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MDS Symbol-Pair Repeated-Root Constacylic Codes of Prime Power Lengths Over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

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ABSTRACT MDS codes have the highest possible error-detecting and error-correcting capability among codes of given length and size. Let p be any prime, and s, m be positive integers. Here, we consider all constacyclic codes of length p^s over the ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ ($u^2 = 0$). The units of the ring $\mathcal{R}$ are of the form $\alpha + u\beta$ and γ, where $\alpha,\ \beta,\ \gamma \in \mathbb{F}_{p^m}^*$, which provides $p^m(p^m-1)$ constacyclic codes. We acquire that the $(\alpha+u\beta)$-constacyclic codes of p^s length over $\mathcal{R}$ are the ideals $\langle(\alpha_0\, x-1)^j\rangle$, $0 \leq j \leq 2\, p^s$, of the finite chain ring $R[x]/\langle x^{p^s} - (\alpha+u\beta)\rangle$ and the γ-constacyclic codes of p^s length over $\mathcal{R}$ are the ideals of the ring $\mathcal{R}[x]/\langle x^{p^s} - \gamma\rangle$ which is a local ring with the maximal ideal $\langle u, x-\gamma_0\rangle$, but it is not a chain ring. In this paper, we obtain all MDS symbol-pair constacyclic codes of length p^s over $\mathcal{R}$. We deduce that the MDS symbol-pair constacyclic codes are the trivial ideal $\langle 1\rangle$ and the Type 3 ideal of γ-constacyclic codes for some particular values of p and s. We also present several parameters including the exact symbol-pair distances of MDS constacyclic symbol-pair codes for different values of p and s.

INDEX TERMS Repeated-root codes, constacyclic codes, MDS codes, symbol-pair distance, finite chain ring.

I. INTRODUCTION

Initially, in the theory of error correcting codes, the message communicated in a noisy channel was divided into information units which were called individual symbols, and the operations of reading and writing were performed on these individual symbol. But due to the recent development of emerging technologies, symbol can be written and read in possible overlapping group. This method was first proposed by Cassuto and Blaum [1] in which the outputs (possibly corrupted) produced by a sequence of read operations are overlapping pairs of adjacent symbols, called pair-read symbols. These pair-read symbols were further developed to compute the symbol-pair distances of the generated codes in order to obtain the optimal codes. The method was further advanced by Cassuto and Blaum [2] in which they studied the pair error coding theory with algebraic cyclic-code constructions and asymptotic bounds on code rates.

After the celebrating result of Cassuto and Blaum [1] and Cassuto and Blaum [2], symbol-pair read channels gains the attraction of many coding theorists. Chee *et al.* [5] established the Singleton-type bound on symbol pair codes, and showed that the length of q-ary MDS symbol pair codes are $\Omega(q^2)$ corresponding to the length $O(q)$ of q-ary classical MDS codes. They also constructed infinite families of optimal symbol-pair codes, where the optimality obtained when the maximum distance of the symbol-pair codes meet the Singleton type bound of symbol-pair codes. Maximum distance separable (MDS) codes are optimal in the sense that they have the highest possible error-detecting and error-correcting capability for given code length and code

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size. This encourages numerous investigations regarding construction of MDS codes with respect to symbol-pair metric like [7], [8], [15], [18], [19].

In particular, Kai *et al.* [17] extended the result of Cassuto and Litsyn [3, Th. 10] for the case of simple-root constacyclic codes. Since then symbol-pair distances over constacyclic codes are an interesting topic to study and under scrutiny in series of papers like [6], [10], [12]–[14], [19]. Constacyclic codes play a significant role in coding theory because of their rich algebraic structure and practical implementations. Repeated-root constacyclic codes were first initiated in the most generality by Castagnoli in [4] and Van Lint in [22]. They found that the repeated-root constacyclic codes have a concatenated construction and are asymptotically bad but the optimal repeated-root constacyclic codes still exist, which motivated the researchers to further study these codes.

For $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$, $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1}) \in \Xi^n$, where Ξ is a code alphabet, the symbol-pair distance is defined in [1] as follows:

$$d_{\text{sp}}(\mathbf{a}, \mathbf{b}) = |i : (a_i, a_{i+1}) \neq (b_i, b_{i+1})|.$$

Then $d_{\text{sp}}(\mathcal{C}) = \min_{\mathbf{a},\mathbf{b}\in\mathcal{C},\, \mathbf{a}\neq\mathbf{b}}\{d_{\text{sp}}(\mathbf{a}, \mathbf{b})\}$ is the symbol-pair distance of code $\mathcal{C}$. Generally, the determination of symbol-pair distnace of a code $\mathcal{C}$ is very difficult. Recently, Dinh et al. computed the Hamming and symbol-pair distances of repeated root constacyclic codes of prime power lengths over $\mathbb{F}_{p^m}$ in [12] and over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ in [14].

In [12], Dinh et al. obtained MDS symbol-pair λ-constacyclic codes of prime power length over $\mathbb{F}_{p^m}$, by satisfying the Singleton bound of symbol-pair codes. Motivated by the concept, in this paper, we determine all MDS symbol-pair constacyclic codes of length p^s over the ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

The paper is organized as follows. In Section 2, we discuss some preliminary results. In Section 3, the MDS symbol-pair constacyclic codes of all length p^s are identified over the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Section 4 contains some examples in which we discuss the parameter of some MDS constacyclic symbol-pair codes for different values of p and s. We conclude the paper in Section 5.

II. PRELIMINARIES

Let $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, $u^2 = 0$ be a finite commutative ring with p^{2m} elements, where p is a prime and m is a positive integer. An ideal generated by one element is called a principal ideal and if all the ideals are principal, then the ring is called principal ideal ring. A local ring is defined if the ring has a unique maximal ideal. Further, a ring is called a chain ring if the set of all ideals of R is linearly ordered under set-theoretic inclusion. From [11], we have the following proposition for the class of finite commutative rings.

Proposition 1 [11]: Let $\mathcal{R}$ be a finite commutative ring, then the following conditions are equivalent:

(i) $\mathcal{R}$ is a local ring and the maximal ideal M of $\mathcal{R}$ is principal, i.e., $M = \langle r\rangle$ for some $r \in \mathcal{R}$,

(ii) $\mathcal{R}$ is a local principal ideal ring,

(iii) $\mathcal{R}$ is a chain ring with ideals $\langle r_i\rangle$, $0 \le i \le N(r)$, where $N(r)$ is the nilpotency of r.

Let λ be an invertible element of $\mathcal{R}$. The λ-constacyclic shift τ_λ on $\mathcal{R}^n$ is defined as

$$\tau_\lambda(c_0, c_1, \ldots, c_{n-1}) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2}).$$

If $\tau_\lambda(\mathcal{C}) = \mathcal{C}$, then $\mathcal{C}$ is called a λ-constacyclic code. In the case $\lambda = 1$, those λ-constacyclic codes are called cyclic codes, and when $\lambda = -1$, then the λ-constacyclic codes are called negacyclic codes.

Consider the polynomial $c(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1}x^{n-1}$ in the ring $\mathcal{R}[x]/\langle x^n - \lambda\rangle$. The polynomial $c(x)$ can be used to express the codeword $c = (c_0, c_1, \ldots, c_{n-1})$ of the code $\mathcal{C}$. And $xc(x)$ corresponds to λ-constacyclic shift of $c(x)$. We have a well-known result about λ-constacyclic codes.

Proposition 2 [9], [16], [20]: A linear code $\mathcal{C}$ is an ideal of $\mathcal{R}[x]/\langle x^n - \lambda\rangle$ if and only if $\mathcal{C}$ is a λ-constacyclic code of length n over $\mathcal{R}$.

So, for any invertible element λ of $\mathbb{F}_{p^m}$, λ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ are precisely the ideals of $\mathbb{F}_{p^m}[x]/\langle x^{p^s} - \lambda\rangle$.

Let α, β, γ be non-zero elements of the field $\mathbb{F}_{p^m}$. In [9], Dinh provided the construction of all constacyclic codes of p^s length over R as follows.

Theorem 3 [9]: Let λ be a unit of the ring $\mathcal{R}$, i.e., λ is of the form $\alpha + u\beta$ or γ, where $0 \neq \alpha, \beta, \gamma \in \mathbb{F}_{p^m}$.

- *If $\lambda = \alpha + u\beta$, then the ring $\mathcal{R}[x]/\langle x^{p^s} - (\alpha + u\beta)\rangle$ is a finite chain ring with maximal ideal $\langle \alpha_0 x - 1\rangle$, and $\langle(\alpha_0 x - 1)^{p^s}\rangle = \langle u\rangle$. The $(\alpha + u\beta)$-constacyclic codes of p^s length over $\mathcal{R}$ are the ideals $\langle(\alpha_0 x - 1)^j\rangle$, $0 \le j \le 2p^s$, of the finite chain ring $R[x]/\langle x^{p^s} - (\alpha + u\beta)\rangle$. The number of codewords in each code $\mathcal{C}_j = \langle(\alpha_0 x - 1)^j\rangle$ is $p^{m(2p^s - j)}$.*
- *If $\lambda = \gamma \in \mathbb{F}^*_{p^m}$, then the ring $R[x]/\langle x^{p^s} - \gamma\rangle$ is a local ring with the maximal ideal $\langle u, x - \gamma_0\rangle$, but it is not a chain ring. The γ-constacyclic codes of p^s length over R, i.e., ideals of the ring $\mathcal{R}[x]/\langle x^{p^s} - \gamma\rangle$, are given by four types*
  - *Type 1 are the trivial ideals, i.e., $\mathcal{C} = \langle 0\rangle$, $\mathcal{C} = \langle 1\rangle$. Number of codewords in theses codes are 1 and $p^{2\,mp^s}$ respectively.*
  - *Type 2 are the principal ideals generated by non-monic polynomials, i.e., $\mathcal{C}_j = \langle u(x - \gamma_0)^j\rangle$, where $0 \le j \le p^s - 1$. In this case, $|\mathcal{C}_j| = p^{m(p^s - j)}$*
  - *Type 3 are the principal ideals generated by monic polynomials, i.e., $\mathcal{C}_j = \langle(x - \gamma_0)^j + u(x - \gamma_0)^t h(x)\rangle$, where $1 \le j \le p^s - 1$, $0 \le t < j$, and either $h(x)$ is 0 or $h(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \gamma\rangle}$. In this case,*

$$|\mathcal{C}_j| = \begin{cases} p^{2m(p^s - j)}, & \text{if } 1 \le j \le p^{s-1} + \lfloor \frac{t}{2} \rfloor \\ p^{m(p^s - t)}, & \text{if } p^{s-1} + \lfloor \frac{t}{2} \rfloor < j \le p^s - 1. \end{cases}$$

  - *Type 4 are the nonprincipal ideals, i.e., $\langle(x - \gamma_0)^j + u(x - \gamma_0)^t h(x), u(x - \gamma_0)^\kappa\rangle$, with $h(x)$ as in Type 3, $\deg h(x) \le \kappa - t - 1$, and $\kappa < T$, where T is the smallest integer such that $u(x - \gamma_0)^T \in \langle(x - \gamma_0)^j +$*

$u(x-\gamma_0)^t h(x)\rangle$; and $T = j$, if $h(x) = 0$, otherwise $T = \min\{j, p^s - j + t\}$. The cardinality of $\mathcal{C}$ is given by $|\mathcal{C}| = p^{m(2p^s - j - \kappa)}$.

In [21, Section 5], Norton and Sălăgean provided the Singleton bound for finite chain ring $\mathcal{R}$ which is given by $|\mathcal{C}| \leq |\mathcal{R}|^{n-d(\mathcal{C})+1}$ or $d(\mathcal{C}) \leq n - \frac{1}{\nu}\sum_{i=0}^{\nu-1}(\nu - i)k_i(\mathcal{C}) + 1$, where ν is the nilpotency index of the fixed generator of the maximal ideal of $\mathcal{R}$ and $k_i(\mathcal{C})$ are the sizes of blocks of columns in standard form of generator matrix. A linear code $\mathcal{C}$ for which $d(\mathcal{C}) = n - k(\mathcal{C}) + 1$ where $k(\mathcal{C}) = \frac{1}{\nu}\sum_{i=0}^{\nu-1}(\nu - i)k_i(\mathcal{C})$, is called an MDS code. Since MDS codes have the best error-correcting capabilities, they form an optimal class of codes making it one of the central topics in the study of error-correcting codes. In [5], Chee et al. initiated the study of the Singleton bound for symbol-pair codes over $\mathbb{F}_{p^m}$. They determined the Singleton bound for any symbol-pair code $\mathcal{C}$ of length n over $\mathbb{F}_{p^m}$ with symbol-pair distance $d_{sp}(\mathcal{C})$ such that $2 \leq d_{sp}(\mathcal{C}) \leq n$, $|\mathcal{C}| \leq p^{m(n-d_{sp}(\mathcal{C})+2)}$ [5, Th. 1]. A symbol-pair code is known as *maximum distance separable code (MDS) symbol-pair code* if it attains the Singleton bound for symbol-pair codes, i.e., $|\mathcal{C}| = p^{m(n-d_{sp}(\mathcal{C})+2)}$. Singleton bound for symbol-pair codes over the finite chain ring $\mathcal{R}$ is as follows.

Theorem 4: Let $\mathcal{C}$ be a symbol-pair code over the finite chain ring $\mathcal{R}$ and let $d_{sp}(\mathcal{C})$ be the minimum symbol-pair distance of $\mathcal{C}$, then $|\mathcal{C}| \leq |\mathcal{R}|^{n-d_{sp}(\mathcal{C})+2}$.

Proof: Let $\mathcal{C}$ be a symbol-pair code over the finite chain ring $\mathcal{R}$. By deleting the last $d_{sp}(\mathcal{C})-2$ coordinates from all the codewords of $\mathcal{C}$, we observe that any $d_{sp}(\mathcal{C}) - 2$ consecutive coordinates contribute at most $d_{sp}(\mathcal{C}) - 1$ to the pair-distance. And since $\mathcal{C}$ has pair-distance $d_{sp}(\mathcal{C})$, the resulting vectors of length $n - d_{sp}(\mathcal{C}) + 2$ remain distinct after deleting the last $d_{sp}(\mathcal{C}) - 2$ coordinates from all codewords. The maximum number of distinct vectors of length $n - d_{sp}(\mathcal{C}) + 2$ over $\mathcal{R}$ is $|\mathcal{R}|^{n-d_{sp}(\mathcal{C})+2}$. Hence, $|\mathcal{C}| \leq |\mathcal{R}|^{n-d_{sp}(\mathcal{C})+2}$. □

Using the results of Theorem 3 and 4 and considering the symbol-pair distances of constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ provided in [14], we compute the MDS symbol-pair constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

III. MDS SYMBOL-PAIR CONSTACYCLIC CODES

In this section, we will use the determination of symbol-pair distance constacyclic codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ in [14, Sec. 4] to identify all MDS symbol-pair constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

A. $(\alpha + u\beta)$-CONSTACYCLIC CODES

Theorem 5: Let $\mathcal{C}_j = \langle(\alpha_0 x - 1)^j\rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s} - (\alpha + u\beta)\rangle}$ be a $(\alpha + u\beta)$-constacylcic code of length p^s over $\mathcal{R}$, for $j \in \{0, 1, \ldots, 2p^s\}$. Then $\mathcal{C}_j$ is a MDS symbol-pair code if and only if $j = 0$, then $d_{sp}(\mathcal{C}_j) = 2$.

Proof: For $(\alpha + u\beta)$-constacyclic codes, we have $|\mathcal{C}| = p^{m(2p^s - j)}$ [9, Th. 4.2]. By Singleton bound, $\mathcal{C}_j$ is the symbol-pair MDS code if and only if $2p^s - j = 2(p^s - d_{sp}(\mathcal{C}_j) + 2)$, i.e., $j = 2d_{sp}(\mathcal{C}_j) - 4$. The symbol-pair distance $d_{sp}(\mathcal{C}_j)$ for all $j \in \{0, 1, \ldots, 2p^s\}$ is established in [14, Th. 11]. We consider cases according to the range of j.

Case 1: $0 \leq j \leq p^s$. Then $d_{sp}(\mathcal{C}_j) = 2$, so obviously, MDS symbol-pair code can be obtained when $j = 0$.

Case 2: $j = 2p^s - p^{s-k} + 1$, where $0 \leq k \leq s-2$. Then $d_{sp}(\mathcal{C}_j) = 3p^k$, and

$$\begin{aligned}
j &= 2p^s - p^{s-k} + 1 \\
&= p^{s-k}(2p^k - 1) + 1 \\
&\geq p^2(2p^k - 1) + 1 \\
&\quad \times (\text{equality when } k = s-2, \text{ or } k = 0) \\
&\geq 4(2p^k - 1) + 1 \\
&\quad \times (\text{equality when } p = 2, \text{ or } k = 0) \\
&= 6p^k + 2p^k - 3 \\
&\geq 6p^k - 1 \\
&\quad \times (\text{equality when } k = 0) \\
&> 2d_{sp}(\mathcal{C}_j) - 4.
\end{aligned}$$

Therefore, no MDS symbol-pair code can be obtained in this cas.

Case 3: $2p^s - p^{s-k} + 2 \leq j \leq 2p^s - p^{s-k} + p^{s-k-1}$, where $0 \leq k \leq s-2$. Then $d_{sp}(\mathcal{C}_j) = 4p^k$, and

$$\begin{aligned}
j &\geq 2p^s - p^{s-k} + 2 \\
&= p^{s-k}(2p^k - 1) + 2 \\
&\geq 4(2p^k - 1) + 2 \\
&\quad \times (\text{equality when } k = s-2 \text{ and } p = 2, \text{ or } k = 0) \\
&= 8p^k - 2 \\
&> 2d_{sp}(\mathcal{C}_j) - 4.
\end{aligned}$$

Therefore, no MDS symbol-pair code can be obtained in this case.

Case 4: $2p^s - p^{s-k} + \delta p^{s-k-1} + 1 \leq j \leq 2p^s - p^{s-k} + (\delta+1)p^{s-k-1}$, where $0 \leq k \leq s-2$ and $1 \leq \delta \leq p-2$. Then $d_{sp}(\mathcal{C}_j) = 2(\delta + 2)p^k$, and

$$\begin{aligned}
j &\geq 2p^s - p^{s-k} + \delta p^{s-k-1} + 1 \\
&= p^{s-k}(2p^k - 1) + \delta p^{s-k-1} + 1 \\
&\geq p^2(2p^k - 1) + \delta p + 1 \\
&\quad \times (\text{equality when } k = s-2, \text{ or } s = 2) \\
&\geq (\delta+2)^2(2p^k - 1) + \delta(\delta+2) + 1 \\
&\quad \times (\text{equality when } \delta = p-2, \text{ or } s = 2) \\
&= 2(\delta+2)^2 p^k - 2\delta - 3 \\
&= 4(\delta+2)p^k + 2\delta(\delta+2)p^k - 2\delta - 3 \\
&\geq 4(\delta+2)p^k + 2\delta(\delta+2) - 2\delta - 3 \\
&\quad \times (\text{equality when } k = 0, \text{ or } s = 2) \\
&= 4(\delta+2)p^k + 2\delta(\delta+1) - 3 \\
&\geq 2d_{sp}(\mathcal{C}_j) + 1 \quad (\text{equality when } \delta = 1) \\
&> 2d_{sp}(\mathcal{C}_j) - 4.
\end{aligned}$$

Therefore, no MDS symbol-pair code can be obtained in this case.

Case 5: $j = 2p^s - p + \delta$, where $0 \leq \delta \leq p-2$. Then $d_{sp}(\mathcal{C}_j) = (\delta+2)p^{s-1}$, and

$$\begin{aligned} j &= 2p^s - p + \delta \\ &= p(2p^{s-1} - 1) + \delta \\ &\geq (\delta+2)(2p^{s-1} - 1) + \delta \\ &\quad \times \text{(equality when } \delta = p-2, \text{ or } s = 1) \\ &= 2(\delta+2)p^{s-1} - 2 \\ &> d_{sp}(\mathcal{C}_j) - 4. \end{aligned}$$

Therefore, no MDS symbol-pair code can be obtained in this case.

Case 6: $j = 2p^s - 1$. Then $d_{sp}(\mathcal{C}_j) = p^s$, and $j = 2p^s - 1 = 2d_{sp}(\mathcal{C}_j) - 1 > d_{sp}(\mathcal{C}_j) - 4$.

Since, $j > p^s + d_{sp}(\mathcal{C}_j) - 2$, no MDS symbol-pair code can be obtained in this case.

Case 7: $j = 2p^s$. Then $d_{sp}(\mathcal{C}_j) = 0$, and $j = 2d_{sp}(\mathcal{C}_j) + 2p^s > 2d_{sp}(\mathcal{C}_j) - 4$.

Since, $j > p^s + d_{sp}(\mathcal{C}_j) - 2$, no MDS symbol-pair code exists in this case.

Thus, we obtain only one MDS symbol-pair $(\alpha + u\beta)$-constacylcic codes of length p^s over $\mathcal{R}$, i.e., $\langle 1 \rangle$. □

Now, we consider the case where the unit $\lambda = \gamma \in \mathbb{F}^*_{p^m}$. From [9], we acquire that for a γ-constacyclic code, there are four types of ideals and the dimension of the code $\mathcal{C}_j$ is varies with each ideal. Here, we will discuss the symbol-pair MDS codes for each type of ideal.

B. γ-CONSTACYCLIC CODES

1) TYPE 1 (TRIVIAL IDEALS)

If $\mathcal{C} = \langle 0 \rangle$, then $|\mathcal{C}| = 1$ and $d_{sp}(\langle 0 \rangle) = 0$. Thus by Singleton bound, $\mathcal{C}$ is a symbol-pair MDS code if and only if $0 = 2(p^s - d_{sp}(\mathcal{C}) + 2)$, i.e., $p^s = -2$, which is not possible.

Again, if $\mathcal{C} = \langle 1 \rangle$, then $|\mathcal{C}| = p^{2mp^s}$ and $d_{sp}(\langle 1 \rangle) = 2$. Thus by Singleton bound, $\mathcal{C}$ is a symbol-pair MDS code if and only if $2p^s = 2(p^s - d_{sp}(\mathcal{C}) + 2)$, i.e., $d_{sp}(\mathcal{C}) = 2$.

Thus, MDS symbol-pair codes for trivial ideals is $\langle 1 \rangle$.

2) TYPE 2 (PRINCIPAL IDEALS GENERATED BY NONMONIC POLYNOMIAL)

Here, we have $\mathcal{C}_j = \langle u(x-\gamma_0)^j \rangle$, where $0 \leq j \leq p^s - 1$ and $|\mathcal{C}| = p^{m(p^s - j)}$. Thus by Singleton bound, $\mathcal{C}_j$ is a symbol-pair MDS code if and only if $p^s - j = 2p^s - 2d_{sp}(\mathcal{C}_j) + 4$, i.e., $j = 2d_{sp}(\mathcal{C}_j) - p^s - 4$. Hence, follows the theorem.

Theorem 6: Let $\mathcal{C}_j = \langle u(x-\gamma_0)^j \rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s} - \gamma \rangle}$ be a $\gamma-$constacyclic code of length p^s over $\mathcal{R}$, for $j \in 0, 1, \ldots, p^s - 1$. Then no MDS symbol-pair constacyclic code exists.

Proof: We get MDS code for $j = 2d_{sp}(\mathcal{C}_j) - p^s - 4$. The symbol-pair distance $d_{sp}(\mathcal{C}_j)$ for all $\kappa \in \{1, \ldots, p^s - 1\}$ of type 2 λ-constacyclic code is established in [14, Th. 12]. Now, we consider the cases according to the range of j.

Case 1: $j = 0$, then $d_{sp}(\mathcal{C}_j) = 2$, and $j = 2d_{sp}(\mathcal{C}_j) - 4 > 2d_{sp}(\mathcal{C}_j) - p^s - 4$. Thus, no MDS symbol-pair constacyclic code exists in this case.

Case 2: $j = p^s - p^{s-k} + 1$, where $0 \leq k \leq s-2$. Then $d_{sp}(\mathcal{C}_j) = 3p^k$, and

$$\begin{aligned} j &= p^s - p^{s-k} + 1 \\ &= p^{s-k}(2p^k - 1) - p^s + 1 \\ &\geq p^2(2p^k - 1) - p^s + 1 \\ &\quad \times \text{(equality when } k = s-2, \text{ or } s = 2) \\ &\geq 4(2p^k - 1) - p^s + 1 \quad \text{(equality when } p = 2) \\ &= 2d_{sp}(\mathcal{C}) - p^s + 2p^k - 3 \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^s - 1 \quad \text{(equality when } k = 0) \\ &> 2d_{sp}(\mathcal{C}_j) - p^s - 4. \end{aligned}$$

Since, $j > 2d_{sp}(\mathcal{C}_j) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 3: $p^s - p^{s-k} + 2 \leq j \leq p^s - p^{s-k} + p^{s-k-1}$, where $0 \leq k \leq s-2$. Then $d_{sp}(\mathcal{C}_j) = 4p^k$, and

$$\begin{aligned} j &= p^s - p^{s-k} + 2 \\ &= p^{s-k}(2p^k - 1) - p^s + 2 \\ &\geq p^2(2p^k - 1) - p^s + 2 \\ &\quad \times \text{(equality when } k = s-2, \text{ or } s = 2) \\ &\geq 4(2p^k - 1) - p^s + 2 \quad \text{(equality when } p = 2) \\ &= 2d_{sp}(\mathcal{C}_j) - p^s - 2 \\ &> 2d_{sp}(\mathcal{C}_j) - p^s - 4. \end{aligned}$$

Since, $j > 2d_{sp}(\mathcal{C}_j) - p^s - 4$, no MDS symbol-pair constacyclic code can exist in this case.

Case 4: $p^s - p^{s-k} + \delta p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + (\delta+1)p^{s-k-1}$, where $0 \leq k \leq s-2$ and $1 \leq \delta \leq p-2$. Then $d_{sp}(\mathcal{C}_j) = 2(\delta+2)p^k$, and

$$\begin{aligned} j &\geq p^s - p^{s-k} + \delta p^{s-k-1} + 1 \\ &= p^{s-k}(2p^k - 1) - p^s + \delta p^{s-k-1} + 1 \\ &\geq p^2(2p^k - 1) - p^s + \delta p + 1 \\ &\quad \times \text{(equality when } k = s-2, \text{ or } s = 2) \\ &\geq (\delta+2)^2(2p^k - 1) - p^s + \delta(\delta+2) + 1 \\ &\quad \times \text{(equality when } \delta = p-2) \\ &= 2(\delta+1)(\delta+2)p^k - p^s + 2(\delta+2)(p^k - 1) + 1 \\ &\geq 4(\delta+2)p^k - p^s + 6(p^k - 1) + 1 \\ &\quad \times \text{(equality when } \delta = 1) \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^s + 1 \quad \text{(equality when } k = 0) \\ &> 2d_{sp}(\mathcal{C}_j) - p^s - 4. \end{aligned}$$

Since, $j > 2d_{sp}(\mathcal{C}_j) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 5: $j = p^s - p + \delta$, where $0 \leq \delta \leq p-2$. Then $d_{sp}(\mathcal{C}_j) = (\delta+2)p^{s-1}$, and

$$\begin{aligned} j &= p^s - p + \delta \\ &= p(2p^{s-1} - 1) - p^s + \delta \end{aligned}$$

$$\begin{aligned}&\geq (\delta+2)(2p^{s-1}-1)-p^s+\delta\\ &\quad \times (\text{equality when } \delta = p-2)\\ &= 2(\delta+2)p^{s-1}-p^s-2\\ &= 2d_{sp}(\mathcal{C})-p^s-2\\ &> 2d_{sp}(\mathcal{C}_j)-p^s-4.\end{aligned}$$

Since, $j > 2d_{sp}(\mathcal{C}_j)-p^s-4$, no MDS symbol-pair constacyclic code exists in this case.

Case 6: $j = p^s-1$. Then $d_{sp}(\mathcal{C}_j) = p^s$, and $j = 2p^s-p^s-1 = 2d_{sp}(\mathcal{C}_j)-p^s-1 > 2d_{sp}(\mathcal{C}_j)-p^s-4$. Thus, no MDS symbol-pair constacyclic code exists in this case.

Hence, MDS symbol-pair code does not exist for γ-constacyclic codes of Type 2. This completes the proof. □

3) TYPE 3 (PRINCIPAL IDEALS GENERATED BY MONIC POLYNOMIAL)

Here, we have $\mathcal{C}_j = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x)\rangle$, where $1 \leq j \leq p^s-1$, $0 \leq t < j$, and either $h(x)$ is 0 or a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\gamma\rangle}$. Thus, we get the following two cases:

Case 1: When $h(x)$ is *0* then, $|\mathcal{C}_j| = p^{2m(p^s-j)}$. Thus by Singleton bound, $\mathcal{C}_j$ is a symbol-pair MDS code if and only if $p^s-j = p^s-d_{sp}(\mathcal{C}_j)+2$, i.e., $j = d_{sp}(\mathcal{C}_j)-2$. Hence, the MDS symbol-pair codes for Type 2 ideals are similar to the MDS λ-constacyclic symbol-pair codes over $\mathbb{F}_{p^m}$. Hence, we have the following theorem:

Theorem 7: Let $\mathcal{C}_j = \langle (x-\gamma_0)^j\rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma\rangle}$ be a $\gamma-$constacyclic code of length p^s over $\mathcal{R}$, for $j \in 1,\dots,p^s-1$. Then $\mathcal{C}_j$ is a MDS symbol-pair code if and only if one of the following conditions holds:

- *If $s = 1$, then $j = \delta$, for $0 \leq \delta \leq p-2$, then $d_{sp}(\mathcal{C}_j) = \delta+2$.*
- *If $s \geq 2$, then*
  - *$j = 1$, $d_{sp}(\mathcal{C}_j) = 3$,*
  - *$j = 2$, $d_{sp}(\mathcal{C}_j) = 4$,*
  - *$s = 2$, $p = 3$, $j = 4$, $d_{sp}(\mathcal{C}_j) = 6$,*
  - *$j = p^s-2$, $d_{sp}(\mathcal{C}_j) = p^s$.*

Case 2: When $h(x)$ is a unit [9] then,

$$|\mathcal{C}_j| = \begin{cases} p^{2m(p^s-j)}, & \text{if } 1 \leq j \leq p^{s-1}+\lfloor \frac{t}{2}\rfloor\\ p^{m(p^s-t)}, & \text{if } p^{s-1}+\lfloor \frac{t}{2}\rfloor < j \leq p^s-1.\end{cases}$$

Therefore, when $1 \leq j \leq p^{s-1}+\lfloor \frac{t}{2}\rfloor$, MDS symbol-pair constacyclic codes can be obtained when $j = d_{sp}(\mathcal{C}_j)-2$, which is similar to the result in case 1. But $j \leq p^{s-1}+\lfloor \frac{t}{2}\rfloor$, where $0 \leq t < j$, which implies that $p^{s-1} \leq j < 2p^{s-1}$, i.e., when $s = 1, j = 1$ and when $s \geq 2, j \geq 2$. Hence, we conclude the following theorem.

Theorem 8: Let $\mathcal{C}_j = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x)\rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma\rangle}$ be a $\gamma-$constacyclic code of length p^s over $\mathcal{R}$, for $j \in 1,\dots,p^s-1$. Then $\mathcal{C}_j$ is a MDS symbol-pair code if and only if one of the following conditions holds:

- *If $s = 1$, then $j = 1$, $d_{sp}(\mathcal{C}_j) = 3$.*
- *If $s \geq 2$, then*
  - *$j = 2$, $d_{sp}(\mathcal{C}_j) = 4$,*
  - *$s = 2$, $p = 3$, $j = 4$, $d_{sp}(\mathcal{C}_j) = 6$,*
  - *$j = p^s-2$, $d_{sp}(\mathcal{C}_j) = p^s$.*

When $p^{s-1}+\lfloor \frac{t}{2}\rfloor < j \leq p^s-1$, i.e., when $0 \leq t < 2j-2p^{s-1}$, MDS symbol-pair constacyclic codes can be obtained when $t = 2d_{sp}(\mathcal{C}_j)-p^s-4$, i.e., when $2j-2p^{s-1} > 2d_{sp}(\mathcal{C}_j)-p^s-4$. In the following theorem we are going to discuss the case when $2j > 2d_{sp}(\mathcal{C}_j)-p^{s-1}(p-2)-4$.

Theorem 9: Let $\mathcal{C}_j = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x)\rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma\rangle}$ be a $\gamma-$constacyclic code of length p^s over $\mathcal{R}$, for $j \in 1,\dots,p^s-1$. Then $\mathcal{C}_j$ is a MDS symbol-pair code if and only if one of the following conditions holds:

- *If $s \geq 1$ and $p \geq 5$, then*
  - *$j = p^s-1$, $d_{sp}(\mathcal{C}_j) = p^s$,*
  - *$j = p^s-2$, $d_{sp}(\mathcal{C}_j) = p^s$.*
- *If $s \geq 2$, then*
  - *$j = 2^s-1$, $d_{sp}(\mathcal{C}_j) = 2^s$,*
  - *$j = 3^s-1$, $d_{sp}(\mathcal{C}_j) = 3^s$,*
  - *$j = 3^s-2$, $d_{sp}(\mathcal{C}_j) = 3^s$.*
- *If $s \geq 3$, then*
  - *$j = 2^s-3$, $d_{sp}(\mathcal{C}_j) = 3\cdot 2^{s-2}$,*
  - *$j = 3^s-5$, $d_{sp}(\mathcal{C}_j) = 2\cdot 3^{s-1}$.*

Proof: Here, the MDS symbol-pair constacyclic code can be obtained if and only if $2j > 2d_{sp}(\mathcal{C}_j)-p^{s-1}(p-2)-4$ and $t = 2d_{sp}(\mathcal{C}_j)-p^s-4 \geq 0$. When $p = 2$, then the condition for a symbol-pair constacyclic code to be MDS becomes $2j > 2d_{sp}(\mathcal{C}_j)-4$ and $t = 2d_{sp}(\mathcal{C}_j)-2^s-4 \geq 0$. The symbol-pair distance $d_{sp}(\mathcal{C}_j)$ for all $j \in \{1,\dots,p^s-1\}$ of type 3 λ-constacyclic code is established in [14, Th. 12]. We consider cases according to the range of j.

Case 1: Here, $j = p^s-p^{s-k}+1$, where $0 \leq k \leq s-2$. And $d_{sp} = 3p^k$, then

$$\begin{aligned}2j &= 2p^s-2p^{s-k}+2\\ &= 2p^{s-k}(p^k-1)+2\\ &\geq 2p^2(p^k-1)+2 \quad (\text{equality when } k=s-2 \text{ or } k=0).\end{aligned}$$

Now, we consider the following sub-cases:

Subcase 1.1: When $p = 2$, we get

$$\begin{aligned}2j &\geq 8(p^k-1)+2\\ &= 6p^k+2p^k-6\\ &= 2d_{sp}(\mathcal{C}_j)-4+2p^k-2\end{aligned}$$

Now, $2j > 2d_{sp}(\mathcal{C}_j)-4$ if and only if $2p^k-2 > 0$, i.e., $k \geq 1$. Thus, equality occurs when $k = s-2$, $k \geq 1$ and $p = 2$ and we have $t = 2d_{sp}(\mathcal{C}_j)-2^s-4 = 2^{s-1}-4$. Now, $2^{s-1}-4 \geq 0$, i.e., $s \geq 3$, satisfying the previous condition. Therefore, MDS symbol-pair constacyclic code is obtained when $k = s-2$, $k \geq 1$, $s \geq 3$ and $p = 2$, i.e., $j = 2^s-3$ and $d_{sp}(\mathcal{C}_j) = 3\cdot 2^{s-2}$, where $s \geq 3$.

Subcase 1.2: When $p \geq 3$, we get

$$\begin{aligned}2j &\geq 18p^k-16 \quad (\text{equality when } p=3)\\ &= 6p^k+12p^k-16\\ &= 2d_{sp}(\mathcal{C}_j)-p^{s-1}(p-2)-4+p^{s-1}(p-2)+12p^k-12\\ &\geq 2d_{sp}(\mathcal{C}_j)-p^{s-1}(p-2)-4+p^{s-1}(p-2)\\ &\quad \times (\text{equality when } k=0)\end{aligned}$$

$$> 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4.$$

Here, $2j > 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4$, with equality $p = 3$ and $k = s-2$. Thus we have $t = 2\cdot 3^{k+1} - 3^s - 4 = -3^{s-1} - 4 < 0$, i.e., a contradiction, since $t \geq 0$. Thus, we can not obtain any MDS symbol-pair constacyclic code in this case.

Case 2: Here, $p^s - p^{s-k} + 2 \leq j \leq p^s - p^{s-k} + p^{s-k-1}$, where $0 \leq k \leq s-2$. And $d_{sp} = 4p^k$. We consider $j = p^s - p^{s-k} + r$, where $2 \leq r \leq p^{s-k-1}$, and we get

$$\begin{aligned} 2j &= 2p^s - 2p^{s-k} + 2r \\ &= 2p^{s-k}(p^k - 1) + 2r \\ &\geq 2p^2(p^k - 1) + 2r \quad (\text{equality when } k = s-2 \text{ or } k = 0). \end{aligned}$$

Now, we consider the following sub-cases:

Subcase 2.1: When $p = 2$, we get

$$\begin{aligned} 2j &\geq 8p^k + 2r - 8 \\ &= 2d_{sp}(\mathcal{C}_j) - 4 + 2r - 4. \end{aligned}$$

Now, $2j > 2d_{sp}(\mathcal{C}_j) - 4$ if and only if $2r - 4 > 0$, i.e., if $r > 2$, which is a contradiction, since for $p = 2$ and $k = s-2$, $r = 2$. Thus, no MDS symbol-pair constacyclic code can be obtained in this case.

Subcase 2.2: When $p \geq 3$, we get

$$\begin{aligned} 2j &\geq 18p^k + 2r - 18 \quad (\text{equality when } p = 3) \\ &= 8p^k + 10p^k + 2r - 18 \\ &= 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) \\ &\quad + 10p^k + 2r - 14 \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) \\ &\quad \times (\text{equality when } k = 0 \text{ and } r = 2) \\ &> 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4. \end{aligned}$$

Thus, $2j > 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4$ with equality $k = s-2$, $p = 3$. Then $t = 8\cdot 3^k - 3^s - 4 = -3^{s-2} - 4 < 0$, which is contradiction, since $t \geq 0$. Thus, no MDS symbol-pair constacyclic code can be obtained in this case.

Case 3: $p^s - p^{s-k} + \delta p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + (\delta + 1)p^{s-k-1}$, where $0 \leq k \leq s-2$ and $1 \leq \delta \leq p-2$. Then $d_{sp} = 2(\delta+2)p^k$. We consider $j = p^s - p^{s-k} + \delta p^{s-k-1} + r$, where $1 \leq r \leq p^{s-k-1}$, and we get

$$\begin{aligned} 2j &= 2p^s - 2p^{s-k} + 2\delta p^{s-k-1} + 2r \\ &= 2p^{s-k}(p^k - 1) + 2\delta p^{s-k-1} + 2r \\ &\geq 2p^2(p^k - 1) + 2\delta p + 2r \\ &\quad \times (\text{equality when } k = s-2 \text{ or } k = 0) \\ &\geq 2(\delta+2)p(p^k - 1) + 2\delta p + 2r \\ &\quad \times (\text{equality when } \delta = p-2) \\ &= 2(\delta+2)p^{k+1} - 4p + 2r. \end{aligned}$$

Now, we consider two sub-cases:

Subcase 3.1: When $p = 2$, we get

$$\begin{aligned} 2j &\geq 4(\delta+2)p^k + 2r - 8 \\ &= 2d_{sp}(\mathcal{C}_j) - 4 + 2r - 4. \end{aligned}$$

Now, $2j > 2d_{sp}(\mathcal{C}_j) - 4$ if and only if $2r - 4 > 0$, i.e., if $r > 2$, which is a contradiction, since for $p = 2$ and $k = s-2$, $1 \leq r \leq 2$. Thus, no MDS symbol-pair constacyclic code can be obtained in this case.

Subcase 3.2: When $p \geq 3$, we get

$$\begin{aligned} 2j &\geq 6(\delta+2)p^k + 2r - 12 \quad (\text{equality when } p = 3) \\ &= 4(\delta+2)p^k + 2(\delta+2)p^k + 2r - 12 \\ &= 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) \\ &\quad + 2(\delta+2)p^k + 2r - 8 \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) \\ &\quad \times (\text{equality when } k = 0,\ \delta = 1 \text{ and } r = 1) \\ &> 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4. \end{aligned}$$

Thus, $2j > 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4$ with the equality $k = s-2$, $p = 3\ \delta = 1$ and $r = 1$. Then $t = 12p^k - p^s - 2 \geq 0$, i.e., $3^{s-1} \geq 4$, i.e., $s \geq 3$. Thus, MDS symbol-pair constacyclic codes can be obtained at $j = 3^s - 5$ and $d_{sp}(\mathcal{C}_j) = 2\cdot 3^{s-1}$, where $s \geq 3$.

Case 4: $j = p^s - p + \delta$, where $0 \leq \delta \leq p-2$. Then $d_{sp} = 2(\delta+2)p^{s-1}$ and,

$$\begin{aligned} 2j &= 2p^s - 2p + 2\delta \\ &= 2p(p^{s-1} - 1) + 2\delta \\ &\geq 2(\delta+2)(p^{s-1} - 1) + 2\delta \quad (\text{equality when } \delta = p-2) \\ &= 2(\delta+2)p^{s-1} - 4 \\ &= 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 \quad (\text{equality when } p = 2). \end{aligned}$$

Thus, $2j > 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4$ with the equality $\delta = p-2$ and $p \geq 3$. Now, $t = p^s - 4 \geq 0$, i.e., when $p = 3$, $s \geq 2$ and when $p \geq 5$, $s \geq 1$. Thus MDS symbol-pair constacyclic code exist when $j = 3^s - 2$, $d_{sp}(\mathcal{C}_j) = 3^s$, where $s \geq 2$ and $j = p^s - 2$, $d_{sp}(\mathcal{C}_j) = p^s$, where $p \geq 5$, $s \geq 1$.

Case 5: Here, $j = p^s - 1$. Then $d_{sp} = p^s$ and,

$$\begin{aligned} 2j &= 2p^s - 2 \\ &= 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 + p^{s-1}(p-2) + 2 \\ &\geq 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4 \end{aligned}$$

Now, $2j > 2d_{sp}(\mathcal{C}_j) - p^{s-1}(p-2) - 4$ and MDS symbol-pair constacyclic codes can be obtained if $t = p^s - 4 \geq 0$, i.e., when $p = 2$, $s \geq 2$, when $p = 3$, $s \geq 2$ and when $p \geq 5$, $s \geq 1$.

This completes the proof. □

4) TYPE 4 (NONPRINCIPAL IDEALS)

Here, we have $\mathcal{C} = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa \rangle$, where $1 \leq j \leq p^s - 1$, $0 \leq t < j$, and either $h(x)$ is either 0 or a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \gamma \rangle}$, $\deg(h) \leq \kappa - t - 1$, and

$$\kappa < T = \begin{cases} j, & \text{if } h(x) = 0 \\ \min\{j, p^s - j + t\}, & \text{if } h(x) \neq 0. \end{cases}$$

In this case, $|\mathcal{C}| = p^{m(2p^s - j - \kappa)}$. Thus by Singleton bound, $\mathcal{C}$ is a symbol-pair MDS code if and only if $2p^s - j - \kappa = 2(p^s -$

$d_{sp}(\mathcal{C}) + 2$), i.e., $\kappa = 2d_{sp}(\mathcal{C}) - 4 - j$. Let $j = p^s - m$, where $1 \le m \le p^s - 1$. Thus, the condition for $\mathcal{C}$ to be a symbol-pair MDS constacyclic code becomes $\kappa = 2d_{sp}(\mathcal{C}) - 4 - p^s + m$. Hence, we can conclude the following theorem:

Theorem 10: Let $\mathcal{C} = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa \rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma \rangle}$ *be a* $\gamma-$*constacyclic code of length* p^s *over* $\mathcal{R}$, *for* $j \in 1, \ldots, p^s - 1$, $0 \le t < j$, *and either* $h(x)$ *is either* 0 *or a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\gamma \rangle}$, $\deg(h) \le \kappa - t - 1$ *and* $\kappa < T$, *where*

$$T = \begin{cases} j, & \text{if } h(x) = 0 \\ \min\{j, p^s - j + t\}, & \text{if } h(x) \neq 0. \end{cases}$$

Then, no MDS symbol-pair constacyclic code exists.

Proof: We get MDS code for $\kappa = 2d_{sp}(\mathcal{C}) - 4 - p^s + m$, where $1 \le m \le p^s - 1$. The symbol-pair distance $d_{sp}(\mathcal{C})$ for all $\kappa \in \{1, \ldots, p^s - 1\}$ of type 4 λ-constacyclic code is established in [14, Th. 12]. Now, we consider the cases according to the range of κ.

Case 1: $\kappa = p^s - p^{s-k} + 1$, where $0 \le k \le s-2$. Then $d_{sp}(\mathcal{C}) = 3p^k$, and

$$\begin{aligned} \kappa &= p^s - p^{s-k} + 1 \\ &= p^{s-k}(2p^k - 1) - p^s + 1 \\ &\ge p^2(2p^k - 1) - p^s + 1 \\ &\quad \times (\text{equality when } k = s-2, \text{ or } s = 2) \\ &\ge 4(2p^k - 1) - p^s + 1 \quad (\text{equality when } p = 2) \\ &= 2d_{sp}(\mathcal{C}) - p^s + 2p^k - 3 \\ &\ge 2d_{sp}(\mathcal{C}) - p^s - 1 \quad (\text{equality when } k = 0). \end{aligned}$$

Now, $\kappa \ge 2d_{sp}(\mathcal{C}) - 4 - p^s + m$ if and only if $3 \ge m$ i.e., equality when $m = 3$. Thus, equality occurs when $p = 2$, $k = s-2$, i.e., $\kappa = 2^s - 3$ and $j = 2^s - 3$, which is a contradiction, since $\kappa < j$. Thus, no MDS symbol-pair constacyclic code exists in this case.

Case 2: $p^s - p^{s-k} + 2 \le \kappa \le p^s - p^{s-k} + p^{s-k-1}$, where $0 \le k \le s-2$. Then $d_{sp}(\mathcal{C}) = 4p^k$, and

$$\begin{aligned} \kappa &= p^s - p^{s-k} + 2 \\ &= p^{s-k}(2p^k - 1) - p^s + 2 \\ &\ge p^2(2p^k - 1) - p^s + 2 \\ &\quad \times (\text{equality when } k = s-2, \text{ or } s = 2) \\ &\ge 4(2p^k - 1) - p^s + 2 \quad (\text{equality when } p = 2) \\ &= 2d_{sp}(\mathcal{C}) - p^s - 2. \end{aligned}$$

Now, $\kappa \ge 2d_{sp}(\mathcal{C}) - 4 - p^s + m$ if and only if $2 \ge m$ i.e., equality when $m = 2$. Thus, equality occurs when $p = 2$, $k = s-2$, i.e., $\kappa = 2^s - 2$ and $j = 2^s - 2$, which is a contradiction, since $\kappa < j$. Thus, no MDS symbol-pair constacyclic code can exist in this case.

Case 3: $p^s - p^{s-k} + \delta p^{s-k-1} + 1 \le \kappa \le p^s - p^{s-k} + (\delta+1)p^{s-k-1}$, where $0 \le k \le s-2$ and $1 \le \delta \le p-2$. Then $d_{sp}(\mathcal{C}) = 2(\delta+2)p^k$, and

$$\begin{aligned} \kappa &\ge p^s - p^{s-k} + \delta p^{s-k-1} + 1 \\ &= p^{s-k}(2p^k - 1) - p^s + \delta p^{s-k-1} + 1 \\ &\ge p^2(2p^k - 1) - p^s + \delta p + 1 \\ &\quad \times (\text{equality when } k = s-2, \text{ or } s = 2) \\ &\ge (\delta+2)^2(2p^k - 1) - p^s + \delta(\delta+2) + 1 \\ &\quad \times (\text{equality when } \delta = p-2) \\ &= 2(\delta+1)(\delta+2)p^k - p^s + 2(\delta+2)(p^k - 1) + 1 \\ &\ge 4(\delta+2)p^k - p^s + 6(p^k - 1) + 1 \quad (\text{equality when } \delta = 1) \\ &\ge 2d_{sp}(\mathcal{C}) - p^s + 1 \quad (\text{equality when } k = 0). \end{aligned}$$

Now, $\kappa \ge 2d_{sp}(\mathcal{C}) - 4 - p^s + m$ if and only if $5 \ge m$ i.e., equality when $m = 5$. Thus, equality occurs when $\delta = 1$, $p = 3$, $k = 0$, $s = 2$, $m = 5$, i.e., $\kappa = 4$ and $j = 4$, which is a contradiction. Thus, no MDS symbol-pair constacyclic code exists in this case.

Case 4: $\kappa = p^s - p + \delta$, where $0 \le \delta \le p-2$. Then $d_{sp}(\mathcal{C}) = (\delta+2)p^{s-1}$, and

$$\begin{aligned} \kappa &= p^s - p + \delta \\ &= p(2p^{s-1} - 1) - p^s + \delta \\ &\ge (\delta+2)(2p^{s-1} - 1) - p^s + \delta \quad (\text{equality when } \delta = p-2) \\ &= 2(\delta+2)p^{s-1} - p^s - 2 \\ &= 2d_{sp}(\mathcal{C}) - p^s - 2. \end{aligned}$$

Now, $\kappa \ge 2d_{sp}(\mathcal{C}) - 4 - p^s + m$ if and only if $2 \ge m$ i.e., equality when $m = 2$. Thus, equality occurs when $\delta = p-2$, i.e., when $\kappa = p^s - 2$ and $j = p^s - 2$. Thus, no MDS symbol-pair constacyclic code exists in this case.

Case 5: $\kappa = p^s - 1$. Then $d_{sp}(\mathcal{C}) = p^s$, and $\kappa = 2p^s - p^s - 1 = 2d_{sp}(\mathcal{C}) - p^s - 1$.

Now, $\kappa \ge 2d_{sp}(\mathcal{C}) - p^s + m - 4$ if and only if $3 \ge m$ i.e., $m = 3$. Thus, $j = p^s - 3 < \kappa$, which is a contradiction. Thus, no MDS symbol-pair constacyclic code exists in this case.

Hence, MDS symbol-pair code does not exist for γ-constacyclic codes of Type 4. This completes the proof. □

Consequently, we have the list of all MDS symbol-pair constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

Theorem 11: All MDS symbol-pair λ-*constacyclic codes of length* p^s *over* $\mathcal{R}$ *are determined as follows:*

- $(\alpha + u\beta)$-*constacyclic codes:* $\mathcal{C} = \langle (\alpha_0 x - 1)^j \rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s} - (\alpha + u\beta) \rangle}$, *where* $0 \le j \le 2p^s$. *Then* $\mathcal{C}$ *is a MDS symbol-pair constacyclic code if and only if* $j = 0$, *i.e.* $\langle 1 \rangle$, *in such case* $d_{sp}(\mathcal{C}) = 2$.
- *For* γ-*constacyclic codes, there are four types of ideals:*
  - *Type 1 (trivial ideals):* $\langle 1 \rangle$ *is the only symbol-pair constacyclic code with* $d_{sp}(\mathcal{C}) = 2$.
  - *Type 2 (principal ideals generated by nonmonic polynomial):* $\mathcal{C} = \langle u(x-\gamma_0)^j \rangle$, *where* $0 \le j \le p^s - 1$. *No MDS symbol-pair constacyclic codes can be obtained in this case.*
  - *Type 3 (principal ideals generated by monic polynomial):* $\mathcal{C} = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x) \rangle$, *where* $1 \le j \le p^s - 1$, $0 \le t < j$, *and either* $h(x)$ *is* 0 *or a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\gamma \rangle}$.
    When $h(x) = 0$, *then* $\mathcal{C}$ *is a MDS symbol-pair code if and only if one of the following conditions holds*:

◇ *If* $s = 1$, *then* $j = \delta$, *for* $0 \leq \delta \leq p-2$, *then* $d_{sp}(\mathcal{C}_j) = \delta + 2$.

◇ *If* $s \geq 2$, *then*

- $j = 1$, $d_{sp}(\mathcal{C}_j) = 3$,
- $j = 2$, $d_{sp}(\mathcal{C}_j) = 4$,
- $s = 2$, $p = 3$, $j = 4$, $d_{sp}(\mathcal{C}_j) = 6$,
- $j = p^s - 2$, $d_{sp}(\mathcal{C}_j) = p^s$.

When $h(x)$ *is a unit and* $1 \leq j \leq p^{s-1} + \lfloor \frac{t}{2} \rfloor$. *Then* $\mathcal{C}$ *is a MDS symbol-pair code if and only if one of the following conditions holds*:

◇ *If* $s = 1$, *then* $j = 1$, $d_{sp}(\mathcal{C}_j) = 3$.

◇ *If* $s \geq 2$, *then*

- $j = 2$, $d_{sp}(\mathcal{C}_j) = 4$,
- $s = 2$, $p = 3$, $j = 4$, $d_{sp}(\mathcal{C}_j) = 6$,
- $j = p^s - 2$, $d_{sp}(\mathcal{C}_j) = p^s$.

When $h(x)$ *is a unit and* $p^{s-1} + \lfloor \frac{t}{2} \rfloor < j \leq p^s - 1$. *Then,* $\mathcal{C}$ *is a MDS symbol-pair code if and only if one of the following conditions holds:*

◇ *If* $s \geq 1$ *and* $p \geq 5$, *then*

- $j = p^s - 1$, $d_{sp}(\mathcal{C}_j) = p^s$,
- $j = p^s - 2$, $d_{sp}(\mathcal{C}_j) = p^s$.

◇ *If* $s \geq 2$, *then*

- $j = 2^s - 1$, $d_{sp}(\mathcal{C}_j) = 2^s$,
- $j = 3^s - 1$, $d_{sp}(\mathcal{C}_j) = 3^s$,
- $j = 3^s - 2$, $d_{sp}(\mathcal{C}_j) = 3^s$.

◇ *If* $s \geq 3$, *then*

- $j = 2^s - 3$, $d_{sp}(\mathcal{C}_j) = 3 \cdot 2^{s-2}$,
- $j = 3^s - 5$, $d_{sp}(\mathcal{C}_j) = 2 \cdot 3^{s-1}$.

- *Type 4 (nonprincipal ideals):* $\mathcal{C} = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa \rangle$, *where* $1 \leq j \leq p^s - 1$, $0 \leq t < j$, *and either* $h(x)$ *is either* 0 *or a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \gamma \rangle}$, $\deg(h) \leq \kappa - t - 1$, *and*
$$\kappa < T = \begin{cases} j, & \text{if } h(x) = 0 \\ \min\{j, p^s - j + t\}, & \text{if } h(x) \neq 0. \end{cases}$$
No MDS symbol-pair constacyclic code can be obtained in this case.

IV. EXAMPLES

In this section, we present some examples of constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

Example 12: Consider the ring $\mathbb{F}_2 + u\mathbb{F}_2$, *where* $p = 2$, $m = 1$. *The units in the ring* $\mathbb{F}_2 + u\mathbb{F}_2$ *of the form* $\alpha + u\beta$ *is* $1 + u$ *and of the form* γ *is* 1. *For* $(1+u)$*-constacyclic codes, the generators are of the form* $\langle (x-1)^j \rangle$, *where* $0 \leq j \leq 2^{s+1}$. *The only MDS symbol-pair constacyclic codes in this case has the parameters* $(2^s, 4^{2^s}, 2)$ *with the Singleton bound* $j = 2d_{sp}(\mathcal{C}_j) - 4$.

We obtain cyclic codes corresponding to the unit $\gamma = 1$. *Different generators of the cyclic codes and their corresponding conditions to be MDS symbol-pair codes are given as follows*:

- *Type 1:* $\langle 0 \rangle$, $\langle 1 \rangle$. *For these codes the condition for MDS code are given by* $p^s = d_{sp}(\mathcal{C}) - 2$ *and* $2 = d_{sp}(\mathcal{C})$. *As mentioned in Section 3, the only MDS symbol-pair constacyclic codes in this case is* $\langle 1 \rangle$ *with the parameters* $(2^s, 4^{2^s}, 2)$.
- *Type 2:* $\langle u(x-1)^j \rangle$, *where* $0 \leq j \leq 2^s - 1$. *The condition for MDS code is given by* $j = 2d_{sp}(\mathcal{C}) - p^s - 4$. *MDS symbol-pair constacyclic codes are non-existent in this case.*
- *Type 3:* $\langle (x-1)^j + u(x-1)^t h(x) \rangle$, *where* $1 \leq j \leq 2^s - 1$, $0 \leq t < j$, *and either* $h(x)$ *is* 0 *or* $h(x)$ *is a unit in* $\frac{\mathbb{F}_2[x]}{\langle x^{2^s} - 1 \rangle}$. *For* $h(x) = 0$, *the MDS code condition is given by* $j = d_{sp}(\mathcal{C}_j) - 2$. *And if* $h(x)$ *is unit, the MDS code condition are* $j = d_{sp}(\mathcal{C}_j) - 2$, *when* $1 \leq j \leq p^{s-1} + \lfloor \frac{t}{2} \rfloor$ *and* $t = 2d_{sp}(\mathcal{C}_j) - p^s - 4$, *when* $p^{s-1} + \lfloor \frac{t}{2} \rfloor < j \leq p^s - 1$. *We present some parameters of MDS codes for* $h(x) = 0$ *in Table 1.*

TABLE 1. Examples of γ**-constacyclic codes over** $\mathbb{F}_2 + u\mathbb{F}_2$**.**

s	γ	Generator	(n, M, d_{sp})
1	1	1	$(2, 2^4, 2)$
2	1	$x-1$	$(4, 2^6, 3)$
2	1	$(x-1)^2$	$(4, 2^4, 4)$
2	1	$(x-1)^3 + u$	$(4, 2^4, 4)$
3	1	$(x-1)$	$(8, 2^{14}, 3)$
3	1	$(x-1)^2$	$(8, 2^{12}, 4)$
3	1	$(x-1)^6$	$(8, 2^4, 8)$
3	1	$(x-1)^7 + u(x-1)^4$	$(8, 2^4, 8)$
3	1	$(x-1)^5 + u$	$(8, 2^8, 6)$
4	1	$x-1$	$(16, 2^{30}, 3)$
4	1	$(x-1)^2$	$(16, 2^{28}, 4)$
4	1	$(x-1)^{14}$	$(16, 2^4, 16)$
4	1	$(x-1)^{15} + u(x-1)^{12}$	$(16, 2^4, 16)$
4	1	$(x-1)^{13} + u(x-1)^4$	$(16, 2^{12}, 12)$

- *Type 4:* $\langle (x-1)^j + u(x-1)^t h(x), u(x-1)^\kappa \rangle$, *with* $h(x)$ *as in Type 3,* $\deg h(x) \leq \kappa - t - 1$, *and* $\kappa < T$, *where* T *is the smallest integer such that* $u(x-1)^T \in \langle (x-1)^j + u(x-1)^t h(x) \rangle$; *and* $T = j$, *if* $h(x) = 0$, *otherwise* $T = \min\{j, 2^s - j + t\}$. *The MDS code condition is given by* $\kappa = 2d_{sp}(\mathcal{C}_j) - j - 4$. *In this case also, no MDS symbol-pair constacyclic code exists.*

Example 13: Consider the ring $\mathbb{F}_4 + u\mathbb{F}_4$, *where* $p = 2$, $m = 2$. *The units in the ring* $\mathbb{F}_4 + u\mathbb{F}_4$ *of the form* $\alpha + u\beta$ *is* $1 + u$ *and of the form* γ *is* 1. *For* $(1+u)$*-constacyclic codes, the generators are of the form* $\langle (x-1)^j \rangle$, *where* $0 \leq j \leq 2^{s+1}$. *The only MDS symbol-pair constacyclic codes in this case has the parameters* $(2^s, 2^{2^{s+2}}, 2)$. *We also provide some parameters of MDS symbol-pair codes in Table 2 for* γ*-constacyclic codes.*

Example 14: Consider the ring $\mathbb{F}_3 + u\mathbb{F}_3$. *Here* $p = 3$, $m = 1$. *The units in the ring* $\mathbb{F}_3 + u\mathbb{F}_3$ *of the form* $\alpha + u\beta$ *are* $1+u$, $1+2u$, $2+u$, $2+2u$ *and of the form* γ *are* 1, 2. *For units* $(1+u)$ *and* $(1+2u)$ *the generators of constacyclic codes are given by* $\langle (x-1)^j \rangle$, *where* $0 \leq j \leq 2 \cdot 3^s$ *and for units* $(2+u)$ *and* $(2+2u)$, *the generators of constacyclic codes are of the form* $\langle (2x-1)^j \rangle$, *where* $0 \leq j \leq 2 \cdot 3^s$. *For* $\gamma = 1$, *we obtain cyclic code of length* 3^s *over* $\mathbb{F}_3 + u\mathbb{F}_3$. *The only MDS symbol-pair* $(\alpha + u\beta)$*-constacyclic code has the*

TABLE 2. Examples of γ-constacyclic codes over $\mathbb{F}_4+u\mathbb{F}_4$.

s	γ	Generator	$(n, M, d_{\rm sp})$
1	1	1	$(2, 2^8, 2)$
2	1	$x-1$	$(4, 2^{12}, 3)$
2	1	$(x-1)^2$	$(4, 2^8, 4)$
2	1	$(x-1)^3+u$	$(4, 2^8, 4)$
3	1	$(x-1)$	$(8, 2^{28}, 3)$
3	1	$(x-1)^2$	$(8, 2^{24}, 4)$
3	1	$(x-1)^6$	$(8, 2^8, 8)$
3	1	$(x-1)^7+u(x-1)^4$	$(8, 2^8, 8)$
3	1	$(x-1)^5+u$	$(8, 2^{16}, 6)$
4	1	$x-1$	$(16, 2^{60}, 3)$
4	1	$(x-1)^2$	$(16, 2^{56}, 4)$
4	1	$(x-1)^{14}$	$(16, 2^8, 16)$
4	1	$(x-1)^{15}+u(x-1)^{12}$	$(16, 2^8, 16)$
4	1	$(x-1)^{13}+u(x-1)^4$	$(16, 2^{24}, 12)$

TABLE 3. Examples of γ-constacyclic codes over $\mathbb{F}_3+u\mathbb{F}_3$.

s	γ	Generator	$(n, M, d_{\rm sp})$
1	1	1	$(3, 3^6, 2)$
1	1	$x-1$	$(3, 3^4, 3)$
1	2	$x-2$	$(3, 3^4, 3)$
1	2	$(x-2)+u$	$(3, 3^6, 3)$
2	1	$x-1$	$(9, 3^{16}, 3)$
2	2	$(x-2)^2$	$(9, 3^{14}, 4)$
2	1	$(x-1)^4$	$(9, 3^{10}, 6)$
2	2	$(x-2)^7$	$(9, 3^4, 9)$
2	1	$(x-1)^8+u(x-1)^5$	$(9, 3^4, 9)$
2	2	$(x-2)^7+u(x-2)^5$	$(9, 3^4, 9)$
3	1	$x-1$	$(27, 3^{52}, 3)$
3	2	$(x-2)^2$	$(27, 3^{50}, 4)$
3	1	$(x-1)^{25}$	$(27, 3^4, 27)$
3	2	$(x-2)^{26}+u(x-2)^{23}$	$(27, 3^4, 27)$
3	1	$(x-1)^{25}+u(x-1)^{23}$	$(27, 3^4, 27)$
3	2	$(x-2)^{22}+u(x-1)^5$	$(27, 3^{22}, 18)$

TABLE 4. Examples of γ-constacyclic codes over $\mathbb{F}_5+u\mathbb{F}_5$.

s	γ	Generator	$(n, M, d_{\rm sp})$
1	1	1	$(5, 5^{10}, 2)$
1	2	$x-2$	$(5, 5^8, 3)$
1	3	$(x-3)^2$	$(5, 5^6, 4)$
1	4	$(x-4)^3$	$(5, 5^4, 5)$
1	1	$(x-1)^4+u(x-1)$	$(5, 5^4, 5)$
1	2	$(x-2)^3+u(x-2)$	$(5, 5^4, 5)$
2	3	$x-3$	$(25, 5^{48}, 3)$
2	4	$(x-4)^2$	$(25, 5^{46}, 4)$
2	1	$(x-1)^{23}$	$(25, 5^4, 25)$
2	2	$(x-2)^{24}+u(x-2)^{21}$	$(25, 5^4, 25)$
2	3	$(x-3)^{23}+u(x-3)^{21}$	$(25, 5^4, 25)$
2	4	$(x-4)^2+u$	$(25, 5^{46}, 4)$

parameters $(3^s, 9^{3^s}, 2)$. *We also provide some parameters of MDS symbol-pair codes in Table 3 for* γ*-constacyclic codes.*

Example 15: Consider the ring $\mathbb{F}_5+u\mathbb{F}_5$. *Here* $p=5$, $m=1$. *The units in the ring* $\mathbb{F}_3+u\mathbb{F}_3$ *of the form* $\alpha+u\beta$ *are* $a+ub$, *where* $a,\ b\in\mathbb{F}_5-\{0\}$ *and of the form* γ *are* $1,\ 2,\ 3,\ 4$. *For units* $a+ub$ *the generators of constacyclic codes are given by* $\langle(ax-1)^j\rangle$, *where* $0\le j\le 2\cdot 5^s$. *For* $\gamma=1$, *we obtain cyclic code of length* 5^s *over* $\mathbb{F}_5+u\mathbb{F}_5$. *The only MDS symbol-pair* $(\alpha+u\beta)$*-constacyclic code has the parameters* $(5^s, 5^{2\cdot 5^s}, 2)$. *We also provide some parameters of MDS symbol-pair codes in Table 4 for* γ*-constacyclic codes.*

V. CONCLUSION

Motivated by the work of Dinh *et al.* [12], we determine all MDS symbol-pair codes among repeated-root constacyclic codes of prime power length over the ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. We know that the units of the ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ are of the form $\alpha+u\beta$ and γ, where $0\neq\alpha,\beta,\gamma\in\mathbb{F}_{p^m}$. MDS symbol-pair codes from $(\alpha+u\beta)$-constacyclic codes $\mathcal{C}_j=\langle(\alpha_0\,x-1)^j\rangle$ over the ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ are obtained when $j=2d_{\rm sp}(\mathcal{C}_j)-4$. We obtained that the only MDS symbol-pair $(\alpha+u\beta)$-constacyclic code is the trivial code $\langle 1\rangle$. For γ-constacyclic codes there are four types of ideals. Type 1 consists of trivial ideals for which we point out that only $\langle 1\rangle$ is a MDS symbol pair code. Type 2 consists of the principal ideals generated by non monic polynomial which are of the form $\mathcal{C}_j=\langle u(x-\gamma_0)^j\rangle$. MDS symbol-pair codes for these codes can not be obtained in this case, with the Singleton bound $j=2d_{\rm sp}(\mathcal{C}_j)-p^s-4$. Type 3 is the principal ideals generated by the monic polynomials which are of the form $\mathcal{C}_j=\langle(x-\gamma_0)^j+u(x-\gamma_0)^t h(x)\rangle$, where $0\le t<j$, and either $h(x)$ is 0 or a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\gamma\rangle}$. The condition for MDS symbol-pair codes varies here and depends on both t and j. We find the constraints on s, j and t to obtain MDS symbol-pair codes for this type. Finally, Type 4 contains non-principal ideals of the form $\mathcal{C}=\langle(x-\gamma_0)^j+u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa\rangle$, where $1\le j\le p^s-1$, $0\le t<j$, and either $h(x)$ is either 0 or a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\gamma\rangle}$, $\deg(h)\le\kappa-t-1$, and

$$\kappa<T=\begin{cases} j, & \text{if } h(x)=0 \\ \min\{j, p^s-j+t\}, & \text{if } h(x)\neq 0. \end{cases}$$

MDS symbol-pair codes in this case depend on κ, which is given by $\kappa=2d_{\rm sp}(\mathcal{C}_j)-p^s+m-4$, for $1\le m\le p^s-1$. We found out that the condition for $\kappa<T$, is contradicted at every interval of κ. Thus, no MDS symbol-pair constacyclic codes can be deduced in this type. Codes satisfying the Singleton bound form an optimal class of codes with respect to symbol-pair metric, and we obtained some parameters of such codes for different types of units.

These results can be further generalized for computing MDS b-symbol constacyclic codes of length p^s over $\mathcal{R}$. Though it is presumed to give a similar conclusion, it will be interesting to observe the outcome for some new MDS constacyclic codes. Similarly, MDS symbol-pair constacyclic codes of length $2p^s$ over $\mathcal{R}$ can be computed to obtain some more optimal codes.

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