IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received September 11, 2019, accepted September 26, 2019, date of publication September 30, 2019,
date of current version October 15, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2944633

ℓ_∞-Gain Analysis of Discrete-Time Positive Singular Systems With Time-Varying Delays

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This work was supported in part by the National Natural Science Foundation of China under Grant 61903258, Grant U1813212, and Grant 51677120, in part by the Shenzhen Government Fund under Grant JCYJ20180305124348603, in part by the Foundation of Chongqing Municipal Key Laboratory of Institutions of Higher Education under Grant [2017]3, in part by the Foundation of Chongqing Development and Reform Commission under Grant 2017[1007], and in part by the Foundation of Chongqing Three Gorges University.

ABSTRACT This paper is concerned with the ℓ_∞-gain analysis problem of discrete-time positive singular systems with time-varying delays. By introducing an auxiliary system, a necessary and sufficient positivity condition is proposed for systems. Then, by using of the positivity and linearity, we investigate the monotonic and asymptotic property of the system with time-varying delays. Then, by resorting to the comparing system with constant delays, an explicit expression of the ℓ_∞-gain of discrete-time singular systems with time-varying delays is given in terms of system matrices. The result shows that ℓ_∞-gain of discrete-time positive singular systems with bounded time-varying delays is insensitive to the magnitude of delays.

INDEX TERMS Positive systems, singular systems, time-delay, ℓ_∞-gain.

I. INTRODUCTION

Singular systems, also called generalized state-space systems, implicit systems, or descriptor systems, are widely employed in different practical engineering systems, such as aircraft control systems, chemical engineering systems and electrical circuit systems [1], [2]. Compared with its counterparts of standard state-space systems, singular system models may provide more precise descriptions of dynamic systems [3], [4]. Practical systems whose state variables take nonnegative values naturally appears in different fields of application ranging from biology and pharmacokinetics [5], to economy and chemistry [6], which are usually referred to as positive systems. For positive singular systems, there are already some results dealing with the fundamental issues as a testimony to the vitality of this area. Some pioneering works on the positivity and stability for continuous-time and discrete-time positive singular systems were proposed in [7] and [8]. It should be emphasized that both works are based on such a common assumption, but it is unnecessary for positive singular systems which was demonstrated in [9].

Since time-delay is often a source of instability and encountered in various engineering systems, the analysis of time-delay positive systems has drawn considerable attention. By proving that stability condition is unrelated to the magnitude of time-delay, the asymptotic stability of positive systems with constant time-delay was first studied in [10]. Inspired by this result, a necessary and sufficient stability condition of the discrete-time positive systems with time-delay was presented in [11]. These results were further extended to cases with bounded time-varying delays in [12] and [13] and the ℓ_∞ and L_∞ gains for positive systems with bounded time-varying delays are analysed in [14].

In many situations, the performance of dynamic systems is portrayed by their input-output relations. For general dynamic systems, the most popular performance measures are the passivity performance and bounded real performance. While it is reasonable to focus on the energy for general dynamic systems, the ℓ_∞-norm accounts for the maximal quantity and is more suitable for positive system rather than ℓ_2-norm representing the square of the energy. For instance, in animal reproduction models, one usually focuses on inspecting the maximal population of the animals in a particular region with a certain amount of food and number of predators. Besides these practical considerations, the positivity property brings a number of input-output features to positive systems based on the integral constraints instead of quadratic integral constraints.

The associate editor coordinating the review of this manuscript and approving it for publication was Guangdeng Zong.

By investigating the monotonic non-decreasing property of positive systems, this result was extended to the investigation of the L_∞-gain and ℓ_∞-gain for the bounded time-varying delay cases in [14]. Then, under certain assumptions on the delays, the conclusion was shown to be still valid in [15] for both discrete-time and continuous-time positive systems with unbounded time-varying delays. Research on the performance measure of positive singular systems is scarce as it is really a new topic. The LMI characterization of the bounded real lemma was studied for positive singular systems in [16] by using the Separating Hyperplane Theorem. The ℓ_∞-gain analysis is rather complicated for the positive singular systems and it remains an unsolved problem for discrete-time positive singular systems with time-varying delays.

In this work, we establish the positivity conditions without any unnecessary assumption compared with the previously reported results. Therefore, our results have wider applicability for the analysis of positivity. By revealing the relationship between constant delay systems and time-varying delay cases, an explicit expression of ℓ_∞-gain is proposed for the discrete-time positive singular systems with time-varying delay. This paper is organized as follows. Some preliminaries are presented in Section II. Positivity analysis of discrete-time positive singular time-delay system are given in Section III. The ℓ_∞-gain of discrete-time positive singular system with bounded time-varying delay is also analysed in Section III. A numerical simulation to prove the theory in this paper is posed in Section IV. In the end, we will conclude the paper in Section V.

Notation: The notation used throughout the paper is standard. A real matrix $A \in \mathbb{R}^{n\times n}$ with all of its entries non-negative is called non-negative matrix and is denoted by $A \succeq 0$ and $A \in \bar{\mathbb{R}}_{+}^{n\times n}$. For two matrices $A,\ B \in \mathbb{R}^{m\times n}$, $A \succeq B$ means that $A - B$ is a non-negative matrix, or equivalently, $a_{ij} \geq b_{ij}$ for $i \in \{1,\ 2,\ \ldots, m\}$ and $j \in \{1,\ 2,\ \ldots, n\}$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated. The ∞-norm of matrix $A \in \mathbb{R}^{m\times n}$ is the maximal absolute row sum, that is, $\| A \|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} \left|[A]_{ij}\right|$. The ∞-norm of a column vector $x \in \mathbb{R}^n$ is $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$, which coincides with the ∞-norm of a matrix. For a vector-valued function $w : \bar{\mathbb{R}}_+ \mapsto \mathbb{R}^n$, the ℓ_∞-norm of a sequence of vector $\{w(k)\}_{k=0}^{\infty}$ is given by $\| w \|_{\ell\infty} = \sup_{k\in\mathbb{N}} \| w(k) \|_\infty$. The ℓ_∞-gain of system (1) is defined as $\sup_{\|w\|_{\ell\infty}=1} \| y \|_{\ell\infty}$.

II. PRELIMINARIES

A discrete-time singular system with bounded time-varying delays is introduced as follows:

$$\begin{cases} Ex(k+1) = Ax(k) + A_d x(k-d(k)) + Bw(k) \\ y(k) = Cx(k) \\ x(s) = \phi(s),\ s = -\overline{d}, -\overline{d}+1, \ldots, 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ and $y(k) \in \mathbb{R}^{n_y}$ are the input and output signals, respectively; Delays $d(k)$ are assumed to be bounded, that is, $1 \leq \underline{d} \leq d(k) \leq \overline{d}$; Matrices A, A_d, B, C are known constant real matrices with appropriate dimensions. The matrix $E \in \mathbb{R}^{n_x\times n_x}$ is supposed to be singular, that is, $\text{rank}(E) = r < n_x$; $\phi(k)$ is the admissible initial condition. Then, some basic definitions and lemmas related to singular time-delay systems are given, which are necessary for the later analysis.

Definition 1 [17]:

(i) The pair (E, A) is said to be regular if $\det(zE - A)$ is not identically zero.

(ii) The pair (E, A) is said to be causal if $\deg\{\det(zE - A)\} = \text{rank}(E)$.

Definition 2 [18]: For any matrix $E \in \mathbb{R}^{n\times n}$, a unique corresponding matrix E^D always exists, called the Drazin inverse of E, satisfying

$$EE^D = E^D E, \quad E^D E E^D = E^D,\ E^D E^{v+1} = E^v,$$

where v is the smallest nonnegative integer such that $\text{rank}(E^v) = \text{rank}(E^{v+1})$ and denoted by $v = \text{ind}(E)$.

Lemma 1 [19]: If the pair (E, A) is regular with $E, A \in \mathbb{R}^{n_x\times n_x}$, we always can find $\eta \in \mathbb{R}$ such that the inverse of $\eta E - A$ exists. It follows that the matrices

$$\hat{E} = (\eta E - A)^{-1}E, \quad \hat{A} = (\eta E - A)^{-1}A$$

commute.

Due to the singularity of the derivative matrix E, system (1) is not intuitive to investigate its positivity. Therefore, we introduce the following technical propositions, which will be employed in the proof of main result.

Proposition 1: Assume that (E, A) is regular and causal. Then, let $x_1(k) = Mx(k)$ and $x_2(k) = (I - M)x(k)$ with $M = \hat{E}^D\hat{E}$ and we have $x_1(k)$ and $x_2(k)$ satisfying

$$\begin{cases} x_1(k+1) = A_1 x_1(k) + A_{d1}x(k-d(k)) + B_1 w(t), \\ 0 = -x_2(k) + A_{d2}x(k-d(k)) + B_2 w(t), \\ y(k) = C(x_1(k) + x_2(k)). \end{cases} \tag{2}$$

for $k \in \mathbb{N}$, where

$$\begin{aligned} A_1 &= \hat{E}^D\hat{A}, \quad A_{d1} = \hat{E}^D\hat{A}_d,\ B_1 = \hat{E}^D\hat{B}, \\ B_2 &= (M - I)\hat{A}^D\hat{B}, \quad A_{d2} = (M - I)\hat{A}^D\hat{A}_d, \\ \hat{E} &= (\eta E - A)^{-1}E, \quad \hat{A} = (\eta E - A)^{-1}A, \\ \hat{A}_d &= (\eta E - A)^{-1}A_d, \quad \hat{B} = (\eta E - A)^{-1}B. \end{aligned}$$

Proof: By following a similar manner of the proof in [20, Lemma 5], one can obtain Proposition 1. □

Lemma 2: The following statements always hold true.

(i) $(M)^2 = M$.

(ii) $MA_1 = A_1 M = A_1$, $MA_{d1} = A_{d1}$.

(iii) $Mx_1(k) = x_1(k)$, $(I - M)x_2(k) = x_2(k)$, $k \in \mathbb{N}$.

(iv) $x(k) = x_1(k) + x_2(k)$, $k \in \mathbb{N}$.

Proof: It is obtained directly from the definition of the Drazin inverse and Lemma 1. □

III. ℓ_∞-GAIN ANALYSIS

First, we introduce the definition for the positivity of system (1) similar to [21, Definition 3.1].

Definition 3: System (1) is said to be positive system if for any initial value $\phi(s) \succeq 0$, $s = -\bar{d}, -\bar{d}+1, \ldots, 0$, and all input $w(k) \succeq 0, \forall k \in \mathbb{N}$, the state trajectory $x(k) \succeq 0$ and the output $y(k) \succeq 0, \forall\, k \in \mathbb{N}$.

Lemma 3: [22] Let W_1, W_2 be matrices with appropriate dimensions. The following statements are equivalent:

(i) $W_1 x \succeq 0$ implies that $W_2 x \succeq 0$,
(ii) there exists $X \succeq 0$ satisfying the equation $W_2 = XW_1$.

The following lemmas and theorems give characterisations of positivity and stability for discrete-time singular system with time-varying delay. First, we will analyse the sufficient and necessary conditions for positivity of system (1).

Theorem 1: Suppose that (E, A) is regular and causal. With the initial conditions $\phi(s) \succeq 0$, for $s = -\bar{d}, \ldots, 0$, the following statements are equivalent:

(i) System (1) is a positive system for any nonnegative initial condition.
(ii) System (2) is a positive system for any nonnegative initial condition.
(iii) If $A_{d2} \succeq 0$, $B_2 \succeq 0$, $C \succeq 0$ and there exists $H \succeq 0$ such that

$$\begin{cases} A_1 = HM, \\ A_{d1} - HA_{d2} \succeq 0, \\ B_1 - HB_2 \succeq 0. \end{cases} \tag{3}$$

Proof: **(i)⇒(iii)**: Suppose that the system is positive for $k \in \mathbb{N}$ and $d(k) \equiv d$. For any $\upsilon \in \mathbb{R}^n$, $x(0) = M\upsilon + A_{d2}x(-d) + B_2 w(0) \succeq 0$ and for $x(0), x(-1), \ldots, x(-d) \succeq 0$, we have

$$\begin{bmatrix} M & A_{d2} & 0 & B_2 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \upsilon \\ x(-d) \\ x(1-d) \\ w(0) \\ w(1) \end{bmatrix} \succeq 0$$

implies that $x(1) = A_1\upsilon + A_{d1}x(-d) + A_{d2}x(1-d) + B_1 w(0) + B_2 w(1) \succeq 0$, which amounts to

$$\begin{bmatrix} A_1 & A_{d1} & A_{d2} & B_1 & B_2 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \upsilon \\ x(-d) \\ x(1-d) \\ w(0) \\ w(1) \end{bmatrix} \succeq 0.$$

Then, Lemma 3 ensures the existence of a matrix

$$\mathcal{H} = \begin{bmatrix} H & H_d & H_{d-1} & H_{w1} & H_{w2} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \succeq 0$$

such that

$$\begin{bmatrix} A_1 & A_{d1} & A_{d2} & B_1 & B_2 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} = \mathcal{H} \begin{bmatrix} M & A_{d2} & 0 & B_2 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

or equivalently,

$$\begin{cases} A_1 = HM, \\ A_{d1} = HA_{d2} + H_d, \\ B_1 = HB_2 + H_{w1}, \\ A_{d2} = H_{d-1}, \\ B_2 = H_{w2}. \end{cases}$$

Since all the matrices H, H_d and H_{d-1} are nonnegative, statement **(iii)** of Theorem 1 readily follows.

(iii)⇒(i): It will be proved that $x_1(k)$, $x_2(k)$ and $x(k)$ are nonnegative for all $k \geq 1$ by induction. Let $\phi(0) = M\upsilon + A_{d2}\phi(-d(0)) + B_2 w(0) \succeq 0$ and $\phi(s) \succeq 0$ for $s = -\bar{d}, \ldots, -1$ be given. By virtue of $H \succeq 0$, $A_{d1} - HA_{d2} \succeq 0$, $B_1 - HB_2 \succeq 0$and $A_1 M = A_1$, we have

$$\begin{aligned} x_1(1) &= A_1 M\upsilon + A_{d1}\phi(-d(0)) + B_1 w(0) \\ &= HM^2\upsilon + A_{d1}\phi(-d(0)) + B_1 w(0) \\ &\succeq HM\upsilon + HA_{d2}\phi(-d(0)) + HB_2 w(0) \\ &= H(M\upsilon + A_{d2}\phi(-d(0)) + B_2 w(0)) = H\phi(0) \succeq 0. \end{aligned}$$

Since $A_{d2} \succeq 0$ and $1 \leq d(k) \leq \bar{d}$, it follows

$$x_2(1) = A_{d2}\phi(1 - d(1)) + B_2 w(1) \succeq 0.$$

And we have $x(1) = x_1(1) + x_2(1) \succeq 0$. Assume now that $x_1(k) = A_1^k M\upsilon + \sum_{j=0}^{k-1} A_1^{k-j-1} A_{d1} x(j - d(j)) + \sum_{j=0}^{k-1} A_1^{k-j-1} B_1 w(j) \succeq 0$ and $x_2(k) = A_{d2}x(k - d(k)) + B_2 w(k) \succeq 0$ for $k \in \{1, \ldots, i\}$, it follows $x(k) = x_1(k) + x_2(k) \succeq 0$ for $k \in \{-\bar{d}, \ldots, i\}$ as initial condition $\phi(\cdot) \succeq 0$. As $w(k) \succeq 0$, we have

$$x_2(i+1) = A_{d2}x(i + 1 - d(i + 1)) + B_2 w(i + 1) \succeq 0.$$

As $H \succeq 0$, we aim to show $x(i+1) \succeq 0$ by proving $x_1(i+1) \succeq Hx_1(i)$.

$$\begin{aligned} x_1(i+1) &= A_1^{i+1}\upsilon + \sum_{j=0}^{i} A_1^{i-j} A_{d1} x(j - d(j)) + \sum_{j=0}^{i} A_1^{i-j} B_1 w(j) \\ &= A_1(A_1^i \upsilon + \sum_{j=0}^{i-1} A_1^{i-j-1} A_{d1} x(j - d(j)) \\ &\quad + \sum_{j=0}^{i-1} A_1^{i-j-1} B_1 w(j)) + A_{d1}x(i - d(i)) + B_1 w(i) \\ &\succeq HM(A_1^i \upsilon + \sum_{j=0}^{i-1} A_1^{i-j-1} A_{d1} x(j - d(j)) \\ &\quad + \sum_{j=0}^{i-1} A_1^{i-j-1} B_1 w(j)). \end{aligned}$$

Noting that $MA_1 = A_1$, we have

$$\begin{aligned} x_1(i+1) &\succeq H(A_1^i \upsilon + \sum_{i=0}^{i-1} A_1^{i-j-1} A_{d1} x(j-d(j)) \\ &\quad + \sum_{j=0}^{i-1} A_1^{i-j-1} B_1 w(j)) \\ &= Hx_1(i) \succeq 0. \end{aligned}$$

By induction, $x_1(k) \succeq 0, x_2(k) \succeq 0$ and $x(k) \succeq 0$ for $k \in \mathbb{N}_+$.

(ii)⇒(i): With $x_1(k) \succeq 0, x_2(k) \succeq 0$ for $k \in \mathbb{N}_+$ and $x(0) = \phi(0) \succeq 0$, it is evident that $x(k) = x_1(k) + x_2(k) \succeq 0$ for $k \in \mathbb{N}$. □

Remark 1: It is easy to see that system (1) is nonnegative if system (2) is nonnegative as $x(k) = x_1(k) + x_2(k)$ but the converse is not necessarily true. However, by Theorem 1, we prove that the positivity properties of them are equivalent, which is an interesting finding.

In the following paragraph, a sufficient condition for stability of positive system (1) is deduced. We will propose some useful lemmas in the beginning, and then give the proof of stability condition.

Lemma 4 [23]: Suppose the pair (E, A) is regular, causal and system (1) is positive; then system (1) with input $w(k) = 0$ is exponential stable if and only if $H - HA_{d2} + A_{d1} + A_{d2}$ is a Schur matrix, or equivalently, there exists a column vector $p \succ 0$, such that $(H - HA_{d2} + A_{d1} + A_{d2} - I)p \prec 0$, where H is given in (3).

In the following discussion, we investigate the ℓ_∞-gain analysis of the positive singular system (1) with time-varying delays. First, some assumptions on singular system (1) are introduced.

Assumption 1: It is assumed that singular system (1) satisfies the following conditions:

- System (1) is positive and exponential stable.
- The initial condition of system (1) is $\phi(s) = 0$, $s = -\bar{d}, -\bar{d}+1, \ldots, 0$.

Under Assumption 1, the ℓ_∞-gain of system (1) is defined as the smallest $\alpha > 0$ such that $\|y\|_{\ell_\infty} \leq \alpha \|w\|_{\ell_\infty}$ holds for all input $w \in \ell_\infty$ and $w(t) \succeq 0$ $(t \geq 0)$, or equivalently, $\sup_{\|w\|_{\ell_\infty}=1} \|y\|_{\ell_\infty}$. In order to simplify further analysis, the following lemma is introduced which can be obtained by resorting to the positivity and linearity of system (1).

Lemma 5: Suppose that $w_1(k) \preceq w_2(k)$ for all $k \in \mathbb{N}$ and the initial condition is $\phi(s) = 0$ $(s = -\bar{d}, -\bar{d}+1, \ldots, 0)$. Let $y_1(k)$ and $y_2(k)$ be the outputs of system (1) under the inputs $w_1(k)$ and $w_2(k)$, resepectively. Then, it follows that $y_1(k) \preceq y_2(k)$, $k \in \mathbb{N}$.

In the light of Lemma 5, it suffices to investigate the following system with a constant input $\bar{w} = \mathbf{1}$ instead of the original nonnegative input signals satisfying $\|w\|_{\ell_\infty} = 1$,

$$\begin{cases} x_1(k+1) = A_1 x_1(k) + A_{d1} x(k - d(k)) + B_1 \bar{w} \\ 0 = -x_2(k) + A_{d2} x(k - d(k)) + B_2 \bar{w} \\ y(k) = C(x_1(k) + x_2(k)) \end{cases} \tag{4}$$

A monotonic property of state trajectory of the positive singular system with delays is proposed in the following.

Lemma 6: Let $d(k) \equiv d \in \mathbb{N}_+$ $(\underline{d} \leq d \leq \bar{d})$, the solutions $x_1(k)$ and $x_2(k)$ of system (4) are monotonically non-decreasing with zero initial condition, that is, $x_1(k+1) \succeq x_1(k)$ and $x_2(k+1) \succeq x_2(k)$ and $x(k+1) \succeq x(k)$.

Proof: With initial condition $x_1(0) = 0$, $x_2(0) = 0$ and $\phi(s) \equiv 0$, it follows that $x_1(1) = B_1\bar{w} \succeq x_1(0)$ and $x_2(1) = B_2\bar{w} \succeq x_2(0)$. Suppose that $x_1(k) \succeq x_1(k-1)$ and $x_2(k) \succeq x_2(k-1)$ hold for $k \in \{0, \ldots, i\}$ with $i \in \mathbb{N}$, it follows that

$$\begin{aligned} x_1(i+1) - x_1(i) &= A_1(x_1(i) - x_1(i-1)) \\ &\quad + A_{d1}(x(i-d) - x(i-1-d)) \succeq 0, \\ x_2(i+1) - x_2(i) &= A_{d2}(x(i-d) - x(i-1-d)) \succeq 0. \end{aligned}$$

By induction and Statement (iv) of Lemma 2, $x_1(k+1) \succeq x_1(k)$, $x_2(k+1) \succeq x_2(k)$ and $x(k+1) \succeq x(k)$ are proved for all $k \in \mathbb{N}$. □

Subsequent to those preliminaries above, the ℓ_∞-gain analysis of singular system (4) with constant delays is presented.

Theorem 2: Let $d(k) \equiv d \in \mathbb{N}_+$ $(\underline{d} \leq d \leq \bar{d})$, the ℓ_∞-gain of stable system (4) is $\sup_{\|w\|_{\ell_\infty}=1} \|y\|_{\ell_\infty} = \|C(I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\|_\infty$.

Proof: By Lemma 2 and Theorem 1, we have

$$\begin{aligned} x_1(k+1) &= HMx_1(k) + A_{d1}x(k-d) + B_1\bar{w} \\ &= Hx_1(k) + A_{d1}x(k-d) + B_1\bar{w} \\ &= H(x(k) - x_2(k)) + A_{d1}x(k-d) + B_1\bar{w}. \end{aligned}$$

As $x_2(k) = A_{d2}x(k-d) + B_2\bar{w}$, we have

$$\begin{aligned} x_1(k+1) &= H(x(k) - A_{d2}x(k-d) - B_2\bar{w}) + A_{d1}x(k-d) \\ &\quad + B_1\bar{w} \\ &= Hx(k) + (A_{d1} - HA_{d2})x(k-d) \\ &\quad + (B_1 - HB_2)\bar{w}. \end{aligned}$$

As $x_2(k+1) = A_{d2}x(k+1-d) + B_2\bar{w}$, it follows that

$$\begin{aligned} x(k+1) &= x_1(k+1) + x_2(k+1) \\ &= Hx(k) + (A_{d1} - HA_{d2})x(k-d) + A_{d2}x(k+1-d) \\ &\quad + (B_1 - HB_2 + B_2)\bar{w}. \end{aligned} \tag{5}$$

By Lemma 6, it follows

$$x(k+1) \preceq (H + A_{d1} - HA_{d2} + A_{d2})x(k) + (B_1 - HB_2 + B_2)\bar{w}. \tag{6}$$

Now we prove that for $k \in \mathbb{N}$, the state trajectory $x(k)$ satisfies that

$$x(k) \preceq \sum_{j=0}^{k-1} (H + A_{d1} - HA_{d2} + A_{d2})^j (B_1 - HB_2 + B_2)\bar{w}.$$

First, under the zero initial condition, it is shown that $x(1) = (B_1 - HB_2 + B_2)\bar{w}$. Assume that

$$x(i) \preceq \sum_{j=0}^{i-1} (H + A_{d1} - HA_{d2} + A_{d2})^j (B_1 - HB_2 + B_2)\bar{w}$$

holds for $k = 1, 2, \ldots, i$, then, we have

$$x(i+1) \preceq (H + A_{d1} - HA_{d2} + A_{d2})x(i) + (B_1 - HB_2 + B_2)\bar{w} = \sum_{j=0}^{i}(H + A_{d1} - HA_{d2} + A_{d2})^j (B_1 - HB_2 + B_2)\bar{w}.$$

By induction, one can conclude that

$$x(k) \preceq \sum_{j=0}^{k-1}(H + A_{d1} - HA_{d2} + A_{d2})^j (B_1 - HB_2 + B_2)\bar{w} = (I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\bar{w}$$

holds for $k \in \mathbb{N}$, which implies that $x(k)$ have upper bounds.

Then, we will show that

$$\lim_{k\to+\infty} x(k) = (I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\bar{w}.$$

It is readily seen that $x(k)$ has an upper bound; hence, $\lim_{k\to+\infty} x(k)$ exists. Letting $k \to +\infty$, it follows that

$$\lim_{k\to+\infty} x(k) = \lim_{k\to+\infty} (H + A_{d1} - HA_{d2} + A_{d2})x(k) + (B_1 - HB_2 + B_2)\bar{w},$$

which leads to

$$\lim_{k\to+\infty} x(k) = (I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\bar{w},$$

as $H - HA_{d2} + A_{d1} + A_{d2}$ is Schur. Then, it is readily seen that

$$\lim_{k\to+\infty} y(k) = C(I - H - A_{d1} + HA_{d2} - A_{d2})^{-1} \times (B_1 - HB_2 + B_2)\bar{w}$$

and ℓ_∞-gain of system (4) is exactly $\|C(I-H-A_{d1}+HA_{d2}-A_{d2})^{-1}(B_1 - HB_2 + B_2)\|_\infty$, which completes the proof.

Corollary 1: Suppose $E - A - A_d$ is nonsingular. For any $d \in \mathbb{N}$, the ℓ_∞-gain of positive singular system (4) with $d(k) = d$ is $\sup_{w_{\ell_\infty}=1}\|y\|_{\ell_\infty} = \|C(E - A - A_d)^{-1}B\|_\infty$.

Proof: Then, we will show that $\lim_{k\to+\infty} x(k) = (E - A - A_d)^{-1}B\bar{w}$ as well when $E - A - A_d$ is nonsingular. From the above proof of Theorem 2, it has been pointed out that $x(k)$ has an upper bound. By Lemma 6, $x(k+1) \succeq x(k)$, it is shown that system (1) with $w(k) = \bar{w} = \mathbf{1}$ monotonically non-decreasing; therefore, $\lim_{k\to+\infty} x(k)$ must exist. By letting $k \to +\infty$ on both side of $Ex(k+1) = Ax(k) + A_d x(k-d) + B\bar{w}$, we have $\lim_{k\to+\infty} x(k) = (E - A - A_d)^{-1}B\bar{w}$. Then, it is readily seen that $\lim_{k\to+\infty} y(k) = C(E - A - A_d)^{-1}B\bar{w}$, which illuminates that ℓ_∞-gain of system (4) is $\|C(E - A - A_d)^{-1}B\|_\infty$, which finishes the proof. □

Before investigating the ℓ_∞-gain of positive singular system with time-varying delays, two constant delay systems with zero initial conditions are introduced corresponding to the upper bound and lower bound of the delays.

$$\begin{cases} \bar{x}_1(k+1) = A_1\bar{x}_1(k) + A_{d1}\bar{x}(k - \underline{d}) + B_1\bar{w} \\ 0 = -\bar{x}_2(k) + A_{d2}\bar{x}(k - \underline{d}) + B_2\bar{w} \\ \bar{y}(k) = C\bar{x}(k) \end{cases} \tag{7}$$

and

$$\begin{cases} \underline{x}_1(k+1) = A_1\underline{x}_1(k) + A_{d1}\underline{x}(k - \bar{d}) + B_1\bar{w} \\ 0 = -\underline{x}_2(k) + A_{d2}\underline{x}(k - \bar{d}) + B_2\bar{w} \\ \underline{y}(k) = C\underline{x}(k) \end{cases} \tag{8}$$

Then, we show that system (4) with time-varying delays is bounded by the above two constant delay systems.

Lemma 7: Suppose that $\bar{x}_1(k), \bar{x}_2(k), \underline{x}_1(k), \underline{x}_1(k)$ and $\bar{y}(k)$, $\underline{y}(k)$ are the state trajectories and outputs of systems (7) and (8). With $\underline{d} \le d(k) \le \bar{d}$, the state trajectories $x_1(k), x_2(k)$ and output $y(k)$ of system (4) satisfy that $\underline{x}_1(k) \preceq x_1(k) \preceq \bar{x}_1(k)$, $\underline{x}_2(k) \preceq x_2(k) \preceq \bar{x}_2(k)$, $\underline{x}(k) \preceq x(k) \preceq \bar{x}(k)$ and $\underline{y}(k) \preceq y(k) \preceq \bar{y}(k)$ for all $k \in \mathbb{N}$.

Proof: Define $e_1(k) \triangleq \bar{x}_1(k) - x_1(k)$, $e_2(k) \triangleq \bar{x}_2(k) - x_2(k)$ and $e(k) \triangleq \bar{x}(k) - x(k)$ then the error system $e_1(k), e_2(k)$ satisfies

$$\begin{cases} e_1(k+1) = A_1e_1(k) + A_{d1}(\bar{x}(k - \underline{d}) - x(k - d(k))) \\ 0 = -e_2(k) + A_{d2}(\bar{x}(k - \underline{d}) - x(k - d(k))) \end{cases}$$

which can be rewritten as

$$\begin{cases} e_1(k+1) = A_1e_1(k) + A_{d1}e(k - d(k) \\ +A_{d1}(\bar{x}(k - \underline{d}) - \bar{x}(k - d(k))) \\ 0 = -e_2(k) + A_{d1}e(k - d(k)) \\ +A_{d2}(\bar{x}(k - \underline{d}) - \bar{x}(k - d(k))) \end{cases} \tag{9}$$

In virtue of Lemma 6, we have $w(k) = \bar{x}(k - \underline{d}) - \bar{x}(k - d(k)) \succeq 0$, $B_1 = A_{d1}$ and $B_2 = A_{d2}$, which implies that the error system is nonnegative. By regarding these two additional items as nonnegative inputs, it follows that $e_1(k) \succeq 0$ and $e_2(k) \succeq 0$ with initial condition $e(s) = 0$ by the positivity of system (9). Therefore, it can be concluded that $\bar{x}_1(k) \succeq x_1(k)$ and $\bar{x}_2(k) \succeq x_2(k)$. Similarly, we can prove that $x_1(k) \succeq \underline{x}_1(k)$ and $x_2(k) \succeq \underline{x}_2(k)$ in the same way, which also implies that $\underline{x}(k) \preceq x(k) \preceq \bar{x}(k)$. With $C \succeq 0$, we obtain that $C(\underline{x}_1(k) + \underline{x}_2(k)) \preceq C(x_1(k) + x_2(k)) \preceq C(\bar{x}_1(k) + \bar{x}_2(k))$ such that $\underline{y}(k) \preceq y(k) \preceq \bar{y}(k)$ for all $k \in \mathbb{N}$. □

The result of ℓ_∞-gain analysis of singular system (4) with time-varying delays is proposed in following theorem.

Theorem 3: For any $d(k) \in \mathbb{N}_+$, $\underline{d} \le d(k) \le \bar{d}$ the ℓ_∞-gain of stable system (4) with time-varying delays is $\sup_{\|w\|_{\ell_\infty}=1}\|y\|_{\ell_\infty} = \|C(I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\|_\infty$.

Proof: By using of Lemma 7, we have that $\underline{y}_i(k) \le y_i(k) \le \bar{y}_i(k)$ for $i = 1, \ldots, p$, where $y(k), \bar{y}(k), \underline{y}(k)$ are the outputs of systems (4), (7) and (8), respectively. It follows that $\sup_{k\in\mathbb{N}} \max_{1\le i\le p} \underline{y}_i(k) \le \sup_{k\in\mathbb{N}} \max_{1\le i\le p} y_i(k) \le \sup_{k\in\mathbb{N}} \max_{1\le i\le p} \bar{y}_i(k)$. By Theorem 2, we have

$$\sup_{k\in\mathbb{N}} \max_{1\le i\le p} \underline{y}_i(k) = \sup_{k\in\mathbb{N}} \max_{1\le i\le p} \bar{y}_i(k) = \|C(I - H - A_{d1} + HA_{d2} - A_{d2})^{-1}(B_1 - HB_2 + B_2)\|_\infty.$$

Therefore, we can conclude that ℓ_∞-gain of singular system (4) is $\|C(I-H-A_{d1}+HA_{d2}-A_{d2})^{-1}(B_1-HB_2+B_2)\|_\infty$.

Corollary 2: Suppose $E-A-A_d$ is nonsingular. For any $d(k)\in\mathbb{N}$, $\underline{d}\le d(k)\le\overline{d}$ the ℓ_∞-gain of stable system (4) with time-varying delays is $\sup_{w_{\ell_\infty}=1}\|y\|_{\ell_\infty}=\|C(E-A-A_d)^{-1}B\|_\infty$.

Proof: The result directly follows from Corollary 1, Theorem 3, and Lemma 7.

Remark 2: Usually, Lyapnunvo-Krasovskii method is employed to investigate the singular system with time-delays. For the time-varying cases, these results sometime are conservative. In this work, a different method based on the positivity of system is proposed and shown good performance for positive singular delay systems.

IV. ILLUSTRATIVE EXAMPLES

In this section, a numerical example is provided to show the effectiveness of the obtained results. Consider the discrete-time singular system (1) with time-varying delays where

$$E=\begin{bmatrix}0.80 & -0.50\\ 0.00 & 0.00\end{bmatrix},\quad A=\begin{bmatrix}-0.80 & 2.00\\ -0.50 & 1.20\end{bmatrix},$$
$$A_d=\begin{bmatrix}-0.15 & 0.10\\ -0.10 & -0.18\end{bmatrix},\quad B=\begin{bmatrix}0.08 & -0.12\\ -0.15 & -0.60\end{bmatrix},$$
$$C=\begin{bmatrix}1.00 & 0.20\\ -0.10 & 1.00\end{bmatrix}.$$

First, by choosing $\eta=3$, one can obtain that $\hat{E}=(\eta E-A)^{-1}E$ and $\hat{A}=(\eta E-A)^{-1}A$ commute, $\hat{A}_d=(\eta E-A)^{-1}A_d$ and $\hat{B}=(\eta E-A)^{-1}B$. Then, it follows

$$\hat{E}=\begin{bmatrix}0.4593 & -0.2871\\ 0.1914 & -0.1196\end{bmatrix},\quad \hat{A}=\begin{bmatrix}0.3780 & -0.8612\\ 0.5742 & -1.3589\end{bmatrix},$$
$$\hat{A}_d=\begin{bmatrix}0.0813 & 0.3589\\ 0.1172 & 0.2995\end{bmatrix},\quad \hat{B}=\begin{bmatrix}0.2971 & 0.9359\\ 0.2488 & 0.8900\end{bmatrix}.$$

Then, it is shown that the system is positive and stable by Theorem 1 and Lemma 4 as

$$M=\hat{E}^D\hat{E}=\begin{bmatrix}1.3521 & -0.8451\\ 0.5634 & -0.3521\end{bmatrix},$$
$$A_1=\hat{E}^D\hat{A}=\begin{bmatrix}0.0762 & -0.0476\\ 0.0317 & -0.0198\end{bmatrix},$$
$$A_{d1}=\hat{E}^D\hat{A}_d=\begin{bmatrix}0.0321 & 0.6832\\ 0.0134 & 0.2847\end{bmatrix},$$
$$A_{d2}=(M-I)\hat{A}^D\hat{A}_d=\begin{bmatrix}0.0704 & 0.1268\\ 0.1127 & 0.2028\end{bmatrix},$$
$$B_1=\hat{E}^D\hat{B}=\begin{bmatrix}0.5637 & 1.5111\\ 0.2349 & 0.6296\end{bmatrix},$$
$$B_2=(M-I)\hat{A}^D\hat{B}=\begin{bmatrix}0.1056 & 0.4225\\ 0.1690 & 0.6761\end{bmatrix}.$$

One can see that M and A_1 are not nonnegative but a positive $H=\begin{bmatrix}0.0514 & 0.0117\\ 0.0117 & 0.0282\end{bmatrix}$ can be found satisfying $A_1=HM$ and

$$A_{d1}-HA_{d2}=\begin{bmatrix}0.0272 & 0.6743\\ 0.0094 & 0.2775\end{bmatrix}\succeq 0,$$

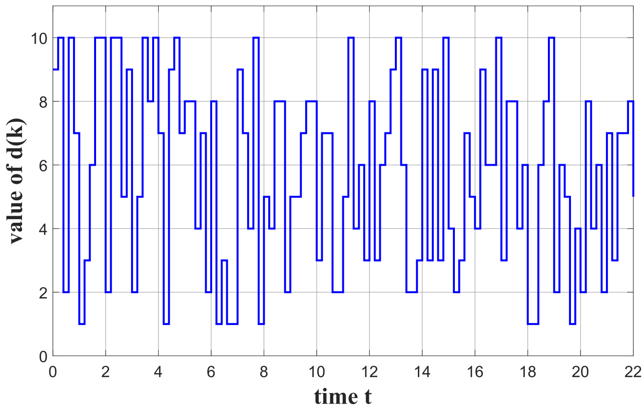

FIGURE 1. Time-varying delay $d(k)$.

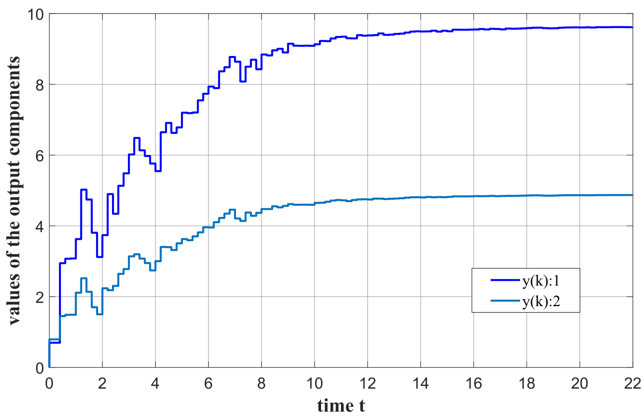

FIGURE 2. Output trajectories of system with input $w(k)=1$ and delay $d(k)$.

$$B_1-HB_2=\begin{bmatrix}0.5563 & 1.4815\\ 0.2289 & 0.6056\end{bmatrix}\succeq 0.$$

which implies that the system (1) is positive. By Lemma 4, system is asymptotically stable for any bounded delay $d(k)$ with $p=\begin{bmatrix}0.50 & 0.50\end{bmatrix}^T$. Bounded time-varying delay $d(k)$ takes value from the set $\{1,2,\ldots,10\}$ with equal probability. Given an input $w(k)=\mathbf{1}$, under zero initial condition, the output trajectory of system is depicted in Figure 2, when $d(k)$ takes value as Figure 1. It can be observed that the limit of the output trajectory is $C(I-H-A_{d1}+HA_{d2}-A_{d2})^{-1}(B_1-HB_2+B_2)\mathbf{1}=\begin{bmatrix}9.6307 & 4.8781\end{bmatrix}^T$, which confirms the results in Theorem 3. It should be pointed out that $C(E-A-A_d)^{-1}B\mathbf{1}=\begin{bmatrix}9.6307 & 4.8781\end{bmatrix}^T$ because $E-A-A_d$ is nonsingular, which satisfies Corollary 2.

V. CONCLUSION

In this paper, the ℓ-gain analysis problem is studied for discrete-time positive singular systems with time-varying delays. By using the Drazin inverse, the positivity criterion is proposed for discrete-time singular time-delay systems with disturbance. Then, an explicit expression of ℓ-gain is presented for systems with the constant delays. It is proved that the trajectories of time-varying delay systems can be bounded by two systems corresponding to the upper bound and lower bound delays. Based on this conclusion, we finally propose the characterization of ℓ-gain for discrete-time positive singular systems with time-varying delays. An example is employed to illuminates the effectiveness of the obtained results.

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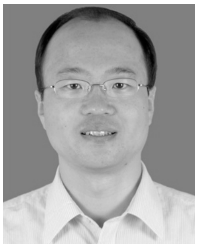

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