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Existence and Global Exponential Stability of Pseudo Almost Periodic Solution for Clifford-Valued Neutral High-Order Hopfield Neural Networks With Leakage Delays

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ABSTRACT In this paper, we are concerned with a class of Clifford-valued neutral high-order Hopfield neural networks with leakage delays. Although the multiplication of Clifford numbers does not satisfy the commutativity, which brings great difficulties to the study of Clifford-valued systems, we have found a method that does not decompose Clifford-valued systems into real-valued systems to study the existence and global exponential stability of pseudo-almost periodic solutions of this class of neural networks. Our results are completely new. Finally, two examples are given to illustrate the effectiveness and feasibility of our main results.

INDEX TERMS Clifford-valued neural network, neutral high-order Hopfield neural network, pseudo almost periodic solution, global exponential stability, leakage delay.

I. INTRODUCTION

It is well known that time delays always exist in real systems. Therefore, in the past decades, neural networks with various types of time delays and their dynamic characteristics have been extensively studied, and many excellent results related to the subject have been obtained. It is worth mentioning that high-order Hopfield neural networks have better performance than low-order Hopfield neural networks in approximation, convergence speed, storage capacity, fault tolerance and so on. Therefore, they have attracted more and more attention. Their dynamics have been extensively studied [1]–[12].

On the one hand, in recent years, it has been found that complex-valued and quaternion-valued neural networks have more advantages than real-valued neural networks in some practical applications. Therefore, there have been many studies on complex-valued and quaternion-valued neural networks [13]–[26]. Clifford algebra is a generalization of real number, complex number and quaternion, which has important and extensive application fields. In particular, in recent years, Clifford-valued neural networks have been found to be superior to real-valued, complex-valued and quaternion-valued neural networks in dealing with high-dimensional spatial data and problems involving spatial transformation [27]. Therefore, the research of Clifford-valued neural networks has become a hot topic. However, due to the noncommutativity of the Clifford numbers' multiplication, the known results on Clifford-valued neural networks are very few [12], [28]–[31]. In particular, there are fewer results about their dynamics. Moreover, most of the existing results are obtained by decomposing Clifford-valued neural networks into real-valued neural networks. However, the results obtained by decomposition method often need to know the explicit expressions of coefficients and activation functions in the network, which brings many difficulties to practical application. Therefore, it is of great theoretical and practical significance to study the dynamics of Clifford-valued neural networks and to explore non-decomposition methods to study Clifford-valued neural networks. At the same time, it is a challenging job.

On the other hand, for autonomous neural networks, the existence and stability of equilibrium points are very

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important dynamics. For non-autonomous systems, the existence and stability of periodic solutions, almost periodic solutions and almost automorphic solutions are as important as the existence and stability of the equilibrium points of autonomous neural networks. At present, there is no direct method to study the almost periodicity of Clifford-valued neutral high-order Hopfield neural networks. In addition, pseudo almost periodicity is a nature generalization of almost periodicity. Besides, as we all know, both the leakage delays and neutral type terms in a system may change the dynamic characteristics of the system [7], [12], [25], [32]–[36], so the main purpose of this paper is to study the existence and global exponential stability of almost periodic solutions for a class of Clifford-valued neutral high-order Hopfield neural networks with leakage delays by direct method. Our results obtained in this paper are new and our methods proposed in this paper can be used to study the problem of almost periodic solutions and almost automorphic solutions for other types of Clifford-valued neural networks.

This paper is organized as follows. In Section 2, we introduce some definitions, lemmas and give a model description. In Section 3, we study the existence of pseudo almost periodic solutions. In Section 4, we discuss the global exponential stability of pseudo almost periodic solutions. In Section 5, an example is given to demonstrate the proposed results. A brief conclusion is drawn in Section 6.

II. PRELIMINARIES AND MODEL DESCRIPTION

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \Big\{ \sum_{A\subseteq\{1,2,\ldots,m\}} a^A e_A, a_A \in \mathbb{R} \Big\},$$

where $e_A = e_{h_1}e_{h_2}\cdots e_{h_\nu}$ with $A = h_1h_2\cdots h_\nu$, $1 \le h_1 < h_2 < \cdots < h_\nu \le m$. Moreover, $e_\emptyset = e_0 = 1$ and e_h, $h = 1, 2, \ldots, m$ are said to be Clifford generators and satisfy $e_p^2 = 1, p = 1, 2, \ldots, s$, $e_p^2 = -1, p = s+1, s+2, \ldots, m$, where $s < m$, and $e_pe_q + e_qe_p = 0$, $p \ne q$, $p, q = 1, 2, \ldots, m$. Let $\Pi = \{\emptyset, 1, 2, \ldots, A, \ldots, 12\cdots m\}$, then it is easy to see that $\mathcal{A} = \big\{\sum_A a^A e_A, a_A \in \mathbb{R}\big\}$, where $\sum_A$ is short for $\sum_{A\in\Pi}$ and $\dim \mathcal{A} = 2^m$.

For $x = \sum_A x^A e_A \in \mathcal{A}$, we define $\|x\|_{\mathcal{A}} = \max_A\{|x^A|\}$ and for $x = (x_1, x_2, \ldots, x_n)^T \in \mathcal{A}^n$, we define $\|x\|_{\mathcal{A}^n} = \max_{1\le p\le n}\{\|x_p\|_{\mathcal{A}}\}$.

The derivative of $x(t) = \sum_A x^A(t)e_A$ is given by $\dot{x}(t) = \sum_A \dot{x}^A(t)e_A$. For more knowledge about Clifford algebra, we refer the reader to [37].

In this paper, we are concerned with the following Clifford-valued neutral high-order Hopfield neural network with delays in the leakage term:

$$\begin{aligned}\dot{x}_i(t) = &-a_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t))\\ &+\sum_{j=1}^n c_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}(t)\\ &\times h_j(x_j(t-\sigma_{ijk}(t)))h_k(x_k(t-\nu_{ijk}(t)))\\ &+\sum_{j=1}^n e_{ij}(t)I_j(\dot{x}_j(t-\mu_{ij}(t))) + J_i(t), \qquad (1)\end{aligned}$$

where $i = 1, 2, \ldots, n$, $x_i(t) \in \mathcal{A}$ corresponds to the state of the ith unit at time t, $a_i(t) \in \mathbb{R}^+$ represents the rate with which the ith unit will reset its potential to the resting state when disconnected from the network and external inputs at time t, $b_{ij}(t) \in \mathcal{A}$ denotes the strength of the jth unit on the ith unit at time t, $d_{ijk}(t) \in \mathcal{A}$ is the second-order synaptic weight of the neural network, $e_{ij}(t) \in \mathcal{A}$ represents the neutral delayed strength of connectivity between cells i and j at time t, $f_j(t), h_j(t), I_j(t) \in \mathcal{A}$ denote activation functions, $J_i(t) \in \mathcal{A}$ is the external input at time t, $\eta_i(t), \sigma_{ijk}(t), \nu_{ijk}(t), \mu_{ij}(t) \in \mathbb{R}^+$ denote the transmission delays.

We will adopt the following notation:

$$\begin{aligned}a_i^- &= \inf_{t\in\mathbb{R}} a_i(t),\ b_{ij}^+ = \sup_{t\in\mathbb{R}}\|b_{ij}(t)\|_{\mathcal{A}},\\ d_{ijk}^+ &= \sup_{t\in\mathbb{R}}\|d_{ijk}(t)\|_{\mathcal{A}},\ e_{ij}^+ = \sup_{t\in\mathbb{R}}\|e_{ij}(t)\|_{\mathcal{A}},\\ \eta_i^+ &= \sup_{t\in\mathbb{R}}\eta_i(t),\ \sigma_{ijk}^+ = \sup_{t\in\mathbb{R}}\sigma_{ijk}(t),\ \nu_{ijk}^+ = \sup_{t\in\mathbb{R}}\sigma_{ijk}(t),\\ \eta^+ &= \max_{1\le i\le n}\{\eta_i^+\},\ \dot{\eta}^+ = \max_{1\le i\le n}\sup_{t\in\mathbb{R}}\dot{\eta}_i(t),\ \mu_{ij}^+ = \sup_{t\in\mathbb{R}}\mu_{ij}(t),\\ \dot{\xi}^+ &= \max\{\sup_{t\in\mathbb{R}}\dot{\eta}_i(t), \sup_{t\in\mathbb{R}}\dot{\sigma}_{ijk}(t), \sup_{t\in\mathbb{R}}\dot{\nu}_{ijk}(t), \sup_{t\in\mathbb{R}}\dot{\mu}_{ij}(t)\}.\end{aligned}$$

The initial value of system (1) is given by

$$x_i(s) = \varphi_i(s) \in \mathcal{A},\quad \dot{x}_i(s) = \dot{\varphi}_i(s) \in \mathcal{A},\ s \in [-\xi, 0],$$

where $\varphi_i \in C^1([-\xi, 0], \mathcal{A})$ and

$$\xi = \max_{1\le i,j\le n}\Big\{\sup_{t\in\mathbb{R}}\{\eta_i(t)\}, \sup_{t\in\mathbb{R}}\{\sigma_{ijk}(t)\}, \sup_{t\in\mathbb{R}}\{\nu_{ijk}(t)\}, \sup_{t\in\mathbb{R}}\{\mu_{ij}(t)\}\Big\}.$$

Throughout the rest of this paper, $BC(\mathbb{R}, \mathcal{A}^n)$ denotes the set of all bounded continuous functions from $\mathbb{R}$ to $\mathcal{A}^n$. Note that $(BC(\mathbb{R}, \mathcal{A}^n), \|\cdot\|_0)$ is a Banach space with the norm

$$\|f\|_0 = \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\|f_i(t)\|_{\mathcal{A}}\Big\},$$

where $f = (f_1, f_2, \ldots, f_i)^T \in BC(\mathbb{R}, \mathcal{A}^n)$.

Definition 1: A function $f \in BC(\mathbb{R}, \mathcal{A}^n)$ is said to be almost periodic, if for every $\varepsilon > 0$, it is possible to find a real number $k = k(\varepsilon) > 0$, for each interval with length $k(\varepsilon)$, there is a number $\tau = \tau(\varepsilon)$ in this interval such that $\|f(t+\tau) - f(t)\|_{\mathcal{A}^n} < \varepsilon$ for all $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AP(\mathbb{R}, \mathcal{A}^n)$.

Let

$$PAP_0(\mathbb{R}, \mathcal{A}^n) = \Big\{f \in BC(\mathbb{R}, \mathcal{A}^n)\Big|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^T\|f(t)\|_{\mathcal{A}}dt = 0\Big\}.$$

Definition 2: A function $f \in BC(\mathbb{R}, \mathcal{A}^n)$ is said to be pseudo almost periodic if it can be expressed as $f = f_1 + f_0$, where $f_1 \in AP(\mathbb{R}, \mathcal{A}^n)$ and $f_0 \in PAP_0(\mathbb{R}, \mathcal{A}^n)$. The collection of all such functions will be denoted by $PAP(\mathbb{R}, \mathcal{A}^n)$.

From the above definition, similar to the proofs of the corresponding results in [38], one can prove the following two lemmas.

Lemma 3: If $\alpha \in \mathbb{R}, f, g \in PAP(\mathbb{R}, \mathcal{A}^n)$, then αf, $f + g, fg \in PAP(\mathbb{R}, \mathcal{A}^n)$.

Lemma 4: Let $f \in C(\mathcal{A}, \mathcal{A}^n)$ satisfy the Lipschitz condition and $\varphi \in PAP(\mathbb{R}, \mathcal{A})$, then $f(\varphi(\cdot)) \in PAP(\mathbb{R}, \mathcal{A}^n)$.

Lemma 5: If $x \in PAP(\mathbb{R}, \mathcal{A}^n)$, $\nu \in AP(\mathbb{R}, \mathbb{R}) \cap C^1(\mathbb{R}, \mathbb{R})$ with $|\nu(t)| \le \nu^+$ and $\dot{\nu}(t) \le \dot{\nu}^+ < 1$, then $x(\cdot - \nu(\cdot)) \in PAP(\mathbb{R}, \mathcal{A}^n)$.

Proof: Since $x \in PAP(\mathbb{R}, \mathcal{A}^n)$, we can write $x = x_1 + x_0$, where $x_1 \in AP(\mathbb{R}, \mathcal{A}^n)$ and $x_0 \in PAP_0(\mathbb{R}, \mathcal{A}^n)$. Consequently, we have

$$x(t - \nu(t)) = x_1(t - \nu(t)) + x_0(t - \nu(t)).$$

From $x_1(\cdot - \nu(\cdot)) \in AP(\mathbb{R}, \mathcal{A}^n)$ it follows that x_1 is uniformly continuous. Therefore, for each $\varepsilon > 0$, there exists a positive constant $\varsigma \in (0, \frac{\varepsilon}{2})$ such that for any $t, s \in \mathbb{R}$ with $|t-s| < \varsigma$,

$$\|x_1(t) - x_1(s)\|_{\mathcal{A}^n} < \frac{\varepsilon}{2}. \tag{2}$$

Since ν and x_1 are almost periodic, for this $\varsigma > 0$, there exists a $k(\varsigma) > 0$ such that in every interval with length $k(\varsigma)$, there is a δ satisfying

$$|\nu(t+\delta) - \nu(t)| < \varsigma, \quad \|x_1(t+\delta) - x_1(t)\|_{\mathcal{A}^n} < \varsigma, \tag{3}$$

for all $t \in \mathbb{R}$. It follows from (2) and (3) that

$$\begin{aligned}
&\|x_1(t+\delta - \nu(t+\delta)) - x_1(t - \nu(t))\|_{\mathcal{A}^n} \\
&\quad \le \|x_1(t+\delta - \nu(t+\delta)) - x_1(t+\delta - \nu(t))\|_{\mathcal{A}^n} \\
&\qquad + \|x_1(t+\delta - \nu(t)) - x_1(t - \nu(t))\|_{\mathcal{A}^n} \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
\end{aligned}$$

which implies that $x_1(\cdot - \nu(\cdot)) \in AP(\mathbb{R}, \mathcal{A}^n)$.

Moreover, let $s = t - \nu(t)$ and noticing that $x_0 \in PAP_0(\mathbb{R}, \mathcal{A}^n) \subset BC(\mathbb{R}, \mathcal{A}^n)$, we find

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|x_0(t - \nu(t))\|_{\mathcal{A}^n} dt \\
&\quad \le \lim_{T\to+\infty} \frac{1}{1-\dot{\nu}^+} \frac{1}{2T} \int_{-T-\nu(-T)}^{T-\nu(T)} \|x_0(s)\|_{\mathcal{A}^n} ds \\
&\quad = \lim_{T\to+\infty} \frac{1}{1-\dot{\nu}^+} \frac{1}{2T} \left(\int_{-T-\nu(-T)}^{-T+\nu(T)} \right. \\
&\qquad \left. + \int_{-T+\nu(T)}^{T-\nu(T)} \right) \|x_0(s)\|_{\mathcal{A}^n} ds \\
&\quad = \lim_{T\to+\infty} \frac{1}{1-\dot{\nu}^+} \frac{T - \nu(T)}{T} \frac{1}{2(T-\nu(T))} \\
&\qquad \times \int_{-T+\nu(T)}^{T-\nu(T)} \|x_0(s)\|_{\mathcal{A}^n} ds = 0,
\end{aligned}$$

which implies that $x_0(\cdot - \nu(\cdot)) \in PAP_0(\mathbb{R}, \mathcal{A}^n)$. Hence, $x(\cdot - \nu(\cdot)) \in PAP(\mathbb{R}, \mathcal{A}^n)$. The proof is complete. ■

Throughout this paper, we make the following assumptions:

(H_1) For $i, j, k = 1, 2, \ldots, n$, $a_i \in AP(\mathbb{R}, \mathbb{R}^+)$, $b_{ij}, c_{ij}, d_{ijk}, e_{ij}, J_i \in PAP(\mathbb{R}, \mathcal{A})$, $\eta_i, \tau_{ij}, \sigma_{ijk}, \nu_{ijk}, \mu_{ij} \in C^1(\mathbb{R}, \mathbb{R}^+) \cap PAP(\mathbb{R}, \mathbb{R})$, $\min\limits_{1\le i\le n} \left\{ \inf\limits_{t\in\mathbb{R}} a_i(t) \right\} > 0$ and $1 - \dot{\xi}^+ > 0$.

(H_2) For $j = 1, 2, \ldots, n$, $f_j, h_j, I_j \in C(\mathcal{A}, \mathcal{A})$ and there exist positive constants $L_j^f, L_j^g, L_j^h, L_j^I, M_i^h$ such that for any $x, y \in \mathcal{A}$,

$$||f_j(x) - f_j(y)||_{\mathcal{A}} \le L_j^f ||x - y||_{\mathcal{A}},$$

$$||g_j(x) - g_j(y)||_{\mathcal{A}} \le L_j^g ||x - y||_{\mathcal{A}},$$

$$||h_j(x) - h_j(y)||_{\mathcal{A}} \le L_j^h ||x - y||_{\mathcal{A}},$$

$$||h_i(x)||_{\mathcal{A}} \le M_i^h, \ \ ||I_j(x) - I_j(y)||_{\mathcal{A}} \le L_j^I ||x - y||_{\mathcal{A}},$$

and $f_j(0) = g_j(0) = h_j(0) = I_j(0) = 0$.

(H_3) $r := \max\limits_{1\le i\le n} \left\{ \frac{1}{a_i^-} U_i, \left(1 + \frac{a_i^+}{a_i^-}\right) U_i \right\} < 1$, where $U_i = a_i^+ \eta_i^+ + \sum\limits_{j=1}^n b_{ij}^+ L_j^f + \sum\limits_{j=1}^n c_{ij}^+ L_j^g + \sum\limits_{j=1}^n \sum\limits_{k=1}^n d_{ijk}^+ (M_k^h L_j^h + M_j^h L_k^h) + \sum\limits_{j=1}^n e_{ij}^+ L_j^I$, $i = 1, 2, \ldots, n$.

III. THE EXISTENCE OF PSEUDO ALMOST PERIODIC SOLUTIONS

Let

$$\mathbb{X} = \left\{ \varphi \in C^1(\mathbb{R}, \mathcal{A}^n) : \varphi, \varphi' \in PAA(\mathbb{R}, \mathcal{A}^n) \right\}.$$

For any $\varphi \in \mathbb{X}$, we define the norm of φ as $\|\varphi\|_{\mathbb{X}} = \max\{\|\varphi\|_0, \|\varphi'\|_0\}$.

Let

$$\begin{aligned}
\varphi_0(t) = \Bigg(&\int_{-\infty}^{t} e^{-\int_s^t a_1(u)du} J_1(s) ds, \\
&\int_{-\infty}^{t} e^{-\int_s^t a_2(u)du} J_2(s) ds, \ldots, \\
&\int_{-\infty}^{t} e^{-\int_s^t a_n(u)du} J_n(s) ds \Bigg)^T
\end{aligned}$$

and take a positive constant $R > \|\varphi_0\|_{\mathbb{X}}$. Define

$$\mathbb{X}_0 = \left\{ \varphi \in \mathbb{X} : \|\varphi - \varphi_0\|_{\mathbb{X}} \le \frac{rR}{1-r} \right\}.$$

Then, for every $\varphi \in X_0$, we have

$$\|\varphi\|_{\mathbb{X}} \le \|\varphi - \varphi_0\|_{\mathbb{X}} + \|\varphi_0\|_{\mathbb{X}} \le \frac{rR}{1-r} + R = \frac{R}{1-r}.$$

Lemma 6: If $G(t) = \int_{-\infty}^t e^{-\int_s^t a(u)du} F(s) ds$, where $a \in AP(\mathbb{R}, \mathbb{R}^+)$ with $\inf\limits_{t\in R} a(t) > 0$, and $F \in PAP(\mathbb{R}, \mathcal{A})$, then $G \in PAP(\mathbb{R}, \mathcal{A})$.

Proof: Since $F \in PAP(\mathbb{R}, \mathcal{A})$, it can be expressed as $F = F^1 + F^0$, where $F^1 \in AP(\mathbb{R}, \mathcal{A})$, $F^0 \in PAP_0(\mathbb{R}, \mathcal{A})$. As a consequent,

$$G(t) = \int_{-\infty}^{t} e^{-\int_s^t a(u)du} F^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a(u)du} F^0(s)ds := G^1(t) + G^0(t).$$

First, we will prove $G^1 \in AP(\mathbb{R}, \mathcal{A})$. Since $a \in AP(\mathbb{R}, \mathbb{R})$, $F^1 \in AP(\mathbb{R}, \mathcal{A})$, for any $\varepsilon > 0$, we can find a number $l(\varepsilon)$ such that in every interval of length l there is a number τ such that

$$|a(t+\tau) - a(t)| < \varepsilon, \quad ||F^1(t+\tau) - F^1(t)||_{\mathcal{A}} < \varepsilon.$$

Hence, we have

$$\begin{aligned}
&\|G^1(t+\tau) - G^1(t)\|_{\mathcal{A}} \\
&= \int_{-\infty}^{t+\tau} e^{-\int_s^{t+\tau} a(u)du} F^1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t a(u)du} F^1(s)ds \\
&\le \int_{-\infty}^{t} e^{-\int_s^t a(u+\tau)du} \|F^1(s+\tau) - F^1(s)\|_{\mathcal{A}} ds \\
&\quad + \int_{-\infty}^{t} ||(e^{-\int_s^t a(u+\tau)du} - e^{-\int_s^t a(u)du})F^1(s)||_{\mathcal{A}} ds \\
&\le \frac{\varepsilon}{a^-} + \frac{\varepsilon}{(a^-)^2} \sup_{t\in\mathbb{R}} ||F^1(t)||_{\mathcal{A}},
\end{aligned}$$

which implies that $G^1 \in AP(\mathbb{R}, \mathcal{A})$.

Next, we will prove that $G^0 \in PAP_0(\mathbb{R}, \mathcal{A})$. Let

$$\Lambda(T) = \frac{1}{2T} \int_{-T}^{T} \left\| \int_{-\infty}^{t} e^{-\int_s^t a(u)du} F^0(s)ds \right\|_{\mathcal{A}} dt,$$

$$\Lambda_1(T) = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} \|e^{-\int_s^t a(u)du} F^0(s)\|_{\mathcal{A}} dsdt,$$

$$\Lambda_2(T) = \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{-T} \|e^{-\int_s^t a(u)du} F^0(s)\|_{\mathcal{A}} dsdt.$$

Then

$$\Lambda(T) \le \Lambda_1(T) + \Lambda_2(T).$$

By Fubini's theorem, we have

$$\begin{aligned}
&\Lambda_1(T) \\
&\le \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} e^{-(t-s)a^-} \|F^0(s)\|_{\mathcal{A}} dsdt \\
&= \frac{1}{2T} \left(\int_{-T}^{T} \int_{0}^{t+T} e^{-\zeta a^-} \|F^0(t-\zeta)\|_{\mathcal{A}} d\zeta \right) dt \\
&\le \frac{1}{2T} \left(\int_{-T}^{T} \int_{0}^{+\infty} e^{-\zeta a^-} \|F^0(t-\zeta)\|_{\mathcal{A}} d\zeta \right) dt \\
&\le \int_{0}^{+\infty} e^{-\zeta a^-} \left(\frac{T+\zeta}{T} \frac{1}{2(T+\zeta)} \int_{-T-\zeta}^{T+\zeta} \|F^0(u)\|_{\mathcal{A}} du \right) d\zeta.
\end{aligned}$$

Since $F^0 \in PAP_0(\mathbb{R}, \mathcal{A})$,

$$\lim_{T\to+\infty} \frac{1}{2(T+\zeta)} \int_{-T-\zeta}^{T+\zeta} \|F^0(u)\|_{\mathcal{A}} du = 0.$$

Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{T\to+\infty} \Lambda_1(T) = 0.$$

On the other hand, we have

$$\begin{aligned}
&\lim_{T\to+\infty} \Lambda_2(T) \\
&\le \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{-T} e^{-(t-s)a^-} \|F^0(s)\|_{\mathcal{A}} dsdt \\
&= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \int_{T+t}^{+\infty} e^{-\zeta a^-} \|F^0(t-\zeta)\|_{\mathcal{A}} d\zeta dt \\
&\le \lim_{T\to+\infty} \frac{\sup_{t\in\mathbb{R}} \|F^0(t)\|_{\mathcal{A}}}{2T} \int_{-T}^{T} \int_{T+t}^{+\infty} e^{-\zeta a^-} d\zeta dt = 0.
\end{aligned}$$

Hence, we have $\lim_{T\to+\infty} \Lambda(T) = 0$, which implies that $G^0 \in PAP_0(\mathbb{R}, \mathcal{A})$. Therefore, $G \in PAP(\mathbb{R}, \mathcal{A}^n)$. The proof is complete. ∎

Theorem 7: Assume that (H_1)-(H_3) hold. Then system (1) has at least one pseudo almost periodic solution in $\mathbb{X}_0$.

Proof: It is easy to check that if $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{X}$ is a solution of the integral equation

$$\begin{aligned}
&x_i(t) \\
&= \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} \Big[a_i(s) \int_{s-\eta_i(s)}^{s} \dot{x}_i(u)du \\
&\quad + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s) g_j(x_j(s-\tau_{ij}(s))) \\
&\quad + \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ijk}(s) h_j(x_j(s-\sigma_{ijk}(s))) \\
&\quad \times h_k(x_k(s-\nu_{ijk}(s))) + \sum_{j=1}^{n} e_{ij}(s) I_j(\dot{x}_j(s-\mu_{ij}(s))) \\
&\quad + J_i(s) \Big] ds, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

then x is also a solution of system (1).

Define an operator $T : \mathbb{X} \to BC(\mathbb{R}, \mathcal{A}^n)$ by

$$T\varphi = (T_1\varphi, T_2\varphi, \cdots, T_n\varphi)^T,$$

where

$$\begin{aligned}
&(T_i\varphi)(t) \\
&= \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} \Big[a_i(s) \int_{s-\eta_i(s)}^{s} \dot{\varphi}_i(u)du \\
&\quad + \sum_{j=1}^{n} b_{ij}(s) f_j(\varphi_j(s)) + \sum_{j=1}^{n} c_{ij}(s) g_j(\varphi_j(s-\tau_{ij}(s))) \\
&\quad + \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ijk}(s) h_j(\varphi_j(s-\sigma_{ijk}(s))) \\
&\quad \times h_k(\varphi_k(s-\nu_{ijk}(s))) + \sum_{j=1}^{n} e_{ij}(s) I_j(\dot{\varphi}_j(s-\mu_{ij}(s))) \\
&\quad + J_i(s) \Big] ds \\
&:= \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du} F_i(s)ds, \quad \varphi \in \mathbb{X}, i = 1, 2, \cdots, n.
\end{aligned}$$

Then by Lemmas 3-5, we have $F_i \in PAP(\mathbb{R}, \mathcal{A})$, $i = 1, 2, \ldots, n$. Therefore, by Lemma 6, we obtain that $T\varphi \in PAP(\mathbb{R}, \mathcal{A}^n)$. In order to prove this theorem, we divide it into the following two steps.

Step 1. We will prove that the mapping T is a self-mapping from $\mathbb{X}_0$ to $\mathbb{X}_0$. In fact, for each $\varphi \in \mathbb{X}_0$, we have

$$\begin{aligned}
&\|T\varphi - \varphi_0\|_0 \\
&\leq \max_{1\leq i\leq n}\left\{\sup_{t\in\mathbb{R}}\left[\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\left(\left\|a_i(s)\right.\right.\right.\right. \\
&\quad\times \left.\int_{s-\eta_i(s)}^{s}\|\dot{\varphi}_i(u)du\right\|_{\mathcal{A}} + \sum_{j=1}^{n}\|b_{ij}(s)f_j(\varphi_j(s))\|_{\mathcal{A}} \\
&\quad+\sum_{j=1}^{n}\|c_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s)))\|_{\mathcal{A}} \\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{n}\|d_{ijk}(s)h_j(\varphi_j(s-\sigma_{ijk}(s))) \\
&\quad\times h_k(\varphi_k(s-\nu_{ijk}(s)))\|_{\mathcal{A}} \\
&\quad\left.\left.\left.+\sum_{j=1}^{n}\|e_{ij}(s)I_j(\dot{\varphi}_j(s-\mu_{ij}(s)))\|_{\mathcal{A}}\right)ds\right]\right\} \\
&\leq \max_{1\leq i\leq n}\left\{\sup_{t\in\mathbb{R}}\left[\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}ds\left(a_i^+\|\varphi'\|_0\eta_i^+\right.\right.\right. \\
&\quad+\sum_{j=1}^{n}b_{ij}^+L_j^f\|\varphi\|_0 + \sum_{j=1}^{n}c_{ij}^+L_j^g\|\varphi\|_0 \\
&\quad\left.\left.\left.+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+M_k^hL_j^h\|\varphi\|_0 + \sum_{j=1}^{n}e_{ij}^+L_j^I\|\varphi'\|_0\right)\right]\right\} \\
&\leq \max_{1\leq i\leq n}\left\{\frac{1}{a_i^-}\left(a_i^+\eta_i^+ + \sum_{j=1}^{n}b_{ij}^+L_j^f + \sum_{j=1}^{n}c_{ij}^+L_j^g\right.\right. \\
&\quad\left.\left.+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+M_k^hL_j^h + \sum_{j=1}^{n}e_{ij}^+L_j^I\right)\right\}\|\varphi\|_{\mathbb{X}} \\
&\leq \frac{rR}{1-r}
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Psi\varphi - \varphi_0)'\|_0 \\
&\leq \max_{1\leq i\leq n}\left\{\sup_{t\in\mathbb{R}}\left[\left\|a_i(t)\int_{t-\eta_i(t)}^{t}\dot{\varphi}_i(u)du\right.\right.\right. \\
&\quad+\sum_{j=1}^{n}b_{ij}(t)f_j(\varphi_j(t)) + \sum_{j=1}^{n}c_{ij}(t)g_j(\varphi_j(t-\tau_{ij}(t))) \\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}(t)h_j(\varphi_j(t-\sigma_{ijk}(t))) \\
&\quad\left.\times h_k(\varphi_k(s-\nu_{ijk}(t))) + \sum_{j=1}^{n}e_{ij}(t)I_j(\dot{\varphi}_j(t-\mu_{ij}(t)))\right\|_{\mathcal{A}} \\
&\quad+\left\|\int_{-\infty}^{t}a_i(t)e^{-\int_s^t a_i(u)du}\left(a_i(s)\int_{s-\eta_i(s)}^{s}\dot{\varphi}_i(u)du\right.\right. \\
&\quad+\sum_{j=1}^{n}b_{ij}(s)f_j(\varphi_j(s)) + \sum_{j=1}^{n}c_{ij}(s)g_j(\varphi_j(s-\tau_{ij}(s))) \\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}(s)h_j(\varphi_j(s-\sigma_{ijk}(s))) \\
&\quad\times h_k(\varphi_k(s-\nu_{ijk}(s))) \\
&\quad\left.\left.\left.\left.+\sum_{j=1}^{n}e_{ij}(s)I_j(\dot{\varphi}_j(s-\mu_{ij}(s)))\right)ds\right\|_{\mathcal{A}}\right]\right\} \\
&\leq \max_{1\leq i\leq n}\left\{\left(a_i^+\eta_i^+ + \sum_{j=1}^{n}b_{ij}^+L_j^f + \sum_{j=1}^{n}c_{ij}^+L_j^g\right.\right. \\
&\quad\left.+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+M_k^hL_j^h + \sum_{j=1}^{n}e_{ij}^+L_j^I\right)\|\varphi\|_{\mathbb{X}} \\
&\quad+\frac{a_i^+}{a_i^-}\left(a_i^+\eta_i^+ + \sum_{j=1}^{n}b_{ij}^+L_j^f + \sum_{j=1}^{n}c_{ij}^+L_j^g\right. \\
&\quad\left.\left.+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+M_k^hL_j^h + \sum_{j=1}^{n}e_{ij}^+L_j^I\right)\|\varphi\|_{\mathbb{X}}\right\} \\
&= \max_{1\leq i\leq n}\left\{\left(1+\frac{a_i^+}{a_i^-}\right)\left(a_i^+\eta_i^+ + \sum_{j=1}^{n}b_{ij}^+L_j^f\right.\right. \\
&\quad\left.\left.+\sum_{j=1}^{n}c_{ij}^+L_j^g + \sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+M_k^hL_j^h + \sum_{j=1}^{n}e_{ij}^+L_j^I\right)\right\}\|\varphi\|_{\mathbb{X}} \\
&\leq \frac{rR}{1-r}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|(T\varphi - \varphi_0)\|_X &= \max_{1\leq i\leq n}\left\{\|(T\varphi - \varphi_0)\|_0, \|(T\varphi - \varphi_0)'\|_0\right\} \\
&\leq \frac{rR}{1-r},
\end{aligned}$$

which implies that $\varphi(\mathbb{X}_0) \subset \mathbb{X}_0$.

Step 2. We will prove that T is a contracting mapping. In fact, for any $\varphi, \psi \in \mathbb{X}_0$, we have that

$$\begin{aligned}
&\|T\varphi - T\psi\|_0 \\
&\leq \max_{1\leq i\leq n}\left\{\sup_{t\in\mathbb{R}}\left[\int_{-\infty}^{t} e^{\int_s^t -a_i(u)du}\right.\right. \\
&\quad\times\left(\left\|a_i(s)\int_{s-\eta_i(s)}^{s}[\dot{\varphi}_i(u) - \dot{\psi}_i(u)]du\right\|_{\mathcal{A}}\right. \\
&\quad+\sum_{j=1}^{n}\left\|b_{ij}(s)\big[f_j(\varphi_j(s)) - f_j(\psi_j(s))\big]\right\|_{\mathcal{A}} \\
&\quad+\sum_{j=1}^{n}\left\|c_{ij}(s)\big[g_j(\varphi_j(s-\tau_{ij}(s)))\right. \\
&\quad\left.-g_j(\psi_j(s-\tau_{ij}(s)))\big]\right\|_{\mathcal{A}} \\
&\quad+\sum_{j=1}^{n}\sum_{k=1}^{n}\left\|d_{ijk}(s)\big[h_j(\varphi_j(s-\sigma_{ijk}(s)))\right. \\
&\quad\times h_k(\varphi_k(s-\nu_{ijk}(s)))
\end{aligned}$$

$$
\begin{aligned}
&-h_j(\varphi_j(s-\sigma_{ijk}(s)))h_k(\varphi_k(s-\nu_{ijk}(s)))\big]\big\|_{\mathcal{A}}\\
&+\sum_{j=1}^{n}\big\|e_{ij}(s)\big[I_j(\dot\varphi_j(s-\mu_{ij}(s)))\\
&-I_j(\dot\psi_j(s-\mu_{ij}(s)))\big]\big\|_{\mathcal{A}}\Big)ds\Big]\Big\}\\
\le{}& \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big[\int_{-\infty}^{t}e^{-a_i^-(t-s)}ds\Big(a_i^+\eta_i^+\|\dot\varphi_j-\dot\psi_j\|_0\\
&+\sum_{j=1}^{n}b_{ij}^+L_j^f\|\varphi_j-\psi_j\|_0+\sum_{j=1}^{n}c_{ij}^+L_j^g\|\varphi_j-\psi_j\|_0\\
&+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+\big[M_j^hL_k^h\|\varphi-\psi\|_0\\
&+M_k^hL_j^h\|\varphi-\psi\|_0\big]+\sum_{j=1}^{n}e_{ij}^+L_j^I\|\dot\varphi_j-\dot\psi_j\|_0\Big)\Big]\\
\le{}& \max_{1\le i\le n}\frac{1}{a_i^-}\Big[a_i^+\eta_i^++\sum_{j=1}^{n}b_{ij}^+L_j^f+\sum_{j=1}^{n}c_{ij}^+L_j^g\\
&+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+(M_k^hL_j^h+M_j^hL_k^h)+\sum_{j=1}^{n}e_{ij}^+L_j^I\Big]\|\varphi-\psi\|_{\mathbb{X}}\\
\le{}& r\|\varphi-\psi\|_{\mathbb{X}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(T\varphi-T\psi)'\|_0\\
\le{}& \max_{1\le i\le n}\Big(1+\frac{a_i^+}{a_i^-}\Big)\Big[a_i^+\eta_i^++\sum_{j=1}^{n}b_{ij}^+L_j^f+\sum_{j=1}^{n}c_{ij}^+L_j^g\\
&+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+(M_k^hL_j^h+M_j^hL_k^h)+\sum_{j=1}^{n}e_{ij}^+L_j^I\Big]\|\varphi-\psi\|_{\mathbb{X}}\\
\le{}& r\|\varphi-\psi\|_{\mathbb{X}}.
\end{aligned}
$$

Hence,

$$\|(T\varphi-T\psi)\|_{\mathbb{X}}\le r\|\varphi-\psi\|_{\mathbb{X}}.$$

Noticing that $r<1$, T is a contraction mapping. Therefore, T has a unique fixed point in $\mathbb{X}_0$, that is, system (1) has a pseudo almost periodic solution in $\mathbb{X}_0$. The proof is complete. ■

IV. GLOBAL EXPONENTIAL STABILITY

In this section, we investigate the global exponential stability of pseudo almost periodic solutions of system (1) by using reduction to absurdity.

Definition 8: Let $x=(x_1,x_2,\dots,x_n)^T$ be a pseudo almost periodic solution of system (1) with the initial value $\varphi=(\varphi_1,\varphi_2,\dots,\varphi_n)^T\in C([-\xi,0],\mathcal{A}^n)$ and $y=(y_1,y_2,\dots,y_n)^T$ be an arbitrary solution of system (1) with the initial value $\psi=(\psi_1,\psi_2,\dots,\psi_n)^T\in C([-\xi,0],\mathcal{A}^n)$, respectively. If there exist positive constants λ and M such that

$$\|x(t)-y(t)\|_1\le M\|\varphi-\psi\|_{\xi}e^{-\lambda t},\quad \forall t>0,$$

where

$$
\begin{aligned}
\|x(t)-y(t)\|_1 &= \max\{\|x(t)-y(t)\|_{\mathcal{A}^n},\|(x(t)-y(t))'\|_{\mathcal{A}^n}\},\\
\|\varphi-\psi\|_{\xi} &= \max\Big\{\sup_{t\in[-\xi,0]}\max_{1\le i\le n}\|\varphi_i(t)-\psi_i(t)\|_{\mathcal{A}},\\
&\qquad \sup_{t\in[-\xi,0]}\max_{1\le i\le n}\|(\varphi_i(t)-\psi_i(t))'\|_{\mathcal{A}}\Big\}.
\end{aligned}
$$

Then the pseudo almost periodic solution x of system (1) is said to be globally exponentially stable.

Theorem 9: Assume that (H_1)-(H_3) hold. Then system (1) has a pseudo almost periodic solution that is globally exponentially stable.

Proof: By Theorem 7, system (1) has a pseudo almost periodic solution, let $x(t)$ be the pseudo almost periodic solution with the initial value $\varphi(t)$ and $y(t)$ be an arbitrary solution with the initial value $\psi(t)$. Taking $z(t)=x(t)-y(t)$, $\phi(t)=\varphi(t)-\psi(t)$, we have

$$
\begin{aligned}
&\dot z_i(t)\\
&\quad= -a_i(t)z_i(t-\eta_i(t))+\sum_{j=1}^{n}b_{ij}(t)\bar f_j(z_j(t))\\
&\qquad+\sum_{j=1}^{n}c_{ij}(t)\bar g_j(z_j(t-\tau_{ij}(t)))\\
&\qquad+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}(t)\bar h_j(z_j(t-\sigma_{ijk}(t)))\bar h_j(z_j(t-\nu_{ijk}(t)))\\
&\qquad+\sum_{j=1}^{n}e_{ij}(t)\bar I_j(\dot z_j(t-\mu_{ij}(t))),\quad i=1,2,\dots,n,
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
\bar f_j(z_j(t)) &= f_j(x_j(t))-f_j(y_j(t)),\\
barg_j(z_j(t-\tau_{ij}(t))) &= g_j(x_j(t-\tau_{ij}(t)))-g_j(y_j(t-\tau_{ij}(t))),\\
&\bar h_j(z_j(t-\sigma_{ijk}(t)))\bar h_j(z_j(t-\nu_{ijk}(t)))\\
&= h_j(x_j(t-\sigma_{ijk}(t)))h_j(x_j(t-\nu_{ijk}(t)))\\
&\quad-h_j(y_j(t-\sigma_{ijk}(t)))h_j(y_j(t-\nu_{ijk}(t))),\\
\bar I_j(\dot z_j(t-\mu_{ij}(t))) &= I_j(\dot x_j(t-\mu_{ij}(t)))-I_j(\dot y_j(t-\mu_{ij}(t))).
\end{aligned}
$$

Let Θ_i and Δ_i be defined, respectively, by

$$
\begin{aligned}
\Theta_i(\omega) &= a_i^- -\omega-\Big(a_i^+\eta_i^+e^{\omega\eta_i^+}+\sum_{j=1}^{n}b_{ij}^+L_j^fe^{\omega}\\
&\quad+\sum_{j=1}^{n}c_{ij}^+L_j^ge^{\omega\tau_{ij}^+}+\sum_{j=1}^{n}\sum_{k=1}^{n}d_{ijk}^+(L_j^he^{\omega\sigma_{ijk}^+}\\
&\quad+L_k^he^{\omega\nu_{ijk}^+})+\sum_{j=1}^{n}e_{ij}^+L_j^Ie^{\omega\mu_{ij}^+}\Big)
\end{aligned}
$$

and

$$\Delta_i(\omega) = a_i^- - \omega - \left(a_i^+ + a_i^-\right)\Bigg(a_i^+\eta_i^+ e^{\omega\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f e^{\omega} + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\omega\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ (L_j^h e^{\omega\sigma_{ijk}^+} + L_k^h e^{\omega\nu_{ijk}^+}) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\omega\mu_{ij}^+}\Bigg),$$

where $i = 1, 2, \ldots, n$. When $\omega = 0$, we get

$$\Theta_i(0) > 0 \quad \text{and} \quad \Delta_i(0) > 0.$$

Since $\omega \in [0, +\infty)$, $\Theta_i(\omega)$, $\Delta_i(\omega)$ are continuous on $[0, +\infty)$ and $\Theta_i(\omega), \Delta_i(\omega) \to -\infty$ as $\omega \to +\infty$, there exist ε_i, $\varepsilon_i^* > 0$ such that $\Theta_i(\varepsilon_i) = \Delta(\varepsilon_i^*) = 0$ and $\Theta_i(\varepsilon) > 0$ for $\varepsilon \in (0, \varepsilon_i)$, $\Delta(\varepsilon_i^*) > 0$ for $\varepsilon \in (0, \varepsilon_i^*)$, $i = 1, 2, \ldots, n$. Take $\alpha = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^*\}$, then we have $\Theta_i(\alpha) \geq 0$, $\Delta_i(\alpha) \geq 0$, $i = 1, 2, \ldots, n$. Hence, for every fixed positive constant $\lambda \in (0, \min\{\alpha, a_1^-, a_2^-, \ldots, a_n^-, \lambda_0\})$, we have $\Theta_i(\lambda) > 0$ and $\Delta_i(\lambda) > 0$ for $i = 1, 2, \ldots, n$. Therefore,

$$\frac{1}{a_i^- - \lambda}\Bigg(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f e^{\lambda} + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ (M_k^h L_j^h e^{\omega\sigma_{ijk}^+} + M_j^h L_k^h e^{\omega\nu_{ijk}^+}) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Bigg) < 1, \tag{5}$$

$$\left(1 + \frac{a_i^+}{a_i^- - \lambda}\right)\Bigg(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f e^{\lambda} + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ (M_k^h L_j^h e^{\omega\sigma_{ijk}^+} + M_j^h L_k^h e^{\omega\nu_{ijk}^+}) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Bigg) < 1, \tag{6}$$

where $i = 1, 2, \ldots, n$. Let $M = \max_{1\leq i\leq n}\left\{\frac{a_i^-}{U_i}\right\}$, then by (H_3), we have $M > 1$ and

$$\frac{1}{M} \leq \frac{1}{a_i^- - \lambda}\Bigg(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f e^{\lambda} + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ (M_k^h L_j^h e^{\omega\sigma_{ijk}^+} + M_j^h L_k^h e^{\omega\nu_{ijk}^+}) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Bigg), \quad i = 1, 2, \ldots, n.$$

By (4), we have

$$\dot{z}_i(t) + a_i(t)z_i(t) = a_i(t)\int_{t-\eta_i(t)}^t \dot{z}_i(s)ds + \sum_{j=1}^n b_{ij}(t)\bar{f}_j(z_j(t)) + \sum_{j=1}^n c_{ij}(t)\bar{g}_j(z_j(t - \tau_{ij}(t))) + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}(t)\bar{h}_j(z_j(t - \sigma_{ijk}(t)))\bar{h}_j(z_j(t - \nu_{ijk}(t))) + \sum_{j=1}^n e_{ij}(t)\bar{h}_j(\dot{z}_j(t - \mu_{ij}(t))), \quad i = 1, 2, \ldots, n. \tag{7}$$

Multiplying both sides of (7) by $e^{\int_0^s a_i(u)du}$ and integrating on $[0, t]$, we have

$$z_i(t) = \phi_i(0)e^{-\int_0^t a_i(u)du} + \int_0^t e^{-\int_s^t a_i(u)du} \times\Bigg[a_i(s)\int_{s-\eta_i(s)}^s \dot{z}_i(u)du + \sum_{j=1}^n b_{ij}(s)\bar{f}_j(z_j(s)) + \sum_{j=1}^n c_{ij}(s)\bar{g}_j(z_j(s - \tau_{ij}(s))) + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}(s)\bar{h}_j(z_j(s - \sigma_{ijk}(s))) \times\bar{h}_j(z_j(s - \nu_{ijk}(s))) + \sum_{j=1}^n e_{ij}(s)\bar{I}_j(\dot{z}_j(s - \mu_{ij}(s)))\Bigg]ds, \ i = 1, 2, \ldots, n. \tag{8}$$

It is easy to see that

$$\|z(t)\|_1 = \|\phi(t)\|_1 \leq \|\phi\|_\xi \leq M\|\phi\|_\xi e^{-\lambda t}, \quad t \in (-\xi, 0].$$

We claim that

$$\|z(t)\|_1 \leq M\|\phi\|_\xi e^{-\lambda t}, t \in [0, +\infty). \tag{9}$$

To prove (9) holds, we show that for any $\beta > 1$, the following inequality holds

$$\|z(t)\|_1 < \beta M\|\phi\|_\xi e^{-\lambda t}, \quad t > 0. \tag{10}$$

If (10) does not hold, then there must be some $t_1 > 0$ such that

$$\|z(t_1)\|_1 = \max_{1\leq i\leq n}\{\|z_i(t_1)\|_0, \|\dot{z}_i(t_1)\|_0\} = \beta M\|\phi\|_\xi e^{-\lambda t_1} \tag{11}$$

and

$$\|z(t)\|_1 < \beta M\|\phi\|_\xi e^{-\lambda t}, \quad t \in [-\xi, t_1). \tag{12}$$

By (5), (8), (11), (12), we have

$$
\begin{aligned}
&\|z_i(t_1)\|_0\\
&\le \|\phi\|_\xi e^{-t_1 a_i^-} + \beta M\|\phi\|_\xi \int_0^{t_1} e^{-(t_1-s)a_i^-}\Big[a_i^+\eta_i^+ e^{\lambda\eta_i^+}\\
&\quad + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+}\\
&\quad + M_j^h L_k^h e^{\lambda\nu_{ijk}^+}\big) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big] e^{-\lambda s} ds\\
&\le \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big[\frac{e^{(\lambda-a_i^-)t_1}}{\beta M} + \frac{1}{a_i^- - \lambda}\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+}\\
&\quad + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+} + M_j^h L_k^h e^{\lambda\nu_{ijk}^+}\big)\\
&\quad + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big)(1 - e^{(\lambda-a_i^-)t_1})\Big]\\
&< \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big[e^{(\lambda-a_i^-)t_1}\Big(\frac{1}{M} - \frac{1}{a_i^- - \lambda}\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+}\\
&\quad + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+} + M_j^h L_k^h e^{\lambda\nu_{ijk}^+}\big)\\
&\quad + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big) + \frac{1}{a_i^- - \lambda}\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f\\
&\quad + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+}\\
&\quad + M_j^h L_k^h e^{\lambda\nu_{ijk}^+}\big) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big)\Big]\\
&\le \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big[\frac{1}{a_i^- - \lambda}\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f\\
&\quad + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+}\\
&\quad + M_j L_k^h e^{\lambda\nu_{ijk}^+}\big) + \sum_{j=1}^n d_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big)\Big]\\
&< \beta M\|\phi\|_\xi e^{-\lambda t_1}, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

Differentiating (8), and by (6), (11) and (12), we have

$$
\begin{aligned}
&\|\dot z_i(t_1)\|_0\\
&\le a_i^+\|\phi\|_\xi e^{-t_1 a_i^-} + a_i^+\eta_i^+ e^{-\lambda(t_1-\eta_i^+)}\beta M\|\phi\|_\xi\\
&\quad + \beta M\|\phi\|_\xi \sum_{j=1}^n b_{ij}^+ L_j^f e^{-\lambda t_1}\\
&\quad + \beta M\|\phi\|_\xi \sum_{j=1}^n c_{ij}^+ L_j^g e^{-\lambda(t_1-\tau_{ij}^+)}\\
&\quad + \beta M\|\phi\|_\xi \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{-\lambda(t_1-\sigma_{ijk}^+)}\\
&\quad + M_j^h L_k^h e^{-\lambda(t_1-\nu_{ijk}^+)}\big) + \beta M\|\phi\|_\xi \sum_{j=1}^n e_{ij}^+ L_j^I e^{-\lambda(t_1-\mu_{ij}^+)}\\
&\quad + \beta M\|\phi\|_\xi \int_0^{t_1} a_i^+ e^{-(t_1-s)a_i^-}\Big[a_i^+\eta_i^+ e^{-\lambda(s-\eta_i^+)}\\
&\quad + \sum_{j=1}^n b_{ij}^+ L_j^f e^{-\lambda s} + \sum_{j=1}^n c_{ij}^+ L_j^g e^{-\lambda(s-\tau_{ij}^+)}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{-\lambda(s-\sigma_{ijk}^+)} + M_j^h L_k^h e^{-\lambda(s-\nu_{ijk}^+)}\big)\\
&\quad + \sum_{j=1}^n e_{ij}^+ L_j^I e^{-\lambda(s-\mu_{ij}^+)}\Big] ds\\
&\le \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big\{\frac{a_i^+ e^{(\lambda-a_i^-)t_1}}{\beta M}\\
&\quad + \Big(1 + a_i^+\int_0^{t_1} e^{(t_1-s)(\lambda-a_i^-)} ds\Big)\Big[a_i^+\eta_i^+ e^{\lambda\eta_i^+}\\
&\quad + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+} + \sum_{j=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+}\\
&\quad + M_j^h L_k^h e^{\lambda\nu_{ijk}^+}\big) + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big]\Big\}\\
&= \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big[\frac{a_i^+ e^{(\lambda-a_i^-)t_1}}{\beta M}\\
&\quad + \Big(1 + \frac{a_i^+}{a_i^- - \lambda}(1 - e^{(\lambda-a_i^-)t_1})\Big)\\
&\quad \times\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+}\\
&\quad + \sum_{j=1}^n\sum_{k=1}^n d_{ijk}^+ \big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^+} + M_j L_k^h e^{\lambda\nu_{ijk}^+}\big)\\
&\quad + \sum_{j=1}^n e_{ij}^+ L_j^I e^{\lambda\mu_{ij}^+}\Big)\Big]\\
&< \beta M\|\phi\|_\xi e^{-\lambda t_1}\Big[a_i^+ e^{(\lambda-a_i^-)t_1}\Big(\frac{1}{M} - \frac{1}{a_i^- - \lambda}\\
&\quad \times\Big(a_i^+\eta_i^+ e^{\lambda\eta_i^+} + \sum_{j=1}^n b_{ij}^+ L_j^f + \sum_{j=1}^n c_{ij}^+ L_j^g e^{\lambda\tau_{ij}^+}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}^{+}\left(M_k^h L_j^h e^{\lambda\sigma_{ijk}^{+}} + M_j L_k^h e^{\lambda\nu_{ijk}^{+}}\right) \\
&+e^{\lambda\mu_{ij}^{+}}\sum_{j=1}^{n} d_{ij}^{+} L_j^I\Big) + \left(1+\frac{a_i^{+}}{a_i^{-}-\lambda}\right) \\
&\times\Big(a_i^{+}\eta_i^{+}e^{\lambda\eta_i^{+}} + \sum_{j=1}^{n} b_{ij}^{+}L_j^f + \sum_{j=1}^{n} c_{ij}^{+}L_j^g e^{\lambda\tau_{ij}^{+}} \\
&+\sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}^{+}\left(M_k L_j^h e^{\lambda\sigma_{ijk}^{+}} + M_j^h L_k^h e^{\lambda\nu_{ijk}^{+}}\right) \\
&+\sum_{j=1}^{n} e_{ij}^{+}L_j^I e^{\lambda\mu_{ij}^{+}}\Big)\Big] \\
\le{}& \beta M\|\phi\|_{\xi} e^{-\lambda t_1}\left(1+\frac{a_i^{+}}{a_i^{-}-\lambda}\right)\Big(a_i^{+}\eta_i^{+}e^{\lambda\eta_i^{+}} + \sum_{j=1}^{n} b_{ij}^{+}L_j^f \\
&+\sum_{j=1}^{n} c_{ij}^{+}L_j^g e^{\lambda\tau_{ij}^{+}} + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}^{+}\big(M_k^h L_j^h e^{\lambda\sigma_{ijk}^{+}} \\
&+M_j L_k^h e^{\lambda\nu_{ijk}^{+}}\big) + \sum_{j=1}^{n} e_{ij}^{+}L_j^I e^{\lambda\mu_{ij}^{+}}\Big) \\
<{}& \beta M\|\phi\|_{\xi} e^{-\lambda t_1}, \quad i=1,2,\ldots,n.
\end{aligned}
$$

Hence,

$$\|z(t_1)\|_1 < \beta M\|\phi\|_{\xi} e^{-\lambda t_1},$$

which contradicts the equality (11), and so (10) holds. Letting $\beta \to 1$, then (9) holds. Hence, the pseudo almost periodic solution of (1) is globally exponentially stable. The proof is complete. ■

Remark 10: When system (1) degenerates into real-valued, complex-valued or quaternion-valued system, Theorems 7 and 9 remain new.

V. EXAMPLES

In this section, we present two examples to show the feasibility of our main results of this paper.

Example 11: In system (1), let $n = m = 2$, $s = 1$, and for $i, j = 1, 2$, take

$$
\begin{aligned}
x_i(t) &= x_i^0(t)e_0 + x_i^1(t)e_1 + x_i^2(t)e_2 + x_i^{12}(t)e_{12} \in \mathcal{A}, \\
f_j(x_j) &= \frac{1}{66}e_0\sin(x_j^0 + x_j^2) + \frac{1}{60}e_1\sin(x_j^1 + x_j^{12}) \\
&\quad + \frac{1}{63}e_2\sin(x_j^0 - x_j^2) + \frac{1}{81}e_{12}\arctan(x_j^1 + x_j^{12}), \\
g_j(x_j) &= \frac{1}{44}e_0\sin(x_j^1 - x_j^{12}) + \frac{1}{40}e_1\sin(x_j^0 + x_j^2) \\
&\quad + \frac{1}{42}e_2\arctan(x_j^1 + x_j^{12}) \\
&\quad + \frac{1}{54}e_{12}\arctan(x_j^0 - x_j^2), \\
h_j(x_j) &= \frac{1}{48}e_0\sin(x_j^0 + x_j^{12}) + \frac{1}{40}e_1\sin(x_j^2 - x_j^1) \\
&\quad + \frac{1}{30}e_2\sin(x_j^1 + x_j^{12}) + \frac{1}{36}e_{12}\sin(x_j^0 + x_j^2), \\
I_j(x_j) &= \frac{1}{50}e_0\sin x_j^2 + \frac{1}{48}e_1\sin(x_j^1 - x_j^{12}) \\
&\quad + \frac{1}{112}e_2\sin(x_j^0 + x_j^2) + \frac{1}{52}e_{12}\sin x_j^1,
\end{aligned}
$$

$$
\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} 1 + 0.1\sin\sqrt{2}t \\ 1.2 + 0.2\cos\sqrt{3}t \end{pmatrix},
$$

$$
\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} 0.15 + 0.02\sin\sqrt{2}t \\ 0.16 + 0.012\cos\sqrt{3}t \end{pmatrix},
$$

$$
\begin{aligned}
&\begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.1e_0\sin\sqrt{6}t + 0.2e_1\sin\sqrt{6}t & 0.13e_0 + 0.1e_{12}\sin\sqrt{7}t \\ 0.15e_0 + 0.1e_1\cos\sqrt{5}t + 0.2e_{12}\cos\sqrt{2}t & 0.11e_0 + 0.2e_2\sin\sqrt{3}t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.16e_0\sin\sqrt{3}t + 0.12e_1\sin\sqrt{3}t & 0.12e_0 + 0.1e_{12}\sin\sqrt{7}t \\ 0.15e_0 + 0.13e_1\cos\sqrt{5}t + 0.12e_{12}\cos 4t & 0.12e_0 + 0.12e_2\sin\sqrt{3}t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} d_{111}(t) & d_{112}(t) \\ d_{121}(t) & d_{122}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.21e_0\sin\sqrt{6}t + 0.12e_1\sin\sqrt{3}t & 0.11e_0 + 0.12e_{12}\sin\sqrt{2}t \\ 0.14e_0 + 0.11e_1\cos\sqrt{5}t + 0.12e_{12}\cos 2\sqrt{5}t & 0.12e_0 + 0.12e_2\sin\sqrt{3}t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} d_{211}(t) & d_{212}(t) \\ d_{221}(t) & d_{222}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.11e_0\sin\sqrt{6}t + 0.12e_1\sin\sqrt{5}t & 0.11e_0 + 0.12e_{12}\sin\sqrt{7}t \\ 0.14e_0 + 0.11e_1\cos\sqrt{5}t + 0.12e_{12}\cos 2\sqrt{3}t & 0.12e_0 + 0.12e_2\sin\sqrt{3}t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.1e_1\sin\sqrt{3}t + 0.02e_2\sin\sqrt{2}t & 0.11e_1 + 0.1e_{12}\sin\sqrt{5}t \\ 0.13e_1\cos\sqrt{5}t + 0.11e_2 + 0.08e_{12}\cos 4t & 0.02e_1 + 0.12e_2\sin\sqrt{11}t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} \tau_{11}(t) & \tau_{12}(t) \\ \tau_{21}(t) & \tau_{22}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.001\sin\sqrt{3}t + 0.1 & 0.3\sin t + 1 \\ 0.2\cos\sqrt{2}t + 1 & 0.001\cos t + 0.01 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\begin{pmatrix} \sigma_{111}(t) & \sigma_{112}(t) \\ \sigma_{121}(t) & \sigma_{122}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.002\sin\sqrt{6}t + 0.01 & 0.2\sin t + 1 \\ 0.1\cos t + 1 & 0.001\sin\sqrt{2}t + 0.01 \end{pmatrix},
\end{aligned}
$$

$$\begin{pmatrix} \sigma_{211}(t) & \sigma_{212}(t) \\ \sigma_{221}(t) & \sigma_{222}(t) \end{pmatrix} = \begin{pmatrix} 0.012\sin\sqrt{3}t+0.02 & 0.2\sin\sqrt{2}+0.8 \\ 0.3\sin 2t+3 & 0.01\sin\sqrt{6}t+0.011 \end{pmatrix},$$

$$\begin{pmatrix} \nu_{111}(t) & \nu_{112}(t) \\ \nu_{121}(t) & \nu_{122}(t) \end{pmatrix} = \begin{pmatrix} 0.02\sin\sqrt{5}t+0.03 & 0.01\cos t+0.2 \\ 0.3\sin t+1 & 0.001\sin\sqrt{7}t+0.01 \end{pmatrix},$$

$$\begin{pmatrix} \nu_{211}(t) & \nu_{212}(t) \\ \nu_{221}(t) & \nu_{222}(t) \end{pmatrix} = \begin{pmatrix} 0.002\sin\sqrt{3}t+0.02 & -0.1\cos t+1 \\ 0.2\sin 2t+3 & 0.01\sin\sqrt{8}t+0.015 \end{pmatrix},$$

$$\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.01\sin\sqrt{5}t+0.12 & 0.1\sin\sqrt{3}t+0.7 \\ 0.1\cos 2t+1 & 0.01\sin\sqrt{2}t+0.11 \end{pmatrix},$$

$$\begin{pmatrix} J_1(t) \\ J_2(t) \end{pmatrix} = \begin{pmatrix} (\frac{1}{15}\sin 2\sqrt{5}t+\frac{2}{1+t^2})e_0+\frac{1}{20}e_1\sin\sqrt{3}t \\ (\frac{1}{15}\sin 2\sqrt{3}t+\frac{3}{1+t^2})e_0+\frac{1}{12}e_1\sin\sqrt{6}t \end{pmatrix} + \begin{pmatrix} \frac{1}{12}e_2\cos\sqrt{3}t+\frac{1}{15}e_{12}\sin\sqrt{7}t \\ \frac{1}{10}e_2\cos\sqrt{7}t+\frac{1}{20}e_{12}\sin^2\sqrt{3}t \end{pmatrix}.$$

By computing, for $j=1,2$, we have $L_j^f=\frac{1}{30}$, $L_j^g=\frac{1}{20}$, $L_j^h=\frac{1}{15}$, $L_j^I=\frac{1}{24}$, $M_j^h=\frac{1}{30}$, $a_1^-=0.9$, $a_2^-=1$, $a_1^+=1.1$, $a_2^+=1.4$, $\eta_1^+=0.17$, $\eta_2^+=0.172$, $b_{11}^+=0.2$, $b_{12}^+=0.13$, $b_{21}^+=0.2$, $b_{22}^+=0.2$, $c_{11}^+=0.16$, $c_{12}^+=0.12$, $c_{21}^+=0.15$, $c_{22}^+=0.12$, $d_{111}^+=0.21$, $d_{112}^+=0.12$, $d_{121}^+=0.14$, $d_{122}^+=0.12$, $d_{211}^+=0.12$, $d_{212}^+=0.12$, $d_{221}^+=0.14$, $d_{222}^+=0.12$, $e_{11}^+=0.1$, $e_{12}^+=0.11$, $e_{21}^+=0.13$, $e_{22}^+=0.12$. So (H_1) and (H_2) are satisfied. Besides, we can get $U_1\approx 0.22$, $U_2\approx 0.28$ and

$$r=\max_{1\le i\le n}\left\{\frac{1}{a_i^-}U_i,\left(1+\frac{a_i^+}{a_i^-}\right)U_i\right\}\approx 0.672<1.$$

Therefore, all of the conditions of Theorem 9 are satisfied. Hence, system (1) has a pseudo almost periodic solution that is globally exponentially stable (see Figures 1-3).

Example 12: In system (1), let $n=2$, $m=3$, $s=1$, and for $i,j=1,2$, take

$$\begin{aligned} x_i(t) &= x_i^0(t)e_0+x_i^1(t)e_1+x_i^2(t)e_2+x_i^{12}(t)e_{12} \\ &\quad +x^{13}e_{13}+x^{23}e_{23}+x^{123}e_{123}\in\mathcal{A}, \\ f_j(x_j) &= \frac{1}{60}e_0e^{-|x_j^2+x_j^0|}+\frac{3}{200}|x_j^1+x_j^{12}|e_1 \\ &\quad +\frac{1}{150}e_2\tanh x_j^2+\frac{1}{68}e_3\sin(x_j^{13}+x_j^1) \\ &\quad +\frac{1}{106}e_{12}\sin x_j^{12}+\frac{3}{250}e_{13}|x_j^2+x_j^{13}| \\ &\quad +\frac{1}{280}e_{23}\tanh x_j^{23}+\frac{1}{152}e_{123}\sin(x_j^{123}+x_j^{23}), \end{aligned}$$

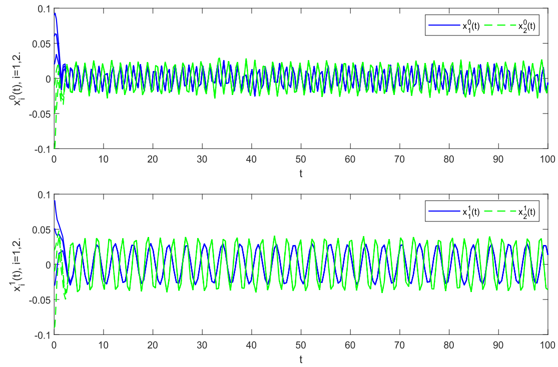

FIGURE 1. Curves of $x_i^0(t)$ and $x_i^1(t)$, $i=1,2$.

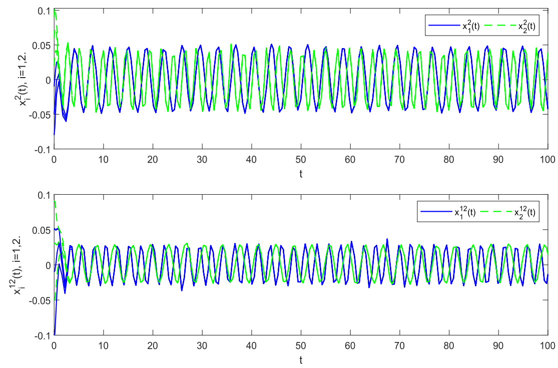

FIGURE 2. Curves of $x_i^2(t)$ and $x_i^{12}(t)$, $i=1,2$.

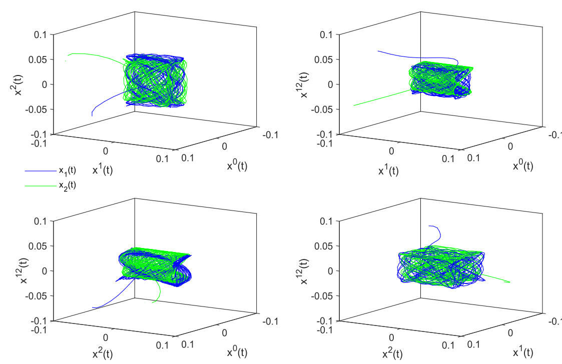

FIGURE 3. Curves of $x^0(t)$, $x^1(t)$, $x^2(t)$ and $x^{12}(t)$ in 3-dimensional space for stable case.

$$\begin{aligned} g_j(x_j) &= \frac{1}{88}e_0\tanh(x_j^3-x_j^{12})+\frac{1}{72}e_1|x_j^0+x_j^2| \\ &\quad +\frac{1}{68}e_2\arctan(x_j^1+x_j^3)+\frac{1}{64}e_3\sin(x_j^0-x_j^2) \\ &\quad +\frac{1}{73}e_{12}\sin(x_j^{13}+x_j^1)+\frac{1}{63}e_{13}\sin x_j^{12} \\ &\quad +\frac{1}{67}e_{23}\tanh x_j^{23}+\frac{1}{82}e_{123}\sin(x_j^{123}-x_j^{23}), \\ h_j(x_j) &= \frac{1}{48}e_0\sin(x_j^0-x_j^3)+\frac{1}{40}e_1\cos(x_j^2-x_j^1) \\ &\quad +\frac{1}{30}e_2\sin(x_j^1+x_j^{12})+\frac{1}{36}e_3\sin(x_j^0+x_j^2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{41}e_{12}\sin(x_j^{13}+x_j^{123})\\
&+\frac{3}{50}\sin(x_j^{3}+x_j^{13})e_{13}+\frac{1}{37}e_{23}\tanh x_j^{23}\\
&+\frac{1}{30}e_{123}\sin(x_j^{123}-x_j^{23}),\\
I_j(x_j) = &\frac{1}{51}e_0\sin(x_j^{2}-x_j^{123})+\frac{1}{48}e_1\sin(x_j^{1}-x_j^{12})\\
&+\frac{1}{112}e_2\sin(x_j^{0}+x_j^{2})+\frac{1}{56}e_3\tanh x_j^{1}\\
&+\frac{3}{50}|x_j^{23}-x_j^{13}|e_{13}+\frac{1}{37}e_{23}\sin x_j^{23}\\
&+\frac{1}{52}e_{123}\sin(x_j^{123}-x_j^{3}),
\end{aligned}
$$

$$
\begin{pmatrix} a_1(t)\\ a_2(t)\end{pmatrix}=\begin{pmatrix}0.9+0.1\cos\sqrt{3}t\\ 1.1+0.1\cos\sqrt{2}t\end{pmatrix},
$$

$$
\begin{pmatrix} \eta_1(t)\\ \eta_2(t)\end{pmatrix}=\begin{pmatrix}0.13+0.02\cos\sqrt{2}t\\ 0.14+0.012\sin\sqrt{3}t\end{pmatrix},
$$

$$
\begin{pmatrix} b_{11}(t) & b_{12}(t)\\ b_{21}(t) & b_{22}(t)\end{pmatrix}
=\begin{pmatrix}0.1e_0\sin\sqrt{6}t+0.2e_1\sin\sqrt{6}t+0.03e_3\sin t & 0.13e_0+0.1e_{12}\sin\sqrt{7}t-0.08e_{23}\cos\sqrt{7}t\\ 0.15e_0+0.1e_{13}\cos\sqrt{5}t+0.14e_{123}\sin\sqrt{2}t & 0.11e_0+0.2e_2\sin\sqrt{3}t-0.07e_{123}\cos\sqrt{5}t\end{pmatrix},
$$

$$
\begin{pmatrix} c_{11}(t) & c_{12}(t)\\ c_{21}(t) & c_{22}(t)\end{pmatrix}
=\begin{pmatrix}0.16e_2\cos\sqrt{3}t+0.12e_3\sin\sqrt{3}t+0.15e_{123}\sin t & 0.11e_0+0.11e_{12}\sin\sqrt{7}t+0.09e_{23}\sin 2t\\ 0.15e_1+0.13e_{12}\cos\sqrt{2}t+0.12e_{13}\cos 2t-0.11e_{23} & 0.12e_0+0.12e_2\sin\sqrt{3}t-0.11e_{123}\sin t\end{pmatrix},
$$

$$
\begin{pmatrix} d_{111}(t) & d_{112}(t)\\ d_{121}(t) & d_{122}(t)\end{pmatrix}
=\begin{pmatrix}0.11e_0+0.1e_1\sin\sqrt{6}t+0.02e_{123}\sin\sqrt{3}t & 0.11e_0+0.12e_{12}\sin\sqrt{2}t+0.08e_3\cos\sqrt{7}t\\ 0.14e_1+0.11e_3\cos\sqrt{5}t+0.12e_{23}\sin\sqrt{5}t & 0.12e_1+0.12e_{12}\sin\sqrt{3}t-0.1e_{123}\cos t\end{pmatrix},
$$

$$
\begin{pmatrix} d_{211}(t) & d_{212}(t)\\ d_{221}(t) & d_{222}(t)\end{pmatrix}
=\begin{pmatrix}0.11e_1\sin\sqrt{3}t+0.12e_{12}\sin\sqrt{5}t+0.07e_{23}\sin 2t & 0.11e_0+0.12e_{12}\sin\sqrt{7}t+0.11e_{23}\sin\sqrt{3}t\\ 0.11e_3\cos\sqrt{5}t+0.12e_{12}\cos 2\sqrt{3}t+0.1e_{23}\cos t & 0.12e_1+0.12e_2\sin\sqrt{3}t+0.11e_{123}\sin t\end{pmatrix},
$$

$$
\begin{pmatrix} e_{11}(t) & e_{12}(t)\\ e_{21}(t) & e_{22}(t)\end{pmatrix}
=\begin{pmatrix}0.1e_1\sin\sqrt{3}t+0.02e_2\sin\sqrt{2}t-0.03e_{23}\sin t & 0.11e_1+0.1e_{23}\sin\sqrt{5}t+0.06e_{23}\cos t\\ 0.11e_2+0.13e_{12}\cos\sqrt{5}t+0.08e_{123}\cos 4t & 0.02e_1+0.12e_2\sin\sqrt{11}t+0.07e_{23}\sin 2t\end{pmatrix},
$$

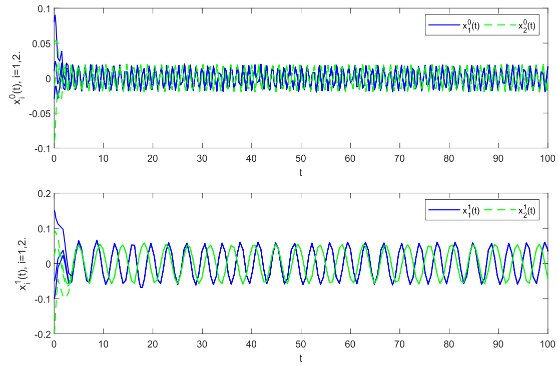

FIGURE 4. Curves of $x_i^0(t)$ and $x_i^1(t)$, $i=1,2$.

$$
\begin{pmatrix} \tau_{11}(t) & \tau_{12}(t)\\ \tau_{21}(t) & \tau_{22}(t)\end{pmatrix}
=\begin{pmatrix}0.02\cos\sqrt{3}t+0.1 & 0.3\sin t+1\\ 0.1\cos\sqrt{2}t+1 & 0.009\cos t+0.01\end{pmatrix},
$$

$$
\begin{pmatrix} \sigma_{111}(t) & \sigma_{112}(t)\\ \sigma_{121}(t) & \sigma_{122}(t)\end{pmatrix}
=\begin{pmatrix}0.002\cos\sqrt{2}t+0.02 & 0.3\sin t+1\\ 0.2\cos t+1 & 0.008\sin\sqrt{2}t+0.02\end{pmatrix},
$$

$$
\begin{pmatrix} \sigma_{211}(t) & \sigma_{212}(t)\\ \sigma_{221}(t) & \sigma_{222}(t)\end{pmatrix}
=\begin{pmatrix}0.015\sin\sqrt{3}t+0.03 & 0.2\cos\sqrt{2}+0.7\\ 0.3\sin 2t+3 & 0.01\sin\sqrt{3}t+0.02\end{pmatrix},
$$

$$
\begin{pmatrix} \nu_{111}(t) & \nu_{112}(t)\\ \nu_{121}(t) & \nu_{122}(t)\end{pmatrix}
=\begin{pmatrix}0.02\sin\sqrt{5}t+0.04 & 0.01\sin t+0.2\\ 0.2\cos t+1 & 0.001\cos\sqrt{5}t+0.01\end{pmatrix},
$$

$$
\begin{pmatrix} \nu_{211}(t) & \nu_{212}(t)\\ \nu_{221}(t) & \nu_{222}(t)\end{pmatrix}
=\begin{pmatrix}0.002\sin\sqrt{2}t+0.02 & -0.1\cos t+1\\ 0.2\sin 3t+3 & 0.01\sin\sqrt{5}t+0.015\end{pmatrix},
$$

$$
\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t)\\ \mu_{21}(t) & \mu_{22}(t)\end{pmatrix}
=\begin{pmatrix}0.01\cos\sqrt{3}t+0.12 & 0.1\sin\sqrt{2}t+0.8\\ 0.1\cos 2t+1 & 0.01\sin\sqrt{2}t+0.16\end{pmatrix},
$$

$$
\begin{pmatrix} J_1(t)\\ J_2(t)\end{pmatrix}
=\begin{pmatrix}(\frac{1}{12}\cos 2\sqrt{7}t+\frac{2}{1+t^2})e_0+\frac{1}{10}e_1\sin\sqrt{3}t\\ (\frac{1}{14}\sin 2\sqrt{5}t+\frac{3}{1+t^2})e_0+\frac{1}{11}e_1\cos\sqrt{2}t\end{pmatrix}
+\begin{pmatrix}\frac{1}{5}e_2\cos\sqrt{3}t+\frac{1}{5}e_{12}\sin\sqrt{2}t\\ \frac{1}{10}e_2\cos\sqrt{3}t+\frac{1}{20}e_{12}\sin^2\sqrt{11}t\end{pmatrix}
+\begin{pmatrix}\frac{1}{11}e_3\cos\sqrt{3}t+\frac{1}{22}e_{23}\sin\sqrt{2}t\\ \frac{1}{17}e_{23}\cos\sqrt{13}t+\frac{1}{30}e_{123}\sin^2\sqrt{13}t\end{pmatrix}.
$$

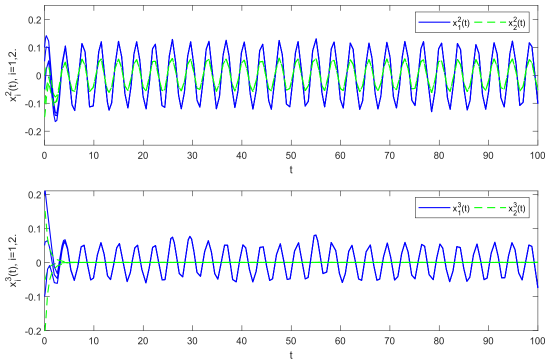

FIGURE 5. Curves of $x_i^2(t)$ and $x_i^3(t)$, $i = 1, 2$.

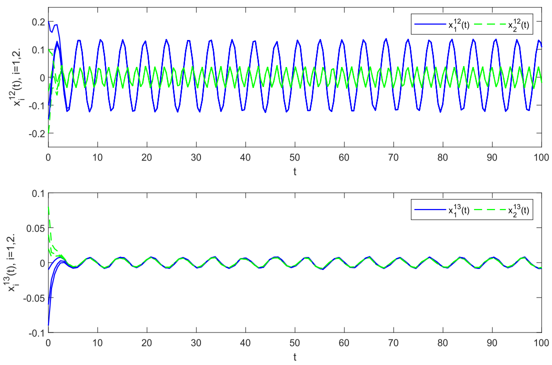

FIGURE 6. Curves of $x_i^{12}(t)$ and $x_i^{13}(t)$, $i = 1, 2$.

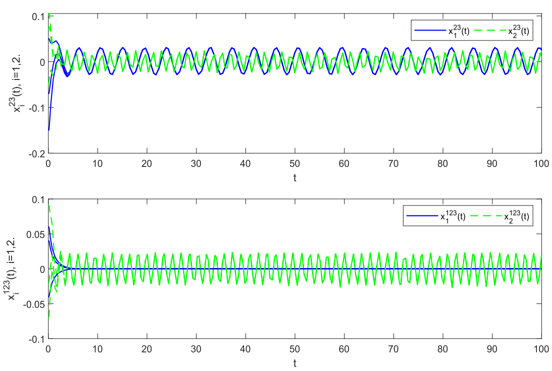

FIGURE 7. Curves of $x_i^{23}(t)$ and $x_i^{123}(t)$, $i = 1, 2$.

By computing, for $j = 1, 2$, we have $L_j^f = \frac{1}{20}, L_j^g = \frac{1}{32}$, $L_j^h = \frac{1}{15}, L_j^I = \frac{1}{24}, M_j^h = \frac{1}{30}, a_1^- = 0.8, a_2^- = 1, a_1^+ = 1$, $a_2^+ = 1.2, \eta_1^+ = 0.15, \eta_2^+ = 0.152, b_{11}^+ = 0.2, b_{12}^+ = 0.13$, $b_{21}^+ = 0.15, b_{22}^+ = 0.2, c_{11}^+ = 0.16, c_{12}^+ = 0.11$, $c_{21}^+ = 0.15, c_{22}^+ = 0.12, d_{111}^+ = 0.11, d_{112}^+ = 0.12$, $d_{121}^+ = 0.14, d_{122}^+ = 0.12, d_{211}^+ = 0.12, d_{212}^+ = 0.12$, $d_{221}^+ = 0.12, d_{222}^+ = 0.12, e_{11}^+ = 0.1, e_{12}^+ = 0.11, e_{21}^+ = 0.13, e_{22}^+ = 0.12$. So (H_1) and (H_2) are satisfied. Besides, we can get $U_1 \approx 0.1772$, $U_2 \approx 0.2198$ and

$$r = \max_{1\leq i\leq n} \left\{ \frac{1}{a_i^-} U_i, \left(1 + \frac{a_i^+}{a_i^-}\right) U_i \right\} \approx 0.484 < 1.$$

Therefore, all of the conditions of Theorem 9 are satisfied. Hence, system (1) has a pseudo almost periodic solution that is globally exponentially stable (see Figures 4-7).

VI. CONCLUSION

In this paper, without decomposing the Clifford-valued systems into real-valued systems, we obtained the existence and global exponential stability of pseudo almost periodic solutions for a class of Clifford-valued neutral high-order Hopfield neural networks with leakage delays. Two examples were given to illustrate the effectiveness and feasibility of our main results. Our results are new and our methods can be used to study the almost periodicity and pseudo almost periodicity for other types of Clifford-valued neural networks.

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