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Dynamic Discount Pricing in Competitive Marketing

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ABSTRACT Discount is a frequently used marketing tool. This paper is devoted to designing an effective dynamic discount pricing (DDP) strategy in competitive marketing. First, we introduce a competitive word-of-mouth (WOM) propagation model with DDP mechanism. On this basis, we model the original problem as an optimal control problem. Next, we derive the optimality system for the optimal control problem and, thereby, propose the concept of competitive DDP strategy. Finally, through comparative experiments, we find that the profit of a competitive DDP strategy is satisfactory. Our findings contribute to maximizing the marketing profit in the presence of commercial competitors.

INDEX TERMS Marketing management, profit maximization, discount pricing, word-of-mouth propagation, optimal control, optimality system.

I. INTRODUCTION

Marketing strategies play a critical role in marketing campaigns; a wise marketing strategy could bring competitive advantage to enterprises [1]. Dynamic pricing as one of the most common marketing strategies has a long history [2]. Discount pricing, by which it means a discount off an earlier price, is a widely adopted form of dynamic pricing [3]. The marketing history has proved that a discounted price is more likely to be accepted by consumers than a merely low price [4]. With the ever-increasing popularity of online social networks (OSNs), people are willing to share with their friends their experiences about their recently purchased products through OSNs [5]. Therefore, the effect of word-of-mouth (WOM) propagation on marketing campaigns must be taken into account in designing discount pricing strategies [6].

A. PROBLEM FORMULATION

Market competition is a universal phenomenon in the business area; competitive industries are like Darwinian arenas where firms struggle to maximize their respective profits, otherwise they would go out of business [7]. When a firm decides to launch a discount marketing campaign to increase its profit, it faces the following problem:

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Dynamic discount pricing (DDP) problem: In the presence of commercial rivals and in view of the effect of WOM propagation, find a dynamic discount pricing (DDP) strategy so that the firm's marketing profit is maximized.

To the best of our knowledge, this is the first time the DDP problem is formulated explicitly.

B. MAIN CONTRIBUTIONS

This paper is devoted to the modeling and analysis of the DDP problem. Our main contributions are listed below.

- In the presence of commercial competitors, we propose a node-level competitive WOM propagation model with dynamic discount pricing (DDP) mechanism. Thereby, we estimate the firm's marketing profit under a DDP strategy. On this basis, we model the DDP problem as an optimal control problem (i.e., the DDP* problem), in which the objective functional stands for the marketing profit under a DDP strategy, each optimal control stands for a DDP strategy that maximizes the marketing profit.
- We derive the optimality system for the DDP* problem. Thereby, we propose the concept of competitive DDP strategy. Through comparative experiments, we find that a competitive DDP strategy is superior to all static discount pricing strategies in terms of the marketing

profit. Therefore, the performance of a competitive DDP strategy is satisfactory.

The subsequent materials of this paper are organized in this fashion: Section 2 reviews the related work. Section 3 reduces the DDP problem to the DDP* problem. Sections 4 derives the optimality system for the DDP* problem and, thereby, proposes the concept of potential DDP strategy. Section 5 examines the performance of a potential DDP strategy. This work is summarized by Section 6.

II. RELATED WORK

This section is dedicated to reviewing the previous work that is closely related to the present paper. First, we mention WOM propagation models. Second, we discuss the optimal control approach to marketing.

A. WOM PROPAGATION MODELS

To deal with the DDP problem, we have to take into full account the effect of WOM propagation on the discount marketing campaign. To this end, we need to establish a WOM propagation model with DDP mechanism. There are three types of WOM propagation models: state-level, degree-level, and node-level. Below is a sketch of this taxonomy.

In a state-level WOM propagation model, all individuals in the WOM propagation network are classified simply based on their states, and the fraction of each class evolves following a differential equation [8]–[11]. As the modeling process neglects the difference between different individuals in terms of network characteristics at all, state-level WOM propagation models can only be used to describe WOM propagation processes on homogeneous networks. Since OSNs are typically heterogeneous, this modeling technique is not applicable to describing WOM propagation on OSNs.

In a degree-level WOM propagation model, all individuals in the WOM propagation network are categorized based on both their states and their network degrees, and the fraction of each class evolves following a differential equation [12], [13]. Although the modeling process considers the difference between different individuals in terms of network degree, it neglects the difference between different individuals in terms of any other network characteristic. As a result, degree-level WOM propagation models can only be used to capture WOM propagation processes on some special heterogeneous networks such as scale-free networks. Since OSNs may admit arbitrary network structures, this modeling technique is not suited to describing WOM propagation on OSNs neither.

In a node-level WOM propagation model, every individual in the WOM propagation network is classified as a few classes based on his state, and the probability of each individual being in each specific state evolves following a differential equation [14]–[17]. Since this modeling process fully considers the difference between different individuals in terms of all network characteristics, node-level WOM propagation models can be used to describe WOM propagation processes on arbitrary networks.

All of the above node-level WOM propagation models are used to characterize the propagation of a single WOM. In many real-world applications, there are multiple competing WOMs. In order to have a better understanding of this phenomenon, it is necessary to introduce and study nodel-level competitive WOM propagation models. Inspired by the node-level conflicting message propagation model proposed in [18], in the present paper we introduce a node-level competing WOM propagation model with DDP mechanism. On this basis, we estimate the firm's marketing profit under a DDP strategy.

B. OPTIMAL CONTROL APPROACH TO MARKETING

Optimal control theory is devoted to finding a control scheme of a state evolution system so that a specific optimality criterion is achieved [19]. In the past decades, optimal control theory has been widely employed in marketing researches such as advertising [20], [21] and influential maximization [22]–[24].

Recently, optimal control theory has been applied to the maximization of marketing profit. [14] introduced a node-level WOM propagation model with an influence-based discount pricing mechanism (see [16]) and, thereby, studied the marketing profit maximization problem. [15] established a node-level positive and negative WOMs mixed propagation model (see [11]) with discount pricing mechanism and, on this basis, dealt with the marketing profit maximization problem. In the above two references, it is implicitly assumed that the firm has no commercial rival. In practice, however, most firms have commercial competitors. Therefore, the results given in these references have very limited applications.

The present paper is devoted to the profit maximization in competitive marketing. Based on the novel competitive WOM propagation model with DDP mechanism proposed by us, we model and study the DDP problem through optimal control approach. As a result, our model and results should be more practical than those presented in the above references.

III. THE MODELING OF THE DDP PROBLEM

This section is dedicated to the modeling of the DDP problem. First, we establish a node-level competitive WOM propagation model with DDP mechanism. Second, we model the DDP problem as an optimal control problem.

A. A NODE-LEVEL COMPETITIVE WOM PROPAGATION MODEL WITH DDP MECHANISM

Let $\mathcal{A}$ denote a firm, $\mathcal{B}$ the set of all commercial virals of $\mathcal{A}$. Suppose $\mathcal{A}$ decides to launch a discount marketing campaign in a given time horizon $[0, T]$. Let $V = \{v_1, v_2, \ldots, v_N\}$ denote the target market of the campaign, where each node v_i stands for a potential customer of the campaign. Let $G = (V, E)$ denote the WOM propagation network over the target market, where each edge $\{v_i, v_j\} \in E$ stands for that the person v_i and the person v_j are friends and hence can recommend their respective favorite goods to each other through the WOM propagation network. Let $\mathbf{A} = (a_{ij})_{N\times N}$ denote the adjacency matrix of G. In practice, the structure of the WOM propagation network is available by employing the network crawling technique.

At each time point in the time horizon $[0, T]$, each and every person in the target market is in one of three possible states: *wavering*, *$\mathcal{A}$-decisive*, and *$\mathcal{B}$-decisive*. Here, a person is wavering if he has not yet decided where to shop, a person is $\mathcal{A}$-decisive if he has decided to shop from $\mathcal{A}$, and a person is $\mathcal{B}$-decisive if he has decided to shop from $\mathcal{B}$. Let $X_i(t) = 0$, 1, and 2 denote that the person v_i is wavering, $\mathcal{A}$-decisive, and $\mathcal{B}$-decisive at time t, respectively. Then the N-dimensional random vector

$$\mathbf{X}(t) = (X_1(t), \ldots, X_N(t)). \tag{1}$$

stands for the state of the target market at time t. Hence, the N-dimensional stochastic process $\{\mathbf{X}(t) : 0 \le t \le T\}$ stands for the evolutionary process of $\mathbf{X}(t)$ over time. In order to describe the stochastic process, we need to introduce a set of notations as follows.

- α: the average rate at which, owing to the effect of $\mathcal{B}$'s average discount, a waverer becomes $\mathcal{B}$-decisive.
- β_{WA}: the average rate at which, owing to the WOM effect of a neighboring $\mathcal{A}$-decider, a waverer becomes $\mathcal{A}$-decisive.
- β_{WB}: the average rate at which, owing to the WOM effect of a neighboring $\mathcal{B}$-decider, a waverer becomes $\mathcal{B}$-decisive.
- β_{BA}: the average rate at which, owing to the WOM effect of a neighboring $\mathcal{A}$-decider, a $\mathcal{B}$-decider becomes $\mathcal{A}$-decisive. Intuitively, we have $\beta_{BA} < \beta_{WA}$.
- β_{AB}: the average rate at which, owing to the WOM effect of a neighboring $\mathcal{B}$-decider, an $\mathcal{A}$-decider becomes $\mathcal{B}$-decisive. Intuitively, we have $\beta_{AB} < \beta_{WB}$.
- δ: the average rate at which, owing to the descent in shopping desire, an $\mathcal{A}$-decider or a $\mathcal{B}$-decider becomes wavering.
- $\theta(t)$: the average discount ratio of all $\mathcal{A}$'s goods at time t.

Additionally, we introduce an assumption as follows.

(A_1) Owing to the effect of $\mathcal{A}$'s discount, a waverer becomes $\mathcal{A}$-decisive at time t at the average rate $f(\theta(t))$, where $f(0) = 0, f$ is increasing. This assumption conforms to the intuition.

In practice, α, β_{WA}, β_{WB}, β_{BA}, β_{AB}, δ, and f can be estimated through well-crafted online questionnaire survey.

We refer to the function θ defined by $\theta(t)$ $(0 \le t \le T)$ as a *dynamic discount pricing (DDP) strategy* of $\mathcal{A}$. For ease in realization, we assume θ is piecewise continuous. Let $PC[0, T]$ denote the set of all piecewise continuous functions defined on $[0, T]$. Then the set of all allowable DDP strategies is

$$\Theta = \{\theta \in PC[0, T] : 0 \le \theta(t) \le 1, 0 \le t \le T\}. \tag{2}$$

Let χ_S denote the characteristic function of the set S. It follows from stochastic process theory [25] that the state transition diagram of the person v_i is as shown in Fig. 1.

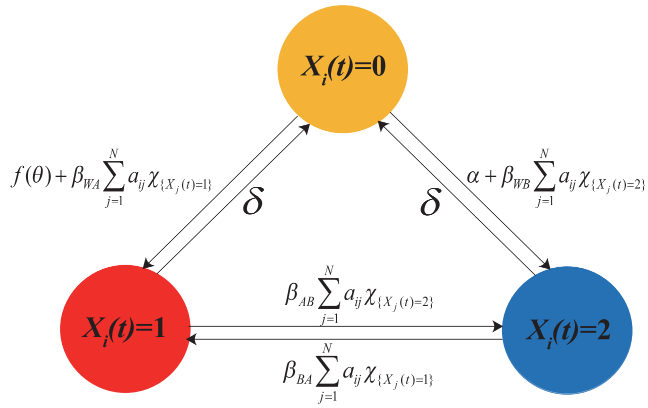

FIGURE 1. State transition diagram of the person v_i at time t.

Let $W_i(t)$, $A_i(t)$ and $B_i(t)$ denote the probabilities of the person v_i being wavering, $\mathcal{A}$-decisive and $\mathcal{B}$-decisive at time t, respectively.

$$\begin{aligned} &W_i(t) = \Pr\{X_i(t) = 0\}, \quad A_i(t) = \Pr\{X_i(t) = 1\}, \\ &B_i(t) = \Pr\{X_i(t) = 2\}. \end{aligned} \tag{3}$$

As $W_i(t) = 1 - A_i(t) - B_i(t)$, the expected state of the target market at time t can be characterized by the vector

$$\mathbf{E}(t) = (A_1(t), \ldots, A_N(t), B_1(t), \ldots, B_N(t)). \tag{4}$$

In practice, $\mathbf{E}(0)$ can be estimated through online questionnaire survey.

Theorem 1: The target market's expected state evolves over time according to the differential equation system (5).

Proof: Let $\mathbb{E}(\cdot)$ denote the mathematical expectation of a random variable. For $1 \le i \le N$, $0 \le t \le T$, v_i becomes $\mathcal{A}$-decisive at time t at the expected rate

$$\mathbb{E}\left(f(\theta(t))+\beta_{WA}\sum_{j=1}^{N} a_{ij}\chi_{\{X_j(t)=1\}}\right) = f(\theta(t))+\beta_{WA}\sum_{j=1}^{N} a_{ij}A_j(t)$$

if $X_i(t) = 0$, v_i becomes $\mathcal{A}$-decisive at time t at the expected rate

$$\mathbb{E}\left(\beta_{BA}\sum_{j=1}^{N} a_{ij}\chi_{\{X_j(t)=1\}}\right) = \beta_{BA}\sum_{j=1}^{N} a_{ij}A_j(t)$$

if $X_i(t) = 2$, and v_i becomes $\mathcal{B}$-decisive at time t at the expected rate

$$\mathbb{E}\left(\beta_{AB}\sum_{j=1}^{N} a_{ij}\chi_{\{X_j(t)=2\}}\right) = \beta_{AB}\sum_{j=1}^{N} a_{ij}B_j(t)$$

if $X_i(t) = 1$, and v_i becomes wavering at time t at the expected rate δ if $X_i(t) = 1$. Hence, the first N equations in this system hold. Similarly, we can prove the second N equations in this system. ■

The differential equation system (5), as shown at top of the next page, is a node-level competitive WOM propagation

$$\begin{cases} \dfrac{dA_i(t)}{dt} = \left[f(\theta(t)) + \beta_{WA}\sum_{j=1}^{N} a_{ij}A_j(t)\right][1 - A_i(t) - B_i(t)] - \left[\beta_{AB}\sum_{j=1}^{N} a_{ij}B_j(t) + \delta\right]A_i(t) \\ \quad + \left[\beta_{BA}\sum_{j=1}^{N} a_{ij}A_j(t)\right]B_i(t), \quad 0 \le t \le T, 1 \le i \le N, \\ \dfrac{dB_i(t)}{dt} = \left[\alpha + \beta_{WB}\sum_{j=1}^{N} a_{ij}B_j(t)\right][1 - A_i(t) - B_i(t)] - \left[\beta_{BA}\sum_{j=1}^{N} a_{ij}A_j(t) + \delta\right]B_i(t) \\ \quad + \left[\beta_{AB}\sum_{j=1}^{N} a_{ij}B_j(t)\right]A_i(t), \quad 0 \le t \le T, 1 \le i \le N, \\ \mathbf{E}(0) = \mathbf{E}_0. \end{cases} \tag{5}$$

model with DDP mechanism. For brevity, we write the WOM propagation model as

$$\begin{cases} \dfrac{d\mathbf{E}(t)}{dt} = \mathbf{f}(\mathbf{E}(t), \theta(t)), \quad 0 \le t \le T, \\ \mathbf{E}(0) = \mathbf{E}_0. \end{cases} \tag{6}$$

In what follows, let

$$\mathbf{E}(t, \theta(t)) = (A_1(t, \theta(t)), \ldots, A_N(t, \theta(t)), \\ B_1(t, \theta(t)), \ldots, B_N(t, \theta(t))), \quad 0 \le t \le T, \tag{7}$$

denote the solution to the WOM propagation model (5) or (6).

B. MODELING THE DDP PROBLEM

In order to model the DDP problem, we have to estimate $\mathcal{A}$'s marketing profit. For now, let p denote $\mathcal{A}$'s average profit per unit time owing to an $\mathcal{A}$-decider, provided there is discount.

Theorem 2: Under the DDP strategy θ, $\mathcal{A}$'s expected marketing profit is

$$\mathcal{P}(\theta) = p\int_0^T \sum_{i=1}^{N} [1 - \theta(t)]A_i(t, \theta(t))dt. \tag{8}$$

Proof: Let $dt > 0$ be an infinitesimal. For $1 \le i \le N$, $0 \le t \le T - dt$, $\mathcal{A}$'s average marketing profit in the infinitesimal time horizon $[t, t + dt)$ owing to v_i is $p[1 - \theta(t)]\,dt$ or 0 according as $X_i(t) = 1$ or not. Hence, $\mathcal{A}$'s expected marketing profit in the infinitesimal time horizon $[t, t + dt)$ owing to v_i is $p[1 - \theta(t)]A_i(t, \theta(t))dt$. Therefore, Eq. (8) follows. ■

$\mathcal{A}$'s goal is to find a DDP strategy θ so that $\mathcal{P}(\theta)$ is maximized. In doing so, the value of p makes no difference. Hence, in what follows we assume $p = 1$. Based on the previous discussions, we model the DDP problem as the following optimal control problem:

$$\text{Maximize } \mathcal{P}(\theta) = \int_0^T \sum_{i=1}^{N} [1 - \theta(t)]A_i(t)dt$$
$$\text{subject to } \begin{cases} \dfrac{d\mathbf{E}(t)}{dt} = \mathbf{f}(\mathbf{E}(t), \theta(t)), \quad 0 \le t \le T, \\ \mathbf{E}(0) = \mathbf{E}_0. \end{cases} \tag{9}$$

We refer to the optimal control problem as the *DDP* problem*. Each instance of the problem is described by the 10-tuple

$$\mathbb{M} = (G, T, \alpha, \beta_{WA}, \beta_{WB}, \beta_{AB}, \beta_{BA}, \delta, f, \mathbf{E}_0). \tag{10}$$

IV. AN APPROACH TO DEALING WITH THE DDP* PROBLEM

In the previous section, we reduced the DDP problem to an optimal control problem, i.e., the DDP* problem. In this section, we devote ourself to dealing with the DDP* problem. First, we derive a necessary condition for optimal control of the DDP* problem. Second, we present the optimality system for the DDP* problem.

A. A NECESSARY CONDITION FOR OPTIMAL CONTROL OF THE DDP* PROBLEM

It follows from optimal control theory [19] that the Hamiltonian for the DDP* problem is as shown in Eq. (11), as shown at top of the next page, where $(\lambda, \mu) = (\lambda_1, \cdots, \lambda_N, \mu_1, \cdots, \mu_N)$ is the adjoint of H. Below we give a necessary condition for optimal control of the DDP* problem.

Theorem 3: Suppose θ is an optimal control of the DDP problem (9), $\mathbf{E}$ is the solution to the corresponding WOM propagation model. Then there exists an adjoint function (λ, μ) such that the system (12) holds. Moreover,*

$$\theta(t) \in \arg\max_{\widetilde{\theta}\in[0,1]} \left\{\left\{\sum_{i=1}^{N} \lambda_i(t)[1 - A_i(t) - B_i(t)]\right\} f(\widetilde{\theta}) - \left[\sum_{i=1}^{N} A_i(t)\right]\widetilde{\theta}\right\}, \quad 0 \le t \le T. \tag{13}$$

Proof: According to the Pontryagin Maximum Principle [19], there exists (λ, μ) such that

$$\begin{cases} \dfrac{d\lambda_i(t)}{dt} = -\dfrac{\partial H(\mathbf{E}(t), \theta(t), \lambda(t), \mu(t))}{\partial A_i}, \\ \dfrac{d\mu_i(t)}{dt} = -\dfrac{\partial H(\mathbf{E}(t), \theta(t), \lambda(t), \mu(t))}{\partial B_i}, \\ 0 \le t \le T, 1 \le i \le N. \end{cases}$$

$$H(\mathbf{E},\theta,\lambda,\mu)=\sum_{i=1}^{N}(1-\theta)A_i$$
$$+\sum_{i=1}^{N}\lambda_i\left[\left(f(\theta)+\beta_{WA}\sum_{j=1}^{N}a_{ij}A_j\right)(1-A_i-B_i)+\left(\beta_{BA}\sum_{j=1}^{N}a_{ij}A_j\right)B_i-\left(\beta_{AB}\sum_{j=1}^{N}a_{ij}B_j+\delta\right)A_i\right]$$
$$+\sum_{i=1}^{N}\mu_i\left[\left(\alpha+\beta_{WB}\sum_{j=1}^{N}a_{ij}B_j\right)(1-A_i-B_i)+\left(\beta_{AB}\sum_{j=1}^{N}a_{ij}B_j\right)A_i-\left(\beta_{BA}\sum_{j=1}^{N}a_{ij}A_j+\delta\right)B_i\right]. \quad (11)$$

$$\begin{cases}\dfrac{d\lambda_i(t)}{dt}=\theta(t)-1+\left[f(\theta(t))+\beta_{WA}\sum_{j=1}^{N}a_{ij}A_j(t)+\beta_{AB}\sum_{j=1}^{N}a_{ij}B_j(t)+\delta\right]\lambda_i(t)\\ \quad-\sum_{j=1}^{N}a_{ij}\left\{\beta_{WA}\left[1-A_j(t)-B_j(t)\right]+\beta_{BA}B_j(t)\right\}\lambda_j(t)\\ \quad+\left[\alpha+(\beta_{WB}-\beta_{AB})\sum_{j=1}^{N}a_{ij}B_j(t)\right]\mu_i(t)+\beta_{BA}\sum_{j=1}^{N}a_{ij}B_j(t)\mu_j(t), \quad 0\le t\le T, 1\le i\le N,\\ \dfrac{d\mu_i(t)}{dt}=\left[f(\theta(t))+(\beta_{WA}-\beta_{BA})\sum_{j=1}^{N}a_{ij}A_j(t)\right]\lambda_i(t)+\beta_{AB}\sum_{j=1}^{N}a_{ij}A_j(t)\lambda_j(t)\\ \quad+\left[\alpha+\beta_{WB}\sum_{j=1}^{N}a_{ij}B_j(t)+\beta_{BA}\sum_{j=1}^{N}a_{ij}A_j(t)+\delta\right]\mu_i(t)\\ \quad-\sum_{j=1}^{N}a_{ij}\left\{\beta_{WB}\left[1-A_j(t)-B_j(t)\right]+\beta_{AB}A_j(t)\right\}\mu_j(t), \quad 0\le t\le T, 1\le i\le N,\\ \lambda(T)=\mu(T)=\mathbf{0}.\end{cases} \quad (12)$$

The first $2N$ equations in Eqs. (12) follow by direct calculations. As the terminal cost is unspecified and the final state is free, the transversality conditions $\lambda(T)=\mu(T)=\mathbf{0}$ hold. Again by the Pontryagin Maximum Principle, we have

$$\theta(t)\in\arg\max_{\widetilde{\theta}\in[0,1]} H(\mathbf{E}(t),\widetilde{\theta},\lambda(t),\mu(t)), \quad 0\le t\le T.$$

This implies Eq. (13). ∎

B. THE OPTIMALITY SYSTEM FOR THE DDP* PROBLEM

According to optimal control theory [19], the system consisting of Eqs. (5), (12), as shown at top of this page, and (13) is referred to as the *optimality system* for the DDP* problem. We refer to the control in each solution to the optimality system as a *competitive control* of the DDP* problem. Since Θ is incomplete [26], the DDP* problem may not admit an optimal control. In the case when the DDP* problem admits an optimal control, it follows from Theorem 3 that the optimal control must be a competitive control. However, the converse is not true. Henceforth, we refer to the DDP strategy represented by a competitive control as a *competitive DDP strategy*

In what follows, we deal with the DDP* problem in this way: First, solve the optimality system to get a competitive DDP strategy. Then, examine the effectiveness of the competitive DDP strategy.

V. THE EFFECTIVENESS OF A COMPETITIVE DDP STRATEGY

In the previous section, we introduced the notion of competitive DDP strategy. In this section, we examine the effectiveness of a competitive DDP strategy in terms of the expected marketing profit through comparative experiments.

A. EXPERIMENT DESIGN

In each of the following experiments, a DDP instance is generated, a competitive DDP strategy for the DDP instance is obtained by solving the associated optimality system, and the competitive DDP strategy is compared with a set of static discount pricing strategies in terms of the expected marketing profit. First, all the experiments are carried out on a PC with inter(R) Core(TM) i5-7600 CPU @3.50GHz and 16GB RAM. Second, all the optimality systems are solved by employing the Euler Forward-Backward scheme [27].

In order to generate a number of DDP instances, we select three online social networks from a dataset named *Graph Embedding with Self Clustering: Facebook* in the widely used

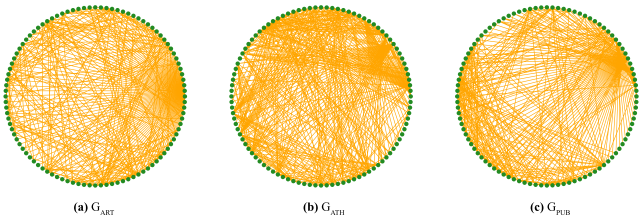

FIGURE 2. **Three subnets of real online social networks.**

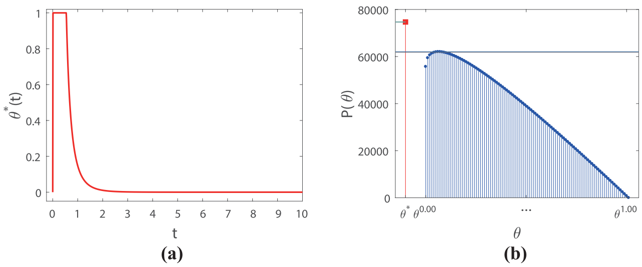

FIGURE 3. **The results in Experiment 1: (a) θ^*, (b) $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$.**

network library SNAP [28], [29]. First, we take a 100-node subnet G_{ART} from an artists network with 50,515 nodes and 819,306 edges. See Fig. 2(a). Second, we extract a 100-node subnet G_{ATH} from an athletes network with 13,866 nodes and 86,858 edges. See Fig. 2(b). Finally, we fetch a 100-node subnet G_{PUB} from a public figures network with 11,565 nodes and 67,114 edges. See Fig. 2(c).

Let $S = \{0, 0.01, 0.02, \ldots, 1\}$. For $a \in S$, let θ_a denote the static discount pricing strategy defined by $\theta_a(t) = a$ $(0 \leq t \leq T)$. Let $\overline{\Theta} = \{\theta_a : a \in S\}$. Finally, let $\mathbf{E}^{(a,b)} = (a, \ldots, a, b, \ldots, b)$, where there are 100 a components and 100 b components.

B. COMPARATIVE EXPERIMENTS

Experiment 1 (Consider the DDP instance):

$$\mathbb{M}_{ART} = \left(G_{ART}, 10, 0.4, 0.2, 0.15, 0.1, 0.15, 0.01, \theta^{0.6}, \mathbf{E}^{(0.1,0.1)}\right).$$

By solving the optimality system, we get the competitive DDP strategy θ^*, which is plotted Fig. 3(a). Fig. 3(b) exhibits $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$. It is seen that $\mathcal{P}(\theta^*) > \mathcal{P}(\theta)$, $\theta \in \overline{\Theta}$. Hence, θ^* is superior to all the static discount pricing strategies in $\overline{\Theta}$ in terms of the expected marketing profit.

Experiment 2 (Consider the DDP instance):

$$\mathbb{M}_{ATH} = \left(G_{ATH}, 10, 0.3, 0.2, 0.15, 0.1, 0.15, 0.01, \theta^{0.3}, \mathbf{E}^{(0.1,0.2)}\right).$$

By solving the optimality system, we get the competitive DDP strategy θ^*, which is plotted in Fig. 4(a). Fig. 4(b) depicts $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$. Again, it is seen that θ^* outperforms all the static discount pricing strategies in $\overline{\Theta}$ in terms of the expected marketing profit.

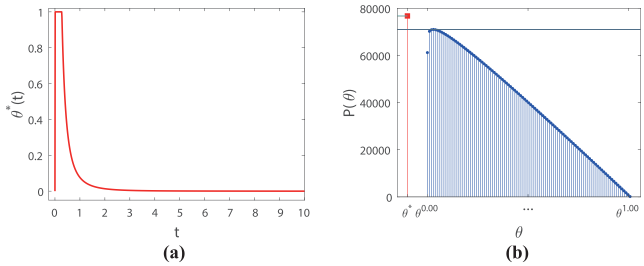

FIGURE 4. **The results in experiment 2: (a) θ^*, (b) $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$.**

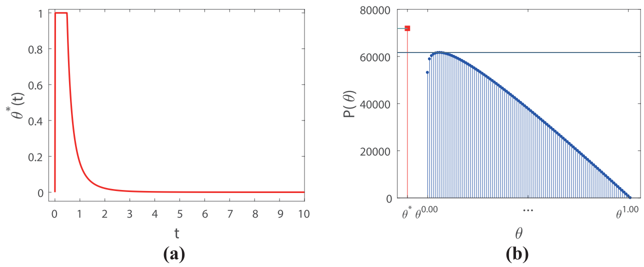

FIGURE 5. **The results in experiment 3: (a) θ^*, (b) $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$.**

Experiment 3 (Consider the DDP instance):

$$\mathbb{M}_{PUB} = \left(G_{PUB}, 10, 0.3, 0.2, 0.15, 0.1, 0.15, 0.01, \theta^{0.5}, \mathbf{E}^{(0.1,0.2)}\right).$$

By solving the optimality system, we get the competitive DDP strategy θ^*, which is plotted in Fig. 5(a). Fig. 5(b) displays $\mathcal{P}(\theta)$ versus θ, $\theta \in \{\theta^*\} \bigcup \overline{\Theta}$. Again, it is seen that θ^* is better than all the static discount pricing strategies in $\overline{\Theta}$ in terms of the expected marketing profit.

From the above three experiments and 100 similar experiments, we find that the competitive DDP strategy is always superior to all the static discount pricing strategies in terms of the expected marketing profit. Therefore, the effectiveness of the competitive DDP strategy is satisfactory.

In practice, the commercial competitors of a firm may well employ DDP strategies. In this case, the proposed scheme is not directly applicable. Nonetheless, we may apply the scheme indirectly in this way: First, divide a larger time horizon into a number of smaller time horizons. Second, apply the scheme to each of these smaller time horizons.

VI. CONCLUSION

This paper has dealt with the problem of developing an effective dynamic discount pricing (DDP) strategy in competitive marketing. Based on a novel WOM propagation model, we have reduced the original problem to an optimal control problem. We have derived the optimality system for the optimal control problem and proposed the concept of competitive DDP strategy. Through comparative experiments, we have found that the effectiveness of a competitive DDP strategy is satisfactory. This work contributes to increasing a firm's marketing profit in competitive marketing.

Toward the direction, there are a number of issues to be addressed. First, estimating the parameters involved in the proposed scheme is a problem. Second, this work should be extended to the situation where the WOM propagation network varies over time [30]–[32]. Next, the methodology used in this paper may be applied to some other issues such as rumor restraint [18], [33]–[36] and active cyber defense [37]–[40]. Last, in the situation where all firms involved in a competitive discount marketing campaign are strategic, the marketing profit maximization problem must be dealt with in the framework of game theory [41]–[43].

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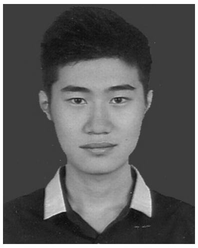

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