IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received September 25, 2019, accepted October 1, 2019, date of publication October 11, 2019, date of current version December 3, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2946977

Modeling the Equilibrium Road Network Capacity

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This work was supported in part by the National Natural Science Foundation of China under Grant 71771013 and Grant 71621001, in part by the Fundamental Research Funds for the Central Universities under Grant 2017JBZ106, and in part by the Center of Cooperative Innovation for Beijing Metropolitan Transportation.

ABSTRACT The aims of this paper are to present a new model to calculate the maximum equilibrium road network capacity and obtain the associated optimal origin-destination (i.e. OD) flow pattern for a general road network with given network topology and link attributes. First, the Static Congested Traffic Assignment (SCTA) is introduced to describe the network where all used routes are in a congested state. Congested Equilibrium Travel Times (CETTs) between OD pairs obtained by the SCTA model are regarded as the corresponding upper limits of actual equilibrium travel times which are called Uncongested Equilibrium Travel Times (UETTs) and derived from the traditional Static Uncongested Traffic Assignment (SUTA) model. The main idea of modeling is that the total OD flows can be maximized by flexibly scaling and adjusting OD flows of each individual OD pairs to make the UETTs of all OD pairs reach their upper limits (i.e. CETTs). Next, a novel equilibrium road network capacity model is built by combining the SUTA and the SCTA models. Then, the equivalency condition and the solution uniqueness of the proposed model are proved. The paper moves on to provide two solution algorithms together with an analysis of the model characteristics. Finally, through three numerical examples, it is demonstrated that the proposed model can obtain the unique equilibrium network capacity with the given network topology and associated link attributes. The optimal OD flow pattern, which leads to the maximum total OD flows, is thus obtained. The findings in the paper can help to improve the utilization of road networks and contribute to land development planning and control.

INDEX TERMS Transportation, equilibrium road network capacity, optimal origin-destination flow pattern, traffic assignment.

I. INTRODUCTION

The imbalance between traffic supply and demand is one of the main causes of traffic congestion [1]. Most existing studies focus on traffic demand prediction and take the supply side as given information [2]–[5]. Relatively fewer researches are reported on how best to alter and upgrade the traffic infrastructure to maximize its utilization and efficiency. Road network capacity is a significant indicator to reflect and evaluate the traffic supply level and the equilibrium relationship between traffic supply and demand. A good network capacity model is highly desirable to analyze the performance of the road network and to obtain the optimal origin-destination (OD) distribution flow pattern which maximizes the utilization of the network. Such a model will be particularly beneficial in combining traffic infrastructure planning with land-use development planning and controls. It can be applied to estimating how much the potential demand can be accommodated by the current road network and investigating the best ways to improve its capacity and to construct the road network which is congruous with the land-use plans. This paper presents a study in searching for a road network capacity modeling method, which is conducive to a quantitative evaluation on how closely the road network matches the land use so that the efficiency and sustainability of the transportation system can be improved.

A. DEFINITION OF THE ROAD NETWORK CAPACITY

According to the conventional theory of the network flow, the network capacity is defined as the maximum possible throughput from a specified origin node to another specified destination node without exceeding the capacity of any link

The associate editor coordinating the review of this manuscript and approving it for publication was Maurice J. Khabbaz.

along the route. Taking an open and multi-OD pair road network into consideration, traffic researchers and engineers proposed a new definition of the network capacity based on the conventional theories of network flow. The road network capacity is defined as the maximum total OD distribution flows (using "OD flow" for short, unit: vehicles per unit time) from some specified origin nodes to other specified destination nodes of the given network when the traffic flow is assigned to the network via a user-equilibrium traffic assignment method [6]–[8]. Based on the equilibrium network capacity, this paper introduces a new concept which will be defined and elaborated subsequently.

B. LITERATURE REVIEW

Existing studies for finding road network capacity can be roughly classified into three categories. The first is from graph theory, such as max-flow min-cut theorem, which can be used to solve the network capacity problem with a fixed demand and route choice proportion given in advance [9]. The second category is the time-space consumption method, which defines the network as a vessel of time-space resources for the vehicle [10]. Thus, the road network capacity here is defined as the maximum number of standard vehicles accommodated by the network in unit time. Both of the two categories of methods do not consider the route choice behavior of road users and user equilibrium. The third category, which considers user equilibrium by adopting the bi-level programming method [6]-[8] and [11], is the most widely used among all the three categories of methods. Extensive research has been carried out to find solutions to these bi-level programming models and many algorithms have been developed as a result.

Asakura [6] proposed a bi-level programming model based on a user equilibrium network and a heuristic solution algorithm. For a real road network, in which the link capacities are given and the OD flow pattern is fixed, a certain additional demand is assigned to the network at each step based on the user equilibrium principle. When a link flow reaches its capacity, travel time on the link will be set to an infinite value. If the minimum travel times between some of the OD pairs equal the infinite value, the algorithm iteration is terminated and then the road network capacity is obtained. Yang *et al.* [8] suggested that the condition of terminating iteration should be relaxed: when the minimum travel times between all of the OD pairs are infinite, the iteration is terminated and the equilibrium network capacity is obtained. Akamatsu and Miyawaki [7] presented a mathematical model, in which the excess demand formulation and the dummy link are introduced. The road network capacity can be obtained by solving a fixed-demand user equilibrium traffic assignment problem. In order to obtain a unique solution, a target OD matrix with a given set of target OD travel costs is introduced in this model. The authors further found that varying travel costs could significantly influence the solution. However, the two models mentioned above are based on a fixed OD flow pattern. Yang *et al.* [8] presented a bi-level optimization formulation, which is based on the combined assignment-distribution model, to obtain the maximum trip generations that the existing network can accommodate. This model takes account of the future travelers' choice behavior on trip destinations instead of using the fixed OD flow pattern. Kasikitwiwat and Chen [13], as well as Chen and Kasikitwiwat [14], proposed three network capacity concepts for different applications. Among the three concepts, the ultimate network capacity is aimed at describing the maximum physical network capacity. The ultimate capacity model is set up based on the model in [8], in which the OD flow pattern can be adjusted to obtain the maximum equilibrium network capacity.

Many further studies and applications based on the road network capacity were reported, such as optimal signal control pattern [11], [15]–[17], road pricing [18]–[21], demand management scheme [22], [23], network design [24]–[33], network capacity reliability [34], [35] and optimal road network topology [36].

C. RESEARCH MOTIVATION

Some existing models [6], [7] need a predefined and fixed OD flow pattern, which may underestimate the maximum OD flows that the road network can handle. Since the OD flows cannot be adjusted by individual OD pairs, the resulting network capacity may be less than the actual value.

Yang *et al.* [8] and Chen and Kasikitwiwat [14] introduced the combined assignment-distribution model, in which OD flows can be adjusted by individual OD pairs, to find the road network capacity. They improved the precision and practice of network capacity models. However, the impedance parameter of trip distribution and the cost function of destination are still needed and to be calibrated and predicted in these models, which is a tough task in practice and limits their applications. Chen and Kasikitwiwat also implied that an OD flow pattern that fits the network topology better will lead to the higher resulting network capacity. The recent research [37] gives further support that the OD flow pattern will affect the maximum total OD flows (i.e. the resulting road network capacity). Therefore, the maximum equilibrium network capacity and the optimal OD flow pattern corresponding to the maximum equilibrium network capacity are quite interesting and worth investigating.

In fact, the maximum equilibrium network capacity should be an inherent physical characteristic of road networks (just like the link capacity). Hence, it should be dependent on the network topology and the link attribute data (i.e. the link performance function and the link capacity, both of which can be derived from the link fundament diagram), but independent of the given OD flow pattern, the impedance parameter of trip distribution, and the cost function of trip destination.

Therefore, the aims of this paper are to research the maximum equilibrium network capacity (defined as the basic network capacity) based on the network topology and the link attribute data and to find the optimal OD flow pattern corresponding to the maximum equilibrium network capacity, when the locations of all origin and destination nodes are determined in advance.

D. MAJOR CONTRIBUTIONS

In order to obtain the equilibrium and unique network capacity, the model is constructed based on the user equilibrium traffic assignment.

First of all, the congested link performance function is introduced to describe the relationship between link flow and travel time under the congested condition based on the congested branch of Fundamental Diagram (FD), followed by introduction to the congested user equilibrium principle and the Static Congested Traffic Assignment (SCTA) model [37], [38], which is different from the traditional static traffic assignment model (i.e. Beckmann's model [39], called the Static Uncongested Traffic Assignment (SUTA) model in this paper). Then, a basic model to calculate the maximum equilibrium network capacity and search for the optimal OD flow pattern is built by combining the SUTA model and SCTA model. Following on from these, the equivalency condition and the solution uniqueness are proven, which demonstrates that a unique equilibrium network capacity can be found. Subsequently, the solution algorithms are described and the characteristics of the basic model are analyzed. In the end, comparisons with other models are made using a simple numerical example and the practicality of the model and its solution algorithms is demonstrated through two well-known networks.

The major contributions of this paper include:

- A novel model is built to obtain the maximum equilibrium network capacity and the optimal OD flow pattern that generates the maximum total OD flows for general road networks with given network topology and link attribute data.
- The equivalency condition and the solution uniqueness of the proposed model are proved.
- The solution algorithms are designed and the characteristics of the proposed model are analyzed.

E. ORGANIZATION

The remainder of this paper is organized as follows. Section II introduces the relative notions and the main idea of modeling. Section III builds the basic model to calculate the basic network capacity and provides the proofs of the equivalency condition and the solution uniqueness of the basic model. The solution algorithms designed are also described. Section IV provides numerical examples to compare with some existing network capacity models and demonstrate the practicality of the proposed model and its solution algorithms. Section V concludes the paper.

II. NOTIONS AND TWO SIMPLE EXAMPLES

A. NOTIONS

Static traffic assignment is the mechanism that calculates the link flow and the link travel time when the given OD flow is time-stable.

Static Uncongested Traffic Assignment (SUTA) is the traditional static traffic assignment with the uncongested link performance function, which is only suitable for the case where the density of all road links is below the critical density (i.e. the uncongested condition).

Static Congested Traffic Assignment (SCTA) is the static traffic assignment with the congested link performance function, which is only suitable for the case where the density of all used road links is larger than the critical density [37], [38] (i.e. the congested condition).

The uncongested condition represents a state where all road links of the network are uncongested links.

The congested condition represents a state where all used road links of the network are congested links.

The uncongested link is the road link whose density is less than the critical density and its travel time and flow rate satisfy a monotonically increasing function, such as BPR function.

The congested link is the road link whose density is larger than the critical density and its travel time and flow rate satisfy a monotonically decreasing function [37], [38].

Link performance function (or travel time/cost function) is the function that depicts the relationship between the link flow and the link travel time, including the uncongested link performance function and the congested link performance function.

Road network topology data used in this paper are the incidence relation data of OD pairs, routes, and links.

OD flow is the flow rate of vehicles from origin to destination (also called OD flow rate, unit: vehicles per unit time).

OD flow pattern is the OD flow matrix or table that describes the OD flow information of all OD pairs.

Uncongested Equilibrium Travel Time (UETT) is the travel time under the uncongested user equilibrium state.

Uncongested user equilibrium is the user equilibrium under the uncongested condition.

Congested Equilibrium Travel Time (CETT) is the travel time under the congested user equilibrium state.

Congested user equilibrium is the user equilibrium under the congested condition.

The basic network capacity is the maximum equilibrium network capacity.

The optimal OD flow patternis the pattern that generates the maximum total OD flows (or the maximum equilibrium network capacity).

It should be emphasized that traffic flows under both congested and uncongested conditions considered are assumed to be stable flow rates in this paper, which means that the link density distributes uniformly and the flow rates are equal on the same link.

B. EXAMPLE 1: THE OPTIMAL OD FLOW PATTERN

This example is used to illustrate the effect of different OD flow distribution proportions on the resulting road network capacity.

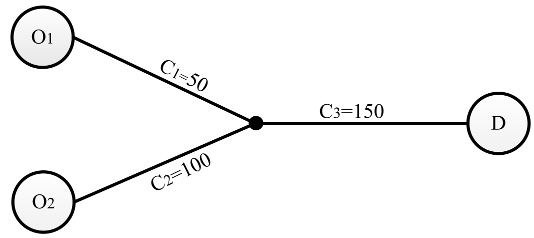

FIGURE 1. A road network with 2 OD pairs and 3 links.

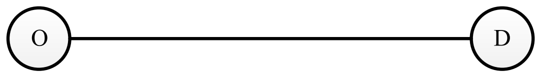

FIGURE 2. A road network with 1 OD pair and 1 link.

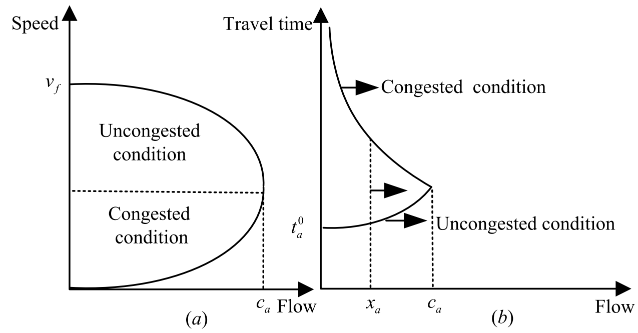

FIGURE 3. The relationship curves on a link. (a) Speed and flow. (b) Travel time and flow.

Consider the simple road network with the origin and destination nodes, as depicted in Figure 1. C_a means the link capacity. With this network topology and its attributes, only when the OD flows of O_1D and O_2D are 50 and 100, respectively, the network achieves the maximum utilization. Any other OD flow patterns will result in underutilization of this network, which is illustrated in the following three cases.

Case 1: If OD flow distribution proportion of O_1D and O_2D is fixed at [1:1], the maximum total OD flow value will be 100. This is because link 1 will reach the capacity at 50, limiting the resulting maximum total OD flows, unless the proportion is allowed to change.

Case 2: If the OD flow distribution proportion of O_1D and O_2D is fixed at [1:3], the resulting maximum total OD flow value is 133.3, as the link 2 will reach its capacity at 100.

Case 3: If the OD flow distribution proportion of O_1D and O_2D is fixed at [1:2], the maximum total OD flow value will be 150 and all links (or routes) capacities are fully utilized. Then, this OD distribution proportion generates the optimal OD flow pattern (i.e. OD flows of O_1D and O_2D are 50 and 100, respectively) and the maximum equilibrium network capacity.

From this simple example, it can be found that the OD flow patterns (or OD flow distribution) will influence the resulting (or calculated) network capacity. If the OD flow pattern is more congruous with the network topology, the resulting network capacity is higher.

C. EXAMPLE 2: THE MAIN IDEA OF MODELING

To illustrate the main idea of our proposed model, let us consider the road network capacity of the simple network with only one link, as depicted in Figure 2.

According to the Fundamental Diagram (FD) of traffic flow [40]–[42], each link possesses a critical point to distinguish two types of traffic conditions: the uncongested condition and the congested condition. Under the uncongested condition, as density increases, the flow (or flow rate) increases until the critical point is reached. Under the congested condition, the flow decreases because the speed decreases more rapidly at the higher density for safe driving. This is the fundamental hypothesis in traffic flow theory and the critical point is defined as the link capacity. The relationship curve of flow vs. speed is depicted in Figure 3a and the corresponding curve of travel time vs. flow, based on the relationship of speed vs. flow, is shown in Figure 3b. It should be noted that the assumption is that the link flow (or outflow) in the congested condition does not stabilize at the link capacity. In Figure 3, v_f is the free flow speed, t_a^0 the travel time at the free flow, x_a the link flow, and c_a the capacity of the link.

From Figure 3, some phenomena can be intuitively observed that a link has two types of link performance functions: the uncongested and the congested link performance functions. And the two curves will intersect at one point where the capacity of the link can be obtained.

Similarly, according to the theory of the Macroscopic Fundamental Diagram [43] and [44], there are three states in a road network: unsaturated (uncongested) state, saturated (capacity) state, and oversaturated (congested) state, in each of which the network outflow correspondingly increases, stabilizes, and decreases with the increasing vehicle accumulation or average network density. Moreover, the average vehicle speed decreases under the uncongested condition and increases under the congested condition with the increasing average flow (or network outflow). Correspondingly, the average travel time of trips respectively increases under the uncongested condition and decreases under the congested condition with the increasing average flow, as depicted in the relationship curve in Figure 3b.

In Figure 3b, we can also find that as the link flow increases, the link flow rate x_a reaches the maximum value (i.e. link capacity) when uncongested travel time of the link is the same as the congested one. In other words, the congested travel time can be regarded as the upper limit because it is the maximum travel time at the same flow rate. Once the uncongested link travel time equals to its upper limit, the corresponding flow is the capacity.

Based on the above analysis, a similar assumption for the network capacity of road networks is proposed: if OD flows of OD pairs increase until the OD Uncongested Equilibrium Travel Times (UETTs) are the same as the OD Congested Equilibrium Travel Times (CETTs), the total OD flows reach the maximum equilibrium network capacity. In this case,

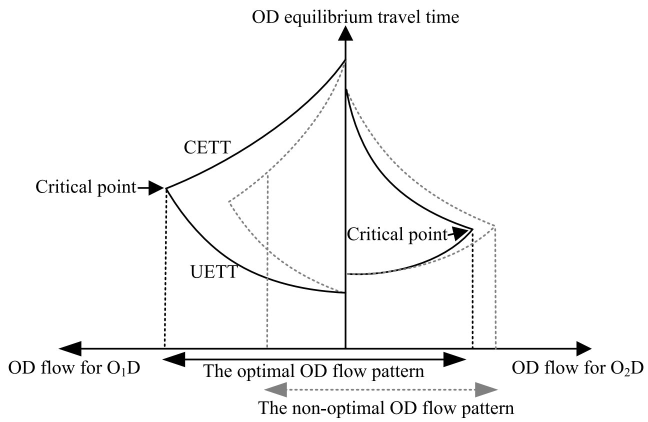

FIGURE 4. The optimal OD flow pattern that maximizes the total OD flows in a network with 2 OD pairs.

the CETT can be defined as an upper limit depending on OD flows. Because all the used routes between one OD pair are in a congested state, the corresponding equilibrium travel time (i.e. CETT) then can be regarded as the maximum equilibrium travel time (or the upper limit) at the same OD flow rate. Thus, once the UETTs of all OD pairs reach their upper limits, the routes between these OD pairs cannot serve more OD flows, and then the maximum equilibrium network capacity can be obtained.

To be specific, when the maximum equilibrium network capacity of a complex road network including multiple OD pairs and routes is to be solved, the two curves in Figure 3b can also be regarded as the relationship of the OD equilibrium travel time vs. the OD flow under the uncongested and the congested conditions, respectively. The UETT (the increasing curve) is obtained at the state where all used routes are uncongested. The CETT (the decreasing curve) is obtained at the state where all used routes are congested. The UETT should not exceed the CETT between the same OD pair. The intersection point of the two curves is the critical point where the UETT equals the CETT between the same OD pair. Consequently, the maximum equilibrium network capacity can be obtained by means of searching for these critical points of all OD pairs in the road network. Take the network in Figure 1 as an example. When the OD flows reach these critical points (i.e. the optimal OD flow pattern, see Figure 4), the capacities of routes between each OD pair can be fully utilized, and hence the total OD flows that the network can serve reaches the maximum value.

Considering that the road network capacity should be a stable and equilibrium value, we adopt the static traffic assignment with user equilibrium method to describe the equilibrium state of networks. In the next section, the SCTA will be introduced to help to search these critical points of all OD pairs.

III. METHODOLOGY

A. STATIC CONGESTED TRAFFIC ASSIGNMENT

In a general road network, the OD UETT under the given OD flows can be calculated by the SUTA model [39], which has been widely used in macroscopic traffic modeling. The CETT under the given OD flows can be calculated by the SCTA model [37] and [38], which is introduced below.

SCTA is applicable to the case in which an over-saturated OD flow pattern is assigned on the road network. The link flows and the link travel time are solved in a high-density and low-flow rate network, where all used links are congested. The model formulation of SCTA describes the case where the flow rate decreases and the travel time increases with the increasing vehicles or density, and hence a decreasing link performance function is used to describe the relationship between the flow rate and the travel time.

SCTA assumes that there are rigid over-saturated demands that desire to go through the congested network. Then, in the network, the behavior of users choosing the shortest routes will result in increasing travel time and decreasing flow rate on these shortest routes because the density increases (every user would like to crowd into the network and complete his/her trip as soon as possible, and hence a higher demand results in a higher density). More users would choose the shortest routes, which can lead to the increasing density but the decreasing speed and flow rate on the shortest routes. Relatively fewer users choose other longer routes, which can relieve the congested condition of these routes to some extent. Therefore, the density decreases and the speed as well as the flow rate increases. Finally, since the users always choose the shortest routes, the travel time on all used routes between the same OD pair is the same and all used routes are in a congested state. Thereby, the congested user equilibrium is produced. Though it rarely occurs that all used routes are in a congested state in the real road network, its equilibrium travel time can serve as an upper limit of the actual equilibrium travel time.

The mathematical model of the SCTA is described in **Appendix A**and the equivalency condition to the congested user equilibrium is described in **Appendix B.**

B. MODEL FORMULATIONS

Under the uncongested condition, the relationship between the UETT u_{rs} and the distribution flow q_{rs} for one OD pair can be described by an increasing function $u_{rs} = T^0_{rs}(q_{rs})$ when the flows of other OD pairs are invariable. Analogously, under the congested condition, the relationship between the CETT v_{rs} and the distribution flow q_{rs} for the same OD pair can be described by a decreasing function $v_{rs} = T^1_{rs}(q_{rs})$ when the flows of other OD pairs are invariable. Accordingly, for one given value of q_{rs}, there are two kinds of equilibrium travel time u_{rs} and v_{rs} under the uncongested and congested conditions, respectively. Moreover, there must be a point of intersection at $q_{rs} = q^c_{rs}$ when $u^c_{rs} = v^c_{rs}$ for each OD pair (see Figure 5), which is the critical point. Furthermore, the maximum equilibrium network capacity (or basic network capacity) $\boldsymbol{C}$ can be calculated by summing the distribution flows of all OD pairs when $u^c_{rs} = v^c_{rs}$ for all OD pairs (i.e. $C = \sum_{r,s} q^c_{rs}$).

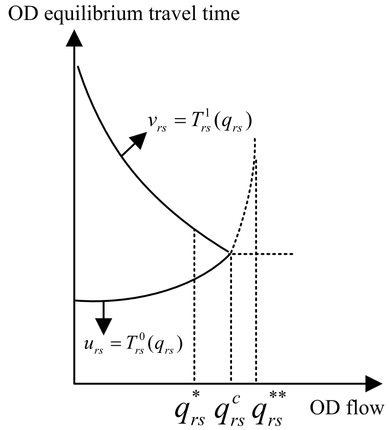

FIGURE 5. The relationship of OD equilibrium travel time and OD flow for one OD pair.

Proposition: In static traffic assignment with user equilibrium, the sum $\sum_{r,s} q^c_{rs}\ (r, s \in N)$ of all OD flows is the basic network capacity of a given road network. For each OD pair, the critical flow q^c_{rs} generates a pair of equal OD equilibrium travel time u^c_{rs} and v^c_{rs}, i.e. $u^c_{rs} = v^c_{rs} \quad \forall r, s$.

Proof: If $C = \sum_{r,s} q^c_{rs}$ is not the basic network capacity of the road network, it can be assumed that the basic network capacity is C^* with $C^* = \sum_{r,s} q^*_{rs}$ and $C^* < C$, or that the basic network capacity is C^{**} with $C^{**} = \sum_{r,s} q^{**}_{rs}$ and $C^{**} > C$.

Case 1: If $\sum_{r,s} q^*_{rs} < C$, the equilibrium travel time u^*_{rs} and v^*_{rs} according to q^*_{rs} can be calculated. There must be $u^*_{rs} < v^*_{rs}$ in some OD pairs (see Figure 5). Since the $T^0_{rs}(q_{rs})$ and $T^1_{rs}(q_{rs})$ are respectively the increasing and decreasing functions, a new pair of v_{rs} and u_{rs} can be found to satisfy $v_{rs} < v^*_{rs}$ and $u_{rs} > u^*_{rs}$, which means that a larger total OD flows can be accommodated by the network. Therefore, $C^* = \sum_{r,s} q^*_{rs}$ is not the basic network capacity of the road network.

Case 2: If $\sum_{r,s} q^{**}_{rs} > C$, there must be $u^{**}_{rs} > v^{**}_{rs}$ in some OD pairs (see Figure 5), which means that the UETT is larger than CETT between the same OD pair (i.e. the UETT exceed its upper limit). The result contradicts the fact. Therefore, $C^{**} = \sum_{r,s} q^{**}_{rs}$ is also not the basic network capacity of the road network.

To sum up, $C = \sum_{r,s} q^c_{rs}$ is the basic network capacity, QED.

According to the statements above, we propose the following model (called the basic model) to search for the OD flow pattern that meets $u^c_{rs} = v^c_{rs}$ of all OD pairs.

$$C = \sum_{r,s} q_{rs}. \tag{1}$$

$$\text{Min}: Z(X^0, X^1) = \sum_a \int_0^{x^0_a} t^0_a(w)dw - \sum_a \int_0^{x^1_a} t^1_a(w)dw, \tag{2}$$

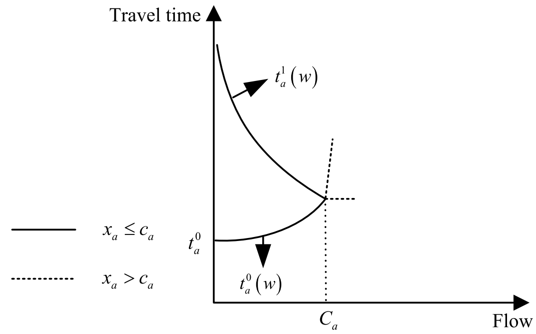

FIGURE 6. Curves of the uncongested and the congested link performance functions.

$$\text{subject to} \sum_k f^{rs}_k = q_{rs}, \tag{3}$$

$$x^0_a = \sum_r \sum_s \sum_k f^{rs}_k \sigma^{rs}_{a,k}, \tag{4}$$

$$f^{rs}_k \ge 0, \tag{5}$$

$$\sum_l g^{rs}_l = q_{rs}, \tag{6}$$

$$x^1_a = \sum_r \sum_s \sum_l g_l{}^{rs} \delta^{rs}_{a,l}, \tag{7}$$

$$g^{rs}_l > 0. \tag{8}$$

In the formulations, $t^0_a(w)$ is the uncongested link performance function and $t^1_a(w)$ is the congested link performance function, which can be derived from the FD; R_{rs} is the set of routes from origin r to destination s in the network; f^{rs}_k represents the flow on route $k \in R_{rs}$ under the uncongested condition and x^0_a represents the flow on link $a \in A$ under the uncongested condition; g^{rs}_l represents the flow on route $l \in R_{rs}$ under the congested condition and x^1_a represents the flow on link $a \in A$ under the congested condition; $\sigma^{rs}_{a,k}$ and $\delta^{rs}_{a,l}$ are link-route incidence indicators, and if link a is on uncongested route k between r and s, $\sigma^{rs}_{a,k} = 1$, else $\sigma^{rs}_{a,k} = 0$, and if link a is on congested route l between r and s, $\delta^{rs}_{a,l} = 1$; else $\delta^{rs}_{a,l} = 0$.

The uncongested link performance function $t^0_a(w)$ and the congested link performance function $t^1_a(w)$ are as follows.

$$t^0_a(w) = \begin{cases} t^0_a(x^0_a) & x^0_a \le c_a \\ t^0_a(c_a) + M\left(x^0_a - c_a\right)/c_a & x^0_a > c_a, \end{cases} \tag{9}$$

$$t^1_a(w) = \begin{cases} t^1_a(x^1_a) & x^1_a \le c_a \\ t^1_a(c_a) & x^1_a > c_a. \end{cases} \tag{10}$$

The curves of the two functions are depicted in Figure 6. To constrain the resulting link flow, the penalty function $M\left(x^0_a - c_a\right)/c_a$ is introduced into the uncongested link performance function Eq.(9). And parameter M is a large constant (i.e. $M \ll c_a$), which can ensure that the over-capacity flow has a large delay and prompt users to choose other unsaturated routes or links as much as possible. Moreover, to ensure that the CETT can be regarded as an upper limit value all along, the congested link performance function

Eq.(10) stabilizes at the capacity-travel time (i.e. $t_a^1(c_a)$) when the congested link flow exceeds its capacity. It should be emphasized that only the uncongested link flow pattern is the link flow result (i.e. actual link flow) that we need, and the congested link flow pattern is regarded as the by-product of the upper limit constraint, for it describes the network where all used routes or links are in a congested state. In terms of practice and management, the network state with congested link flow pattern is not acceptably safe.

In this programming, q_{rs} is variable. It should be noted that all origin and destination nodes of the network are determined in advance while the distribution flow of each OD pair is variable. Eq.(1) means that the basic network capacity equals the resulting total OD flows of all OD pairs. The objective function (2) is to minimize the difference between the sum of integrals of all link performance functions under the uncongested condition and that under the congested condition. Eq.(3) is the flow conservation constraints under the uncongested condition. Eq.(4) is the incidence relationship between link flows and route flows under the uncongested condition. Eq.(5) means the non-negative constraint of uncongested route flows. Eq.(6) is the flow conservation constraints under the congested condition. Eq.(7) is the incidence relationship between link flows and route flows under the congested condition. Eq.(8) means the non-negative constraint of congested route flows. Eqs.(3), (4), (6), and (7) describe the incidence relation of OD pairs, routes, and links, i.e. the road network topology data used in this paper.

The first part of the objective function (2) and constraint conditions (3)-(5) are identical with formulations of the SUTA model with user equilibrium (i.e. Beckmann model [39]). And the second part of the objective function (2) and constraint conditions (6)-(8) are identical with the formulations of the SCTA model with user equilibrium, which is described in **Appendix A**and**Appendix B**. The SCTA model can help to restrain the OD actual equilibrium travel time (i.e. UETT) and then flexibly adjust the OD flow pattern to maximize the total OD flows.

The link flow pattern x_a^c and the OD flow pattern q_{rs}^c can be calculated by solving the basic model. The basic network capacity can be calculated by $C = \sum_{r,s} q_{rs}^c$.

C. EQUIVALENCY CONDITION AND UNIQUENESS CONDITION

The Lagrangian equation of the minimization problem with respect to the equality constraints (3) and (6) can be formulated as follows.

$$L = Z(X^0, X^1) + \sum_{rs} u_{rs}\left(q_{rs} - \sum_k f_k^{rs}\right) - \sum_{rs} v_{rs}\left(q_{rs} - \sum_l g_l^{rs}\right), \quad (11)$$

subject to Eqs.(3)-(8).

The first-order conditions of Eq.(11) are:

$$f_k^{rs}\frac{\partial L}{\partial f_k^{rs}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial f_k^{rs}} \geq 0 \quad \forall k, r, s,$$
$$\text{i.e. } f_k^{rs}(c_k^{rs} - u_{rs}) = 0 \quad \text{and } c_k^{rs} - u_{rs} \geq 0 \quad \forall k, r, s, \quad (12)$$

$$g_l^{rs}\frac{\partial L}{\partial g_l^{rs}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial g_l^{rs}} \geq 0 \quad \forall l, r, s,$$
$$\text{i.e. } g_l^{rs}(d_l^{rs} - v_{rs}) = 0 \quad \text{and } d_l^{rs} - v_{rs} \leq 0 \quad \forall l, r, s, \quad (13)$$

$$q_{rs}\frac{\partial L}{\partial q_{rs}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial q_{rs}} \geq 0 \quad \forall r, s,$$
$$\text{i.e. } q_{rs}(u_{rs} - v_{rs}) = 0 \quad \text{and } u_{rs} - v_{rs} \geq 0 \quad \forall r, s, \quad (14)$$

$$\frac{\partial L}{\partial u_{rs}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial v_{rs}} = 0,$$
$$\text{i.e. } q_{rs} - \sum_k f_k^{rs} = 0 \quad \text{and } q_{rs} - \sum_k g_k^{rs} = 0 \quad \forall r, s, \quad (15)$$

where c_k^{rs} is the travel time on route k between OD pair (r, s) under the uncongested condition; u_{rs} is the Lagrange multiplier for equality constraint (3) and can also be regarded as the OD UETT; d_l^{rs} is the travel time on route l between OD pair (r, s) under the congested condition; v_{rs} is the Lagrange multiplier for constraint (6) and can also be regarded as the OD CETT.

The first-order conditions above are the necessary condition for the optimal solution of the basic model. In other words, if the first-order conditions are the same as the user equilibrium conditions, the solution of the basic model satisfies the user equilibrium conditions.

Eq.(12) means that the uncongested link flow results X^0 satisfy the uncongested user equilibrium principle (i.e. Wardrop's principle) [45]. If $f_k^{rs} > 0$, $c_k^{rs} = u_{rs}$; If $f_k^{rs} = 0$, $c_k^{rs} \geq u_{rs}$. It describes that at uncongested user equilibrium state, for each OD pair, the travel time on all used routes is equal, either shorter than or equal to the travel time on any unused route.

Eq.(13) means that the congested link flow results X^1 satisfy the congested user equilibrium, which represents the network where all used links are in a congested state. Since $g_l^{rs} > 0$, $d_l^{rs} = v_{rs}$ and the travel times of all congested routes between the same OD pair are equal. It shows that at congested user equilibrium state, the travel time on all used routes (also congested routes) connecting each OD pair will be equal [37], [38].

Eq.(14) indicates that there must be $u_{rs} = v_{rs}$ if $q_{rs} > 0$. That is to say, when the optimum solution is obtained, u_{rs} must equal v_{rs} between each OD pair (i.e. $u_{rs}^c = v_{rs}^c$ for all OD pairs), which satisfies the **Proposition**above. Therefore, the basic model (2)-(8) can obtain critical points of all OD pairs and calculate the basic network capacity.

Eq.(15) is the flow conservation constraint of each OD pair.

In summary, the first-order conditions (12)-(15) indicate that the proposed basic model can obtain the basic network capacity and that its optimal solution satisfies the user equilibrium principle.

The basic model can obtain the basic network capacity (C), the trip generation rate ($O_r = \sum_s q_{rs}$) and the trip completion rate ($D_s = \sum_r q_{rs}$), the resulting optimal OD flow pattern$\left(\cdots, q_{rs}^c, \cdots\right)$, and the resulting equilibrium link flow pattern (X^0, X^1), in which the basic network capacity, $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$, and the equilibrium link flow pattern are all unique, but the optimal OD flow pattern may not be unique.

To be specific, the basic model has one unique solution of the link flow pattern, whose proof is summarized in **Appendix C**. $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$ calculated by the unique link flow pattern are unique and the sum of all OD flows (i.e. the basic network capacity) is also unique. It should be noted that the optimal OD flow patterns generating the basic network capacity are not necessarily unique, whose proof is summarized in **Appendix D**. In other words, the resulting optimal OD flow pattern may be unique or multiple, which depends on the incidence relation of OD pairs and links, as shown in **Appendix D**. Surely, we can also guarantee a unique optimal OD flow pattern through extending the basic model. For example, if there are multiple OD flow patterns corresponding to the basic network capacity for a given network, the maximum entropy model can be introduced to choose the unique one with maximum entropy in these OD flow patterns, which will be further studied in future work.

In this paper, we focus on the basic network capacity and which OD flow pattern can reach the basic network capacity based on the road network itself, so the resulting optimal OD flow patterns may not go in-line with the existing travel behavior pattern. In other words, the resulting optimal OD flow pattern here is just the OD flow pattern that generates the maximum total OD flows (or the basic network capacity) but not the OD flow pattern that describes the existing travel behavior best. Since the real travel behavior of travelers will affect the practical network capacity, in the future, we will have a thorough consideration of that in the research of the practical network capacity, which can be further developed based on the basic model.

The fact that the basic network capacity, $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$, and the resulting equilibrium link flow pattern are all unique is quite useful in macroscopic transportation planning. For a unique basic network capacity, it can be used to evaluate the equilibrium relationship of road traffic supply and demand. For a unique $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$ of each node (i.e. traffic zone), it can be used to guide and control the land-use development intensity to match with the existing road network. Put it another way, we can guide the $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$ of traffic zones rather than directly adjust the OD flow pattern. For a unique link flow pattern, it can be used to identify the critical links that limit the road network capacity and optimize the road network design. These are potential applications of the basic model and are further illustrated in Section IV.

D. SOLUTION ALGORITHMS

Two methods are designed to solve the basic network capacity problem. The first one is a direct mathematical algorithm of the basic model (2)-(8), which is called the convex combinations method and needs to be specifically designed for the basic model. The second one is a solution method by network representation, which is to modify the original network and the basic model so that an existing and sophisticated algorithm (i.e. a standard user equilibrium algorithm) can be applied directly. Both of the two solution methods can be applied to general road networks. However, they have different characteristics. The first solution method is newly designed for the basic model, which is more suitable for large-scale networks. But in the application process, its algorithm codes need to be embedded in the traffic planning software. The second solution method, which adopts the existing algorithm whose codes has been embedded in the traffic planning software by modifying the original network, is easier to understand and more convenient for application. But it adds the number of OD pairs and links and hence is more suitable for middle and small-sized road networks instead of larger-scale networks as in the first solution method.

1) CONVEX COMBINATIONS METHOD

The proposed basic model (2)-(8) is a convex programming problem (see **Appendix C**) and can be solved by the convex combinations method which is widely adopted to solve the macroscopic traffic model. The modified convex combinations method for the basic model is designed as follows.

The main idea of the convex combinations method is summarized below.

a) Change the nonlinear objective function approximately into the linear function based on a given feasible solution.
b) Solve the linear programming to obtain the direction of steepest descent.
c) Search for the optimal move-size based on the direction of steepest descent and obtain the next feasible solution.
d) Repeat the above steps until the optimal solution is obtained.

The detailed solution procedure is designed as follows.

a) At *nth* iteration, the linear objective function to obtain the direction of steepest descent is

$$\min: \; Z^n(f_k^{rs}, g_k^{rs}) = \sum_{rs}\sum_k \frac{\partial Z^n(f_k^{rs}, g_k^{rs})}{\partial f_k^{rs}} h_k^{rs,n} + \sum_{rs}\sum_k \frac{\partial Z^n(f_k^{rs}, g_k^{rs})}{\partial g_k^{rs}} p_k^{rs,n}, \tag{16}$$

where $h_k^{rs,n}, p_k^{rs,n}$ are the auxiliary variables corresponding to f_k^{rs}, g_k^{rs} at step n.

b) The linear programming can be written as

$$\min: \; Z^n(f_k^{rs}, g_k^{rs}) = \sum_{rs}\sum_k c_k^{rs,n} h_k^{rs,n} - \sum_{rs}\sum_k d_k^{rs,n} p_k^{rs,n}, \tag{17}$$

$$\text{subject to } h_k^{rs,n} \geq 0 \quad \forall k \in R_{rs}, r \in N, s \in N, \tag{18}$$

$$p_k^{rs,n} \geq 0 \quad \forall k \in R_{rs}, r \in N, s \in N, \tag{19}$$

$$\sum_k h_k^{rs,n} \leq \bar{q}_{rs} \quad \forall k \in R_{rs}, r \in N, s \in N, \tag{20}$$

$$\sum_k p_k^{rs,n} \leq \bar{q}_{rs} \quad \forall k \in R_{rs}, r \in N, s \in N, \tag{21}$$

where $c_k^{rs,n}$ and $d_k^{rs,n}$ are the travel time of route k between OD pair (r, s) at step n under the uncongested condition and the congested condition respectively, and $\bar{q}_{rs}$ is the upper limit of OD flows and is introduced only for computational reasons.

To solve the linear programming (17)-(21) and then obtain $h_k^{rs,n}$ and $p_k^{rs,n}$, set

$$c_i^{rs,n} = \min_k \left\{c_k^{rs,n}\right\} = u^{rs,n} \quad \forall i, k \in R_{rs}, r, s \in N, \tag{22}$$

$$d_j^{rs,n} = \max_k \left\{d_k^{rs,n}\right\} = v^{rs,n} \quad \forall j, k \in R_{rs}, r, s \in N. \tag{23}$$

If $c_i^{rs,n} < d_j^{rs,n}$,

$$\begin{aligned} h_i^{rs,n} &= \bar{q}_{rs}, \quad h_k^{rs,n} = 0 \quad \forall k \neq i \in R_{rs}, \\ p_j^{rs,n} &= \bar{q}_{rs}, \quad p_k^{rs,n} = 0 \quad \forall k \neq j \in R_{rs}. \end{aligned} \tag{24}$$

If $c_i^{rs,n} < d_j^{rs,n}$,

$$h_k^{rs,n} = 0, \quad p_k^{rs,n} = 0 \quad \forall k \in R_{rs}. \tag{25}$$

If $c_i^{rs,n} = d_j^{rs,n}$, either Eq.(24) or Eq.(25) can be adopted.

Then the auxiliary link flow and the auxiliary OD flow are as follows.

$$y_a^{0,n} = \sum_{rs} \sum_k h_k^{rs,n} \sigma_{a,k}^{rs} \quad \forall a \in A, k \in R_{rs}, r, s \in N, \tag{26}$$

$$y_a^{1,n} = \sum_{rs} \sum_k p_k^{rs,n} \delta_{a,k}^{rs} \quad \forall a \in A, k \in R_{rs}, r, s \in N, \tag{27}$$

$$w_{rs}^n = \sum_k h_k^{rs,n} = \sum_k p_k^{rs,n} \quad \forall k \in R_{rs}, r, s \in N, \tag{28}$$

where $y_a^{0,n}$ and $y_a^{1,n}$ are the auxiliary variables corresponding to x_a^0 and x_a^1 at step n; w_{rs}^n are auxiliary variables corresponding to q_{rs} at step n.

Hence, the descent direction is a vector including $\left(y_a^{0,n} - x_a^{0,n}\right)$, $\left(y_a^{1,n} - x_a^{1,n}\right)$, and $\left(w_{rs}^n - q_{rs}^n\right)$.

c) The move size θ^n can be calculated by

$$\min Z(\theta^n) = \sum_a \int_0^{x_a^{0,n} + \theta\left(y_a^{0,n} - x_a^{0,n}\right)} t_a^0(w) dw - \sum_a \int_0^{x_a^{1,n} + \theta^n\left(y_a^{1,n} - x_a^{1,n}\right)} t_a^1(w) dw, \tag{29}$$

$$\text{subject to } 0 \leq \theta \leq 1. \tag{30}$$

Thus,

$$x_a^{0,n+1} = x_a^{0,n} + \theta\left(y_a^{0,n} - x_a^{0,n}\right), \tag{31}$$

$$x_a^{1,n+1} = x_a^{1,n} + \theta\left(y_a^{1,n} - x_a^{1,n}\right), \tag{32}$$

$$q_{rs}^{n+1} = q_{rs}^n + \theta\left(w_{rs}^n - q_{rs}^n\right). \tag{33}$$

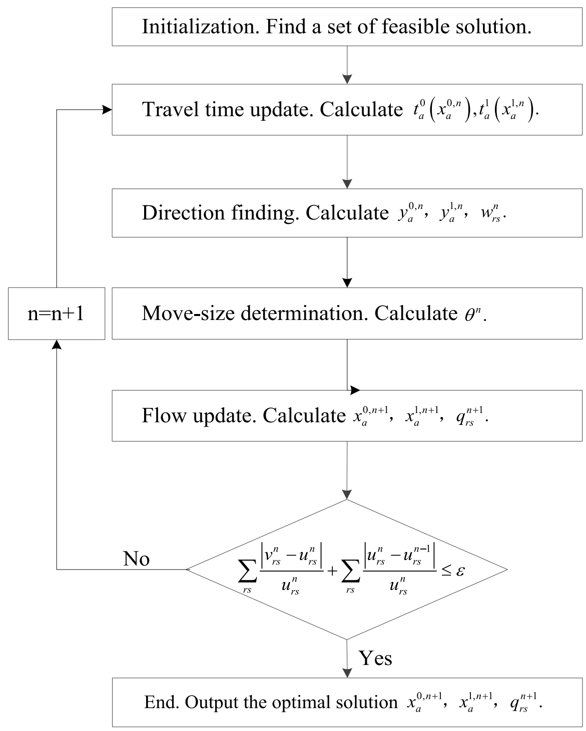

FIGURE 7. Flow diagram of the modified convex combinations method.

d) If $\sum_{rs} \frac{\left|v_{rs}^n - u_{rs}^n\right|}{u_{rs}^n} + \sum_{rs} \frac{\left|u_{rs}^n - u_{rs}^{n-1}\right|}{u_{rs}^n} \leq \varepsilon \forall r, s \in N$, where ε is a predetermined tolerance, the algorithm stops. Otherwise, let $n = n + 1$, and then the algorithm continues.

Based on the above arguments, this solution algorithm is described as follows, whose flow diagram is depicted in Figure 7.

Step 0: Initialization. Find a set of feasible $\{q_{rs}^n\}$, $\{x_a^{0,n}\}$, $\{x_a^{1,n}\}$. Set n =1.

Step 1: Travel time update. Calculate $t_a^{0,n} = t_a^0\left(x_a^{0,n}\right)$, $t_a^{1,n} = t_a^1\left(x_a^{1,n}\right)$, $\forall a$ by using Eqs.(9)-(10).

Step 2: Direction finding. Find $\{h_k^{rs,n}\}$ and $\{p_k^{rs,n}\}$ that minimize programming (17)-(21) and set

$$y_a^{0,n} = \sum_{rs} \sum_k h_k^{rs,n} \sigma_{a,k}^{rs} \quad \forall a \in A, k \in R_{rs}, r, s \in N,$$

$$y_a^{1,n} = \sum_{rs} \sum_k p_k^{rs,n} \delta_{a,k}^{rs} \quad \forall a \in A, k \in R_{rs}, r, s \in N,$$

$$w_{rs}^n = \sum_k h_k^{rs,n} = \sum_k p_k^{rs,n} \quad \forall k \in R_{rs}, r, s \in N.$$

Step 3: Move-size determination. Find θ^n by solving programming (29)-(30).

Step 4: Flow update. Set

$$x_a^{0,n+1} = x_a^{0,n} + \theta^n\left(y_a^{0,n} - x_a^{0,n}\right) \quad \forall a \in A,$$

$$x_a^{1,n+1} = x_a^{1,n} + \theta^n\left(y_a^{1,n} - x_a^{1,n}\right) \quad \forall a \in A,$$

$$q_{rs}^{n+1} = q_{rs}^n + \theta^n\left(w_{rs}^n - q_{rs}^n\right) \quad \forall r, s \in N.$$

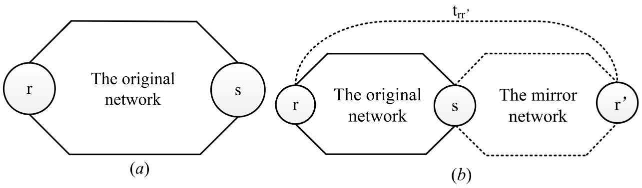

FIGURE 8. The zero-cost overflow network representation. (a) The original network. (b) The modified network.

Step 5: Convergence test. If $\sum_{rs} \frac{|v_{rs}^n - u_{rs}^n|}{u_{rs}^n} + \sum_{rs} \frac{|u_{rs}^n - u_{rs}^{n-1}|}{u_{rs}^n} \leq \varepsilon$, then the algorithm stops; otherwise let $n = n + 1$, and go to step 1.

2) THE METHOD BY NETWORK REPRESENTATION

This method consists of two steps. Firstly, a mirror network and a zero-cost overflow link are added to the original network. Secondly, a standard user equilibrium algorithm [46], [47] is applied to the modified network and the basic network capacity of the original network is obtained.

The original network of specific OD pairs is depicted in Figure 8a. Some modifications are made for the original network in Figure 8b. A set of dummy nodes r' and a dummy network connecting sr' are added, which are mirrored by the original network. The performance function for these dummy links is $-t_a^1(x_a^1) \quad a \in A$. A set of zero-cost overflow link rr' is also added to the dummy network and its performance function is $t_{rr'}(q_{rr'}) = 0 \quad r \in N$.

An excess-demand formulation to solve the equilibrium network capacity problem is also proposed as follows.

$$C = \sum_{r,r'} q_{rr'}, \tag{34}$$

$$\min: \ Z(X) = \sum_a \int_0^{x_a^0} t_a^0(w)dw - \sum_a \int_0^{x_a^1} t_a^1(w)dw + \sum_{rs} \int_0^{x_{rr'}} t_{rr'}(w)dw, \tag{35}$$

$$\text{subject to } \sum_k f_k^{rr'} + x_{rr'} = \bar{q}_{rr'}, \tag{36}$$

$$\sum_k f_k^{rr'} = q_{rr'}, \tag{37}$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rr'} \varphi_{a,k}^{rr'}, \tag{38}$$

$$f_k^{rr'} \geq 0, \tag{39}$$

$$x_{rr'} \geq 0. \tag{40}$$

In the formulations, $\bar{q}_{rr'}$ is a large given constant of each OD pair (r, r'); $x_{rr'}$ is the excess demand of each OD pair (r, r'); $q_{rr'}$ is the actual OD flow that we should solve. $R_{rr'}$ is the set of routes from origin r to destination r' in the network. If link a is on route k between r and r', $\varphi_{a,k}^{rr'} = 1$; otherwise, $\varphi_{a,k}^{rr'} = 0$. The meanings of other notations are the same as above.

The programming (34)-(40) can be solved by a standard user equilibrium algorithm [46] and [47] in the modified network. It can be found that the equilibrium travel time through the modified network is zero if the given $\bar{q}_{rr'}$ is large enough, the equilibrium travel time through the original network is equal to u_{rs}, and the equilibrium travel time through the dummy network is $-v_{sr'}$.

Since $t_{rr'}$ equals zero, $u_{rs} - v_{sr'}$ must be equal to zero at the equilibrium state. Thus, if the modified network is at equilibrium state, there must be

$$u_{rs} = v_{sr'} \quad \forall r, s \in N. \tag{41}$$

Consequently, if the given $\bar{q}_{rr'}$ is large enough to ensure that $x_{rr'}$ is larger than zero, the basic network capacity can be obtained (i.e. $C = \sum_{rs} q_{rs} = \sum_{rr'} q_{rr'}$).

E. MODEL CHARACTERISTICS

The basic model, which does not need a predetermined OD flow pattern that may influence the resulting network capacity, can develop the most potential capacity of the existing network by automatically adjusting the distribution flow of each OD pair based on the network topology data and the link attribute data.

One main merit of this basic model is that the basic network capacity depends on the network topology and the link attribute, but is independent of the OD flow pattern or other impedance parameters. In other words, a given OD flow pattern and other impedance parameters are not necessary to calculate the maximum equilibrium network capacity, which improves the practicability of the proposed model. Meanwhile, when the unique equilibrium network capacity is obtained, the corresponding OD flow pattern can also be obtained, which is the most congruous with the network topology to maximize the total OD flows.

The other main merit is that, without the link capacity constraint in the basic model, the sophisticated convex combinations method can be easily used to solve this model. Besides, the properties of computational convenience and the solution uniqueness of the basic model can be retained, which further ensures the practicability of the proposed model.

Essentially, the proposed methodology is based on the static traffic assignment method and the experienced convex combinations method (i.e. Frank-Wolfe algorithm), both of which are widely applied to transportation planning and real-world macroscopic road networks because of their ease of use and computational convenience. Therefore, theoretically, we believe that the practical strengths in realistic case studies of the proposed method can be warranted. In Section IV, a well-known and common test network, Sioux–Falls network, is used to test the basic model and its solution algorithms.

However, some idealized assumptions still exist in the basic model. For example, the signalized intersection and the level of service are not considered in the basic model so that

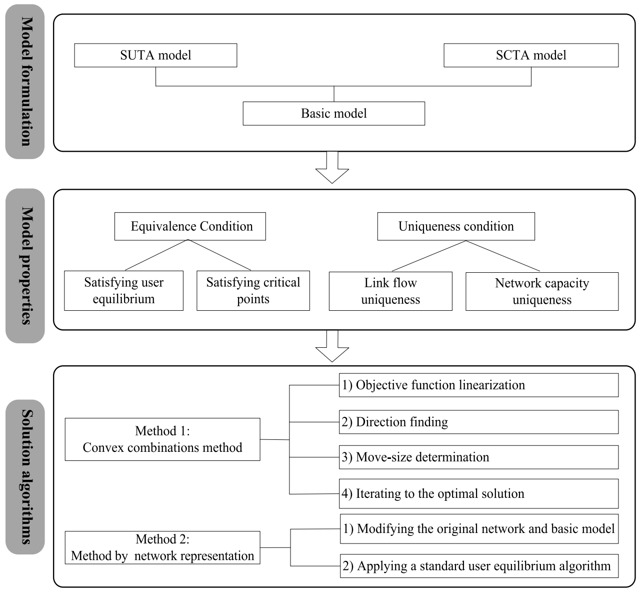

FIGURE 9. Methodology summary.

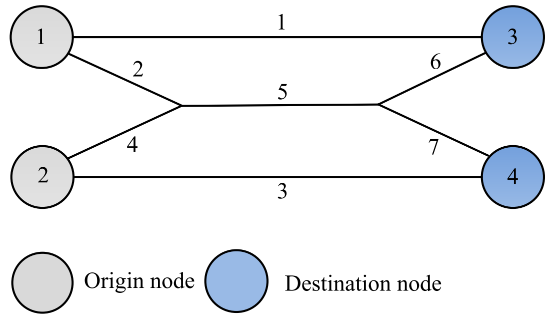

FIGURE 10. A test network.

TABLE 1. Related attribute data of road links.

Link number	t_a^0	t_a^m	c_a
1	10	32.84	100
2	4	13.14	80
3	12	39.41	80
4	4	13.14	80
5	5	16.42	120
6	5	16.42	50
7	4	13.14	50

the maximum equilibrium network capacity can be acquired. In spite of that, they can be easily considered in the future to solve practical network capacity problems.

F. METHODOLOGY SUMMARY

In order to demonstrate the overall methodology more clearly, a methodology summary is depicted in Figure 9.

The overall methodology is composed of three parts: the basic model formulation, model properties (the equivalence condition and the solution uniqueness), and the solution algorithms.

First, this basic model formulation consists of two parts, i.e. the SUTA model [39] and SCTA model (see **Appendix A.**). Then, we derive the first-order conditions of the basic model to prove the equivalence condition where the optimal solutions satisfy the user equilibrium and the critical points condition. And we prove the solution uniqueness of the basic model (see **Appendix C.**). Finally, we provide two methods to solve the basic network capacity problem. One is a direct mathematical solution algorithm of the basic model, called the convex combinations method, which needs to be specifically designed for the basic model. The other is an indirect method called the method by network representation.

IV. NUMERICAL EXAMPLES AND DISCUSSION

In numerical examples, acomparison with other modelsis made, and potential applicatios, as well as computational efficiency of the proposed model,areinvestigated.

A. A SMALL TEST NETWORK

Consider a test network in Figure 10, which is used in Chen and Kasikitwiwat [14] and Yang *et al.* [8]. In order to compare with other models [8], [14], the related attribute data of links used in this test network are basically the same as those in Chen and Kasikitwiwat [14] except the maximum travel time. The maximum travel time is a new index adopted in the basic model and its value is hypothetical in this paper, which will not change the basic model results. The test network consists of 7 links, 4 OD pairs (origin node 1, 2 and destination node 3, 4), and 6 distinct routes (including route 1 (link 1), route 2 (links 2-5-6), route 3 (links 2-5-7), route 4 (links 4-5-6), route 5 (links 4-5-7), and route 6 (link 3)), which are the network topology data of this test network. Table 1 shows the related road link attribute data. t_a^0 is the free-flow travel time. t_a^m is the maximum travel time under the jammed condition. c_a is the link capacity. The uncongested link performance function uses Eq.(42), which is based on the BPR function. The congested link performance function uses Eq.(43), which refers to **Appendix A**. Its parameters 0.3 and -4 in Eq.(43), which are assumed to test the model and solution algorithms, can also be acquired by fitting realistic traffic data. Besides, the two link-performance functions can use the travel time functions derived from the FD. It should be noted that the two function curves of each link intersect at the link-capacity point.

$$t_a^0\left(x_a^0\right) = \begin{cases} t_a^0\left[1+0.15(x_a^0/c_a)^4\right] & x_a^0 \le c_a \\ t_a^0\left[1+0.15\right] + M\left(x_a^0 - c_a\right)/c_a & x_a^0 > c_a, \end{cases} \tag{42}$$

$$t_a^1\left(x_a^1\right) = \begin{cases} t_a^m\left(1+0.3x_a^1/c_a\right)^{-4} & x_a^1 \le c_a \\ t_a^m\left(1+0.3\right)^{-4} & x_a^1 > c_a. \end{cases} \tag{43}$$

It is obvious that the proposed basic model needs the network topology data depicted in Figure 10 and link attribute data shown in Table 1 and Eqs.(42)-(43). And the results of the basic model are shown in Table 2, including the link-level data (link flow pattern $\boldsymbol{X^0}$), OD-level data (trip

TABLE 2. The results of the basic model.

Link	1	2	3	4	5	6	7
X^0	100.23	57.57	80.05	42.47	100.04	50.02	50.02
q_{13}				111.86			
q_{14}				45.94			
q_{23}				38.39			
q_{24}				84.13			
O_1				157.80			
O_2				122.52			
D_3				150.25			
D_4				130.07			
C				280.32			

TABLE 3. Comparison results of different network capacity models.

Capacity concepts	Network capacity	OD flow (1-3)	OD flow (1-4)	OD flow (2-3)	OD flow (2-4)
Reserve capacity	227.92	82.88	20.72	20.72	103.60
Practical capacity	257.58	100.80	37.03	31.36	88.39
Ultimate capacity	262.54	99.97	38.04	43.75	80.78
Basic capacity	280.32	111.86	45.94	38.39	84.13

generation rate O_r, trip completion rate D_s and the OD flow pattern q_{rs}), and network-level data (the unique basic network capacity C).

It can be verified that the uncongested equilibrium link flows, which are calculated by the basic model and at the same time, are identical with the results of Beckmann's model, satisfy the uncongested user equilibrium principle (i.e. Wardrop's principle). In addition, link 1, link 3, link 6, and link 7 are critical links that limit the basic network capacity, which can only be enhanced by expanding the capacities of the four critical links because all critical links are saturated. More importantly, if the road infrastructure investment is limited, these critical links can be expanded in priority. This information can be applied to the road network optimization design, especially in the case of limited land resource and investment.

Tables 3, 4 and Figure 11 provide the comparison results of other existing network capacity models including the reserve network capacity model, the ultimate network capacity model, and the practical network capacity model [14]. The relevant results of these three network capacity models in Table 3 and Table 4 come from [14].

All of the four models aim to search for the maximum throughput (i.e. the maximum total OD flows) that the network can handle. Moreover, they all are based on the deterministic user equilibrium principle without considering the signalized intersection. The main difference between the four models is the way that determines the OD flow patterns.

- The OD flow pattern in the reserve capacity model is fixed and given in advance. Based on this fixed OD flow pattern, the reserve network capacity is defined as the maximum throughput that the network can accommodate.
- The OD flow pattern in the ultimate capacity model, which is influenced by the impedance parameter that reflects the sensitivity of travelers to travel time, is changed by all travelers' destination choice. To be specific, travelers can choose the destination according to the total trip time. If the impedance parameter is higher, the effect of trip time to travelers' destination choice is larger. The ultimate network capacity is defined as the maximum throughput that the network can accommodate, in which all the OD flow can be scaled and adjusted by individual OD pairs.
- The OD flow pattern in the practical capacity model, which is influenced by both the destination cost function and the impedance parameter that reflects the sensitivity of travelers to travel time, is changed by the future travelers' destination choice. The practical network capacity is defined as the summation of the current OD flow and the future OD flow that the network can accommodate, in which only the future OD flow can be scaled and adjusted by individual OD pairs.
- The OD flow pattern in our basic capacity model is determined by the network topology data and the link attribute data.

TABLE 4. Link flow-capacity ratios of all links.

Link number	1	2	3	4	5	6	7
Reserve capacity model	0.83	0.26	1.00	0.89	0.54	0.41	0.89
Practical capacity model	1.00	0.45	0.99	0.84	0.65	0.65	0.91
Ultimate capacity model	1.00	0.47	1.00	0.91	0.69	0.88	0.78
Basic capacity model	1.00	0.72	1.00	0.85	0.83	1.00	1.00

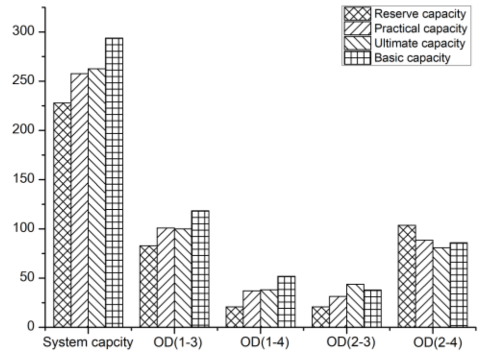

FIGURE 11. Comparison results of different network capacity models.

According to the definitions, both the ultimate network capacity model and the basic network capacity model are built to estimate the maximum equilibrium network capacity, but their resulting network capacity is different because the impedance parameter in the ultimate capacity model influences its resulting OD flow pattern.

Table 3 shows that our basic network capacity is the largest one among the four network capacities. It implies that the OD flow pattern obtained by the basic model fits the test network topology structure best (i.e. the optimal OD flow pattern). This information can be used to guide macroscopic traffic management or the land-use development for adjusting O_r

and $\boldsymbol{D_s}$, which can ensure that the corresponding OD flow pattern is more congruous with the existing network topology and hence the road network can serve more trips without link capacity enhancement.

Table 4 presents the link flow-capacity ratios of different models. Most of the link flow-capacity ratios calculated by the basic model are larger than those of the other three models except link 4, which means that the network is utilized better under this OD flow pattern. Furthermore, the capacities of all routes are fully utilized (i.e. every route includes at least one saturated link), so no more OD flow could be assigned to the test network. In other words, any other OD flow patterns cannot improve the resulting network capacity of the given network. It further illustrates the OD flow pattern calculated by the basic model is the optimal OD flow pattern. By comparison, in the results of the other three models, some routes are unsaturated and can be assigned more OD flows. Take the results of the ultimate capacity model as an example. Route 2 (2-5-6), route 3 (2-5-7), route 4 (4-5-6) and route 5 (4-5-7) can serve more OD flows if the OD flow pattern is optimized because their links are unsaturated.

The simple numerical example shows that both the basic equilibrium network capacity and the optimal OD flow pattern, which can provide important information for road network optimization design and the future land-use development plan, can be obtained by using the proposed model and its solution algorithms. To be specific, link-level data can help to identify the critical links that limit the network capacity, which is beneficial to road network optimization design; OD-level data can conduce to the macroscopic traffic management or land-use development control to guide the $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$ and make the resulting OD flow pattern coordinate with the existing network topology; Network-level data can contribute to evaluating the macroscopic equilibrium relationship of road traffic supply and demand.

B. NGUYEN-DUPUIS NETWORK

To demonstrate the practicability of the proposed model, the well-known Nguyen-Dupuis network [48] depicted in Figure 12 is used to test the basic model. Its network topology data are revealed in Figure 12. The link attribute data are given in Table 5. The uncongested link performance and the congested link performance functions employ Eqs.(42) and (43), respectively.

The results of the proposed model applied to Nguyen-Dupuis network are shown in Tables 6 and 7.

From Table 6, it can be found that links 1, 2, 3, 4, 11, 13, 15, 16, and 19 are critical links which limit the basic network capacity. The basic network capacity can be improved much by enhancing their capacities directly. This information can be applied to the road network optimization design.

We can further find that all the links directly connecting O_1 and O_4 (links 1, 2, 3, and 4) and all the links directly connecting D_2 and D_3 (links 11, 15, 16 and 19) are both saturated, which means that each route of all OD pairs is fully utilized under this resulting OD flow pattern. Any other OD flow patterns cannot improve the resulting basic network capacity of the given network unless the capacities of critical links are enhanced. It demonstrates that the calculated basic network capacity is the maximum equilibrium network capacity and the resulting OD flow pattern is the optimal OD flow pattern.

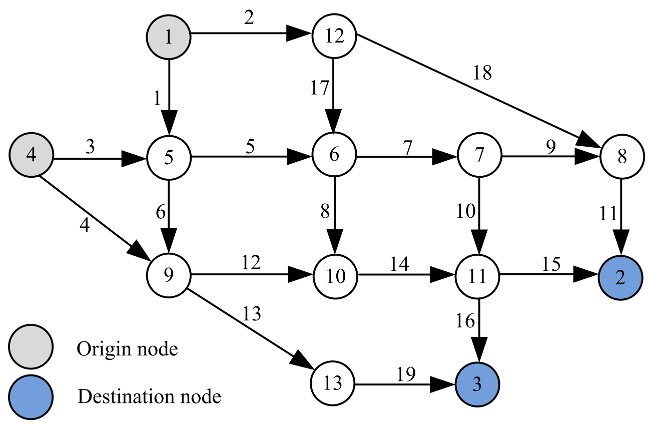

FIGURE 12. Nguyen-Dupuis network.

TABLE 5. Related data of links in Nguyen-Dupuis network.

Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles/ min)	Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles /min)
1	7	22.99	75	11	12	39.41	75
2	9	29.56	75	12	10	32.85	135
3	9	29.56	75	13	9	29.56	50
4	15	49.27	50	14	3	9.85	100
5	6	19.71	85	15	9	29.56	75
6	9	29.56	100	16	7	22.99	75
7	4	13.14	75	17	5	16.42	40
8	13	42.70	50	18	14	45.98	75
9	8	26.28	50	19	11	36.13	50
10	11	36.13	55				

TABLE 6. Link flow results of Nguyen-Dupuis network.

Link	x_a^0 (vehicles/min)	x_a^0 / c_a	Link number	x_a^0 (vehicles/min)	x_a^0 / c_a
1	75.16	1.00	11	75.25	1.00
2	75.14	1.00	12	72.37	0.54
3	75.14	1.00	13	50.03	1.00
4	50.15	1.00	14	98.98	0.99
5	66.44	0.78	15	75.17	1.00
6	72.25	0.72	16	75.14	1.00
7	64.95	0.87	17	13.52	0.34
8	26.62	0.53	18	61.62	0.82
9	13.63	0.27	19	50.03	1.00
10	51.32	0.93			

TABLE 7. The OD flow and network capacity results of Nguyen-Dupuis network.

OD pair number	OD (1-2)	OD (1-3)	OD (4-2)	OD (4-3)
OD flow	80.48	69.82	69.94	55.35
Trip generation rate	O_1: 150.30 (vehicles/min) O_4: 125.29 (vehicles/min)			
Trip completion rate	D_2: 150.42 (vehicles/min) D_3: 125.17 (vehicles /min)			
Basic network capacity (C)	275.59 (vehicles/min)			

TABLE 8. Related attribute data of links in Sioux Falls network.

Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles/ min)	Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles/ min)	Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles/ min)	Link	t_a^0 (min)	t_a^m (min)	c_a (vehicles /min)
1	6	19.71	75	21	10	32.85	50	41	5	16.43	75	61	4	13.14	75
2	4	13.14	125	22	5	16.43	50	42	4	13.14	100	62	6	19.71	50
3	6	19.71	75	23	5	16.43	50	43	6	19.71	125	63	5	16.43	50
4	5	16.43	100	24	10	32.85	100	44	5	16.43	50	64	6	19.71	100
5	4	13.14	75	25	3	9.86	75	45	3	9.86	75	65	2	6.57	75
6	4	13.14	125	26	3	9.86	75	46	3	9.86	75	66	3	9.86	75
7	4	13.14	75	27	5	16.43	50	47	5	16.43	50	67	3	9.86	75
8	4	13.14	75	28	6	19.71	75	48	4	13.14	50	68	5	16.43	75
9	2	6.57	100	29	4	13.14	50	49	2	6.57	50	69	2	6.57	75
10	6	19.71	75	30	8	26.28	125	50	3	9.86	75	70	4	13.14	50
11	2	6.57	100	31	6	19.71	50	51	8	26.28	50	71	4	13.14	50
12	4	13.14	100	32	5	16.43	125	52	2	6.57	100	72	4	13.14	100
13	5	16.43	50	33	6	19.71	75	53	2	6.57	75	73	2	6.57	50
14	5	16.43	100	34	4	13.14	75	54	2	6.57	75	74	4	13.14	50
15	4	13.14	75	35	4	13.14	50	55	3	9.86	50	75	3	9.86	125
16	2	6.57	100	36	6	19.71	125	56	4	13.14	75	76	2	6.57	100
17	3	9.86	75	37	3	9.86	55	57	3	9.86	75				
18	2	6.57	100	38	3	9.86	125	58	2	6.57	50				
19	2	6.57	75	39	4	13.14	125	59	4	13.14	50				
20	3	9.86	50	40	4	13.14	100	60	4	13.14	100				

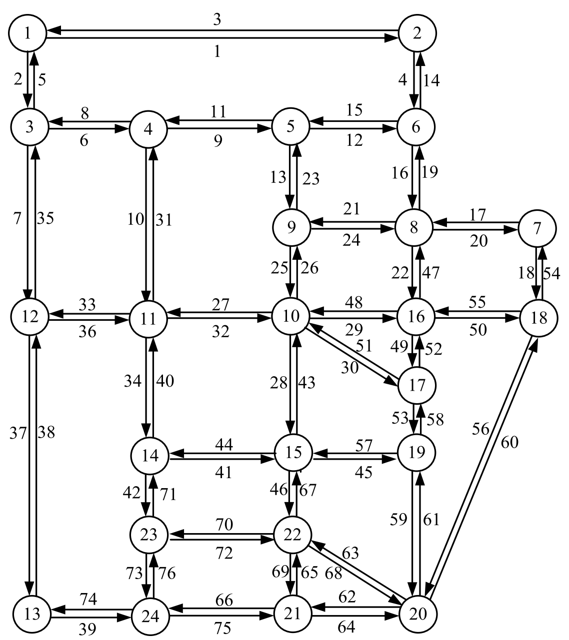

FIGURE 13. Sioux Falls network.

TABLE 9. The given OD pair information.

Sequence number	OD pair	Sequence number	OD pair
1	1-18	12	14-16
2	7-3	13	11-21
3	20-12	14	10-20
4	12-21	15	24-10
5	12-2	16	19-9
6	7-20	17	18-5
7	1-10	18	20-15
8	21-7	19	10-4
9	13-18	20	22-24
10	13-4	21	10-8
11	2-12	22	4-2

TABLE 10. The OD flow results of Sioux Falls network.

OD pair	OD flow (vehicles/min)	OD pair	OD flow (vehicles/min)
1-18	50.10	14-16	50.14
7-3	45.11	11-21	75.07
20-12	94.02	10-20	125.09
12-21	54.93	24-10	100.02
12-2	25.14	19-9	22.53
7-20	53.29	18-5	80.25
1-10	57.00	20-15	33.94
21-7	54.68	10-4	26.74
13-18	45.68	22-24	50.06
13-4	48.46	10-8	72.87
2-12	68.44	4-2	99.91
Basic network capacity (C)	1333.46 (vehicles/min)		

This OD-level information can help to guide the macroscopic traffic management or the land-use development control.

C. SIOUX-FALLS NETWORK

To verify the practicality of the proposed model further, the basic model is applied to another well-known test network, Sioux-Falls network, which is depicted in Figure 13. Its network topology data are revealed in Figure 13. The link attribute data of Sioux-Falls network are given in Table 8. The link performance functions employ Eqs.(42) and (43). We assume that there are 22 OD pairs in the network, and the location information of given OD pairs is shown in Table 9.

The results of the proposed basic model applied to Sioux-Falls network are shown in Tables 10 and 11.

From Table 11, we can find the critical links of this network. Moreover, it can be verified that all of the effective routes between the given OD pairs are saturated (i.e. every

TABLE 11. Link flow results of Sioux Falls network.

Link	x_a^0 (vehicles/min)	x_a^0 / c_a	Link	x_a^0 (vehicles/min)	x_a^0 / c_a	Link	x_a^0 (vehicles/min)	x_a^0 / c_a	Link	x_a^0 (vehicles/min)	x_a^0 / c_a
1	75.19	1.00	21	20.33	0.41	41	50.13	0.67	61	52.64	0.70
2	125.49	1.00	22	3.28	0.07	42	75.00	0.75	62	50.11	1.00
3	68.44	0.91	23	50.13	1.00	43	72.40	0.58	63	25.21	0.50
4	50.07	1.00	24	22.85	0.23	44	43.91	0.88	64	100.11	1.00
5	25.13	0.34	25	0.21	0.00	45	50.13	0.67	65	0.11	0.00
6	82.12	0.66	26	75.18	1.00	46	75.03	1.00	66	74.91	1.00
7	68.44	0.91	27	26.70	0.53	47	50.01	1.00	67	75.10	1.00
8	45.12	0.60	28	75.03	1.00	48	29.77	0.60	68	75.07	1.00
9	100.17	1.00	29	50.19	1.00	49	0.01	0.00	69	75.09	1.00
10	56.79	0.76	30	50.11	0.40	50	3.42	0.05	70	25.27	0.51
11	45.14	0.45	31	50.10	1.00	51	0.02	0.00	71	50.12	1.00
12	99.96	1.00	32	107.13	0.86	52	50.13	0.50	72	100.11	1.00
13	0.21	0.00	33	43.91	0.59	53	50.12	0.67	73	50.04	1.00
14	99.93	1.00	34	75.00	1.00	54	75.00	1.00	74	50.11	1.00
15	75.26	1.00	35	50.20	1.00	55	29.77	0.60	75	125.25	1.00
16	50.10	0.50	36	23.54	0.19	56	53.29	0.71	76	100.00	1.00
17	45.11	0.60	37	54.93	1.00	57	75.14	1.00			
18	100.12	1.00	38	98.73	0.79	58	50.15	1.00			
19	75.26	1.00	39	100.47	0.80	59	50.12	1.00			
20	46.83	0.94	40	94.04	0.94	60	100.21	1.00			

effective route of each OD pair includes at least one saturated link and the definition of effective routes refers to Dial [49]). Therefore, no more OD flows can be assigned to the test network, which demonstrates that the calculated basic network capacity is the maximum equilibrium network capacity and the resulting OD flow pattern is the most congruous with the network topology. Hence, it is the optimal OD flow pattern.

In addition, the solution algorithm (i.e. the convex combinations method) was run on a computer with Core i7@3.4Ghz running Windows 10 64-bit 8G RAM. It takes 830.64 CPU seconds to reach the user equilibrium with a convergence criterion in Step 5 of $\varepsilon = 10^{-2}$, and the resulting link flow error is less than 10^{-7}. It suggests that the basic model, as well as its solution algorithms, can be used on large-scale road networks.

V. CONCLUSION

In this paper, a basic model to compute the maximum equilibrium network capacity and searching for the optimal OD flow pattern is established based on the SUTA (Static Uncongested Traffic Assignment) and SCTA (Static Congested Traffic Assignment), in which the network topology data and link attribute data are needed and the sum of OD flows can be maximized by flexibly scaling and adjusting the individual OD pair flows. Furthermore, the equivalency and solution uniqueness of the basic model are proved. In addition, the solution algorithms of the basic model are designed, and the characteristics of the basic model are analyzed as well. Finally, a test network in Chen and Kasikitwiwat [14] is used to compare the basic model with some existing network capacity models and to analyze its potential applications. Two well-known test networks are employed to further demonstrate the practicality of the proposed model.

The outcomes of numerical examples indicate that a unique basic network capacity and link flow pattern, as well as the optimal OD flow pattern, can be obtained by the basic model. Moreover, the basic network capacity calculated by the basic model is the maximum equilibrium network capacity, and any other OD flow patterns cannot improve the basic network capacity of the test networks unless the capacities of critical links are enhanced. The test networks are utilized to the utmost extent under the optimal OD flow pattern. In addition, the proposed model does not need a preset and fixed OD flow pattern or other impedance parameters anymore, which further improves the practicability of the network capacity model.

By applying the basic model, the basic network capacity, critical links, and the optimal OD flow pattern are obtained, all of which are significant to road network optimization design and macroscopic traffic management or land-use development control. There should be the following potential applications.

- Evaluating whether the existing OD flow pattern matches the existing road network topology or not.
- Formulating the appropriate land-use development or traffic restraint policy for managing $\boldsymbol{O_r}$ and $\boldsymbol{D_s}$ in the existing network and making full use of the existing network capacity.
- Optimizing the existing road network for accommodating the current or future OD flow pattern, such as adding some new road links or expanding the capacity of critical links.
- Constructing the most suitable new road network in a newly-built city according to the future land-use

development which determines the future OD flow pattern.

More valuable extensions of the basic model are worth exploring in the future. For example, the signalized intersection, the level of service, and the real OD flow pattern will be considered to obtain a practical equilibrium network capacity. In addition, the land-use development model and road investment model can also be further incorporated into the basic model for the integrated design of land use and road networks, which is beneficial to the coordinated development between land use and transportation system as well as the sustainability of cities.

APPENDIX A. SCTA MODEL

$$\text{Min}: P(X^1) = -\sum_a \int_0^{x_a^1} t_a^1(w)dw$$

$$\text{Or max}: \bar{P}(X^1) = \sum_a \int_0^{x_a^1} t_a^1(w)dw \tag{A.1}$$

$$\text{subject to} \sum_l g_l^{rs} = q_{rs}, \tag{A.2}$$

$$x_a^1 = \sum_r \sum_s \sum_l g_l^{rs} \delta_{a,l}^{rs}, \tag{A.3}$$

$$g_l^{rs} > 0. \tag{A.4}$$

The meanings of notations are the same as the body text. $t_a^1(x_a)$ is the congested link performance function, which is monotonically decreasing with the flow rate, referring to [38]:

$$t_a^1(x_a) = t_a^m (1 + \alpha x_a / c_a)^\beta. \tag{A.5}$$

In Eq.(A.5), α and β, which can be obtained by fitting realistic traffic data, are model parameters and satisfy $\alpha > 0$ and $\beta < 0$; x_a is the flow rate on the congested link a; $c_a$is the capacity of link a; t_a^m is the maximum travel time under the jammed condition, which is used to simplify the calculation and establish a well-defined problem formulation.

APPENDIX B. EQUIVALENCY CONDITION

The Lagrangian equation of the maximization programming (A.1) with respect to the equality constraints (A.2) is

$$L = \bar{P}(X^1) + \sum_{rs} v_{rs}\left(q_{rs} - \sum_l g_l^{rs}\right). \tag{B.1}$$

The following first-order conditions must hold:

$$g_l^{rs}\frac{\partial L}{\partial g_l^{rs}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial g_l^{rs}} \le 0 \quad \forall l \in R_{rs}, r, s \in N. \tag{B.2}$$

Eq.(B.2) can be rewritten as:

$$\begin{cases} g_l^{rs}\left(c_l^{rs} - v_{rs}\right) = 0 \\ c_l^{rs} - v_{rs} \le 0, \end{cases} \quad \forall l \in R_{rs}, r, s \in N. \tag{B.3}$$

Eq.(B.3) holds for each used route between all OD pairs of the road network. Since all used routes must be congested routes and have positive flow rates, $g_l^{rs} > 0$ must hold for all congested routes of the road network, so $c_l^{rs} = v_{rs}$ for all used routes. Eq.(B.3) suggests that the travel times of all used routes (i.e. congested routes) between each OD pair are equal, which describes the congested user equilibrium state. The general first-order conditions of the SCTA model (A.1)-(A.4) satisfy the congested user equilibrium, and thereby the optimal solution of the SCTA model must satisfy the congested user equilibrium principle.

APPENDIX C. UNIQUENESS CONDITION FOR THE LINK FLOW

In order to show that the programming has a unique solution with respect to link flows (X^0, X^1), it is sufficient to prove that the objective function is strictly convex with respect to (X^0, X^1) and that the feasible region is convex.

The derivative of $Z(X^0, X^1)$ is taken with respect to the uncongested and the congested flows on the mth and nth links, respectively.

The first-order partial derivatives are

$$\frac{\partial Z(X^0, X^1)}{\partial x_m^0} = t_m^0\left(x_m^0\right), \tag{C.1}$$

$$\frac{\partial Z(X^0, X^1)}{\partial x_m^1} = t_m^1\left(x_m^1\right). \tag{C.2}$$

The second-order partial derivatives are

$$\frac{\partial Z^2(X^0, X^1)}{\partial x_m^0 \partial x_m^1} = \frac{\partial Z^2(X^0, X^1)}{\partial x_m^1 \partial x_m^0} = 0, \tag{C.3}$$

$$\frac{\partial^2 Z}{\partial x_m^0 \partial x_n^0} = \frac{\partial t_m\left(x_m^0\right)}{\partial x_n^0} = \begin{cases} \frac{dt_m\left(x_m^0\right)}{dx_n^0} for & m = n \\ 0 & m \ne n, \end{cases} \tag{C.4}$$

$$\frac{\partial^2 Z}{\partial x_m^1 \partial x_n^1} = \frac{\partial t_m\left(x_m^1\right)}{\partial x_n^1} = \begin{cases} \frac{dt_m\left(x_m^1\right)}{dx_n^1} for & m = n \\ 0 & m \ne n. \end{cases} \tag{C.5}$$

The Hessian matrix is

$$\nabla^2 Z\left(X^0, X^1\right) = \begin{bmatrix} \frac{dt_1^0\left(x_1^0\right)}{dx_1^0} & & & & & \\ & \cdots & & & & \\ & & \frac{dt_A^0\left(x_1^0\right)}{dx_A^0} & & & \\ & & & -\frac{dt_1^1\left(x_1^1\right)}{dx_1^1} & & \\ & & & & \cdots & \\ & & & & & -\frac{dt_A^1\left(x_1^1\right)}{dx_A^1} \end{bmatrix}.$$

A is the number of links.

$$\nabla^2 T_a(x) = \begin{bmatrix} t_0'\left(x_a^0\right)\pi_1^{a0}\pi_1^{a0} - t_1'\left(x_a^1\right)\pi_1^{a1}\pi_1^{a1} & \cdots & t_0'\left(x_a^0\right)\pi_1^{a0}\pi_R^{a0} - t_1'\left(x_a^1\right)\pi_1^{a1}\pi_R^{a1} \\ \cdots & \cdots & \cdots \\ t_0'\left(x_a^0\right)\pi_R^{a0}\pi_1^{a0} - t_1'\left(x_a^1\right)\pi_R^{a1}\pi_1^{a1} & \cdots & t_0'\left(x_a^0\right)\pi_R^{a0}\pi_R^{a0} - t_1'\left(x_a^1\right)\pi_R^{a1}\pi_R^{a1} \end{bmatrix}.$$

Since $t_1^0\left(x_a^0\right)$ is monotonically increasing and $t_1^1\left(x_a^1\right)$ is monotonically decreasing, $\nabla^2 Z\left(X^0, X^1\right)$ is strictly positive definite and the objective function is strictly convex. Furthermore, since the feasible region defined by linear equality constraints is convex, this programming is a convex programming problem and has a unique minimum.

APPENDIX D. NON-UNIQUENESS CONDITION FOR THE OD FLOW

We prove that the objective function is not strictly convex with respect to the OD flow.

The incidence relations between the link flow and the OD flow are:

$$x_a^0 = \sum_{rs} \pi_{rs}^{a0} q_{rs}, \tag{D.1}$$

$$x_a^1 = \sum_{rs} \pi_{rs}^{a1} q_{rs}, \tag{D.2}$$

where π_{rs}^{a0} and π_{rs}^{a1} are the ratio of the partial flow from OD pair rs on link a to the total flow on link a under the uncongested condition and the congested condition, respectively, and $0 \le \pi_{rs}^{a0} \le 1$, $0 \le \pi_{rs}^{a1} \le 1$, $\sum_{rs} \pi_{rs}^{a0} = 1$, and $\sum_{rs} \pi_{rs}^{a1} = 1$.

With respect to mth OD pair flow (i.e. q_{rs}^m), the first-order partial derivative is

$$\begin{aligned} \frac{\partial Z(X^0, X^1)}{\partial q_{rs}^m} &= \frac{\partial Z(X^0, X^1)}{\partial x_a^0}\frac{\partial x_a^0}{\partial q_{rs}^m} + \frac{\partial Z(X^0, X^1)}{\partial x_a^1}\frac{\partial x_a^1}{\partial q_{rs}^m} \\ &= t_0\left(x_a^0\right)\pi_{rs}^{a0} - t_1\left(x_a^1\right)\pi_{rs}^{a1}. \end{aligned} \tag{D.3}$$

With respect to mth and nth OD pair flow (i.e. q_{rs}^m and q_{rs}^n), the second-order partial derivative is

$$\begin{aligned} \frac{\partial Z^2(X^0, X^1)}{\partial q_{rs}^m \partial q_{rs}^n} &= \frac{\partial\left[t_0\left(x_a^0\right)\pi_m^{a0} - t_1\left(x_a^1\right)\pi_m^{a1}\right]}{\partial x_a^0}\frac{\partial x_a^0}{\partial q_{rs}^n} \\ &\quad + \frac{\partial\left[t_0\left(x_a^0\right)\pi_m^{a0} - t_1\left(x_a^1\right)\pi_m^{a1}\right]}{\partial x_a^1}\frac{\partial x_a^1}{\partial q_{rs}^n} \\ &= t_0'\left(x_a^0\right)\pi_m^{a0}\pi_n^{a0} - t_1'\left(x_a^1\right)\pi_m^{a1}\pi_n^{a1} \\ \frac{\partial Z^2(X^0, X^1)}{\partial q_{rs}^m \partial q_{rs}^n} &= \frac{\partial\left[t_0\left(x_a^0\right)\pi_m^{a0} - t_1\left(x_a^1\right)\pi_m^{a1}\right]}{\partial x_b^0}\frac{\partial x_b^0}{\partial q_{rs}^n} \\ &\quad + \frac{\partial\left[t_0\left(x_a^0\right)\pi_m^{a0} - t_1\left(x_a^1\right)\pi_m^{a1}\right]}{\partial x_b^1}\frac{\partial x_b^1}{\partial q_{rs}^n} = 0, \end{aligned}$$

when $a \ne b$.

The Hessian matrix is

$$\nabla^2 Z\left(q_{rs}\right) = \begin{bmatrix} \nabla^2 T_1(x) & & \\ & \cdots & \\ & & \nabla^2 T_A(x) \end{bmatrix}. \tag{D.4}$$

This is a $(A \times R) \times (A \times R)$ diagonal matrix, where $\nabla^2 T_a(x)$ is a $(R \times R)$ matrix shown at the top of this page. R represents the number of OD pairs.

If link a $(a = 1, 2, 3 \cdots, A)$ does not connect the mth OD pair, its π_m^{a0} and π_m^{a1} $(m = 1, 2, 3 \cdots, R)$ are zero, and hence the leading principal minor of $\nabla^2 Z\left(q_{rs}\right)$ cannot be guaranteed to be positive. The Hessian matrix with respect to q_{rs} is not strictly convex definite. Hence, the objective function with respect to q_{rs} is not strictly convex and the optimal OD flow pattern is not necessarily unique.

ACKNOWLEDGMENT

The authors would like to thank Prof. Junyi ZHANG from Hiroshima University for lots of insightful discussions and thank editors and the anonymous reviewers for their constructive comments.

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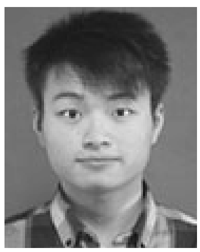

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