer hardware performance, we believe that the existing problems can be solved step by step via GA-based algorithms. And it is expected that the GA-based theories and algorithms will be a competitive alternative in applications of signal and image processing.

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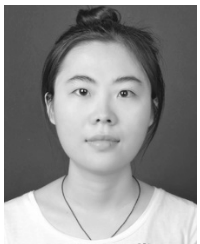

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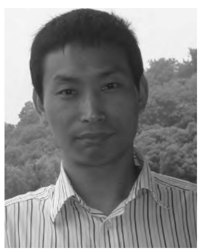

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