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On Distance-Based Topological Descriptors of Subdivision Vertex-Edge Join of Three Graphs

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ABSTRACT The analysis of networks and graphs through topological descriptors carries out a useful role to derive their underlying topologies. This process has been widely used in biomedicine, cheminformatics, and bioinformatics, where assessments based on graph invariants have been made available for effectively communicating with the various challenging schemes. In the studies of quantitative structure-activity relationships (QSARs) and quantitative structure-property relationships (QSPRs), graph invariants are used to approximate the biological activities and properties of chemical compounds. In this paper, we give the results related to the eccentric-connectivity index, connective eccentricity index, total-eccentricity index, average eccentricity index, Zagreb eccentricity indices, eccentric geometric-arithmetic index, eccentric atom-bond connectivity index, eccentric adjacency index, modified eccentric-connectivity index, eccentric distance sum, Wiener index, Harary index, hyper-Wiener index and degree distance index of a new graph operation named as ''subdivision vertex-edge join'' of three graphs.

INDEX TERMS Topological indices, degree, distance, eccentricity, subdivision vertex-edge join.

I. INTRODUCTION

Graph theory concerned with a lot of applications in a number of domains of chemistry such as Quantitative structure-activity and property relationships, isomer enumeration, prediction of biological activities, topological characterization, graph polynomials for structural analysis, quantum chemistry, NMR spectroscopy, nuclear spin statistics, spectroscopy, proteomics, statistical and other procedures for forecast of toxicity of chemical structures and so on [7], [9], [11], [13], [15], [17], [19], [20], [31], [43], [45], [46]. The QSAR/QSPR studies made use of connection among molecular connectivity and the properties of chemical compounds, therefore a fundamental graph-theoretical characterizations set up the principles for computer-aided predictive toxicology and drug discovery. As a result, successful uses of QSAR/QSPR studies have stimulated the emergence of several topological indices of chemical structures [10]–[14], [16]–[20], [34], [43]–[45], [47]. The intermolecular interactions rely on the distance and degree criterions and moreover, numerous physico-chemical properties of chemical structures have been proven to correlate with topological characteristics as decent initial points. However, one may require sophisticated bio-descriptors and quantum chemical in addition to quantum molecular dynamics simulations for extra precise forecasts of biological and chemical characteristics, due to computationally extensive nature of such approaches, topological techniques have constructed valuable implementations because of the comparatively easy process with which they can be determined. Many properties such as receptor binding propensity, toxicity, protein-drug interactions, dermal penetrations, drug metabolomics, guest-host interactions, etc., rely on the intermolecular interactions, pore sizes, structural parameters, electrostatic and electronic properties various of which rely on fundamental topological parameter distances and therefore topological descriptors are much more appealing initiating objects to any statistical approximation for securing structure-activity connections.

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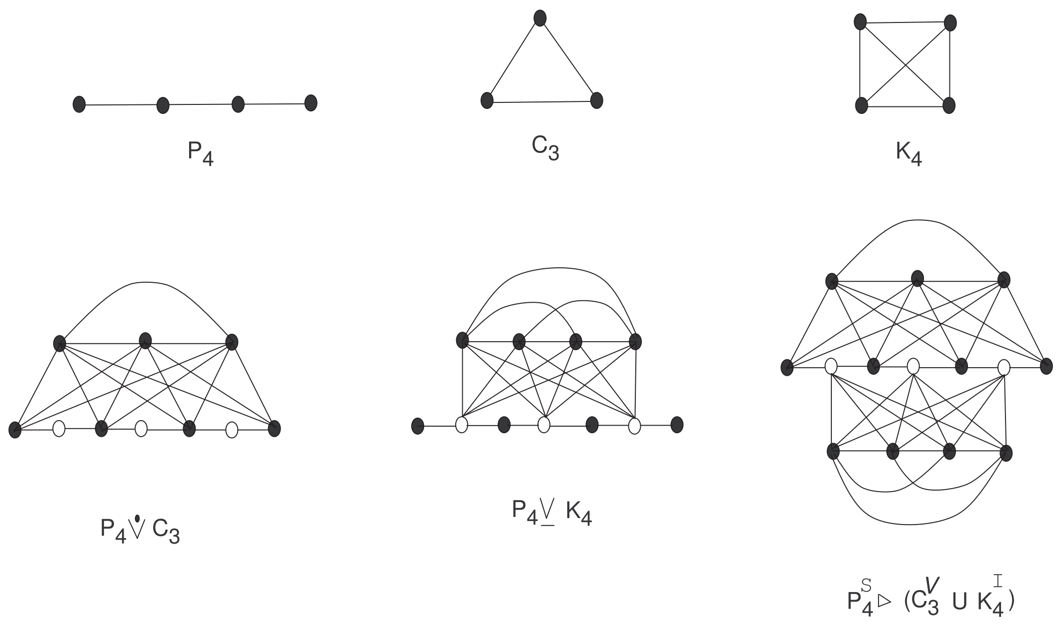

FIGURE 1. $P_4\dot{\vee}C_3$, $P_4\underline{\vee}K_4$ and $P_4^{\mathcal{S}} \rhd (C_3^{\mathcal{V}} \cup K_4^{\mathcal{I}})$.

Throughout the manuscript, all considered graphs are simple and connected. For a graph $\mathcal{H}$, $\mathcal{V}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ appear for vertex and edge sets, respectively, and n and m stands for the order and size of $\mathcal{H}$, respectively. An edge with end vertices h_i and h_j is recognized by $h_ih_j \in \mathcal{E}(\mathcal{H})$. For $h \in \mathcal{V}(\mathcal{H})$, the number of edges whose an end vertex h is called the degree of h in $\mathcal{H}$ and it is denoted by $\deg_{\mathcal{H}}(h)$. A (y_1, y_n)-path of n-vertices is described as a graph whose vertex and edge sets are $\{y_1, \ldots, y_n\}$ and $\{y_iy_{i+1} : 1 \leq i \leq n-1\}$, respectively. The notions K_n, P_n and C_n are commonly used for complete graph, path and cycle, respectively. The distance among two vertices $a, c \in \mathcal{V}(\mathcal{H})$ is represented by $d_{\mathcal{H}}(a, c)$ and explained as the length of shortest (a, c)-path in $\mathcal{H}$. For $a \in \mathcal{V}(\mathcal{H})$, the eccentricity $ec_{\mathcal{H}}(a)$ is specified as the largest distance among a and any other vertex in $\mathcal{H}$.

Recently, a new graph operation has been initiated by Wen et al. in [51], they named it as the subdivision vertex-edge join (SVE-join). For a graph $\mathcal{H}_q$, $\mathcal{S}(\mathcal{H}_q)$ is the subdividing graph of $\mathcal{H}_q$ whose vertex set has two portions, one the primary vertices $\mathcal{V}(\mathcal{H}_q)$, another, represented by $\mathcal{I}(\mathcal{H}_q)$, the inserting vertices that are end vertices of the edges of $\mathcal{H}_q$. Let $\mathcal{H}_r$ and $\mathcal{H}_s$ be the other two disjoint graphs. The SVE-join of $\mathcal{H}_q$ with $\mathcal{H}_r$ and $\mathcal{H}_s$, expressed by $\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})$, is the graph containing of $\mathcal{S}(\mathcal{H}_q)$, $\mathcal{H}_r$ and $\mathcal{H}_s$, all vertex-disjoint, and connecting the l-th vertex of $\mathcal{V}(\mathcal{H}_q)$ to every vertex in $\mathcal{V}(\mathcal{H}_r)$ and l-th vertex of $\mathcal{I}(\mathcal{H}_q)$ to each vertex in $\mathcal{V}(\mathcal{H}_s)$. It can be observed that $\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})$ is $\mathcal{H}_q\dot{\vee}\mathcal{H}_r$ (is attained from $\mathcal{S}(\mathcal{H}_q)$ and $\mathcal{H}_r$ by linking each vertex of $\mathcal{V}(\mathcal{H}_q)$ to every vertex of $\mathcal{V}(\mathcal{H}_r)$ [38]) if $\mathcal{H}_s$ is the null graph, and is $\mathcal{H}_q\underline{\vee}\mathcal{H}_s$ (is attained from $\mathcal{S}(\mathcal{H}_q)$ and $\mathcal{H}_s$ by linking each vertex of $\mathcal{E}(\mathcal{H}_q)$ to every vertex of $\mathcal{V}(\mathcal{H}_s)$ [38]) if $\mathcal{H}_r$ is the null graph. The graphs $P_4\dot{\vee}C_3$, $P_4\underline{\vee}K_4$ and $P_4^{\mathcal{S}} \rhd (C_3^{\mathcal{V}} \cup K_4^{\mathcal{I}})$ are illustrated in Figure 1.

Sharma *et al.* [49] introduced the well-known eccentricity-based index of a graph $\mathcal{H}$. They defined it as follows:

$$\xi^c(\mathcal{H}) = \sum_{h \in \mathcal{V}(\mathcal{H})} \deg_{\mathcal{H}}(h) ec_{\mathcal{H}}(h). \tag{1}$$

Gupta *et al.* [28] gave the concept of connective eccentricity index of $\mathcal{H}$ as follows:

$$\xi^{ce}(\mathcal{H}) = \sum_{h \in \mathcal{V}(\mathcal{H})} \frac{\deg_{\mathcal{H}}(h)}{ec_{\mathcal{H}}(h)}. \tag{2}$$

If we use only the eccentricities of vertices of $\mathcal{H}$ in (1), then we can describe the total-eccentricity index as follows [6]:

$$\tau(\mathcal{H}) = \sum_{h \in \mathcal{V}(\mathcal{H})} ec_{\mathcal{H}}(h). \tag{3}$$

The mean value of the eccentricities of elements of $\mathcal{V}(\mathcal{H})$ is said to be the average eccentricity $aveg(\mathcal{H})$ of it [50], that is

$$aveg(\mathcal{H}) = \frac{1}{n} \sum_{h \in \mathcal{V}(\mathcal{H})} ec_{\mathcal{H}}(h) = \frac{\tau(\mathcal{H})}{n}. \tag{4}$$

The eccentricity versions of Zagreb indices of $\mathcal{H}$ were given in [26] as follows:

$$\mathcal{M}_1(\mathcal{H}) = \sum_{h_{p_1} \in \mathcal{V}(\mathcal{H})} ec^2_{\mathcal{H}}(h_{p_1}),$$
$$\mathcal{M}_2(\mathcal{H}) = \sum_{h_{p_1}h_{p_2} \in \mathcal{E}(\mathcal{H})} ec_{\mathcal{H}}(h_{p_1}) ec_{\mathcal{H}}(h_{p_2}). \tag{5}$$

The eccentric form of geometric-arithmetic index [27] of $\mathcal{H}$ is as follows:

$$\mathcal{GA}_{ec}(\mathcal{H}) = \sum_{h_{p_1}h_{p_2} \in \mathcal{E}(\mathcal{H})} \frac{2\sqrt{ec_{\mathcal{H}}(h_{p_1}) ec_{\mathcal{H}}(h_{p_2})}}{ec_{\mathcal{H}}(h_{p_1}) + ec_{\mathcal{H}(h_{p_2})}}. \tag{6}$$

The eccentric form of atom-bond connectivity index [22] of $\mathcal{H}$ is as follows:

$$\mathcal{ABC}_{ec}(\mathcal{H}) = \sum_{h_{p_1}h_{p_2}\in\mathcal{E}(\mathcal{H})} \sqrt{\frac{ec_{\mathcal{H}}(h_{p_1}) + ec_{\mathcal{H}}(h_{p_2}) - 2}{ec_{\mathcal{H}}(h_{p_1})ec_{\mathcal{H}}(h_{p_2})}}. \quad (7)$$

The eccentric adjacency index is given by Gupta et al. in [29] as follows:

$$\xi^{ad}(\mathcal{H}) = \sum_{h\in\mathcal{V}(\mathcal{H})} \frac{S_{\mathcal{H}}(h)}{ec_{\mathcal{H}}(h)}. \quad (8)$$

The modified type of eccentric-connectivity index [5] of $\mathcal{H}$ is given in the following way:

$$\xi_c(\mathcal{H}) = \sum_{h\in\mathcal{V}(\mathcal{H})} S_{\mathcal{H}}(h)ec_{\mathcal{H}}(h). \quad (9)$$

The eccentric distance sum index was first presented in [30] as follows:

$$\xi^{ds}(\mathcal{H}) = \sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} (ec_{\mathcal{H}}(h_{p_1}) + ec_{\mathcal{H}}(h_{p_2}))d_{\mathcal{H}}(h_{p_1}, h_{p_2})$$
$$= \sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})} ec_{\mathcal{H}}(h_{p_1})D_{\mathcal{H}}(h_{p_1}). \quad (10)$$

where $D_{\mathcal{H}}(h_{p_1}) = \sum_{h_{p_2}\in\mathcal{V}(\mathcal{H})} d_{\mathcal{H}}(h_{p_1}, h_{p_2})$.

The Wiener index is a distance-based graph descriptor described by [21] as follows:

$$W(\mathcal{H}) = \sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} d_{\mathcal{H}}(h_{p_1}, h_{p_2})$$
$$= \frac{1}{2}\sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})} D_{\mathcal{H}}(h_{p_1})$$
$$= \frac{1}{2}\sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})}\sum_{h_{p_2}\in\mathcal{V}(\mathcal{H})} d_{\mathcal{H}}(h_{p_1}, h_{p_2}). \quad (11)$$

The Harary index of $\mathcal{H}$ is explained as the sum of reciprocals of distances among all the unordered pairs of its vertices as follows: [42]:

$$H(\mathcal{H}) = \sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} \frac{1}{d_{\mathcal{H}}(h_{p_1}, h_{p_2})}$$
$$= \frac{1}{2}\sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})}\sum_{h_{p_2}\in\mathcal{V}(\mathcal{H})} \frac{1}{d_{\mathcal{H}}(h_{p_1}, h_{p_2})}. \quad (12)$$

As an extension of the Wiener index, Randić put forward the hyper-Wiener index as:

$$WW(\mathcal{H}) = \frac{1}{2}\sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} \Big(d_{\mathcal{H}}(h_{p_1}, h_{p_2}) + d^2_{\mathcal{H}}(h_{p_1}, h_{p_2})\Big),$$

where

$$A(\mathcal{H}) = \sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} d^2_{\mathcal{H}}(h_{p_1}, h_{p_2})$$

and

$$DD_{\mathcal{H}}(h_{p_1}) = \sum_{h_{p_2}\in\mathcal{V}(\mathcal{H})} d^2_{\mathcal{H}}(h_{p_1}, h_{p_2}),$$

then

$$WW(\mathcal{H}) = \frac{1}{2}(W(\mathcal{H}) + A(\mathcal{H}))$$
$$= \frac{1}{4}\sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})} D_{\mathcal{H}}(h_{p_1})$$
$$+ \frac{1}{4}\sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})} DD_{\mathcal{H}}(h_{p_1}). \quad (13)$$

Dobrynin and Kochetova [21] introduced the degree-distance index of a graph $\mathcal{H}$ as follows:

$$DD(\mathcal{H}) = \sum_{\{h_{p_1},h_{p_2}\}\subseteq\mathcal{V}(\mathcal{H})} (\deg_{\mathcal{H}}(h_{p_1}) + \deg_{\mathcal{H}}(h_{p_2}))d_{\mathcal{H}} \times (h_{p_1}, h_{p_2})$$
$$= \sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})} \deg_{\mathcal{H}}(h_{p_1})D_{\mathcal{H}}(h_{p_1})$$
$$= \sum_{h_{p_1}\in\mathcal{V}(\mathcal{H})}\sum_{h_{p_2}\in\mathcal{V}(\mathcal{H})} \deg_{\mathcal{H}}(h_{p_1})d_{\mathcal{H}}(h_{p_1}, h_{p_2}). \quad (14)$$

For the in depth study of these descriptors and other famous topological descriptors, we recommended the reader to [1]–[4], [23]–[25], [32], [33], [35]–[37], [39]–[41], [52]. Now we state certain properties of the subdivision vertex-edge join of three graphs in the next lemma.

Lemma 1: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be graphs. Then we have:

1) $|\mathcal{V}(\mathcal{H}^{\mathcal{S}}_q \rhd (\mathcal{H}^{\mathcal{V}}_r \cup \mathcal{H}^{\mathcal{I}}_s))| = n_1 + m_1 + n_2 + n_3$, and $|\mathcal{E}(\mathcal{H}^{\mathcal{S}}_q \rhd (\mathcal{H}^{\mathcal{V}}_r \cup \mathcal{H}^{\mathcal{I}}_s))| = 2m_1 + n_1n_2 + m_1n_3 + m_2 + m_3$.
2) $\deg_{\mathcal{H}^{\mathcal{S}}_q \rhd (\mathcal{H}^{\mathcal{V}}_r \cup \mathcal{H}^{\mathcal{I}}_s)}(h)$

$$= \begin{cases} \deg_{\mathcal{H}_q}(h) + n_2, & \text{if } h \in \mathcal{V}(\mathcal{H}_q), \\ n_3 + 2, & \text{if } h \in \mathcal{I}(\mathcal{H}_q), \\ \deg_{\mathcal{H}_r}(z) + n_1, & \text{if } h \in \mathcal{V}(\mathcal{H}_r), \\ \deg_{\mathcal{H}_s}(z) + m_1, & \text{if } h \in \mathcal{V}(\mathcal{H}_s). \end{cases}$$

3) $d_{\mathcal{H}^{\mathcal{S}}_q \rhd (\mathcal{H}^{\mathcal{V}}_r \cup \mathcal{H}^{\mathcal{I}}_s)}(h_{p_1}, h_{p_2})$

$$= \begin{cases} 0, & \text{if } h_{p_1} = h_{p_2}, \\ 1, & \text{if } h_{p_1}h_{p_2} \in \mathcal{E}(\mathcal{H}_i), \quad i = 2, 3, \\ & \text{or } h_{p_1}h_{p_2} \in \mathcal{E}(\mathcal{S}(\mathcal{H}_q)), \quad h_{p_1} \in \mathcal{V}(\mathcal{H}_q), \\ & h_{p_2} \in \mathcal{I}(\mathcal{H}_q) \quad \text{or } h_{p_1} \in \mathcal{V}(\mathcal{H}_q) \\ & \text{and } h_{p_2} \in \mathcal{V}(\mathcal{H}_r), \\ & \text{or } h_{p_1} \in \mathcal{I}(\mathcal{H}_q) \quad \text{and } h_{p_2} \in \mathcal{V}(\mathcal{H}_s), \\ 2, & h_{p_1}h_{p_2} \notin \mathcal{E}(\mathcal{H}_i), \quad i = 2, 3, \\ & \text{or } h_{p_1}, h_{p_2} \in \mathcal{V}(\mathcal{H}_q), \quad \text{or} \quad h_{p_1}, h_{p_2} \in \mathcal{I}(\mathcal{H}_q), \\ & \text{or } h_{p_1} \in \mathcal{I}(\mathcal{H}_q) \quad \text{and } h_{p_2} \in \mathcal{V}(\mathcal{H}_r), \\ & \text{or } h_{p_1} \in \mathcal{V}(\mathcal{H}_q) \quad \text{and } h_{p_2} \in \mathcal{V}(\mathcal{H}_r). \\ 3, & \text{if } h_{p_1}h_{p_2} \notin \mathcal{E}(\mathcal{S}(\mathcal{H}_q)), \\ & h_{p_1} \in \mathcal{V}(\mathcal{H}_q) \quad \text{and } h_{p_2} \in \mathcal{I}(\mathcal{H}_q) \\ & h_{p_1} \in \mathcal{V}(\mathcal{H}_r) \quad \text{and } h_{p_2} \in \mathcal{V}(\mathcal{H}_s). \end{cases}$$

4) $ec_{\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})}(h)$
$$= \begin{cases} 2, & \text{if } h \in \mathcal{V}(\mathcal{H}_q) \quad \text{or } h \in \mathcal{I}(\mathcal{H}_q), \\ 3, & \text{if } h \in \mathcal{V}(\mathcal{H}_r) \quad \text{or } h \in \mathcal{V}(\mathcal{H}_s). \end{cases}$$

5) $S_{\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})}(h)$
$$= \begin{cases} S_{\mathcal{H}_r}(h) + n_1 \deg_{\mathcal{H}_r}(h) \\ +2m_1 + n_1 n_2, & \text{if } h \in \mathcal{V}(\mathcal{H}_r), \\ S_{\mathcal{H}_s}(h) + m_1 \deg_{\mathcal{H}_s}(h) \\ +m_1(n_3+2), & \text{if } h \in \mathcal{V}(\mathcal{H}_s), \\ \deg_{\mathcal{H}_q}(h)(n_3+2) \\ +2m_2 + n_1 n_2, & \text{if } h \in \mathcal{V}(\mathcal{H}_q), \\ \deg_{\mathcal{H}_q}(h_{p_1}) \\ +\deg_{\mathcal{H}_q}(h_{p_2}) \\ +2n_2 + 2m_3 + m_1 n_3, & \text{if } h = h_{p_1}h_{p_2} \in \mathcal{I}(\mathcal{H}_q), \end{cases}$$

II. MAIN RESULTS

This section provides the results associated to various distance based indices of the subdivision vertex-edge join of graphs. In next theorem, we present the formulae for eccentric connectivity and connective eccentricity indices of subdivision vertex-edge join for three graphs.

Theorem 1: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

1) $\xi^c(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = 5m_1n_3+5n_1n_2+8m_1+6m_2+6m_3$.
2) $\xi^{ce}(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = \frac{1}{6}(5m_1n_3+5n_1n_2+12m_1+4m_2+4m_3)$.

Proof:

1) By using Lemma 1 in Equation (1), we get
$$\begin{aligned}\xi^c(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) &= \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} 3(\deg_{\mathcal{H}_s}(h_s)+m_1) \\ &\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} 3(\deg_{\mathcal{H}_r}(h_r)+n_1) \\ &\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} 2(\deg_{\mathcal{H}_q}(h_q)+n_2) \\ &\quad + \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} 2(n_3+2) \\ &= 3(2m_3+m_1n_3)+3(2m_2+n_1n_2) \\ &\quad +2(2m_1+n_1n_2)+2m_1(n_3+2) \\ &= 5m_1n_3+5n_1n_2+8m_1+6m_2 \\ &\quad +6m_3.\end{aligned}$$

2) By using Lemma 1 in Equation (2), we get
$$\begin{aligned}\xi^{ce}(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) &= \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \frac{\deg_{\mathcal{H}_s}(h_s)+m_1}{3} \\ &\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \frac{\deg_{\mathcal{H}_r}(h_r)+n_1}{3} \\ &\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \frac{\deg_{\mathcal{H}_q}(h_q)+n_2}{2} \\ &\quad + \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} \frac{n_3+2}{2} \\ &= \frac{1}{3}(2m_3+m_1n_3)+\frac{1}{3}(2m_2 \\ &\quad +n_1n_2)+\frac{1}{2}(2m_1+n_1n_2) \\ &\quad +\frac{1}{2}m_1(n_3+2) \\ &= \frac{1}{6}(5m_1n_3+5n_1n_2+12m_1 \\ &\quad +4m_2+4m_3).\end{aligned}$$

This complete the proof. □

Now, we set up the precise values of the total eccentricity and average eccentricity indices of subdivision vertex-edge join for three graphs.

Theorem 2: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

1) $\tau(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = 3n_3+3n_2+2n_1+2m_1$.
2) $aveg(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = \frac{3n_3+3n_2+2n_1+2m_1}{n_1+n_2+n_3+m_1}$.

Proof:

1) By using Lemma 1 in Equation (3), we get
$$\begin{aligned}\tau(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) &= \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} 3 + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} 3 \\ &\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} 2 + \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} 2 \\ &= 3n_3+3n_2+2n_1+2m_1.\end{aligned}$$

2) By using Lemma 1 in Equation (4), we obtain
$$\begin{aligned}aveg(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) &= \frac{\tau(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))}{n_1+n_2+n_3+m_1} \\ &= \frac{3n_3+3n_2+2n_1+2m_1}{n_1+n_2+n_3+m_1}.\end{aligned}$$

This complete the proof. □

Theorem 3: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then

1) $\mathcal{M}_1(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = 9n_3+9n_2+4n_1+4m_1$.
2) $\mathcal{M}_2(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})) = 9m_3+9m_2+6n_1n_2+6n_3m_1+8m_1$.

Proof:

1) By using Lemma 1 in Equation (5), we get
$$\mathcal{M}_1(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$$

$$= \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} (3)^2 + \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} (3)^2 + \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} (2)^2 + \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} (2)^2 = 9n_3 + 9n_2 + 4n_1 + 4m_1.$$

2) By using Lemma 1 in Equation (5), we obtain

$$\mathcal{M}_2(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \sum_{h_{s_1}h_{s_2} \in \mathcal{E}(\mathcal{H}_s)} 3 \times 3 + \sum_{h_{r_1}h_{r_2} \in \mathcal{E}(\mathcal{H}_r)} 3 \times 3 + \sum_{h_{q_1} \in \mathcal{V}(\mathcal{H}_q)} \sum_{h_{r_1} \in \mathcal{V}(\mathcal{H}_r)} 2 \times 3 + \sum_{h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)} \sum_{h_{s_1} \in \mathcal{V}(\mathcal{H}_s)} 2 \times 3 + \sum_{\substack{h_{q_1}h'_{q_1} \in \mathcal{E}(\mathcal{S}(\mathcal{H}_q)),\\ h_{q_1} \in \mathcal{V}(\mathcal{H}_q), h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)}} 2 \times 2 = 9m_3 + 9m_2 + 6n_1n_2 + 6n_3m_1 + 8m_1.$$

This complete the proof. □

Theorem 4: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

1) $\mathcal{GA}_{ecc}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = m_2 + m_3 + 2m_1 + \frac{2\sqrt{6}}{5}(n_1n_2 + m_1n_3)$.
2) $\mathcal{ABC}_{ecc}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \frac{2}{3}(m_2+m_3) + \frac{1}{\sqrt{2}}(n_1n_2 + m_1n_3 + 2m_1)$.

Proof:

1) By using Lemma 1 in Equation (6), we obtain

$$\mathcal{GA}_{ecc}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \sum_{h_{s_1}h_{s_2} \in \mathcal{E}(\mathcal{H}_s)} \frac{2\sqrt{3 \times 3}}{3+3} + \sum_{h_{r_1}h_{r_2} \in \mathcal{E}(\mathcal{H}_r)} \frac{2\sqrt{3 \times 3}}{3+3} + \sum_{h_{q_1} \in \mathcal{V}(\mathcal{H}_q)} \sum_{h_{r_1} \in \mathcal{V}(\mathcal{H}_r)} \frac{2\sqrt{2 \times 3}}{2+3} + \sum_{h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)} \sum_{h_{s_1} \in \mathcal{V}(\mathcal{H}_s)} \frac{2\sqrt{2 \times 3}}{2+3} + \sum_{\substack{h_{q_1}h'_{q_1} \in \mathcal{E}(\mathcal{S}(\mathcal{H}_q)),\\ h_{q_1} \in \mathcal{V}(\mathcal{H}_q), h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)}} \frac{2\sqrt{2 \times 2}}{2+2} = m_2 + m_3 + \frac{2\sqrt{6}}{5}n_1n_2 + \frac{2\sqrt{6}}{5}m_1n_3 + 2m_1.$$

2) By using Lemma 1 in Equation (7), we obtain

$$\mathcal{ABC}_{ecc}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \sum_{h_{s_1}h_{s_2} \in \mathcal{E}(\mathcal{H}_s)} \sqrt{\frac{3+3-2}{3 \times 3}} + \sum_{h_{r_1}h_{r_2} \in \mathcal{E}(\mathcal{H}_r)} \sqrt{\frac{3+3-2}{3 \times 3}} + \sum_{h_{q_1} \in \mathcal{V}(\mathcal{H}_q)} \sum_{h_{r_1} \in \mathcal{V}(\mathcal{H}_r)} \sqrt{\frac{2+3-2}{2 \times 3}} + \sum_{h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)} \sum_{h_{s_1} \in \mathcal{V}(\mathcal{H}_s)} \sqrt{\frac{2+3-2}{2 \times 3}} + \sum_{\substack{h_{q_1}h'_{q_1} \in \mathcal{E}(\mathcal{S}(\mathcal{H}_q)),\\ h_{q_1} \in \mathcal{V}(\mathcal{H}_q), h'_{q_1} \in \mathcal{I}(\mathcal{H}_q)}} \sqrt{\frac{2+2-2}{2 \times 2}} = \frac{2}{3}m_2 + \frac{2}{3}m_3 + \frac{1}{\sqrt{2}}n_1n_2 + \frac{1}{\sqrt{2}}m_1n_3 + \left(\frac{1}{\sqrt{2}}\right)2m_1.$$

This complete the proof. □

Theorem 5: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$\xi^{ad}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \frac{1}{6}(3\mathcal{M}_1(\mathcal{H}_q) + 2\mathcal{M}_1(\mathcal{H}_r) + 2\mathcal{M}_1(\mathcal{H}_s)) + \frac{5}{3}(n_1m_2 + m_1n_2 + m_1m_3 + n_3m_1) + \frac{1}{6}n_1n_2(2n_2 + 3n_1) + \frac{1}{6}n_3m_1(2n_3 + 3m_1) + 2m_1.$$

Proof: By Lemma 1 in Equation (8), we get

$$\xi^{ad}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) = \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} \frac{S_{\mathcal{H}_s}(h_s) + m_1 \deg_{\mathcal{H}_s}(h_s) + m_1(n_3+2)}{3} + \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} \frac{S_{\mathcal{H}_r}(h_r) + n_1 \deg_{\mathcal{H}_r}(h_r) + 2m_1 + n_1n_2}{3} + \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} \frac{\deg_{\mathcal{H}_q}(h_q)(n_3+2) + 2m_2 + n_1n_2}{2} + \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} \frac{\deg_{\mathcal{H}_q}(h_{q_1}) + \deg_{\mathcal{H}_q}(h_{q_2}) + 2n_2 + 2m_3 + m_1n_3}{2}$$

$$= \frac{1}{3}(\mathcal{M}_1(\mathcal{H}_s) + 2m_1m_3 + n_3m_1(n_3+2)) + \frac{1}{3}(\mathcal{M}_1(\mathcal{H}_r) + 2n_1m_2 +, 2n_2m_1 + n_1n_2^2) + \frac{1}{2}\left(2m_1(n_3+2) + 2m_2n_1 + n_1^2n_2\right) + \frac{1}{2}(\mathcal{M}_1(\mathcal{H}_q) + 2n_2m_1 + 2m_3m_1 + m_1^2n_3)$$

After simplification we acquire required result. This complete the proof. □

Theorem 6: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$\begin{aligned}\xi_c(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) &= 2\mathcal{M}_1(\mathcal{H}_q) + 3\mathcal{M}_1(\mathcal{H}_r) + 3\mathcal{M}_1(\mathcal{H}_s) \\ &\quad + 10(n_1m_2 + m_1n_2 + m_1m_3 + n_3m_1) \\ &\quad + n_1n_2(2n_1 + 3n_2) + n_3m_1(3n_3 + 2m_1) \\ &\quad + 8m_1.\end{aligned}$$

Proof: By Lemma 1 in Equation (9), we get

$$\begin{aligned}&\xi_c(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) \\ &= \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} 3(S_{\mathcal{H}_s}(h_s) + m_1 \deg_{\mathcal{H}_s}(h_s) + m_1(n_3+2)) \\ &\quad + \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} 3(S_{\mathcal{H}_r}(h_r) + n_1 \deg_{\mathcal{H}_r}(h_r) + 2m_1 + n_1n_2) \\ &\quad + \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} 2(\deg_{\mathcal{H}_q}(h_q)(n_3+2) + 2m_2 + n_1n_2) \\ &\quad + \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} 2(\deg_{\mathcal{H}_q}(h_{q_1}) + \deg_{\mathcal{H}_q} \\ &\quad \times (h_{q_2}) + 2n_2 + 2m_3 + m_1n_3) \\ &= 3\left(\mathcal{M}_1(\mathcal{H}_s) + 2m_1m_3 + n_3m_1(n_3+2)\right) 3(\mathcal{M}_1(\mathcal{H}_r) \\ &\quad + 2n_1m_2 + 2n_2m_1 + n_1n_2^2) \\ &\quad + 2\left(2m_2(n_3+2) + 2m_2n_1 + n_1^2n_2\right) + 2(\mathcal{M}_1(\mathcal{H}_q) \\ &\quad + 2n_2m_1 + 2m_3m_1 + m_1^2n_3)\end{aligned}$$

After simplification we get required result. This complete the proof. □

Theorem 7: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$\begin{aligned}\xi^{ds}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) &= 6n_3(n_3-1) + 6n_2(n_2-1) + 4n_1(n_1-1) \\ &\quad + 5n_1n_2 + 10n_2m_1 + 4m_1^2 + 18n_2n_3 \\ &\quad + 5m_1n_3 + 10n_1n_3 + 12n_1m_1 + 4m_1^2 - 20m_1 \\ &\quad - 6m_2 - 6m_3.\end{aligned}$$

Proof: By Lemma 1 in Equation (10), we get

$$\begin{aligned}&\xi^{ds}(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) \\ &= \Bigg(\sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} \sum_{h \in N_{\mathcal{H}_s}(h_s) \cup \mathcal{I}(\mathcal{H}_q)} 3 \times 1 \\ &\quad + \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} \sum_{h \in (\mathcal{V}(\mathcal{H}_s) \backslash N_{\mathcal{H}_s}(h_s)) \cup \mathcal{V}(\mathcal{H}_q)} 3 \times 2 \\ &\quad + \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} 3 \times 3 \Bigg) \\ &\quad + \Bigg(\sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} \sum_{h \in N_{\mathcal{H}_r}(h_r) \cup \mathcal{V}(\mathcal{H}_q)} 3 \times 1 \\ &\quad + \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} \sum_{h \in (\mathcal{V}(\mathcal{H}_r) \backslash N_{\mathcal{H}_r}(h_r)) \cup \mathcal{I}(\mathcal{H}_q)} 3 \times 2 \\ &\quad + \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} 3 \times 3 \Bigg) \\ &\quad + \Bigg(\sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} \sum_{h \in N_{\mathcal{S}(\mathcal{H}_q)}(h_q) \cup \mathcal{V}(\mathcal{H}_r)} 2 \times 1 \\ &\quad + \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} \sum_{h \in (\mathcal{V}(\mathcal{H}_q) \backslash \{h_q\}) \cup \mathcal{V}(\mathcal{H}_s)} 2 \times 2 \\ &\quad + \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q) \backslash N_{\mathcal{S}(\mathcal{H}_q)}(h_q)} 2 \times 3 \Bigg) \\ &\quad + \Bigg(\sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} \sum_{h \in N_{\mathcal{S}(\mathcal{H}_q)}(h'_q) \cup \mathcal{V}(\mathcal{H}_s)} 2 \times 1 \\ &\quad + \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} \sum_{h \in (\mathcal{I}(\mathcal{H}_q) \backslash \{h'_q\}) \cup \mathcal{V}(\mathcal{H}_r)} 2 \times 2 \\ &\quad + \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} \sum_{h_q \in \mathcal{V}(\mathcal{H}_q) \backslash N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)} 2 \times 3 \Bigg) \\ &= \Bigg(3 \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} (\deg_{\mathcal{H}_s}(h_s) + m_1) + 6 \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} (n_3 - 1 \\ &\quad - \deg_{\mathcal{H}_s}(h_s) + n_1) + 9 \sum_{h_s \in \mathcal{V}(\mathcal{H}_s)} n_2 \Bigg) \\ &\quad + \Bigg(3 \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} (\deg_{\mathcal{H}_r}(h_r) + n_1) \\ &\quad + 6 \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} (n_2 - 1 - \deg_{\mathcal{H}_r}(h_r) + m_1) + 9 \sum_{h_r \in \mathcal{V}(\mathcal{H}_r)} n_3 \Bigg) \\ &\quad + \Bigg(2 \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} (\deg_{\mathcal{H}_q}(h_q) + n_2) + 4 \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} (n_1 - 1 + n_3) \\ &\quad + 6 \sum_{h_q \in \mathcal{V}(\mathcal{H}_q)} (m_1 - \deg_{\mathcal{H}_q}(h_q)) \Bigg) + \Bigg(2 \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} (n_3 + 2) \\ &\quad + 4 \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} (m_1 - 1 + n_2) + 6 \sum_{h'_q \in \mathcal{I}(\mathcal{H}_q)} (n_1 - 2) \Bigg) \\ &= 3(2m_3 + m_1n_3) + 6(n_3^2 - n_3 - 2m_3 + n_1n_3) + 9n_2n_3 \\ &\quad + 3(2m_2 + n_1n_2) + 6(n_2^2 - n_2 - 2m_2 + n_2m_1) + 9n_2n_3 \\ &\quad + 2(2m_1 + n_1n_2) + 4(n_1^2 - n_1 + n_1n_3) + 6(n_1m_1 - 2m_1) \\ &\quad + 2m_1(n_3 + 2) + 4m_1(m_1 - 1 + n_2) + 6m_1(n_1 - 2) \\ &= 6n_3(n_3-1) + 6n_2(n_2-1) + 4n_1(n_1-1) + 5n_1n_2 \\ &\quad + 10n_2m_1 + 4m_1^2 + 18n_2n_3 + 5m_1n_3 + 10n_1n_3 \\ &\quad + 12n_1m_1 + 4m_1^2 - 20m_1 - 6m_2 - 6m_3.\end{aligned}$$

This complete the proof. □

Theorem 8: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$\begin{aligned}W(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}})) &= (n_1^2 + n_2^2 + n_3^2 + m_1^2) - (n_1 + n_2 + n_3 \\ &\quad + 5m_1) - (m_2 + m_3) + n_1n_2 + 3n_2n_3\end{aligned}$$

$$+2n_2m_1+m_1n_3+2n_1n_3+3n_1m_1.$$

Proof: By Lemma 1 in Equation (11), we get

$$\begin{aligned}
&W(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}}))\\
&= \frac{1}{2}\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h\in N_{\mathcal{H}_s}(h_s)\cup\mathcal{I}(\mathcal{H}_q)} 1\\
&\quad + \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h\in(\mathcal{V}(\mathcal{H}_s)\setminus N_{\mathcal{H}_s}(h_{s_1}))\cup\mathcal{V}(\mathcal{H}_q)} 2\\
&\quad + \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} 3\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h\in N_{\mathcal{H}_r}(h)\cup\mathcal{V}(\mathcal{H}_q)} 1\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h\in(\mathcal{V}(\mathcal{H}_r)\setminus N_{\mathcal{H}_r}(h_r))\cup\mathcal{I}(\mathcal{H}_q)} 2\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h\in\mathcal{V}(\mathcal{H}_s)} 3\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h_q)\cup\mathcal{V}(\mathcal{H}_r)} 1\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h\in(\mathcal{V}(\mathcal{H}_q)\setminus\{h_q\})\cup\mathcal{V}(\mathcal{H}_s)} 2\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h_q)} 3\\
&\quad + \Bigg(\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} \sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)\cup\mathcal{V}(\mathcal{H}_s)} 1\\
&\quad + \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} \sum_{h\in(\mathcal{I}(\mathcal{H}_q)\setminus\{h'_q\})\cup\mathcal{V}(\mathcal{H}_r)} 2\\
&\quad + \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)} 3\Bigg)\\
&= \frac{1}{2}\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} (\deg_{\mathcal{H}_s}(h_s)+m_1)\\
&\quad +2\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} (n_3-1-\deg_{\mathcal{H}_s}(h_s)+n_1)\\
&\quad +3\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} n_2 + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} (\deg_{\mathcal{H}_r}(h_r)\\
&\quad +n_1)+2\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} (n_2-1-\deg_{\mathcal{H}_r}(h_r)\\
&\quad +m_1)+3\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} n_3 + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} (\deg_{\mathcal{H}_q}(h_q)\\
&\quad +n_2)+2\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} (n_1-1+n_3)\\
&\quad +3\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} (m_1-\deg_{\mathcal{H}_q}(h_q))\\
&\quad +\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} (n_3+2)+2\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} (m_1\\
&\quad -1+n_2)+3\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)} (n_1-2)\Bigg)\\
&= \frac{1}{2}\Big(2m_3+m_1n_3+2n_3^2-2n_3-4m_3+2n_1n_3+3n_2n_3\\
&\quad +2m_2+n_1n_2+2n_2^2-2n_2-4m_2+2n_2m_1+3n_2n_3\\
&\quad +2m_1+n_1n_2+2n_1^2-2n_1+2n_1n_3+3n_1m_1-6m_1\\
&\quad +m_1(n_3+2)+2m_1(m_1-1+n_2)+3m_1(n_1-2)\Big)\\
&= (n_1^2+n_2^2+n_3^2+m_1^2)-(n_1+n_2+n_3+5m_1)\\
&\quad -(m_2+m_3)+n_1n_2+3n_2n_3+2n_2m_1+m_1n_3\\
&\quad +2n_1n_3+3n_1m_1.
\end{aligned}$$

This complete the proof. □

Theorem 9: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$\begin{aligned}
&H(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}}))\\
&\quad = \frac{1}{2}\Bigg(\frac{13}{6}m_1+m_2+m_3+2n_1n_2+\frac{1}{2}n_1(n_1\\
&\quad -1)+\frac{1}{2}n_2(n_2-1)+\frac{1}{2}n_3(n_3-1)\\
&\quad +m_1\left(n_2+2n_3+\frac{2}{3}n_1\right)+n_3\left(\frac{2}{3}n_2+\frac{1}{2}n_1\right)\\
&\quad +\frac{1}{2}m_1^2\Bigg).
\end{aligned}$$

Proof: By Lemma 1 in Equation (12), we get

$$\begin{aligned}
&H(\mathcal{H}_q^{\mathcal{S}} \rhd (\mathcal{H}_r^{\mathcal{V}} \cup \mathcal{H}_s^{\mathcal{I}}))\\
&= \frac{1}{2}\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h\in N_{\mathcal{H}_s}(h_s)\cup\mathcal{I}(\mathcal{H}_q)} \frac{1}{1}\\
&\quad + \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h\in(\mathcal{V}(\mathcal{H}_s)\setminus N_{\mathcal{H}_s}(h_s))\cup\mathcal{V}(\mathcal{H}_q)} \frac{1}{2}\\
&\quad + \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \frac{1}{3}\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h\in N_{\mathcal{H}_r}(h_r)\cup\mathcal{V}(\mathcal{H}_q)} \frac{1}{1}\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h\in(\mathcal{V}(\mathcal{H}_r)\setminus N_{\mathcal{H}_r}(h_r))\cup\mathcal{I}(\mathcal{H}_q)} \frac{1}{2}\\
&\quad + \sum_{h_r\in\mathcal{V}(\mathcal{H}_r)} \sum_{h_s\in\mathcal{V}(\mathcal{H}_s)} \frac{1}{3}\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h_q)\cup\mathcal{V}(\mathcal{H}_r)} \frac{1}{1}\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h\in(\mathcal{V}(\mathcal{H}_q)\setminus\{h_q\})\cup\mathcal{V}(\mathcal{H}_s)} \frac{1}{2}\\
&\quad + \sum_{h_q\in\mathcal{V}(\mathcal{H}_q)} \sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h_q)} \frac{1}{3}
\end{aligned}$$

$$+\Big(\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)\cup\mathcal{V}(\mathcal{H}_s)}\frac{1}{1}$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in(\mathcal{I}(\mathcal{H}_q)\setminus\{h'_q\})\cup\mathcal{V}(\mathcal{H}_r)}\frac{1}{2}$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)}\frac{1}{3}\Big)$$
$$=\frac{1}{2}\Big(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg_{\mathcal{H}_s}(h_s)+m_1)$$
$$+\frac{1}{2}\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(n_3-1-\deg_{\mathcal{H}_s}(h_s)+n_1)$$
$$+\frac{1}{3}\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}n_2+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg_{\mathcal{H}_r}(h_r)+n_1)$$
$$+\frac{1}{2}\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(n_2-1-\deg_{\mathcal{H}_r}(h_r)+m_1)$$
$$+\frac{1}{3}\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}n_3+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)$$
$$+\frac{1}{2}\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(n_1-1+n_3)+\frac{1}{3}\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(m_1-\deg_{\mathcal{H}_q}(h_q))$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_3+2)+\frac{1}{2}\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(m_1-1+n_2)$$
$$+\frac{1}{3}\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_1-2)\Big)$$
$$=\frac{1}{2}\Big(2m_3+m_1n_3+\frac{1}{2}(n_3^2-n_3-2m_3+n_1n_3)+\frac{1}{3}n_2n_3$$
$$+2m_2+n_1n_2+\frac{1}{2}(n_2^2-n_2-2m_2+n_2m_1)+\frac{1}{3}n_2n_3$$
$$+2m_1+n_1n_2+\frac{1}{2}n_1(n_1-1+n_3)+\frac{1}{3}n_1m_1-\frac{2}{3}m_1$$
$$+m_1(n_3+2)+\frac{1}{2}m_1(m_1-1+n_2)+\frac{1}{3}m_1(n_1-2)\Big)$$
$$=\frac{1}{2}\Big(\frac{13}{6}m_1+m_2+m_3+2n_1n_2+\frac{1}{2}n_1(n_1-1)$$
$$+\frac{1}{2}n_2(n_2-1)+\frac{1}{2}n_3(n_3-1)+m_1\left(n_2+2n_3+\frac{2}{3}n_1\right)$$
$$+n_3\left(\frac{2}{3}n_2+\frac{1}{2}n_1\right)+\frac{1}{2}m_1^2\Big).$$

This complete the proof. □

Theorem 10: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$WW(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$$
$$=\frac{3}{2}(n_1^2+n_2^2+n_3^2+m_1^2)-\frac{3}{2}(n_1+n_2$$
$$+n_3)+\frac{23}{2}m_1-2(m_2+m_3)+n_1n_2$$
$$+6n_2n_3+3n_2m_1+m_1n_3+3n_1n_3$$
$$+6n_1m_1.$$

Proof: By Lemma 1 in Equation (13), we need to find only $A(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$.

$$A(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$$
$$=\frac{1}{2}\left(\sum_{h_1\in\mathcal{V}(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))}DD_{\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})}(h_1)\right)$$
$$=\frac{1}{2}\Big(\sum_{h_1\in\mathcal{V}(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))}$$
$$\sum_{h_2\in\mathcal{V}(\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))}d^2_{\mathcal{H}_q^{\mathcal{S}}\rhd(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}})}(h_1,h_2)\Big)$$
$$=\frac{1}{2}\Big(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h\in N_{\mathcal{H}_s}(h_s)\cup\mathcal{I}(\mathcal{H}_q)}(1)^2$$
$$+\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h\in(\mathcal{V}(\mathcal{H}_s)\setminus N_{\mathcal{H}_s}(h_s))\cup\mathcal{V}(\mathcal{H}_q)}(2)^2$$
$$+\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(3)^2+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h_r\in N_{\mathcal{H}_r}(h)\cup\mathcal{V}(\mathcal{H}_q)}(1)^2$$
$$+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h\in(\mathcal{V}(\mathcal{H}_r)\setminus N_{\mathcal{H}_r}(h_r))\cup\mathcal{I}(\mathcal{H}_q)}(2)^2$$
$$+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(3)^2$$
$$+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h_q)\cup\mathcal{V}(\mathcal{H}_r)}(1)^2$$
$$+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h\in(\mathcal{V}(\mathcal{H}_q)\setminus\{h_q\})\cup\mathcal{V}(\mathcal{H}_s)}(2)^2$$
$$+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h_q)}(3)^2$$
$$+\Big(\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)\cup\mathcal{V}(\mathcal{H}_s)}(1)^2$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in(\mathcal{I}(\mathcal{H}_q)\setminus\{h'_q\})\cup\mathcal{V}(\mathcal{H}_r)}(2)^2$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)\setminus N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)}(3)^2\Big)$$
$$=\frac{1}{2}\Big(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg_{\mathcal{H}_s}(h_s)+m_1)+4\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(n_3-1$$
$$-\deg_{\mathcal{H}_s}(h_s)+n_1)+9\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}n_2+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg_{\mathcal{H}_r}(h_r)$$
$$+n_1)+4\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(n_2-1-\deg_{\mathcal{H}_r}(h_r)+m_1)$$
$$+9\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}n_3+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)$$
$$+4\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(n_1-1+n_3)+9\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(m_1-\deg_{\mathcal{H}_q}(h_q))$$

$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_3+2)+4\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(m_1-1+n_2)$$
$$+9\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_1-2)\Bigg)$$
$$=\frac{1}{2}\Big(2m_3+m_1n_3+4n_3^2-4n_3-8m_3+4n_1n_3$$
$$+9n_2n_3+2m_2+n_1n_2+4n_2^2-4n_2-8m_2+4n_2m_1$$
$$+9n_2n_3+2m_1+n_1n_2+4n_1^2-4n_1+4n_1n_3+9n_1m_1$$
$$-18m_1+m_1(n_3+2)+4m_1(m_1-1+n_2)$$
$$+9m_1(n_1-2)\Big)$$
$$=2(n_1^2+n_2^2+n_3^2+m_1^2)-2(n_1+n_2+n_3+5m_1)$$
$$-3(m_2+m_3)+n_1n_2+9n_2n_3+4n_2m_1+m_1n_3$$
$$+4n_1n_3+9n_1m_1. \quad (15)$$

Therefore by using Theorem 8 and (15) in (13), we get the required result. This complete the proof. □

Theorem 11: Let $\mathcal{H}_q$, $\mathcal{H}_r$ and $\mathcal{H}_s$ be three graphs. Then we have

$$DD(\mathcal{H}_q^{\mathcal{S}}\triangleright(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$$
$$=-\mathcal{M}_1(\mathcal{H}_s)-\mathcal{M}(\mathcal{H}_r)-2\mathcal{M}_1(\mathcal{H}_q)$$
$$+n_1n_2(n_1+n_2-4)+4(n_1-1)(m_1$$
$$+m_2+m_3)+3m_1n_3(m_1+n_3-2)$$
$$+6(m_1n_2-n_2m_1+n_3m_2+n_2m_3)$$
$$+4n_3m_3+3n_1n_2(n_3+m_1)+5n_3m_1(n_1$$
$$+n_2)+2m_1(5m_1-6).$$

Proof: By Lemma 1 in Equation (14), we get

$$DD(\mathcal{H}_q^{\mathcal{S}}\triangleright(\mathcal{H}_r^{\mathcal{V}}\cup\mathcal{H}_s^{\mathcal{I}}))$$
$$=\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h\in N_{\mathcal{H}_s}(h_s)\cup\mathcal{I}(\mathcal{H}_q)}1(\deg_{\mathcal{H}_s}(h_s)+m_1)$$
$$+\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h\in(\mathcal{V}(\mathcal{H}_s)\backslash N_{\mathcal{H}_s}(h_s))\cup\mathcal{V}(\mathcal{H}_q)}2(\deg_{\mathcal{H}_s}(h_s)+m_1)$$
$$+\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}3(\deg_{\mathcal{H}_s}(h_s)+m_1)\Bigg)$$
$$+\Bigg(\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h\in N_{\mathcal{H}_r}(h_r)\cup\mathcal{V}(\mathcal{H}_q)}1(\deg_{\mathcal{H}_r}(h_r)+n_1)$$
$$+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h\in(\mathcal{V}(\mathcal{H}_r)\backslash N_{\mathcal{H}_r}(h_r))\cup\mathcal{I}(\mathcal{H}_q)}2(\deg_{\mathcal{H}_r}(h_r)+n_1)$$
$$+\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}3(\deg_{\mathcal{H}_r}(h_r)+n_1)\Bigg)$$
$$+\Bigg(\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h_q)\cup\mathcal{V}(\mathcal{H}_r)}1(\deg_{\mathcal{H}_q}(h_q)+n_2)$$
$$+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h\in(\mathcal{V}(\mathcal{H}_q)\backslash\{h_q\})\cup\mathcal{V}(\mathcal{H}_s)}2(\deg_{\mathcal{H}_q}(h_q)+n_2)$$
$$+\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)\backslash N_{\mathcal{S}(\mathcal{H}_q)}(h_q)}2(\deg_{\mathcal{H}_q}(h_q)+n_2)\Bigg)$$
$$+\Bigg(\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)\cup\mathcal{V}(\mathcal{H}_s)}1(n_3+2)$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h\in(\mathcal{I}(\mathcal{H}_q)\backslash\{h'_{q_1}\})\cup\mathcal{V}(\mathcal{H}_r)}2(n_3+2)$$
$$+\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)\backslash N_{\mathcal{S}(\mathcal{H}_q)}(h'_q)}3(n_3+2)\Bigg)$$
$$=\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg_{\mathcal{H}_s}(h_s)+m_1)^2+2\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg_{\mathcal{H}_s}(h_s)+m_1)$$
$$(n_3-1-\deg_{\mathcal{H}_s}(h_s)+n_1)+3\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}n_2(\deg_{\mathcal{H}_s}(h_s)+m_1)\Bigg)$$
$$+\Bigg(\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg_{\mathcal{H}_r}(h_r)+n_1)^2$$
$$+2\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg_{\mathcal{H}_r}(h_r)+n_1)(n_2-1-\deg_{\mathcal{H}_r}(h_r)+m_1)$$
$$+3\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}n_3(\deg_{\mathcal{H}_r}(h_r)+n_1)\Bigg)$$
$$+\Bigg(\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)^2$$
$$+2\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)(n_1-1+n_3)$$
$$+3\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)(m_1-\deg_{\mathcal{H}_q}(h_q))\Bigg)$$
$$+\Bigg(\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_3+2)^2+2\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_3+2)(m_1-1+n_2)$$
$$+3\sum_{h'_q\in\mathcal{I}(\mathcal{H}_q)}(n_3+2)(n_1-2)\Bigg)$$
$$=\Bigg(\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg^2_{\mathcal{H}_s}(h_s)+m_1^2+2m_1\deg_{\mathcal{H}_s}(h_s))$$
$$+2\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}((n_3-1+n_1)(\deg_{\mathcal{H}_s}(h_s)+m_1)$$
$$-\deg^2_{\mathcal{H}_s}(h_s)-m_1\deg_{\mathcal{H}_s}(h_s))$$
$$+3n_2\sum_{h_s\in\mathcal{V}(\mathcal{H}_s)}(\deg_{\mathcal{H}_s}(h_s)+m_1)\Bigg)$$
$$+\Bigg(\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg^2_{\mathcal{H}_r}(h_r)+n_1^2+2n_1\deg_{\mathcal{H}_r}(h_r))$$
$$+2\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}((n_2-1+m_1)(\deg_{\mathcal{H}_r}(h_r)+n_1)-\deg^2_{\mathcal{H}_r}(h_r)$$
$$-n_1\deg_{\mathcal{H}_r}(h_r))+3n_3\sum_{h_r\in\mathcal{V}(\mathcal{H}_r)}(\deg_{\mathcal{H}_r}(h_r)+n_1)\Bigg)$$
$$+\Bigg(\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg^2_{\mathcal{H}_q}(h_q)+n_2^2+2n_2\deg_{\mathcal{H}_q}(h_q))$$

$$
\begin{aligned}
&+2(n_1-1+n_3)\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(\deg_{\mathcal{H}_q}(h_q)+n_2)\\
&+3\sum_{h_q\in\mathcal{V}(\mathcal{H}_q)}(m_1\deg_{\mathcal{H}_q}(h_q)\\
&-\deg^2_{\mathcal{H}_q}(h_q)+n_2m_1-n_2\deg_{\mathcal{H}_q}(h_q))\Big)\\
&+\Big(m_1(n_3+2)^2+2m_1(n_3+2)(m_1-1+n_2)\\
&+3m_1(n_3+2)(n_1-2)\Big)\\
=\ &\mathcal{M}_1(\mathcal{H}_s)+m_1^2n_3+4m_1m_3+2(n_3+n_1-1)\\
&\times(2m_3+m_1n_3)-2\mathcal{M}_1(\mathcal{H}_s)-4m_1m_3+6n_2m_3\\
&+3n_2m_1n_3+\mathcal{M}_1(\mathcal{H}_r)+n_1^2n_2+4n_1m_2\\
&+2(n_2-1+n_1)-2\mathcal{M}_1(\mathcal{H}_r)-4n_1m_2+6n_3m_2\\
&+3n_1n_2n_3+\mathcal{M}(\mathcal{H}_q)+n_1n_2^2+4n_2m_1+2(n_1+n_2-1)\\
&\times(2m_1+n_1n_2)+3(2m_1^2-\mathcal{M}_1(\mathcal{H}_q)+n_1n_2m_1-2n_2m_1)\\
&+m_1(n_3+2)(n_3+2+2m_1-2+2n_2+3n_1+6)\\
=\ &-\mathcal{M}_1(\mathcal{H}_s)-\mathcal{M}(\mathcal{H}_r)-2\mathcal{M}_1(\mathcal{H}_q)+n_1n_2(n_1+n_2-4)\\
&+4(n_1-1)(m_1+m_2+m_3)+3m_1n_3(m_1+n_3-2)\\
&+6(m_1n_2-n_2m_1+n_3m_2+n_2m_3)+4n_3m_3\\
&+3n_1n_2(n_3+m_1)+5n_3m_1(n_1+n_2)+2m_1(5m_1-6).
\end{aligned}
$$

This complete the proof. □

III. CONCLUSION

The analysis of graphs and networks through structural properties is a research topic with growing significance. One of the approaches in investigating structural characteristics is discussing the quantitative measure that encodes structural statistics of the entire network by a real number. In this article, we have provided the results related to the eccentric-connectivity, connective eccentricity, total-eccentricity, average eccentricity, Zagreb eccentricity, eccentric geometric-arithmetic, eccentric atom-bond connectivity, eccentric adjacency, modified eccentric-connectivity, eccentric distance sum, Wiener, Harary, hyper-Wiener and degree distance indices of subdivision vertex-edge join of graphs.

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