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Multivalued Fixed Point Theorems for F_ρ-Contractions With Applications to Volterra Integral Inclusion

AFRAH AHMAD NOMAN ABDOU AND JAMSHAID AHMAD
Department of Mathematics, University of Jeddah, Jeddah 21589, Saudi Arabia
Corresponding author: Jamshaid Ahmad (jkhan@uj.edu.sa)

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ABSTRACT The aim of this article is to define the notion of F_ρ-contraction and obtain some new fixed point theorems for a new class of contractive conditions in the context of complete metric spaces. The obtained results extend and improve the well-known results of literature by means of this new class of contractions. As application, we discuss the electric circuit equation and apply our result to solve the second order differential equation arising in it. To rationalize the notions and outcome, the existence of solution for a certain Volterra-type integral inclusion is also obtained.

INDEX TERMS F_ρ-contraction, fixed point, multivalued mappings, complete metric space, volterra-type integral inclusion.

I. INTRODUCTION AND LITERATURE REVIEW

In nonlinear analysis, the theory of fixed points plays one of the chief and important part and many applications in computing science, physical science and Engineering. In 1922, Stefan Banach [1] established a prominent fixed point result for contractive mappings in complete metric spaces. Following the Banach contraction principle Nadler [2] initiated the notion of multi-valued contractions utilizing the Hausdorff metric and proved that a multi-valued contraction owns a fixed point in a complete metric space. Let $(\mathcal{S},\sigma)$ be a metric space. For $\zeta \in \mathcal{S}$ and $A \subseteq \mathcal{S}$, we indicate $\sigma(\zeta,A) = \inf\{\sigma(\zeta,\xi) : \xi \in A\}$. Let us indicate by $N(\mathcal{S})$, the class of all nonempty subsets of $\mathcal{S}$, $2^{\mathcal{S}}$, the class of all nonempty subsets of $\mathcal{S}$, $CL(\mathcal{S})$, the family of all nonempty closed subsets of $\mathcal{S}$, $CB(\mathcal{S})$, the family of all nonempty closed and bounded subsets of $\mathcal{S}$ and $K(\mathcal{S})$, the family of all compact subsets of $\mathcal{S}$. Let H be the Hausdorff-Pompeiu metric induced by metric σ on $\mathcal{S}$, that is,

$$H(A,B) = \max\{\sup_{\zeta\in A}\sigma(\zeta,B), \sup_{\xi\in B}\sigma(\xi,A)\}$$

for every $A, B \in CB(\mathcal{S})$. For more details in this direction, we refer the reader to [3]–[5].

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Now, following the lines in [6], we denote by $\mathcal{E}$ the set of all continuous mappings $\rho : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ satisfying the following assertions:

(ϱ_1) $\rho(1,1,1,2,0), \rho(1,1,1,0,2), \rho(1,1,1,1,1) \in (0,1]$,

(ϱ_2) ρ is sub-homogeneous, that is, for all $(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5) \in (\mathbb{R}^+)^5$ and $\alpha \geq 0$, we have $\rho(\alpha\zeta_1,\alpha\zeta_2,\alpha\zeta_3,\alpha\zeta_4,\alpha\zeta_5) \leq \alpha\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)$;

(ϱ_3) ρ is a non-decreasing function, that is, for $\zeta_i, \xi_i \in \mathbb{R}^+$, $\zeta_i \leq \xi_i$, $i = 1,\ldots,5$, we have

$$\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5) \leq \rho(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$$

and if $\zeta_i, \xi_i \in \mathbb{R}^+$, $i = 1,\ldots,4$, then

$$\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,0) \leq \rho(\xi_1,\xi_2,\xi_3,\xi_4,0)$$

and

$$\rho(\zeta_1,\zeta_2,\zeta_3,0,\zeta_4) \leq \rho(\xi_1,\xi_2,\xi_3,0,\xi_4).$$

The following lemma of [7] is needed in the sequel.

Lemma 1.1: If $\rho \in \mathcal{E}$ and $\zeta, \xi \in \mathbb{R}^+$ are such that

$$\zeta < \max\{\rho(\xi,\xi,\zeta,\xi+\zeta,0), \rho(\xi,\xi,\zeta,0,\xi+\zeta), \rho(\xi,\zeta,\xi,\xi+\zeta,0), \rho(\xi,\zeta,\xi,0,\xi+\zeta)\},$$

then $\zeta < \xi$.

Wardowski [8] introduced and studied a new contraction called F -contraction to prove a fixed point result as a generalization of the Banach contraction principle.

Definition 1: Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying the following conditions:

(F_1) F is strictly increasing;

(F_2) for all sequence $\{\zeta_n\}\subseteq R^+$, $\lim_{n\to\infty}\zeta_n=0\iff\lim_{n\to\infty}F(\zeta_n)=-\infty$;

(F_3) $\exists\, 0<r<1$ so that $\lim_{\zeta\to 0^+}\zeta^r F(\zeta)=0$.

Consistent with Wordowski [8], we denote by $\digamma$ the set of all functions $F:\mathbb{R}^+\to\mathbb{R}$ satisfying conditions (F_1), (F_2) and (F_3).

Definition 2 [8]: Let $(\mathcal{S},\sigma)$ be a metric space. A self-mapping $\mathcal{G}$ on $\mathcal{S}$ is called an F-contraction if there exist some $F\in\digamma$ and $\tau>0$ such that

$$\sigma(\mathcal{G}\zeta,\mathcal{G}\xi)>0\Longrightarrow\tau+F\big(\sigma(\mathcal{G}\zeta,\mathcal{G}\xi)\big)\le F\big(\sigma(\zeta,\xi)\big)$$

for $\zeta,\xi\in\mathcal{S}$.

Lemma 1.2 [7]: Let $(\mathcal{S},\sigma)$ be a metric space and $A,B\in CL(\mathcal{S})$ with $H(A,B)>0$. Then, for every $h>1$ and for each $a\in A$, there exists $b=b(a)\in B$ such that $\sigma(a,b)<hH(A,B)$.

In this paper, we define the notion of F_ρ-contraction and establish some generalized fixed point theorem in the setting of complete metric spaces. As application of our main result, we discuss electric circuit equation and the existence of solution for a certain Volterra-type integral inclusion.

II. MATERIALS AND METHODS

In this article, we utilize the family $\mathcal{E}$ of all continuous mappings $\varrho:(\mathbb{R}^+)^5\to\mathbb{R}^+$ and the class $\digamma$ of all functions $F:\mathbb{R}^+\to\mathbb{R}$ to define the notion of F_ϱ -contraction. Some generalized fixed point theorems for multivalued mapping $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$ involving Housdorff metric defined by

$$H(A,B)=\max\left\{\sup_{\zeta\in A}\sigma(\zeta,B),\sup_{\xi\in B}\sigma(\xi,A)\right\}$$

for all $A,B\in CB(\mathcal{S})$, have been established. To verify the effectiveness and applicability of our main results, the solutions of Volterra-type integral inclusion are also discussed.

We take ζ_0 as an arbitrary point in complete metric space $\mathcal{S}$ and use the hypothesis given in the statement of our main theorem to prove $\{\zeta_n\}$ is a Cauchy sequence in $\mathcal{S}$. Then by using the completeness of $(\mathcal{S},\sigma)$, we get the convergence of $\{\zeta_n\}$ which converges to a point $\zeta^*\in\mathcal{S}$. Subsequently we prove that ζ^* is a fixed point of mpping $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$.

III. RESULTS

Definition 3: Let $(\mathcal{S},\sigma)$ be a metric space. A multivalued mapping $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$ is said to be F_ρ-contraction, if there exist $F\in\digamma$, $\rho\in\mathcal{E}$ and $\tau>0$ so that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\le F\left(\rho\left(\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \sigma(\zeta,\mathcal{G}\xi),\sigma(\xi,\mathcal{G}\zeta)\end{matrix}\right)\right)\tag{3.1}$$

$\forall\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$.

From now to onward, we take $(\mathcal{S},\sigma)$ as a complete metric space.

Remark 1: If $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$ is F_ρ-contraction, then by (3.1), we get

$$\begin{aligned}F(H(\mathcal{G}\zeta,\mathcal{G}\xi))&\le F\left(\rho\left(\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \sigma(\zeta,\mathcal{G}\xi),\sigma(\xi,\mathcal{G}\zeta)\end{matrix}\right)\right)\\&\quad-2\tau\\&<F\left(\rho\left(\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \sigma(\zeta,\mathcal{G}\xi),\sigma(\xi,\mathcal{G}\zeta)\end{matrix}\right)\right)\end{aligned}$$

By (F_1), we have

$$H(\mathcal{G}\zeta,\mathcal{G}\xi)<\rho\left(\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \sigma(\zeta,\mathcal{G}\xi),\sigma(\xi,\mathcal{G}\zeta)\end{matrix}\right)$$

for all $\zeta,\xi\in\mathcal{S}$ with $\mathcal{G}\zeta\ne\mathcal{G}\xi$.

Theorem 1: Let $\mathcal{G}:\mathcal{S}\to CL(\mathcal{S})$ be an F_ρ-contraction. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Let $\zeta_0\in\mathcal{S}$ be an arbitrary and $\zeta_1\in\mathcal{G}\zeta_0$. If $\zeta_1=\zeta_0$ or $\zeta_1\in\mathcal{G}\zeta_1$ then ζ_1 is a fixed point of $\mathcal{G}$ and so the proof is finished. So we suppose that $\zeta_1\ne\zeta_0$ or $\zeta_1\notin\mathcal{G}\zeta_1$. Then $\sigma(\zeta_1,\mathcal{G}\zeta_1)>0$ and hence $H(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1)>0$. From (3.1), we get

$$\begin{aligned}&2\tau+F\left(\sigma(\zeta_1,\mathcal{G}\zeta_1)\right)\\&\quad\le 2\tau+F\left(H\left(\mathcal{G}\zeta_0,T\zeta_1\right)\right)\\&\quad\le F\left(\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\mathcal{G}\zeta_0),\sigma(\zeta_1,\mathcal{G}\zeta_1),\\ \sigma(\zeta_0,\mathcal{G}\zeta_1),\sigma(\zeta_1,\mathcal{G}\zeta_0)\end{matrix}\right)\right)\\&\quad\le F\left(\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\zeta_1),\sigma(\zeta_1,\mathcal{G}\zeta_1),\\ \sigma(\zeta_0,\mathcal{G}\zeta_1),0\end{matrix}\right)\right)\end{aligned}$$

and so

$$\sigma(\zeta_1,\mathcal{G}\zeta_1)<\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\zeta_1),\sigma(\zeta_1,\mathcal{G}\zeta_1),\\ \sigma(\zeta_0,\mathcal{G}\zeta_1),0\end{matrix}\right)$$

Then Lemma 1.1 gives that $\sigma(\zeta_1,\mathcal{G}\zeta_1)<\sigma(\zeta_0,\zeta_1)$. Thus, we obtain

$$\begin{aligned}&2\tau+F\left(\sigma(\zeta_1,\mathcal{G}\zeta_1)\right)\\&\quad\le 2\tau+F\left(H\left(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1\right)\right)\\&\quad\le F\left(\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\mathcal{G}\zeta_0),\sigma(\zeta_1,\mathcal{G}\zeta_1),\\ \sigma(\zeta_0,\mathcal{G}\zeta_1),\sigma(\zeta_1,\mathcal{G}\zeta_0)\end{matrix}\right)\right)\\&\quad\le F\left(\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\zeta_1),\sigma(\zeta_1,\mathcal{G}\zeta_1),\\ \sigma(\zeta_0,\mathcal{G}\zeta_1),0\end{matrix}\right)\right)\\&\quad< F\left(\rho\left(\begin{matrix}\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\zeta_1),\sigma(\zeta_0,\zeta_1),\\ 2\sigma(\zeta_0,\zeta_1),0\end{matrix}\right)\right)\\&\quad\le F\left(\sigma(\zeta_0,\zeta_1)\rho\left(1,1,1,2,0\right)\right)\\&\quad\le F\left(\sigma(\zeta_0,\zeta_1)\right)\end{aligned}$$

Thus

$$2\tau+F\left(\sigma(\zeta_1,\mathcal{G}\zeta_1)\right)\le F\left(\sigma(\zeta_0,\zeta_1)\right)\tag{3.2}$$

Since $F\in\digamma$ is continuous from the right function, so there exists a real number $h>1$ such that

$$F\left(hH\left(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1\right)\right)<F\left(H\left(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1\right)\right)+\tau.\tag{3.3}$$

Next as

$$\sigma(\zeta_1,\mathcal{G}\zeta_1)\le H\left(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1\right)<hH\left(\mathcal{G}\zeta_0,\mathcal{G}\zeta_1\right)\tag{3.4}$$

by Lemma 1.2, there exists $\zeta_2 \in \mathcal{G}\zeta_1$ (obviously, $\zeta_2 \neq \zeta_1$) such that

$$\sigma(\zeta_1, \zeta_2) \leq \sigma(\zeta_1, \mathcal{G}\zeta_1). \tag{3.5}$$

Thus by (3.3), (3.4) and (3.5), we have

$$F(\sigma(\zeta_1, \zeta_2)) \leq F(hH(\mathcal{G}\zeta_0, \mathcal{G}\zeta_1)) < F(H(\mathcal{G}\zeta_0, \mathcal{G}\zeta_1)) + \tau \tag{3.6}$$

which implies by (3.2) that

$$\begin{aligned} 2\tau + F(\sigma(\zeta_1, \zeta_2)) &\leq 2\tau + F(H(\mathcal{G}\zeta_0, \mathcal{G}\zeta_1)) + \tau \\ &\leq F(\sigma(\zeta_0, \zeta_1)) + \tau \end{aligned}$$

Thus we have

$$\tau + F(\sigma(\zeta_1, \zeta_2)) \leq F(\sigma(\zeta_0, \zeta_1)). \tag{3.7}$$

From (3.1), we get

$$\begin{aligned} &2\tau + F(\sigma(\zeta_2, \mathcal{G}\zeta_2)) \\ &\leq 2\tau + F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) \\ &\leq F\left(\rho\left(\begin{array}{c}\sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1)\end{array}\right)\right) \\ &\leq F\left(\rho\left(\begin{array}{c}\sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), 0\end{array}\right)\right) \end{aligned}$$

and so

$$\sigma(\zeta_2, \mathcal{G}\zeta_2) < \rho\left(\begin{array}{c}\sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), 0\end{array}\right).$$

Then Lemma 1.1 gives that $\sigma(\zeta_2, \mathcal{G}\zeta_2) < \sigma(\zeta_1, \zeta_2)$. Thus, we obtain

$$\begin{aligned} &2\tau + F(\sigma(\zeta_2, \mathcal{G}\zeta_2)) \\ &\leq 2\tau + F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) \\ &\leq F\left(\rho\left(\begin{array}{c}\sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1)\end{array}\right)\right) \\ &\leq F\left(\rho\left(\begin{array}{c}\sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \zeta_2), \\ 2\sigma(\zeta_1, \zeta_2), 0\end{array}\right)\right) \\ &\leq F(\sigma(\zeta_1, \zeta_2)\rho(1, 1, 1, 2, 0)) \\ &\leq F(\sigma(\zeta_1, \zeta_2)). \end{aligned}$$

Thus we get

$$2\tau + F(\sigma(\zeta_2, \mathcal{G}\zeta_2)) \leq F(\sigma(\zeta_1, \zeta_2)) \tag{3.8}$$

Since $F \in \digamma$ is continuous from the right function, so there exists a real number $h > 1$ such that

$$F(hH(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) < F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) + \tau. \tag{3.9}$$

Next as

$$\sigma(\zeta_2, \mathcal{G}\zeta_2) \leq H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) < hH(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) \tag{3.10}$$

by Lemma 1.1, there exists $\zeta_3 \in \mathcal{G}\zeta_2$ (obviously, $\zeta_3 \neq \zeta_2$) such that

$$\sigma(\zeta_2, \zeta_3) \leq \sigma(\zeta_2, \mathcal{G}\zeta_2). \tag{3.11}$$

Thus by (3.9), (3.10) and (3.11), we have

$$F(\sigma(\zeta_2, \zeta_3)) \leq F(hH(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) < F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) + \tau \tag{3.12}$$

which implies by (3.8) that

$$\begin{aligned} 2\tau + F(\sigma(\zeta_2, \zeta_3)) &\leq 2\tau + F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) + \tau \\ &\leq F(\sigma(\zeta_1, \zeta_2)) + \tau. \end{aligned}$$

Thus we have

$$\tau + F(\sigma(\zeta_2, \zeta_3)) \leq F(\sigma(\zeta_1, \zeta_2)). \tag{3.13}$$

So pursuing in this way we obtain $\{\zeta_n\}$ in $\mathcal{S}$ such that $\zeta_{n+1} \in \mathcal{G}\zeta_n$ and

$$\tau + F(\sigma(\zeta_n, \zeta_{n+1})) \leq F(\sigma(\zeta_{n-1}, \zeta_n)) \tag{3.14}$$

for all $n \in \mathbb{N}$. Therefore by (3.14), we have

$$\begin{aligned} (\sigma(\zeta_n, \zeta_{n+1})) &\leq F(\sigma(\zeta_{n-1}, \zeta_n)) - \tau \\ &\leq F(\sigma(\zeta_{n-2}, \zeta_{n-1})) - 2\tau \\ &\leq \cdots \leq F(\sigma(\zeta_0, \zeta_1)) - n\tau. \end{aligned} \tag{3.15}$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} F(\sigma(\zeta_n, \zeta_{n+1})) = -\infty$ that jointly with (F_2) gives

$$\lim_{n\to\infty} \sigma(\zeta_n, \zeta_{n+1}) = 0.$$

Thus from (F_3), $\exists\, r \in (0, 1)$ so that

$$\lim_{n\to\infty} [\sigma(\zeta_n, \zeta_{n+1})]^r F(\sigma(\zeta_n, \zeta_{n+1})) = 0. \tag{3.16}$$

By (3.15) and (3.16), we obtain

$$\begin{aligned} &[\sigma(\zeta_n, \zeta_{n+1})]^r F(\sigma(\zeta_n, \zeta_{n+1})) - [\sigma(\zeta_n, \zeta_{n+1})]^r F(\sigma(\zeta_0, \zeta_1)) \\ &\leq [\sigma(\zeta_n, \zeta_{n+1})]^r [F(\sigma(\zeta_0, \zeta_1)) - n\tau] \\ &\quad - [\sigma(\zeta_n, \zeta_{n+1})]^r F(\sigma(\zeta_0, \zeta_1)) \\ &\leq -n\tau[\sigma(\zeta_n, \zeta_{n+1})]^r \leq 0. \end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} n[\sigma(\zeta_n, \zeta_{n+1})]^r = 0. \tag{3.17}$$

Hence $\lim_{n\to\infty} n^{\frac{1}{r}} \sigma(\zeta_n, \zeta_{n+1}) = 0$, which implies that $\sum_{n=1}^{\infty} \sigma(\zeta_n, \zeta_{n+1})$ converges. Hence the sequence $\{\zeta_n\}$ is Cauchy in $\mathcal{S}$. As $(\mathcal{S}, \sigma)$ is a complete metric space, so there exists a $\zeta^* \in \mathcal{S}$ such that

$$\lim_{n\to\infty} \zeta_n = \zeta^*. \tag{3.18}$$

Now, we prove that $\zeta^* \in \mathcal{G}\zeta^*$. Assume on the contrary that $\zeta^* \notin \mathcal{G}\zeta^*$, then $\exists\, n_0 \in \mathbb{N}$ and a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\sigma(\zeta_{n_k+1}, \mathcal{G}\zeta^*) > 0$ for all $n_k \geq n_0$. Now, using (3.1) with $\zeta = \mathcal{S}_{n_k+1}$ and $\xi = \zeta^*$. Taking Remark 1 into account, we have

$$\begin{aligned} &\sigma(\zeta_{n_k+1}, \mathcal{G}\zeta^*) \\ &\leq H(\mathcal{G}\zeta_{n_k}, \mathcal{G}\zeta^*) \\ &\leq \rho\left(\begin{array}{c}\sigma(\zeta_{n_k}, \zeta^*), \sigma(\zeta_{n_k}, \mathcal{G}\zeta_{n_k}), \sigma(\zeta^*, \mathcal{G}\zeta^*), \\ \sigma(\zeta_{n_k}, \mathcal{G}\zeta^*), \sigma(\zeta^*, \mathcal{G}\zeta_{n_k})\end{array}\right) \end{aligned}$$

$$\leq \rho\begin{pmatrix} \sigma(\zeta_{n_k},\zeta^*), \sigma(\zeta_{n_k},\zeta_{n_k+1}), \sigma(\zeta^*,\mathcal{G}\zeta^*), \\ \sigma(\zeta_{n_k},\mathcal{G}\zeta^*), \sigma(\zeta^*,\zeta_{n_k+1}) \end{pmatrix}$$

Taking $n\to\infty$, we get

$$\sigma(\zeta^*,\mathcal{G}\zeta^*) \leq \rho\left(0,0,\sigma(\zeta^*,\mathcal{G}\zeta^*),\sigma(\zeta^*,\mathcal{G}\zeta^*),0\right)$$

which implies by Lemma 1.1 that

$$0<\sigma(\zeta^*,\mathcal{G}\zeta^*)<0$$

which is a contradiction. Hence $\sigma(\zeta^*,\mathcal{G}\zeta^*)=0$. Since $\mathcal{G}\zeta^*$ is closed, we deduce that $\zeta^*\in\overline{\mathcal{G}\zeta^*}$. Thus $\zeta^*\in\mathcal{G}\zeta^*$. □

IV. CONSEQUENCES

In all these consequences, we assume $F\in\digamma$ is a continuous from the right function.

Corollary 1: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\leq F\left(\sigma(\zeta,\xi)\right)$$

for all $\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exist $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Consider $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\zeta_1$. Then the result follows from Theorem 1. □

Corollary 2: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\leq F\left(\sigma(\zeta,\mathcal{G}\zeta)+\sigma(\xi,\mathcal{G}\xi)\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\zeta_2+\zeta_3$ inTheorem 1. □

Corollary 3: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\leq F\left(\sigma(\zeta,\mathcal{G}\xi)+\sigma(\xi,\mathcal{G}\zeta)\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\zeta_4+\zeta_5$ in Theorem 1. □

Corollary 4: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ and non-negative real numbers α,β,γ with $\alpha+\beta+\gamma\leq 1$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\leq F\left(\alpha\sigma(\zeta,\xi)+\beta\sigma(\zeta,\mathcal{G}\zeta)+\gamma\sigma(\xi,\mathcal{G}\xi)\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\alpha\zeta_1+\beta\zeta_2+\gamma\zeta_3$ in Theorem 1. □

Corollary 5: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ and non-negative real number $\alpha\in(0,1]$ and $L\geq 0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi))\leq F\left(\alpha\sigma(\zeta,\xi)+L\sigma(\xi,\mathcal{G}\zeta)\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\alpha\zeta_1+L\zeta_5$ in Theorem 1. □

Corollary 6: [10] Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ and non-negative real numbers $\alpha,\beta,\gamma,\delta$ and L with $\alpha+\beta+\gamma+\delta+2L\leq 1$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi)) \leq F\begin{pmatrix}\alpha\sigma(\zeta,\xi)+\beta\sigma(\zeta,\mathcal{G}\zeta)+\gamma\sigma(\xi,\mathcal{G}\xi)\\ +\delta\sigma(\zeta,\mathcal{G}\xi)+L\sigma(\xi,\mathcal{G}\zeta)\end{pmatrix}$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\alpha\zeta_1+\beta\zeta_2+\gamma\zeta_3+\delta\zeta_4+L\zeta_5$ in Theorem 1. □

Corollary 7: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi)) \leq F\left(\max\left\{\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \frac{\sigma(\zeta,\mathcal{G}\xi)+\sigma(\xi,\mathcal{G}\zeta)}{2}\end{matrix}\right\}\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\max\left\{\zeta_1,\zeta_2,\zeta_3,\frac{\zeta_4+\zeta_5}{2}\right\}$ in Theorem 1. □

If we take self mapping $\mathcal{G}:\mathcal{S}\to\mathcal{S}$ in the above Corollary, then we get the following result.

Corollary 8: Let $(\mathcal{S},\sigma)$ be a complete metric space and let $\mathcal{G}:\mathcal{S}\to\mathcal{S}$. Suppose that there exists some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(\sigma(\mathcal{G}\zeta,\mathcal{G}\xi)) \leq F\left(\max\left\{\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \frac{\sigma(\zeta,\mathcal{G}\xi)+\sigma(\xi,\mathcal{G}\zeta)}{2}\end{matrix}\right\}\right)$$

for all $\zeta,\xi\in\mathcal{S}$ with $\sigma(\mathcal{G}\zeta,\mathcal{G}\xi)>0$.Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*=\mathcal{G}\zeta^*$.

Corollary 9: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi)) \leq F\left(\max\left\{\begin{matrix}\sigma(\zeta,\xi),\sigma(\zeta,\mathcal{G}\zeta),\sigma(\xi,\mathcal{G}\xi),\\ \sigma(\zeta,\mathcal{G}\xi),\sigma(\xi,\mathcal{G}\zeta)\end{matrix}\right\}\right) \tag{4.1}$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho\in\mathcal{E}$ given by $\rho(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)=\max\{\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5\}$ in Theorem 1. □

Corollary 10: Let $\mathcal{G}:\mathcal{S}\to CB(\mathcal{S})$. Suppose that there exist some $F\in\digamma$ and $\tau>0$ such that

$$2\tau+F(H(\mathcal{G}\zeta,\mathcal{G}\xi)) \leq F\left(\max\left\{\sigma(\zeta,\xi),\frac{\sigma(\zeta,\mathcal{G}\zeta)+\sigma(\xi,\mathcal{G}\xi)}{2},\frac{\sigma(\zeta,\mathcal{G}\xi)+\sigma(\xi,\mathcal{G}\zeta)}{2}\right\}\right)$$

$\forall\,\zeta,\xi\in\mathcal{S}$ with $H(\mathcal{G}\zeta,\mathcal{G}\xi)>0$. Then there exists $\zeta^*\in\mathcal{S}$ such that $\zeta^*\in\mathcal{G}\zeta^*$.

Proof: Considering $\rho \in \mathcal{E}$ given by $\rho(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) = \max\left\{\zeta_1, \frac{\zeta_2+\zeta_3}{2}, \frac{\zeta_4+\zeta_5}{2}\right\}$ in Theorem 1. □

V. APPLICATIONS

A. APPLICATION TO ELECTRIC CIRCUIT EQUATION

In this subsection we apply our result to solve electric circuit equation which is in the form of second order differential equation. It is conventional that electric circuit contains an electromotive force E, a resistor R, capacitance C and an inductance L. If the current I is the rate of change of electric charge Q with respect to time t, $I = \frac{dQ}{dt}$. We are familiar with the following relations:

(i) $V = IR$;
(ii) $V = \frac{Q}{C}$;
(iii) $V = L\frac{dI}{dt}$.

Now by Kirchhoffs voltage law, the sum of these voltage drops is equal to the supplied voltage, i.e.,

$$IR + \frac{Q}{C} + L\frac{dI}{dt} = V(t)$$

or

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t)$$
$$Q(0) = 0, \quad Q^{/}(0) = 0. \tag{5.1}$$

The Green function associated to (5.1) is given by

$$G(t,s) = \begin{cases} -se^{\tau(s-t)}, & 0 \le s \le t \le 1 \\ -te^{\tau(s-t)}, & 0 \le t \le s \le 1, \end{cases} \tag{5.2}$$

where $\tau > 0$ is a constant, calculated in terms of R and L. Let $\mathcal{S} = C([0,a], \mathbb{R}^+)$ be the set of all non negative real valued functions defined on $[0,a]$. For an arbitrary $\zeta \in \mathcal{S}$, we define

$$\|\zeta\|_\tau = \sup_{t\in[0,a]} \left\{|\zeta(t)|\, e^{-2\tau t}\right\}. \tag{5.3}$$

Define $\sigma : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ by

$$\sigma(\zeta,\xi) = \|\zeta - \xi\|_\tau = \sup_{t\in[0,a]} \left\{|\zeta(t) - \xi(t)|\, e^{-2\tau t}\right\}. \tag{5.4}$$

Then clearly $(\mathcal{S}, \sigma)$ is a metric space. We now state and the prove the result for the existence of a solution of the LCR-circuit equation of the second order differential equation:

Theorem 2: Let $\mathcal{G} : C([0,a]) \to C([0,a])$ be self mapping such that the following condition hold:

there exists a function $K : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ such that

$$|K(t,s,\zeta) - K(t,s,\xi)| \le \tau^2 e^{-2\tau} M(\zeta,\xi)$$

where

$$M(\zeta,\xi) = \max\left\{\begin{array}{c} \sigma(\zeta,\xi), \sigma(\zeta,\mathcal{G}\zeta), \sigma(\xi,\mathcal{G}\xi), \\ \dfrac{\sigma(\zeta,\mathcal{G}\xi) + \sigma(\xi,\mathcal{G}\zeta)}{2} \end{array}\right\}$$

for all $t, s \in [0,a]$, $\zeta, \xi \in C([0,a])$ and $\tau > 0$. Then equation (5.1) has a solution.

Proof: Above problem is equivalent to the integral equation

$$\zeta(t) = \int_0^t G(t,s)K(t,s,\zeta(s))\sigma s, \tag{5.5}$$

$t \in [0,a]$. Consider $\mathcal{G} : C([0,a]) \to C([0,a])$ defined by

$$\mathcal{G}(\zeta(t)) = \int_0^t G(t,s)K(t,s,\zeta(s))\sigma s \tag{5.6}$$

for $t \in [0,a]$ and $a > 0$. Then clearly ζ^* is a solution of (5.5), if and only if ζ^* is a fixed point of $\mathcal{G}$. Now

$$\begin{aligned} &|\mathcal{G}(\zeta(t)) - \mathcal{G}(\xi(t))| \\ &\quad\le \int_0^t G(t,s)\,|K(t,s,\zeta(s)) - K(t,s,\xi(s))|\,\sigma s \\ &\quad\le \int_0^t G(t,s)\tau^2 e^{-2\tau} M(\zeta,\xi)\sigma s \end{aligned}$$

which implies

$$\begin{aligned} &|\mathcal{G}(\zeta(t)) - \mathcal{G}(\xi(t))| \\ &\le \int_0^t \tau^2 e^{-2\tau} e^{2\tau s} e^{-2\tau s} M(\zeta,\xi) G(t,s)\sigma s \\ &\le \tau^2 e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau \times \int_0^t e^{2\tau s} G(t,s)\sigma s \\ &\le \tau^2 e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau \left[-\frac{e^{2\tau t}}{\tau^2}\left(2\tau t - \tau t e^{-\tau t} + e^{-\tau t} - 1\right)\right]. \end{aligned}$$

This implies that

$$\begin{aligned} &|\mathcal{G}(\zeta(t)) - \mathcal{G}(\xi(t))|\, e^{-2\tau t} \\ &\qquad\le e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau \times \left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right). \end{aligned}$$

Thus

$$\begin{aligned} &\|\mathcal{G}(\zeta(t)) - \mathcal{G}(\xi(t))\|_\tau \\ &\qquad\le e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau \times \left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right). \end{aligned}$$

Evidently, $\left(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right) \le 1$, so we have

$$\|\mathcal{G}(\zeta(t)) - \mathcal{G}(\xi(t))\|_\tau \le e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau$$

that is

$$\sigma\left(\mathcal{G}(\zeta), \mathcal{G}(\xi)\right) \le e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau\,.$$

Taking logarithm both sides, we get

$$\ln\left(\sigma\left(\mathcal{G}(\zeta), \mathcal{G}(\xi)\right)\right) \le \ln\left(e^{-2\tau}\, \|M(\zeta,\xi)\|_\tau\right),$$

which implies

$$2\tau + \ln\left(\sigma\left(\mathcal{G}(\zeta), \mathcal{G}(\xi)\right)\right) \le \ln\left(\|M(\zeta,\xi)\|_\tau\right).$$

Hence all the conditions conditions of result 8 are satisfied by taking $F(t) = \ln(t)$ for $t > 0$ and $\mathcal{G}$ has a fixed point which is the solution of differential equation arising in electric circuit equation. □

B. APPLICATION TO VOLTERRA-TYPE INTEGRAL INCLUSION

The aim of this subsection is to apply the established results to obtain the existence of solutions for a determined Volterra-type integral inclusion. Consider the Volterra-type integral inclusion as

$$\zeta(t) \in f(t) + \int_a^t K(t,s,\zeta(s))\sigma s, \quad t \in [a,b] \tag{5.7}$$

where $f \in C[a,b]$ is a given real-valued function and $K : [a,b]\times[a,b]\times\mathbb{R} \to K_{cv}(\mathbb{R})$ and $K_{cv}(\mathbb{R})$ indicates the class of nonempty compact and convex subsets of $\mathbb{R}$ and $\zeta \in C[a,b]$ is the unknown function.

Let $C([a,b],\mathbb{R})$ be endowed with the metric

$$\sigma(\zeta,\xi) = (\max_{t\in[a,b]} |\zeta(t)-\xi(t)|) = \max_{t\in[a,b]} |\zeta(t)-\xi(t)| \tag{5.8}$$

for all $\zeta,\xi \in C[a,b]$. Subsequently $(C[a,b],\sigma)$ is a complete metric space.

We will suppose the following conditions:

(A_1) for each $\zeta \in C[a,b]$, $K : [a,b] \times [a,b] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $K(t,s,\zeta(s))$ is lower semicontinuous in $[a,b]\times [a,b]$,

(A_2) there exists some continuous function $l : [a,b] \times [a,b] \to [0,+\infty)$ such that

$$\begin{aligned}|k_\zeta(t,s) - k_\xi(t,s)| \\ \le l(t,s)\{\max\{&|\zeta(s)-\xi(s)|, |\zeta(s)-K(t,s,\zeta(s))|,\\ &|\xi(s)-K(t,s,\xi(s))|, |\zeta(s)-K(t,s,\xi(s))|,\\ &|\xi(s)-K(t,s,\zeta(s))|\}\}\end{aligned}$$

for all $t,s \in [a,b]$, $\zeta,\xi \in C[a,b]$.

(A_3) there exists some $\tau > 0$ such that

$$\sup_{t\in[a,b]} \int_a^t l(t,s)\sigma s \le e^{-2\tau}.$$

Theorem 3: With the assertions (A_1)–(A_3), the integral inclusion (5.7) has a solution in $C[a,b]$.

Proof: Let $\mathcal{S} = C[a,b]$. Define the multivalued mapping $\mathcal{G} : \mathcal{S} \to CB(\mathcal{S})$ by

$$\mathcal{G}\zeta = \left\{\xi \in \mathcal{S} : \xi(t) \in f(t) + \int_a^t K(t,s,\zeta(s))\sigma s, \quad t \in [a,b]\right\}.$$

It is simple and direct that the set of solutions of the integral inclusion (5.7) synchronizes with the set of fixed points of $\mathcal{G}$. Thus, we have to show that with the stated conditions, $\mathcal{G}$ has at least one fixed point in $\mathcal{S}$. For it, we shall examine that the conditions of Corollary 9 are satisfied. □

Let $\zeta \in \mathcal{S}$ be arbitrary. For the multivalued operator $K_\zeta(t,s) : [a,b] \times [a,b] \to K_{cv}(\mathbb{R})$, it act in accordance with the Michael's selection theorem that there exists a continuous function $k_\zeta(t,s) : [a,b] \times [a,b] \to \mathbb{R}$ such that $k_\zeta(t,s) \in K_\zeta(t,s)$ for all $t,s \in [a,b]$. It follows that $f(t) + \int_a^t k_\zeta(t,s)\sigma s \in \mathcal{G}\zeta$. Hence $\mathcal{G}\zeta \ne \emptyset$. It is an obvious matter to prove that $\mathcal{G}\zeta$ is closed, and so specific aspects are excluded (see also [13]). Moreover, since f is continuous on $[a,b]$ and $K_\zeta(t,s)$ is continuous on $[a,b] \times [a,b]$, so their ranges are bounded. It follows that $\mathcal{G}\zeta$ is also bounded. Hence $\mathcal{G}\zeta \in CB(\mathcal{S})$.

We now analyze that (2.19) holds for $\mathcal{G}$ on $\mathcal{S}$ with some $\tau > 0$ and $F \in \digamma$, i.e.,

$$\begin{aligned}&2\tau + F(H(\mathcal{G}\zeta_1,\mathcal{G}\zeta_2))\\ &\quad\le F\left(\max\left\{\begin{matrix}\sigma(\zeta_1,\zeta_2), \sigma(\zeta_1,\mathcal{G}\zeta_1), \sigma(\zeta_2,\mathcal{G}\zeta_2),\\ \sigma(\zeta_1,\mathcal{G}\zeta_2), \sigma(\zeta_2,\mathcal{G}\zeta_1)\end{matrix}\right\}\right)\end{aligned}$$

for $\zeta_1,\zeta_2 \in \mathcal{S}$. Let $\xi_1 \in \mathcal{G}\zeta_1$ be arbitrary such that

$$\xi_1(t) \in f(t) + \int_a^t K(t,s,\zeta_1(s))\sigma s$$

for $t \in [a,b]$ holds. It implies that $\forall\, t,s \in [a,b]$, $\exists\, k_{\zeta_1}(t,s) \in K_{\zeta_1}(t,s) = K(t,s,\zeta_1(s))$ such that

$$\xi_1(t) = f(t) + \int_a^t k_{\zeta_1}(t,s)\sigma s$$

for $t \in [a,b]$. For all $\zeta_1,\zeta_2 \in \mathcal{S}$, it follows from (A_2) that

$$\begin{aligned}&H(K(t,s,\zeta_1) - K(t,s,\zeta_2)\\ &\quad\le l(t,s)\left\{\begin{matrix}\max\{|\zeta_1(s)-\zeta_2(s)|, |\zeta_1(s)-K(t,s,\zeta_1(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_2(s))|, |\zeta_1(s)-K(t,s,\zeta_2(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_1(s))|\}\end{matrix}\right\}.\end{aligned}$$

This implies that $\exists\, z(t,s) \in K_{\zeta_2}(t,s)$ such that

$$\begin{aligned}&\left|k_{\zeta_1}(t,s) - z(t,s)\right|\\ &\quad\le l(t,s)\left\{\begin{matrix}\max\{|\zeta_1(s)-\zeta_2(s)|, |\zeta_1(s)-K(t,s,\zeta_1(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_2(s))|, |\zeta_1(s)-K(t,s,\zeta_2(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_1(s))|\}\end{matrix}\right\}.\end{aligned}$$

for all $t,s \in [a,b]$.

Now, we can deal with the multivalued mapping U defined by

$$\begin{aligned}U(t,s) = K_{\zeta_2}(t,s) \cap \{u \in \mathbb{R} : &\left|k_{\zeta_1}(t,s) - u\right|\\ &\le l(t,s)|\zeta_1(s)-\zeta_2(s)|\}.\end{aligned}$$

Hence, by (A_1), U is lower semicontinuous, it implies that there exists a continuous mapping $k_{\zeta_2}(t,s) : [a,b]\times[a,b] \to \mathbb{R}$ such that $k_{\zeta_2}(t,s) \in U(t,s)$ for $t,s \in [a,b]$. Then $\xi_2(t) = f(t) + \int_a^t k_{\zeta_1}(t,s)\sigma s$ satisfies that

$$\xi_2(t) \in f(t) + \int_a^t K(t,s,\zeta_2(s))\sigma s, \quad t \in [a,b].$$

$t \in [a,b]$. That is $\xi_2 \in \mathcal{G}\zeta_2$ and

$$\begin{aligned}&|\xi_1(t) - \xi_2(t)|\\ &\quad\le \int_a^t \left|k_{\zeta_1}(t,s) - k_{\zeta_2}(t,s)\right|\sigma s\\ &\quad\le \int_a^t l(t,s)|\zeta_1(s)-\zeta_2(s)|\sigma s\\ &\quad\le \max_{t\in[a,b]}\left(\int_a^t l(t,s)\left\{\begin{matrix}\max\{|\zeta_1(s)-\zeta_2(s)|,\\ |\zeta_1(s)-K(t,s,\zeta_1(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_2(s))|,\\ |\zeta_1(s)-K(t,s,\zeta_2(s))|,\\ |\zeta_2(s)-K(t,s,\zeta_1(s))|\}\end{matrix}\right\}\sigma s\right)\end{aligned}$$

$$\leq e^{-2\tau} \max \left\{ \begin{array}{c} \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1) \end{array} \right\}$$

for all $t, s \in [a, b]$. Hence, we have

$$\sigma(\xi_1, \xi_2) \leq e^{-2\tau} \max \left\{ \begin{array}{c} \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1) \end{array} \right\}$$

Interchanging the roles of ζ_1 and ζ_2 , we obtain that

$$H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2) \leq e^{-2\tau} \max \left\{ \begin{array}{c} \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1) \end{array} \right\}$$

Taking natural log on both side, we have

$$2\tau + \ln(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) \leq \ln \left(\max \left\{ \begin{array}{c} \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1) \end{array} \right\} \right)$$

Taking $F \in \digamma$ defined by $F(t) = \ln(t)$ for $t > 0$, we have

$$2\tau + F(H(\mathcal{G}\zeta_1, \mathcal{G}\zeta_2)) \leq F \left(\max \left\{ \begin{array}{c} \sigma(\zeta_1, \zeta_2), \sigma(\zeta_1, \mathcal{G}\zeta_1), \sigma(\zeta_2, \mathcal{G}\zeta_2), \\ \sigma(\zeta_1, \mathcal{G}\zeta_2), \sigma(\zeta_2, \mathcal{G}\zeta_1) \end{array} \right\} \right).$$

All other conditions of Corollary 9 immediately follows by the hypothesis by taking the function $\rho \in \mathcal{E}$ given by $\rho(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) = \max\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ and the given integral inclusion (5.7) has a solution.

VI. CONCLUSION

In this article, we have defined the notion of F_ρ-contractions to establish new fixed point results for a new class of contractive conditions in the context of complete metric spaces. The given results extended and improved the well-known results of Banach, Kanan, Chatterjea, Reich, Hardy-Rogers, Berinde and Ćirić by means of this new class of contractions. As application of our main results, the existence of solution of second order differential equation and a certain Volterra-type integral inclusion is also investigated. Our results are new and significantly contribute to the existing literature in fixed point theory.

AUTHOR CONTRIBUTIONS

Both authors contributed equally and significantly in writing this article. All authors read and approved the final article.

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AFRAH AHMAD NOMAN ABDOU received the Ph.D. degree from King Abdulaziz University, Jeddah, Saudi Arabia. She is currently a Full Professor of mathematics with the Department of Mathematics, University of Jeddah. She published more than 70 articles in high standard journals. She is one of the most influential Muslim scientific minds in the world.

JAMSHAID AHMAD received the Ph.D. degree in functional analysis from the COMSATS Institute of Informational Technology, Islamabad, Pakistan. Since 2015, he has been an Assistant Professor with the Department of Mathematics, University of Jeddah. Seven M.Sc. students have been completed their degree under his supervision. He was declared a productive scientist by the Pakistan Council for Science and Technology, in 2015. He has published more than 70 articles in International Reputed Journals.