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An Explicit Method for Stability Analysis of 2D Systems Described by Transfer Function

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ABSTRACT In this paper, we propose a method to test the stability of two-dimensional (2D) linear discrete systems described by transfer function. And a complete region of disturbance parameters is solved to ensure the considered system with perturbations stability. Different from any other traditional algebraic methods, the algorithm of this method is explicit, which a large number of higher-order polynomial iterative operation doesn't need to be carried out. By the fractional linear transformations, new condition is given in Hurwitz theorem, which doesn't involve fractional calculation. The method is non-conservative for stability analysis and solving stable parameter region of 2D systems. It simplifies some existing methods. The computational cost is reduced. And it is better to solve the stability problems for 2D systems with disturbances. Examples illustrate the efficiency for testing the stability of higher-order 2D system and solving the robust stability problem of uncertain 2D systems.

INDEX TERMS 2D systems, stability analysis, robust stability, explicit algorithm, transfer function.

I. INTRODUCTION

Two-dimensional systems, traced back to the mid 1970s [1]–[3], are a class of dynamic systems which propagate information in two independent directions [4]–[6]. Recently, 2D systems have attracted a lot of attention due to their widespread applications in various areas such as image and video processing, multi-dimensional digital filtering, batch process, thermal process, signal processing, and sensor networks [7]–[11]. The methods of stability and robust stability of 2D system can be widely applied in these practical applications to keep them in normal operation. Different from 1D systems, 2D systems depends on two independent variables and the available related 2D preliminaries are insufficient. That makes the analysis and synthesis of 2D systems much more complicated and difficult than the 1D systems [4], [12], especially for uncertain 2D systems. The characteristic of stability is one of the main research problems in control theory. For the above factors, the stability of 2D systems has received extensive interests.There are a lot of results on the stability of 2D systems have been proposed [13]–[18]. Note that, with focus on applied researches, we refer the readers to some practical works in [18]–[21]. Although lots of significant results on the stability of 2D discrete linear systems have been derived now, there are many limitations in these results. Hence, it is meaningful to study the stability and robust stability of 2D systems.

The methods for 2D systems described by transfer function to analyze the stability may be divided into two classes. One is finite verification algorithm in frequency domain. For this class of methods, some are sufficient [22] and some are necessary [23], [24]. The other one is algebraic, and the conditions of the results of this class are sufficient and necessary, such as Sturm's theorem criterion [25], [26], the Hurwitz-Schur algorithm stability criterion [27], [28] and the method of polynomial discriminant system theory [29]. While due to the complexity of the structure of 2D system, the stability analysis of 2D systems is complicated and difficult, especially for higher-order systems.

Note that one of the algebraic methods, the method of polynomial discriminant system theory, is based on Jury array [29], [30]. It transforms the conditions of testing stability into new conditions of checking whether a polynomial has real roots [30]. For this new problem, it can be solved using the discriminant system method in [31]. This method in detecting 2D system stability is simpler than other traditional algebraic methods. However, the testing stability steps include Jury Array. This calculation of Jury Array [29], [30] is recursive and involves a higher-order matrix fractional computation. It's difficult to check the 2D systems with

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uncertain parameters. The process of this method is not tractable since the computation of Jury array is implicit and increases rapidly with the increasing of variables.

This paper shows a concise method. Different from the above method based on Jury array, this process doesn't include Jury array. By the fractional linear transformations, new condition is given in Hurwitz theorem, which doesn't involve fractional and recursive calculation. The method is non-conservative for stability analysis and solving stable parameter region of uncertain 2D systems. It simplifies the existing method. The computational cost is reduced. And it is better to solve the robust stability problems for 2D systems with disturbances without fractional calculation. We change the problem of stability analysis to a new problem whether the polynomials are Hurwitz stable by fractional linear transformations. Then, the new problem can be easily dealt with by the discriminant system of polynomial in [31]. This method is more efficient and explicit without fractional computation and iterative operation.

The rest of the paper is organized as follows. The problems that need to be solved is specified by mathematical description in section II. In section III, a transformation from the Schur Stability criterion to Hurwitz theorem and two efficient algorithms for analyzing the stability and robust stability are presented. In Section IV, examples are given for proving the effectiveness of the algorithms. And it gives a comparison between the method of this paper and the method based on the traditional algebraic methods to illustrate the former is more efficient. Conclusions are given in Section V.

II. MATHEMATICAL DESCRIPTION

The given recursive model for 2D linear discrete system is as follow:

$$\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}y(k-i,l-j)=\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}u(k-i,l-j), \quad (1)$$

where $u(k,l)$ and $y(k,l)$ are 2D input signals and 2D output signals, respectively. a_{ij} and b_{ij} are constants. The transfer function of this system is given by:

$$H(z_1,z_2)=\frac{A(z_1,z_2)}{B(z_1,z_2)}, \quad (2)$$

where $A(z_1,z_2)=\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}z_1^i z_2^j$, $B(z_1,z_2)=\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}z_1^i z_2^j$.

Remark 1: We take a recursive model as an example to analyze the stability. If one transfer function of other linear discrete model is given, then we can also use this method to analyze stability of the system.

The stability of the system (1) depends on whether denominator polynomial $B(z_1,z_2)$ of the transfer function (2) is Schur stable, that is, whether the denominator polynomial satisfies the following definition.

Definition 1 [29]: The 2D polynomial $B(z_1,z_2)$ is called Schur stable, if $B(z_1,z_2)$ satisfies

$$B(z_1,z_2)\neq 0, \quad \forall (z_1^i z_2^j): |z_1|\leq 1, |z_2|\leq 1. \quad (3)$$

Remark 2: We assume that $H(z_1,z_2)$ in this paper doesn't have any nonessential singularities of the second type. Specifically, there is no (z_1^*,z_2^*) which satisfies $A(z_1^*,z_2^*)=B(z_1^*,z_2^*)=0$. So it's obvious that the proposed method has no the disadvantages of the bilinear applications for 2D systems described by transfer function in [32]. We can further get the discussion as follows.

The BIBO stability of the system is equivalent to Schur stability [23]. It is difficult to determine the Schur stability of a 2D polynomial by directly using condition (3). Literature [23] simplified this determination process, $H(z_1,z_2)$ is Schur stable if and only if

$$B(z_1,0)\neq 0, \quad \forall z_1: |z_1|\leq 1, \quad (4)$$

and

$$B(z_1,z_2)\neq 0, \quad \forall (z_1^i z_2^j): |z_1|=1, |z_2|\leq 1. \quad (5)$$

Through variable substitution, the condition (4) is equivalent to a new condition whether the real polynomial is Hurwitz stable, which can be solved by a lot of methods such as Hurwitz criterion. It's easily tractable.

Stability detection of a 2D linear system is mainly focused on testing condition (5). Many literatures have studied this problem and put forward different test methods. The stability analysis method of the polynomial discriminant system theory makes fractional transformation of variable z_1 in $B(z_1,z_2)$. Condition (5) is turned into checking whether a polynomial has real roots. In the calculation of Jury array, the computational complexity of this algorithm will be increased with the increasing number of variables. We have to think about whether the denominator is zero in the fractional calculation. If the system contains parameters, the fraction analysis will be too complicated to carry out. Based on the stability analysis method of the polynomial discriminant system theory, this paper does linear fraction transformation to the variable z_2 in condition (5). The new condition can be simply tested by the Hurwitz criterion which does't involve Jury array.

For discussing the robust stability in the next section, we call the recursive model of 2D linear discrete systems with uncertain parameters as follows:

$$\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}c_{ij}y(k-i,l-j)=\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}u(k-i,l-j) \quad (6)$$

where c_{ij} is uncertain parameter, $c_{ij}\in R$.

The transfer function is shown as follows:

$$\widetilde{H}(z_1,z_2)=\frac{M(z_1,z_2)}{N(z_1,z_2)}, \quad (7)$$

where $M(z_1,z_2)=\sum_{j=0}^{n} a_{ij}z_1^i z_2^j$, $N(z_1,z_2)=\sum_{j=0}^{n} b_{ij}c_{ij}z_1^i z_2^j$. we assume that $\widetilde{H}(z_1,z_2)$ has no the second type nonessential singularities.

III. MAIN RESULTS

A. STABILITY ANALYSIS

The aim of this subsection is to get a new condition by linear fraction transformation based on condition (5). An algorithm for analyzing stability of 2D discrete linear systems is presented.

Let

$$F(s, z_2) = (1 - js)^m B(\frac{1 + js}{1 - js}, z_2), \tag{8}$$

where the degrees of $F(s, z_2)$ in s and z_2 are m and n, respectively. It's easy to know that condition (5) is equivalent to

$$B(-1, z_2) \neq 0, \quad \forall z_2 : |z_2| \leq 1, \tag{9}$$

and

$$F(s, z_2) \neq 0, \quad \forall s : s \in R, \ \forall z_2 : |z_2| \leq 1. \tag{10}$$

Thus, the Schur theorem is changed to the conditions (4), (9) and (10). If a transfer function's denominator polynomial of the 2D linear discrete system satisfies the three conditions, the proposed system is stable. Now we further analyze and simplify these three conditions (4), (9) and (10).

Making the fractional linear transformations of z_1 in condition (4) and z_2 in condition (9)

$$z_1 = \frac{1 - x}{1 + x}, \quad z_2 = \frac{1 - y}{1 + y},$$

we can obtain, respectively

$$Y_1(x, 0) = (1 + x)^m B(\frac{1 - x}{1 + x}, 0), \tag{11}$$

and

$$Y_2(-1, y) = (1 + y)^m B(-1, \frac{1 - y}{1 + y}). \tag{12}$$

By the transformation, the unit disk $d_1 \equiv (z_1, |z_1| \leq 1)$ in the z_1 plane is mapped into $d'_1 \equiv (x, Re(x) \geq 0)$ in the x plane, and the unit disk $d_2 \equiv (z_2, |z_2| \leq 1)$ in the z_2 plane is mapped into $d'_2 \equiv (y, Re(y) \geq 0)$ in the y plane. Therefore, conditions (4) and (9), respectively, are equivalent to

$$Y_1(x, 0) \neq 0, \quad \forall x : Re(x) > 0, \tag{13}$$

$$Y_2(-1, y) \neq 0, \quad \forall y : Re(y) > 0. \tag{14}$$

$Y_1(x, 0)$, $Y_2(-1, y)$ are real coefficient polynomials in x and y, respectively. It's obvious that conditions (13) and (14) is the problems whether $Y_1(x, 0)$ and $Y_2(-1, y)$ are Hurwitz stable. This problem can be solved by the Hurwitz criterion in the following lemma.

Lemma 1 [33]: Let $f(\tau) = a_n\tau_n + a_{n-1}\tau_{n-1} + \ldots + a_0 (a_n > 0)$ be a real coefficient polynomial, where $a_i, i = 0, \ldots n$ are real. The $n \times n$ Hurwitz matrix is constructed from $f(\tau)$, as follow:

$$H_f = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix},$$

we denote the order principal minor of H_f by $\triangle_k(f)$, $k = 1, 2 \ldots, n$, respectively. The necessary and sufficient condition for the roots' real part of $f(\tau)$ to be negative is $\triangle_k(f) > 0, k = 1, 2 \ldots, n$.

Similar to condition (4) and (9) processing method, condition (10) is changed as follows.

Making the fractional linear transformation of z_2 in condition (10)

$$z_2 = \frac{1 - z}{1 + z},$$

we obtain

$$Y_3(s, z) = (1 + z)^n F(s, \frac{1 - z}{1 + z}). \tag{15}$$

By the transformation, the unit disk $d_3 \equiv (z_2, |z_2| \leq 1)$ in the z_2 plane is mapped into $d'_3 \equiv (z, Re(z) \geq 0)$ in the z plane. Condition (10) is changed into

$$Y_3(s, z) \neq 0, \quad \forall s : s \in R, \ \forall z : Re(z) \geq 0. \tag{16}$$

$Y_3(s, z)$ can be represented as a complex coefficient polynomial in z. We represent condition (16) as follows,

$$Y_3(s, z) = \sum_{i=0}^{n} a_i(s) z^i \neq 0, \quad \forall s : s \in R, \ \forall z : Re(z) \geq 0, \tag{17}$$

condition (17) is equivalent to whether $Y_3(s, z)$ is Hurwitz stable. We recall the following lemma.

Lemma 2 [33]: Let $f(\tau)$ be a complex coefficient polynomial and satisfy $f(j\tau) = b_n\tau_n + b_{n-1}\tau_{n-1} + \ldots + b_0 + j(a_n\tau_n + a_{n-1}\tau_{n-1} + \ldots + a_0)$, $a_n \neq 0$, where a_i and b_i, $i = 0, \ldots n$ are real. The $2n \times 2n$ Hurwitz matrix is constructed from $f(\tau)$, as follow:

$$H_f = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & 0 \\ b_n & b_{n-1} & b_{n-2} & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \\ 0 & 0 & 0 & \cdots & b_0 \end{bmatrix},$$

we denote the $2k_{th}$ order principal minor determinant of H_f by $\triangle_{2k}(f), k = 1, 2 \ldots, n$, respectively. If $\triangle_{2k}(f) \neq 0$, the necessary and sufficient condition for the roots' real part of $f(\tau)$ to be negative is $\triangle_{2k}(f) > 0, k = 1, 2 \ldots, n$.

To sum up, we change the testing condition of Schur theorem to a new one through the technique of fractional linear transformations. Then we have the following theorem.

Theorem 1: If $H(z_1, z_2)$ has no the second type nonessential singularities, then the system is BIBO stable if and only if the following conditions are satisfied,

a. $Y_1(x, 0) \neq 0, \forall x : Re(x) > 0$
b. $Y_2(-1, y) \neq 0, \forall y : Re(y) > 0$
c. $Y_3(s, z) \neq 0, \forall s : s \in R, \forall z : Re(z) \geq 0$

Proof: These three conditions **a,b,c** of Theorem 1 are conditions (13), (14) and (17), respectively. According to the above discussions, Schur theorem is changed into conditions (4) and (5) in [24]. Making the change of variables according to equation (8), the condition (5) is equivalent to condition (9) and condition (10). Conditions (4) and (9), respectively, are equivalent to (13) and (14). Making the change of variables according to equation (15), we know that condition (10) is equivalent to the condition (17). So Schur stability criterion of the 2D system is equivalent to these three conditions of Theorem 1. We can obtain a conclusion that Theorem 1 is satisfied which means the system is Schur stable, and the system is BIBO stable if $H(z_1, z_2)$ has no nonessential singularities of the second type.

Based on the above results, we show an algorithm for testing stability of 2D linear discrete system as follows:

Algorithm 1 2DStabilityTest

Input: A transfer function $H(z_1, z_2) = A(z_1, z_2)/B(z_1, z_2)$.

Output: True if the considered 2D system is stable. False otherwise.
Step1. Calculate $Y_1(x, 0)$, and check condition $Y_1(x, 0) \neq 0, \forall x : Re(x) > 0$ by Lemma 1, if not satisfied, output False.
Step2. Calculate $Y_2(-1, y)$, and check condition $Y_2(-1, y) \neq 0, \forall y : Re(y) > 0$ by Lemma 2, if not satisfied, output False.
Step3. Calculate $Y_3(js, z)$, and mark it as

$$Y_3(z) = q_n z^n + q_{n-1} z^{n-1} + \ldots + q_0 + j(p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0),$$

where p_i, q_i are real coefficient polynomials in s. Check whether $Y_3(s, z) \neq 0, \forall s : s \in R, \forall z : Re(z) \geq 0$ is satisfied by Lemma 2, if not satisfied, output False.
Step4. Output True.

In this subsection, we show the process of getting the efficient algorithm for analyzing stability of 2D linear discrete system. In next subsection, we apply the above process to solve the robust stability problem of 2D linear discrete system.

B. ROBUST STABILITY

Consider the robust stability of uncertain 2D linear discrete system (6). We recall the process of stability analysis to deal with the robust stability problem.

Denote

$$\widetilde{F}(s, z_2) = (1 - js)^m N(\frac{1 + js}{1 - js}, z_2),$$
$$L_1(x, 0) = (1 + x)^m N(\frac{1 - x}{1 + x}, 0),$$
$$L_2(-1, y) = (1 + y)^m N(-1, \frac{1 - y}{1 + y}),$$
$$L_3(s, z) = (1 + z)^n \widetilde{F}(s, \frac{1 - z}{1 + z})$$

Theorem 2: If $\widetilde{H}(z_1, z_2)$ has no the second type nonessential singularities, then the 2D linear discrete system with perturbations is BIBO stable if and only if these conditions are satisfied

a. $L_1(x, 0) \neq 0, \forall x : Re(x) > 0$
b. $L_2(-1, y) \neq 0, \forall y : Re(y) > 0$
c. $L_3(s, z) \neq 0, \forall s : s \in R, \forall z : Re(z) \geq 0$

We give an algorithm for stability analysis of 2D linear discrete systems with uncertain parameters.

Algorithm 2 2DRobustStabilityTest

Input: The denominator $N(z_1, z_2)$ of $\widetilde{H}(z_1, z_2)$.
Output: A disturbance parameters region of uncertain 2D system.
Step1. Calculate $L_1(x, 0)$, and solve $L_1(x, 0) \neq 0, \forall x : Re(x) > 0$ based on lemma 1 for getting the value range of c_{ij}. If not satisfied,output False.
Step2. Calculate the polynomial $L_2(-1, y)$, and solve $L_2(-1, y) \neq 0, \forall y : Re(y) > 0$ based on lemma 2 for getting the value range of c_{ij}.
Step3. Calculate $L_3(s, z)$, and mark it as
$L_3(z) = q_n z_n + q_{n-1} z_{n-1} + \ldots + q_0 + j(p_n z_n + p_{n-1} z_{n-1} + \ldots + p_0)$
where q_i and p_i, $i = 0, \ldots n$ are real coefficient polynomials in s and c_{ij}. Solve $L_3(s, z) \neq 0, \forall s : s \in R, \forall z : Re(z) \geq 0$ based on Lemma 2 to obtain the value range of c_{ij}.
Step4. According to the obtained solutions in Step1,2,3 of c_{ij}, get the stable parameter region of uncertain 2D system.

This subsection shows the process of getting the stable parameter region of uncertain 2D system. In the next section, we compare the proposed results with the exiting method based on iterative algorithms in [29], [30]. The following examples illustrate the efficiency for testing the stability and solving the robust stability problem as comparing the exiting methods.

IV. EXAMPLES

Example 1: Consider a 2D digital filter with the following denominator polynomial:

$$B(z_1, z_2) = \frac{1}{4} z_2^4 + (\frac{1}{2} + \frac{1}{4} z_1) z_2 + \frac{1}{4} z_1^2 + \frac{1}{2} z_1 + 1 \quad (18)$$

Now we follow the algorithm 2DStabilityTest to analyze the stability of this system.

Step 1. Calculate the polynomial $Y_1(x, 0)$,

$$Y_1 = (x+1)^2 B(\frac{1-x}{1+x}, 0) = \frac{3}{4}x^2 + \frac{3}{2}x + \frac{7}{4}.$$

According to Lemma 1, construct the Hurwitz matrix H_{Y_1} of $Y_1(x, 0)$ as follow,

$$H_{Y_1} = \begin{pmatrix} \frac{3}{2} & 0 \\ \frac{3}{4} & \frac{7}{4} \end{pmatrix},$$

we can get the principal minor determinant $\triangle_k(Y_1) > 0$, $k = 1, 2$. From Lemma 1, we know that the polynomial Y_1 is Hurwitz stable, hence the condition $Y_1(x, 0) \neq 0, \forall x : Re(x) > 0$ is satisfied.

Step2. Calculate the polynomial $Y_2(-1, y)$,

$$\begin{aligned} Y_2(-1, y) &= (1+y)^4 B(-1, \frac{1-y}{1+y}) \\ &= \frac{3}{4}y^4 + \frac{3}{2}y^3 + 6\,y^2 + \frac{5}{2}y + \frac{5}{4}. \end{aligned}$$

According to Lemma 1, construct the Hurwitz matrix of $Y_2(-1, y)$ as follow,

$$H_{Y_2} = \begin{pmatrix} \frac{3}{2} & \frac{5}{2} & 0 & 0 \\ \frac{3}{4} & 6 & \frac{5}{4} & 0 \\ 0 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & \frac{3}{4} & 6 & \frac{5}{4} \end{pmatrix},$$

we can get the principal minor determinant $\triangle_k(Y_2) > 0$, $k = 1, 2, 3, 4$. From Lemma 1, we know the polynomial Y_2 is Hurwitz stable, hence the condition $Y_2(-1, y) \neq 0, \forall y : Re(y) > 0$ is satisfied.

Step3. Calculate the polynomial $Y_3(s, z)$

$$\begin{aligned} Y_3(s, z) &= (1-js)^2(1+z)^4 F(\frac{1+js}{1-js}, \frac{1-z}{1+z}) \\ &= -\frac{5}{4}s^2 + \frac{11}{4} - \frac{5}{2}s^2 z - \frac{3}{4}s^2 z^4 - \frac{3}{2}s^2 z^3 - 6s^2 z^2 \\ &\quad + \frac{15}{2}z + \frac{5}{4}z^4 + \frac{9}{2}z^3 + 12z^2 \\ &\quad + j(-sz^4 - 2sz^3 - 12sz^2 - 6sz - 3s) \end{aligned} \tag{19}$$

According to (19), we can get $Y_3(jz)$. Therefore, we can draw H_{Y_3} as shown at the bottom of this page.

According to H_{Y_3}, we obtain the $2k_{th}$ order principal minor determinant as follows:

$$\begin{aligned} \triangle_2(Y_3) &= 9s^4 - 26s^2 + 45 \\ \triangle_4(Y_3) &= 513s^8 - 3372s^6 + 10726s^4 - 16428s^2 + 16065 \\ \triangle_6(Y_3) &= 2565s^{12} - 18936s^{10} + 78189s^8 - 212824s^6 \\ &\quad + 459887S^4 - 548208s^2 + 498015 \\ \triangle_8(Y_3) &= 3375s^{16} - 23328s^{14} + 116748s^{12} - 415696s^{10} \\ &\quad + 1193986s^8 - 2275648s^6 + 4223644s^4 \\ &\quad - 4243536s^2 + 4281255, \end{aligned}$$

we can get $\triangle_{2k}(Y_3) > 0$, $k = 1, 2, 3, 4$ using Maple program. From Lemma 2, the polynomial Y_3 is Hurwitz stable. The condition $Y_3(s, z) \neq 0, \forall s : s \in R, \forall z : Re(z) \geq 0$ is satisfied. To sum up, it can be concluded that the filter is stable by Theorem 1.

Remark 3: To verify the stability of system (18) in the above example, the main steps verifying condition (5) based on the method of polynomial discriminant system theory in [29], [30] are as follows:

$$\begin{aligned} F'(s, z_2) &= (1-js)^m z_2^n B(\frac{1+js}{1-js}, \frac{1}{z_2}) \\ &= -\frac{1}{4}s^2 - \frac{1}{2}js + \frac{1}{4} - \frac{1}{4}s^2 z_2^3 - jsz_2^3 + \frac{3}{4}z_2^3 \\ &\quad - \frac{3}{4}s^2 z_2^4 - \frac{3}{2}sz_2^4 \end{aligned}$$

According to the literature [29], [30], the polynomial array is as follow:

where $a'_{0,0}(s) = a_k(s)$, $k = 0, \ldots, 4$, and

$$a'_{1,k}(s) = \begin{vmatrix} a'_{0,4} & a'_{0,k} \\ \bar{a}'_{0,0} & \bar{a}'_{0,4-k} \end{vmatrix}, \quad k = 0, \ldots, 3$$

$$a'_{2,k}(s) = \begin{vmatrix} a'_{1,0} & a'_{1,3-k} \\ \bar{a}'_{1,n-1} & \bar{a}'_{1,k} \end{vmatrix}, \quad k = 0, 1, 2$$

$$a'_{i,k}(s) = \frac{1}{a'_{i-2,0}} \begin{vmatrix} a'_{i-1,0} & a'_{i-1,5-k-i} \\ \bar{a}'_{i-1,5-i} & \bar{a}'_{i-1,k} \end{vmatrix}, \quad i = 3, 4, \quad k = 0, \ldots, 4-i$$

From the above calculation, we have

$$a'_{0,0}(s) = -\frac{1}{4}s^2 - \frac{1}{2}js + \frac{1}{4}$$

$$a'_{1,0}(s) = \frac{1}{2}s^4 - \frac{1}{2}s^2 + 3$$

$$H_{Y_3} = \begin{pmatrix} -4s & 6s^2-18 & 48s & -10s^2+30 & -12s & 0 & 0 & 0 \\ -3s^2+5 & -8s & 24s^2-48 & 24s & -5s^2+11 & 0 & 0 & 0 \\ 0 & -4s & 6s^2-18 & 48s & -10s^2+30 & -12s & 0 & 0 \\ 0 & -3s^2+5 & -8s & 24s^2-48 & 24s & -5s^2+11 & 0 & 0 \\ 0 & 0 & -4s & 6s^2-18 & 48s & -10s^2+30 & -12s & 0 \\ 0 & 0 & -3s^2+5 & -8s & 24s^2-48 & 24s & -5s^2+11 & 0 \\ 0 & 0 & 0 & -4s & 6s^2-18 & 48s & -10s^2+30 & -12s \\ 0 & 0 & 0 & -3s^2+5 & -8s & 24s^2-48 & 24s & -5s^2+11 \end{pmatrix}.$$

$$a'_{2,0}(s) = \frac{63}{256}s^8 - \frac{35}{64}s^6 + \frac{401}{128}s^4 - \frac{199}{64}s^2 + \frac{2295}{256}$$

$$a'_{3,0}(s) = \frac{1}{3+\frac{1}{2}s^4-\frac{1}{2}s^2}\Big(\frac{14913}{8192}s^{12} - \frac{41113}{8192}s^{10} + \frac{495}{8192}s^{16} - \frac{2229}{8192}s^{14} + \frac{143131}{8192}s^8 - \frac{242559}{8192}s^6 + \frac{536091}{8192}s^4 - \frac{458739}{8192}s^2 + \frac{328941}{4098}\Big)$$

$$a'_{4,0}(s) = \frac{1}{\frac{2295}{256}+\frac{63}{256}s^8-\frac{401}{128}s^4-\frac{199}{64}s^2}\frac{1}{(3+\frac{1}{2}s^4-\frac{1}{2}s^2)^2} \times \Big(\frac{88429322025}{16777216} + \frac{288252658621}{67108864}s^{12} - \frac{15018704471}{2097152}s^{10} + \frac{64418967191}{67108864}s^{16} - \frac{71992866621}{33554432}s^{14} + \frac{718628304351}{67108864}s^8 - \frac{420879585159}{33554432}s^6 + \frac{879661401381}{67108864}s^4 - \frac{61217797587575}{16777216}s^2 + \frac{585445205}{67108864}s^{24} - \frac{1179020801}{33554432}s^{22} + \frac{19976247}{67108864}s^{28} - \frac{14798043}{8388608}s^{26} + \frac{8262577431}{67108864}s^{20} - \frac{6121779915}{16777216}s^{18} + \frac{212625}{67108864}s^{32} - \frac{1183707}{33554432}s^{30}\Big).$$

According to the results, the second line $a'_{1,k}(s)(k = 0, \ldots, 3)$ of Table1 is obtained by the first line $a'_{0,k}(s)(k = 0, \ldots, 4)$, which is polynomial matrix computation in s. Similarly, the final line $a'_{4,0}(s)$ is obtained via recurrence. In this process, we need to compute 10 polynomial matrices, which involves complex polynomial matrix calculation. As for the higher-order system, the calculation is more cumbersome. When $k \geq 3$, the process involves fractional calculation. The test condition $a'_{n,0}(s)(s \in R)$ is established for any real number s. While all the numerical values of parameter s need to make sure that the denominator isn't zero. As for higher-order system, a large number of higher-order polynomial iterative operations need to be carried out. If the system with uncertain parameters, the computation is more complex. Based on the above discussions, it's obvious that the method in this paper is tractable and has less computation compared with the methods based on Jury Array.

TABLE 1. $F'(s, z_2)$**Jury array.**

ith row	z_2^0	z_2^1	z_2^2	z_2^3	z_2^4
0	$a'_{0,0}(s)$	$a'_{0,1}(s)$	$a'_{0,2}(s)$	$a'_{0,3}(s)$	$a'_{0,4}(s)$
1	$a'_{1,0}(s)$	$a'_{1,1}(s)$	$a'_{1,2}(s)$	$a'_{1,3}(s)$	
2	$a'_{2,0}(s)$	$a'_{2,1}(s)$	$a'_{2,2}(s)$		
3	$a'_{3,0}(s)$	$a'_{3,1}(s)$			
4	$a'_{4,0}(s)$				

The robust stability problem of 2D system with uncertain parameters is shown in the proposed Algorithm 2 of this paper, and a complete parameter region of disturbance parameters is given to ensure the considered system with perturbations stability. In order to make this procedure clear, we illustrate the solving process of disturbance parameters by the following example.

Example 2: Consider an uncertain 2D digital filter with the following denominator polynomial:

$$N(z_1, z_2) = \frac{1}{2}c_2z_1z_2 - c_1z_2 - \frac{1}{4}c_2z_1 + z_1z_2 - \frac{1}{2}z_1 + 1.$$

Now we follow the algorithm 2DRobustStabilityTest to analyze the stability of the system with uncertain parameters c_1, c_2.

Step 1. Calculate the polynomial $L_1(x, 0)$,

$$L_1(x,0) = \frac{1}{2} + \frac{3}{2}x + \frac{1}{4}c_2x - \frac{1}{4}c_2 \neq 0, \quad \forall x : Re(x) > 0. \tag{20}$$

Using Lemma 1 to solve (20), we can get

$$\frac{c_2-2}{c_2+6} < 0. \tag{21}$$

Step 2. Calculate the polynomial $L_2(-1, y)$,

$$L_2(-1, y) = \frac{3}{4}c_2y - \frac{1}{4}c_2 + c_1y - c_1 + \frac{1}{2} + \frac{5}{2}y \neq 0, \quad \forall y : Re(y) > 0. \tag{22}$$

Using Lemma 1 to solve (22), we can get

$$\frac{4c_1+c_2-2}{4c_1+3c_2+10} < 0. \tag{23}$$

Step 3. Calculate the polynomial $L_3(s, z)$,

$$L_3(s,z) = -4c_1z + 3c_2z + 4c_1 - c_2 + 2z - 6 + j(4c_1sz + 3c_2sz - 4c_1s - c_2s + 10sz + 2s \neq 0, \quad \forall s : s \in R,\ \forall z : Re(z) \geq 0. \tag{24}$$

According to (24), we can get $L_3(jz)$ as follows:

$$L_3(s, jz) = -4c_1sz - 3c_2sz - 10sz + 4c_1 - c_2 - 6 + j(-4c_1s - 4c_1z - c_2s + 2s + 2z).$$

From the above, using Lemma 2 based on $L_3(jz)$ to solve (24), we can get

$$\begin{cases} -(4c_1 - c_2 - 6)(4c_1 - 3c_2 - 2) > 0, \\ -(4c_1 + 3c_2 + 10)(4c_1 + c_2 - 2) > 0 \end{cases} \tag{25}$$

Step 4. From (21)(23) and (25), we have the solutions as follows

$$\begin{cases} 4c_1 - c_2 - 6 < 0, \\ 4c_1 - 3c_2 - 2 > 0 \\ 4c_1 + 3c_2 + 10 > 0 \\ 4c_1 + c_2 - 2 < 0 \end{cases} \tag{26}$$

By solving conditions of Thoerem 2 for getting uncertain parameters, we get the the stable parameter region (26). It is shown in Figure 1.

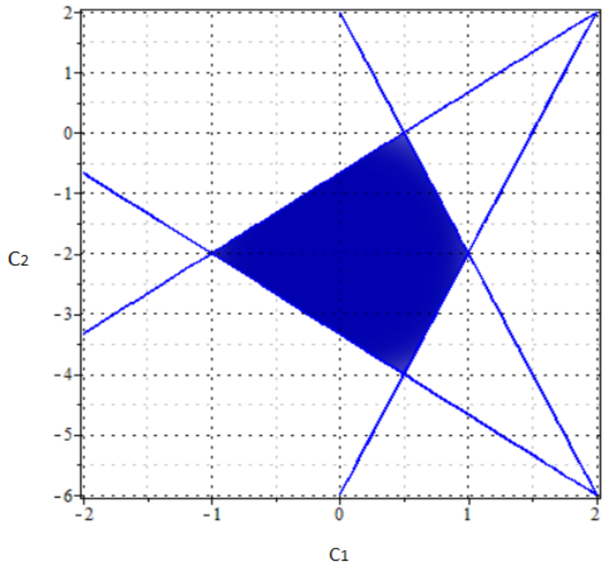

FIGURE 1. **The solved region of uncertain parameters.**

Remark 4: For better showing the correctness of the results in this example, we give some simulations of the trajectories of the system outputs with different uncertain parameters. Figure 2 shows the output of the proposed system with the uncertain parameters of the solved results shown in Figure 1. And Figure 3 shows the output of the proposed system with the uncertain parameters beyond the solved region shown in Figure 1. It's obvious that the output in Figure 2 is stable and the output in Figure 3 is divergent and not stable. The simulations illustrate that a complete region of disturbance parameters is solved to ensure the considered system with perturbations stability.

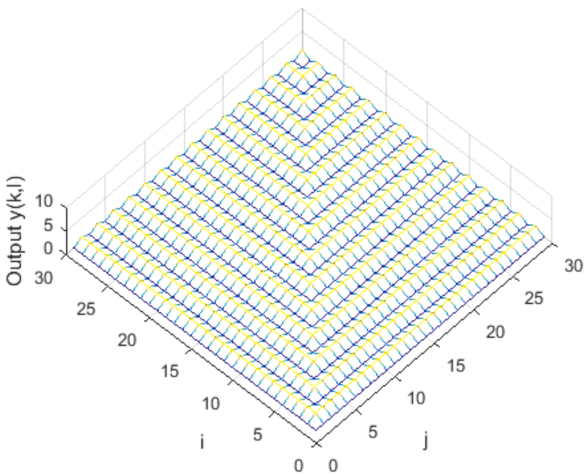

FIGURE 2. **Output response of the proposed system with $c_1 = 0, c_2 = -2$ in the solved region.**

Remark 5: For better showing the efficiency of the proposed method, we give a comparison of computation between the method of this paper and the existing methods in [29], [30] shown in Table 2, where m and n ($m > n$) are the variables' orders of the denominator polynomial,respectively. The data of Table 2 indicates that the amount of computation of the proposed method is significantly reduced. In the calculation process, the proposed method doesn't involve any fractional calculation and iterative operation. After transformation, the calculation of the proposed Hurwitz criterion is explicit. Hence, compared with the traditional algebraic methods, our method has less computation.

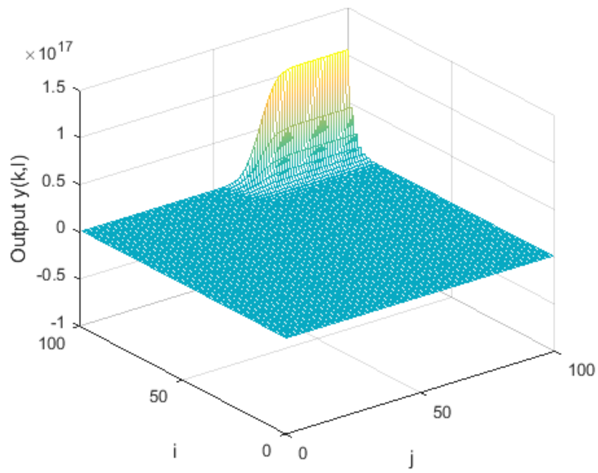

FIGURE 3. **Output response of the proposed system with $c_1 = 0, c_2 = -4$ beyond the solved region.**

TABLE 2. **The comparison of computation.**

Methods	The maximum order of the polynomials to be analyzed in the process of testing the stability	The number of polynomials for testing condition (5)
The existing methods	$2^n \cdot m$	$\frac{(1+n)\cdot n}{2}$
The method in this paper	$2m \cdot n$	n

V. CONCLUSION

This paper, a 2D recursive model as an example, has proposed two efficient and explicit algorithms for the stability and robust stability. The algorithms are based on Hurwitz's theorem. The proposed algorithms are concise and effective to analyze the stability and robust stability of the 2D linear discrete system, and they can solve the cumbersome problems of the iterative computation and the fractional calculation in the other algebraic stability tests. Note that a complete region of disturbance parameters can be obtained to ensure the stability of 2D system with perturbations. We change the testing stability condition based on Schur theorem to a new one based on Hurwitz theorem through the technique of fractional linear transformations. Then, the new problem can be solved by the discriminant systems of polynomial.

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