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Radio Number for Generalized Petersen Graphs $P(n, 2)$

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ABSTRACT Let G be a connected graph and $d(\mu, \omega)$ be the distance between any two vertices of G. The diameter of G is denoted by $diam(G)$ and is equal to $\max\{d(\mu, \omega);\ \mu, \omega \in G\}$. The radio labeling (RL) for the graph G is an injective function $F : V(G) \to N \cup \{0\}$ such that for any pair of vertices μ and ω $|F(\mu) - F(\omega)| \geq diam(G) - d(\mu, \omega) + 1$. The span of radio labeling is the largest number in $F(V)$. The radio number of G, denoted by $rn(G)$ is the minimum span over all radio labeling of G. In this paper, we determine radio number for the generalized Petersen graphs, $P(n, 2)$, $n = 4k+2$. Further the lower bound of radio number for $P(n, 2)$ when $n = 4k$ is determined.

INDEX TERMS Diameter, radio number, generalized Petersen graph.

I. INTRODUCTION

In graph theory, a graph labeling is the assignment of labels, generally represented by whole numbers, to edges as well as vertices of a graph [37]. Most graph labelings follow their sources to labelings exhibited by Alex Rosa in his 1967 paper [40] Rosa recognized three kinds of labelings, which he called $\alpha-$, $\beta-$, and ρ-labelings [41]. β-labelings were later renamed graceful by S. W. Golomb and the name has been mainstream since. Afterward, various kinds of graph labelings have been characterized and numerous papers have been composed on various graph labelings until now, for example [4]–[10] and the references there in.

One of the intriguing and significant graph labeling in graph theory is RL "which is spurred by the channel assignment issue presented by Hale [11]". In telecommunication system to radio network, the interference constraints between a couple of transmitters assume an indispensable job. For the transmitters of radio system, we look to allot channels with the end goal that the system satisfies all the interference constraints. The task of assigning channels to the transmitters is prevalently known as channel assignment problem which was presented by Hale [11]. For radio system on the off chance that we accept that the frequencies are uniformly distributed in the spectrum then the frequency span determine the bandwidth allocation for the assignment. For this situation, the obstruction between two transmitters is firmly related with the geographic location of the transmitters. Prior designer of radio systems considered just the two-level interference, in particular, major and minor. They arranged a couple of transmitters as close transmitters if the interference level between them is major and close transmitters if the interferene level between them is minor.

To take care of the channel assignment problem, the interference graph is created and task of channels assignment changed over into graph labeling (a graph labeling is a task of labeling every vertex as per certain standard). In interference graph, the transmitters are spoken to by the vertices, and two vertices are joined by an edge if relating transmitters have the significant (major) interference while two transmitters having minor interference are at distance 2, and there is no interference between transmitters they are at distance 3 or more. As it

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were, very close transmitters are spoken to by neighboring vertices, and close transmitters are spoken to by the vertices which are at distance two apart. Roberts [12] suggested that a couple of transmitters which has minor interference must get various channels and a couple of transmitters which has significant interference must get channels that are at least 2. Inspired through this issue Griggs and Yeh [13] presented $L(2, 1)$-labeling in which channels are connected with the nonnegative whole numbers.

Definition 1: "A distance two labeling (or $L(2, 1)$-labeling) of a graph is a function F from vertex set G to the set of nonnegative integers such that the following conditions are satisfied: (1) if $|F(\mu) - F(\omega)| \geq 2$, if $d(\mu, \omega) = 1$ (2) if $|F(\mu) - F(\omega)| \geq 1$, if $d(\mu, \omega) = 2$."

The span of F can be defined as $\max\{|F(\mu) - F(\omega)| : \mu, \omega \in V(G)\}$. The λ-number for a graph G is denoted by $\lambda(G)$, and is the minimum span of a $L(2, 1)$-labeling of G. The $L(2, 1)$-labeling has been studied by many scientists, for example Yeh [14], Sakai [15], Chang and Kuo [16], Vaidya *et al.* [17], and Vaidya and Bantva [18].

In that case, as time passed, it has been seen that the interference among transmitters may go past two levels. RL expands the number of interference level considered in $L(2, 1-$labeling from two to the biggest possible-the diameter of G. The diameter of G is represented by diam(G) or just by d is the most extreme distance among all pairs of vertices in G. Inspired through the issue of channel task of FM radio stations, Chartrand *et al.* [19] presented the idea of radio labeling of graphs as follows.

Definition 2: "A radio labeling F of G is an assignment of positive integers to the vertices of G satisfying

$$|F(\mu) - F(\omega)| \geq d + 1 - d_G(\mu, \omega), \quad \forall \mu, \omega \in V(G).$$

The radio number denoted by $rn(G)$ is the minimum span of a radio labeling for G. Note that when diam(G) is two then radio labeling and distance two labeling are identical."

The radio labeling is actually an assignment of allocating frequencies to AM/FM radio channel suggested by Chartrand *et al.* [19] in such a way that there is no disturbance in the signals received due to nearby or geographically closed radio stations.

Examining the radio number of a graph is a fascinating task. So far the radio number is known distinctly for bunch of graph families. Liu and Zhu [20] have given the radio number for paths and cycles. Liu and Xie [21], [22] additionally studied the radio labeling for square of paths and cycles while Liu [23] has given a lower bound for radio number of trees and exhibited a class of trees accomplishing the lower bound.

Notice that the development of radio system as per certain standard is comparable to stating that the extension of interference graphs by methods for explicit graph operation. The extension of existing system and to decide the radio number for the extended system is likewise a fascinating task [24]–[27]. Simultaneously, it is an essential issue to relate the radio number of existing system with the extended system. In this paper, we register the radio number for peterson graphs.

Definition 3: Let $n \geq 3$ be a positive integer and let $m \in \{1, 2, \ldots, n-1\}$. The generalized Petersen graph $P(n, m)$ has its vertex and edge set as $V(P(n, m)) = \{u_i : i \in Z_n\} \cup \{u'_i : i \in Z_n\}$ and $E(P(n, m)) = \{u_i u_{i+1} : i \in Z_n\} \cup \{u'_i u'_{i+m} : i \in Z_n\} \cup \{u_i u'_i : i \in Z_n\}$. Obviously $m \leq \lfloor \frac{n}{2} \rfloor$ because of obvious isomorphism $P(n, m) \cong P(n, n-m)$.

Peterson graphs has been largely studied in past years [29]–[31], for example, spectrum of generalized Petersen graphs has been studied in [31]. Coloring and Tutte polynomial of Peterson graphs have been given in [32] and [33] respectively. Metric dimension of some classes of Peterson graph has been computed in [34]. For more properties, we refer [35], [36]. In this paper, we aim to study radio labeling for generalized peterson graphs. The main results of this paper are:

Theorem 1: For the generalized Petersen graphs $P(n, 2)$, $n = 4k + 2$, $k \geq 3$

$$rn(P(n, 2)) = \begin{cases} \dfrac{4k^2 + 21k + 8}{2}, & \text{for even } k; \\ \dfrac{4k^2 + 25k + 9}{2}, & \text{for odd } k. \end{cases}$$

Theorem 2: For the generalized Petersen graphs $P(n, 2)$, $n = 4k$, $k \geq 5$

$$rn(P(n, 2)) \geq \begin{cases} \dfrac{4k^2 + 11k}{2}, & \text{for even } k; \\ \dfrac{4k^2 + 15k - 1}{2}, & \text{for odd } k. \end{cases}$$

Note that, for a generalized Petersen graph, $P(n, m)$ $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, the vertex set is

$$V(G) = \{\alpha_i, \beta_i : i = 1, 2, \ldots n\}$$

and the edge set is

$E(G) = \{\alpha_i \alpha_{i+1}, \beta_i \beta_{i+m}, \alpha_i \beta_i|$ with indices taken modulo $n\}$.

The following remark is useful in proving our main theorems.

Remark 1 [8]: For the generalized Petersen graphs $P(n, 2)$, $n > 6$,

$$diam(P(n, 2)) = \begin{cases} \dfrac{n}{4} + 2, & \text{if } n = 4k; \\ \dfrac{n-2}{4} + 3, & \text{if } n = 4k + 2. \end{cases}$$

II. A LOWER BOUND FOR $P(N, 2)$, $N = 4K + 2$

In this section, the lower bound for rn$P(n, 2)$ where $n = 4k+2$ is determined. Here,

$$V(G) = \{\alpha_i, \beta_i : i = 1, 2, \ldots n\}$$

and an edge set $E(G) = \{\alpha_i \alpha_{i+1}, \beta_i \beta_{i+2}, \alpha_i \beta_i|$ with indices taken modulo $n\}$.

The vertex set can be divided into two classes. The vertices that lies on inner cycle are called as interior vertices and the vertices that lies on the outer cycle are called exterior vertices.

Note that, $diam(P(n,2)) = \frac{n-2}{4} + 3 = k + 3$ when $n = 4k + 2$.

Lemma 1: Let $P(n, 2)$ be the generalized Petersen graphs for $n = 4k + 2$, then the following statements holds:

i. *For each exterior vertex α_1, there exist exactly one vertex at distance equal to diameter of $P(n, 2)$.*
ii. *For each interior vertex β_1, there exist exactly one vertex at distance equal to diameter of $P(n, 2)$.*

Proof:

i. We show that $d(\alpha_1, \alpha_{2k+2}) = k + 3$. Since $n = 4k + 2$, there are equal vertices on the left and right half of cycle. So, the path starting from α_1 to α_{2k+2} has length $k + 3$ as
$\alpha_1 \to \beta_{2(0)+1} \to \beta_{2(1)+1} \to \beta_{2(2)+1} \to \ldots\beta_{2(k)+1} \to \alpha_{2k+1} \to \alpha_{2k+2}$.
ii. $d(\beta_1, \beta_{2k+2}) = k + 3$
$\beta_1 \to \beta_{2(1)+1} \to \beta_{2(2)+1} \to \ldots\beta_{2(k)+1} \to \alpha_{2k+1} \to \alpha_{2k+2} \to \beta_{2k+2}$. ■

Lemma 2: Let α, β, γ are 3 vertices lies on the exterior cycle of $P(n, 2)$, where $n = 4k + 2$ then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d + 2.$$

Proof: By Lemma 1, $d(\alpha_1, \alpha_{2k+2}) = k + 3 = d$. Now $d(\alpha_{2k+2}, \alpha_{4k-5}) = k - 1$ and a path of length $k - 1$ between α_{2k+2} to α_{4k-5} is

$\alpha_{2k+2} \to \beta_{2(k+1)} \to \beta_{2(k+2)} \to \beta_{2(k+3)}\ldots \to \beta_{2(2k-3)} = \beta_{4k-6} \to \alpha_{4k-6} \to \alpha_{4k-5}$

and $d(\alpha_{4k-5}, \alpha_1) = 6$ because

$\alpha_{4k-5} \to \beta_{4k-5} \to \beta_{4k-3} \to \beta_{4k-1} \to \beta_{4k+1} \to \beta_{4k+3} = \beta_1 \to \alpha_1$

Therefore, $d(\alpha_1, \alpha_{2k+2}) + d(\alpha_{2k+2}, \alpha_{4k-5}) + d(\alpha_{4k-5}, \alpha_1) = (k+3)+(k-1)+6 = (k+3)+(k+3)+2 = 2d + 2$

So if α, β, γ are 3 vertices on exterior cycle of $P(n, 2)$ then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d + 2.$$

■

Lemma 3: If α, β, γ are 3 vertices lies on the interior cycles of $P(n, 2)$, $n = 4k + 2$, then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d.$$

Proof: By Lemma 1, $d(\beta_1, \beta_{2k+2}) = k + 3 = d$.

Now $d(\beta_{2k+2}, \beta_{4k-5}) = k - 1$ and a path of length $k - 1$ between β_{2k+2} to β_{4k-5} is

$\beta_{2k+2} = \beta_{2(k+1)} \to \beta_{2(k+2)} \to \beta_{2(k+3)}\ldots \to \beta_{2(2k-3)} = \beta_{4k-6} \to \alpha_{4k-6} \to \alpha_{4k-5} \to \beta_{4k-5}$

and $d(\beta_{4k-5}, \beta_1) = 4$ as

$\beta_{4k-5} \to \beta_{4k-3} \to \beta_{4k-1} \to \beta_{4k+1} \to \beta_{4k+3} = \beta_1$

Therefore, $d(\beta_1, \beta_{2k+2}) + d(\beta_{2k+2}, \beta_{4k-5}) + d(\beta_{4k-5}, \beta_1) = (k + 3) + (k - 1) + 4 = (k + 3) + (k + 3) = 2d$

Thus if α, β, γ are 3 vertices lies on interior cycles $P(n, 2)$ then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d.$$

■

Lemma 4: Let α, β, γ be three vertices of $P(n, 2)$, $n = 4k + 2$ such that 2 vertices are on the exterior cycle and 1 vertex lies on the interior cycle then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d.$$

Proof: By Lemma 1, $d(\alpha_1, \alpha_{2k+2}) = k + 3 = d$.

For any vertex β_1 that lies on the interior cycle, we have exactly 1 vertex α_{2k+2} that lies on the exterior cycle at a distance $d - 1$. i.e $d(\beta_1, \alpha_{2k+2}) = d - 1$,

Therefore, $d(\alpha_1, \alpha_{2k+2}) + d(\alpha_{2k+2}, \beta_1) + d(\beta_1, \alpha_1) = d + (d - 1) + 1 = 2d$

Thus if α, β, γ are 3 vertices such that 2 of them are at exterior cycle and 1 of them is at interior cycle of $P(n, 2)$, $n \equiv 2(mod\ 4)$, then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d.$$

■

Lemma 5: If F is a RL of $P(n, 2)$, $n = 4k+2$, $k \geq 3$. Then we have following statements:

i. *Let $\{\mu_i : 1 \leq i \leq n\}$ represents the vertex set of exterior cycle and $F(\mu_i) < F(\mu_j)$ whenever $i < j$.*
Then $|F(\mu_{i+2}) - F(\mu_i)| \geq \phi(n)$, where
$$\phi(n) = \begin{cases} \frac{k}{2} + 2, & \text{for even } k \\ \frac{k+1}{2} + 2, & \text{for odd } k. \end{cases}$$
ii. *Let $\{\omega_i : 1 \leq i \leq n\}$ is the vertex set of interior cycles and $F(\omega_i) < F(\omega_j)$ whenever $i < j$. Then $|F(\omega_{i+2}) - F(\omega_i)| \geq \psi(n)$, where*
$$\psi(n) = \begin{cases} \frac{k}{2} + 3, & \text{for even } k; \\ \frac{k+1}{2} + 3, & \text{for odd } k. \end{cases}$$

Proof:

i. Consider $\{\mu_i, \mu_{i+1}, \mu_{i+2}\}$ are any 3 vertices of exterior cycle of $P(n, 2)$, $n = 4k+2$. By applying radio condition to every pair of vertex set $\{\mu_i, \mu_{i+1}, \mu_{i+2}\}$ and take the sum of the following three inequalities.
$|F(\mu_{i+1}) - F(\mu_i)| \geq \text{diam}(G) - d(\mu_{i+1}, \mu_i) + 1$
$|F(\mu_{i+2}) - F(\mu_{i+1})| \geq \text{diam}(G) - d(\mu_{i+2}, \mu_{i+1}) + 1$
$|F(\mu_{i+2}) - F(\mu_i)| \geq \text{diam}(G) - d(\mu_{i+2}, \mu_i) + 1$
$|F(\mu_{i+1}) - F(\mu_i)| + |F(\mu_{i+2}) - F(\mu_{i+1})| + |F(\mu_{i+2}) - F(\mu_i)| \geq 3\text{diam}(G) + 3 - d(\mu_{i+1}, \mu_i) - d(\mu_{i+2}, \mu_{i+1}) - d(\mu_{i+2}, \mu_i)$
By omitting absolute sign because $F(\mu_i) < F(\mu_{i+1}) < F(\mu_{i+2})$ and by using Lemma 2, we get:
$2[F(\mu_{i+2}) - F(\mu_i)] \geq 3 + 3d - (2d + 2) = d + 1$
$[F(\mu_{i+2}) - F(\mu_i)] \geq \frac{d+1}{2} = \frac{k+3+1}{2} = \frac{k+4}{2} = \frac{k}{2} + 2$
Thus,
$$\phi(n) = \begin{cases} \frac{k}{2} + 2, & \text{for even } k; \\ \frac{k+1}{2} + 2, & \text{for odd } k. \end{cases}$$
ii. Consider $\{\omega_i, \omega_{i+1}, \omega_{i+2}\}$ are 3 vertices of exterior cycles of $P(n, 2)$, $n = 4k + 2$. By applying radio

condition to every pair in the same way as we did in above and utilizing Lemma 3, we have,
$2[F(\omega_{i+2}) - F(\omega_i)] \geq 3 + 3d - 2d = d + 3$
$[F(\mu_{i+2}) - F(\mu_i)] \geq \frac{d+3}{2} = \frac{k+3+3}{2} = \frac{k+6}{2} = \frac{k}{2} + 3$
Thus

$$\psi(n) = \begin{cases} \frac{k}{2} + 3, & \text{for even } k; \\ \frac{k+1}{2} + 3, & \text{for odd } k. \end{cases}$$

■

Theorem 3: For $P(n, 2)$, with $n = 4k + 2$ and $k \geq 3$, we have

$$rn(P(n,2)) \geq \begin{cases} \frac{4k^2 + 21k + 8}{2}, & \text{for even } k; \\ \frac{4k^2 + 25k + 9}{2}, & \text{for odd } k. \end{cases}$$

Proof: $P(n, 2)$ has $2n$ vertices. First we divide the vertex set into two classes $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$. Let F be the RL for $P(n, 2)$. We order the vertices of $P(n, 2)$ that lies on exterior cycle by $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ with $F(\mu_i) < F(\mu_{i+1})$ and the vertices that lies on the interior cycle by $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ with
$F(\omega_i) < F(\omega_{i+1})$. We have $d = \frac{n-2}{4} + 3 = k + 3$.
For $i = 1, 2, 3, \ldots n - 1$, set $d_i = d(\mu_i, \mu_{i+1})$ and $F_i = F(\mu_{i+1}) - F(\mu_i)$
Then $F_i \geq d - d_i + 1$ for all i.
By using Lemma 5(i), the span of RL F of $P(n, 2)$ for vertices of exterior cycle is given by

$$\begin{aligned} F(\mu_n) &= \sum_{i=1}^{n-1} F_i = F_1 + F_2 + F_3 + \ldots. + F_{n-2} + F_{n-1} \\ &= [F(\mu_2) - F(\mu_1)] + [F(\mu_3) - F(\mu_2)] + \ldots \\ &\quad + [F(\mu_{n-1}) - F(\mu_{n-2})] + [F(\mu_n) - F(\mu_{n-1})] \\ &= (F_1 + F_2) + (F_3 + F_4) + (F_5 + F_6) + \ldots \\ &\quad + (F_{n-3} + F_{n-2}) + F_{n-1} \\ &= \sum_{i=1}^{\frac{n-2}{2}} (F_{2i-1} + F_{2i}) + F_{n-1} \\ &\geq \frac{n-2}{2}\phi(n) + 1 \end{aligned}$$

$$F(\mu_n) \geq \begin{cases} \frac{n-2}{2}.(\frac{k}{2} + 2) + 1, & \text{for even } k; \\ \frac{n-2}{2}.(\frac{k+1}{2} + 2) + 1, & \text{for odd } k. \end{cases}$$

$$F(\mu_n) \geq \begin{cases} k^2 + 4k + 1, & \text{for even } k; \\ k^2 + 5k + 1, & \text{for odd } k. \end{cases}$$

Using Lemma 4 and Lemma 5(ii) to vertices $\mu_{n-1}, \mu_n, \omega_1$ such that
$F(\mu_{n-1}) < F(\mu_n) < F(\omega_1)$, then

$$|F(\omega_1) - F(\mu_{n-1})| \geq \begin{cases} \frac{k}{2} + 3, & \text{for even } k; \\ \frac{k+1}{2} + 3, & \text{for odd } k. \end{cases}$$

$$F(\omega_1) \geq \begin{cases} F(\mu_{n-1}) + \frac{k}{2} + 3 = k^2 + 4k + \frac{k}{2} + 3, & \text{for even } k; \\ f(\mu_{n-1}) + \frac{k+1}{2} + 3 = k^2 + 5k + \frac{k+1}{2} + 3, & \text{for odd } k. \end{cases}$$

By using lemma 5(ii), the span of RL F' of $P(n, 2)$ for the vertices that lies on interior cycles is

$$\begin{aligned} F(\omega_n) - F(\omega_1) &= \sum_{i=1}^{n-1} F'_i = (F'_1 + F'_2) + (F'_3 + F'_4) + \ldots \\ &\quad + (F'_{n-3} + F'_{n-2}) + F'_{n-1} \\ &= \sum_{i=1}^{\frac{n-2}{2}} (F'_{2i-1} + F'_{2i}) + F'_{n-1} \\ &\geq \frac{n-2}{2}\psi(n) + 1 \end{aligned}$$

$$F(\omega_n) - F(\omega_1) \geq \begin{cases} \frac{n-2}{2}.(\frac{k}{2} + 3) + 1, & \text{for even } k; \\ \frac{n-2}{2}.(\frac{k+1}{2} + 3) + 1, & \text{for odd } k. \end{cases}$$

$$F(\omega_n) \geq \begin{cases} k^2 + 6k + 1 + F(\omega_1), & \text{for even } k; \\ k^2 + 7k + 1 + f(\omega_1), & \text{for odd } k. \end{cases}$$

$$F(\omega_n) \geq \begin{cases} \frac{4k^2 + 21k + 8}{2}, & \text{for even } k; \\ \frac{4k^2 + 25k + 9}{2}, & \text{for odd } k. \end{cases}$$

Hence

$$rn(P(n,2)) \geq \begin{cases} \frac{4k^2 + 21k + 8}{2}, & \text{for even } k; \\ \frac{4k^2 + 25k + 9}{2}, & \text{for odd } k. \end{cases}$$

■

III. AN UPPER BOUND FOR $P(N, 2)$, $N = 4K + 2$

In order to complete our proof for the Theorem 1, we remain to give RL of $P(n, 2)$ having span exactly equal to our desired number. The required labeling can be generated with the help of following three sequences:

- the distance gap sequence (DGS)

$$D = (d_1, d_2, d_3, \ldots., d_{n-1})$$
$$D' = (d'_1, d'_2, d'_3, \ldots, d'_{n-1})$$

- the color gap sequence (CGS)

$$F = (f_1, f_2, f_3, \ldots., f_{n-1})$$
$$F' = (f'_1, f'_2, f'_3, \ldots, f'_{n-1})$$

- the vertex gap sequences (VGS)

$$T = (t_1, t_2, t_3, \ldots., t_{n-1})$$
$$T' = (t'_1, t'_2, t'_3, \ldots, t'_{n-1})$$

We have two cases
Case 1: When k is even.

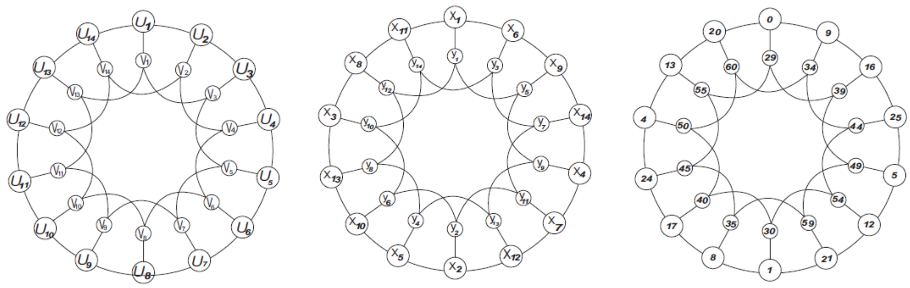

FIGURE 1. **Ordinary labeling and radio labeling for $P(14, 2)$.**

The DGS are

$$d_i = \begin{cases} k+3, & \text{for odd } i; \\ \frac{k}{2}+3, & \text{for even } i. \end{cases}$$

and

$$d'_i = \begin{cases} k+3, & \text{for odd } i; \\ \frac{k}{2}+2, & \text{for even } i. \end{cases}$$

For every i, we have
$d(\mu_i, \mu_{i+1}) = d_i$,
$d(\omega_i, \omega_{i+1}) = d'_i$
and

$$d' = d(\mu_n, \omega_1) = \frac{k}{2}+2.$$

The CGS are

$$f_i = \begin{cases} 1, & \text{for odd } i; \\ \frac{k}{2}+1, & \text{for even } i. \end{cases}$$

$$f'_i = \begin{cases} 1, & \text{for odd } i; \\ \frac{k}{2}+2, & \text{for eeven } i. \end{cases}$$

$$f' = \frac{k}{2}+2.$$

The VGS are

$$t_i = \begin{cases} 2k, & \text{for odd } i; \\ k, & \text{for even } i. \end{cases}$$

$$t'_i = \begin{cases} 2k, & \text{for odd } i; \\ k-2, & \text{for even } i. \end{cases}$$

Take t_i as the number of vertices lies between μ_i and μ_{i+1} on exterior cycle and t'_i are number of vertices lies between ω_i and ω_{i+1} on interior cycles.

Consider $\Gamma, \Gamma' : \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, n\}$ be defined as $\Gamma(1) = 1$ and $\Gamma'(1) = 1$

$$\Gamma(i+1) = \Gamma(i) + t_i + 1 (\mathrm{mod}\, n)$$
$$\Gamma'(i+1) = \Gamma'(i) + t'_i + 1 (\mathrm{mod}\, n)$$

We are to show that for every sequence given below, the corresponding Γ, Γ' are permutations.

Let $\mu_i = \alpha_{\Gamma(i)}$ for $i = 1, 2, 3, \ldots, n$
$\omega_i = \beta_{\Gamma'(i)}$ for $i = 1, 2, 3, \ldots, n$.

Then $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ is an ordering of the vertices of $P(n, 2)$ lies on exterior cycle and $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ is an ordering of the vertices of $P(n, 2)$ lies on interior cycle.

$f(\mu_1) = 0$
$f(\mu_{i+1}) = f(\mu_i) + f_i$
Then for $i = 1, 2, 3, \ldots, 2k+1$

$$\Gamma(2i-1) = (i-1)(2k+1) + (i-1)(k+1) + 1 (\mathrm{mod}\, n)$$
$$\Gamma(2i) = i(2k+1) + (i-1)(k+1) + 1 (\mathrm{mod}\, n)$$

and

$$\Gamma'(2i-1) = (i-1)(2k+1) + (i-1)(k-1) + 1 (\mathrm{mod}\, n)$$
$$\Gamma'(2i) = i(2k+1) + (i-1)(k-1) + 1 (\mathrm{mod}\, n)$$

We prove that Γ and Γ' are permutations.

Note that $g.c.d.(n, k) = 2$ and $3k+2 \equiv -k (\mathrm{mod}\, n)$ Thus, $(3k+2)(i-i') \equiv k(i'-i) \not\equiv 0 (\mathrm{mod}\, n)$, if $0 < i-i' < \frac{n}{2}$. Because if it does so then $k(i'-i) \equiv k.0 (\mathrm{mod}\, n)$ as $g.c.d.(n, k) = 2$, we have $i'-i \equiv 0 (\mathrm{mod}\, \frac{n}{2})$ which is impossible when $0 < i-i' < \frac{n}{2}$.

This implies that, $\Gamma(2i-1) \neq \Gamma(2i'-1)$ and $\Gamma(2i) \neq \Gamma(2i')$, if $i \neq i'$

If $\Gamma(2i) = \Gamma(2i'-1)$, then we get:
$i(2k+1) + (i-1)(k+1) + 1 = (i'-1)(2k+1) + (i'-1)(k+1) + 1$
$i(2k+1+k+1) = i'(2k+1+k+1) - (2k+1)$
$(i-i')(3k+2) = -2k-1 \equiv 2k+1 (mod\ n)$.
It follows that,

$$2(i'-i)k \equiv 0 (\mathrm{mod}\, n)$$

As k is even therefore, $g.c.d.(2k, n) = 2$, and

$$i'-i \equiv 0 (\mathrm{mod}\, \frac{n}{2}).$$

But this is not possible.

Now, to show Γ' is a permutation.

Since $g.c.d(n, k) = 2$ and $3k \equiv -k-2 (\mathrm{mod}\, n)$. Thus, $(i-i')3k \equiv (k+2)(i-i') \not\equiv 0 (\mathrm{mod}\, n)$ if $0 < i-i' < \frac{n}{2}$

This implies that $\Gamma(2i-1) \neq \Gamma'(2i'-1)$ and $\Gamma'(2i) \neq \Gamma'(2i'-1)$ if $i \neq i'$.

If $\Gamma'(2i) = \Gamma'(2i' - 1)$, then similarly we get:
$i(2k+1) + (i-1)(k-1) + 1 = (i'-1)(2k+1) + (i'-1)(k-1) + 1$
$i(2k+1+k-1) = i'(2k+1+k-1) - (2k+1)$
$(i-i')3k = -2k-1 \equiv 2k+1(mod\ n)$.
Thus,

$$2(k+2)(i'-i) \equiv 0(\bmod n)$$

As k is even and $g.c.d.(2k+4, n) = 2$, it follows that $i' - i \equiv 0(\bmod \frac{n}{2})$. But this is impossible.
The span of RL F is equal to

$$\begin{aligned} &F_1 + F_2 + F_3 +, \ldots, F_{n-2} + F_{n-1} + F' + F'_1 \\ &\quad + F'_2 + F'_3 +, \ldots, F'_{n-2} + F'_{n-1} \\ &= [(F_1 + F_3 + F_5 +, \ldots, +F_{n-1})] \\ &\quad + [(F_2 + F_4 + F_6 +, \ldots, +F_{n-2})] + F' \\ &\quad + [(F'_1 + F'_3 + F'_5 +, \ldots, +F'_{n-1})] \\ &\quad + [(F'_2 + F'_4 + F'_6 +, \ldots, +F'_{n-2})] \\ &= \frac{n}{2}(1) + \frac{n-2}{2}(\frac{k}{2}+1) + \frac{k}{2} + 2 \\ &\quad + \frac{n}{2}(1) + \frac{n-2}{2}(\frac{k}{2}+2) \\ &= \frac{4k^2 + 21k + 8}{2} \end{aligned}$$

Case 2: When k is odd.
The DGS are

$$d_i = \begin{cases} k+3, & \text{for odd } i; \\ \dfrac{k+1}{2} + 2, & \text{for even } i. \end{cases}$$

and

$$d'_i = \begin{cases} k+3, & \text{for odd } i; \\ \dfrac{k+1}{2} + 1, & \text{for even } i. \end{cases}$$

$$d' = d(\mu_n, \omega_1) = \frac{k+1}{2} + 1.$$

The CGS are

$$f_i = \begin{cases} 1, & \text{for odd } i; \\ \dfrac{k+1}{2} + 1, & \text{for even } i. \end{cases}$$

$$f'_i = \begin{cases} 1, & \text{for odd } i; \\ \dfrac{k+1}{2} + 2, & \text{for even } i. \end{cases}$$

$$f' = \frac{k+1}{2} + 2.$$

The VGS are

$$t_i = \begin{cases} 2k, & \text{for odd } i; \\ k-1, & \text{for even } i. \end{cases}$$

$$t'_i = \begin{cases} 2k, & \text{for odd } i; \\ k-3, & \text{for even } i. \end{cases}$$

Let $\theta, \theta' : \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, n\}$ are defined as $\Delta(1) = 1$ and $\Delta'(1) = 1$

$$\begin{aligned} \Delta(i+1) &= \Delta(i) + t_i + 1(\bmod n) \\ \Delta'(i+1) &= \Delta'(i) + t'_i + 1(\bmod n) \end{aligned}$$

Then for $i = 1, 2, 3, \ldots, 2k+1$

$$\begin{aligned} \Delta(2i-1) &= (i-1)(2k+1) + (i-1)k + 1(\bmod n) \\ \Delta(2i) &= i(2k+1) + (i-1)k + 1(\bmod n) \end{aligned}$$

and

$$\begin{aligned} \Delta'(2i-1) &= (i-1)(2k+1) + (i-1)(k-2) + 1(\bmod n) \\ \Delta'(2i) &= i(2k+1) + (i-1)(k-2) + 1(\bmod n), \end{aligned}$$

We will prove that Δ and Δ' are permutations.
Note that $g.c.d.(n, k) = 1$ and $3k+1 \equiv -k-1(\bmod n)$. Thus,
$(3k+1)(i-i') \equiv (k+1)(i'-i) \not\equiv 0(\bmod n)$ when $0 < i - i' < \frac{n}{2}$.
This implies that $\Delta(2i-1) \neq \Delta(2i'-1)$ and $\Delta(2i) \neq \Delta(2i')$ if $i \neq i'$.
If $\Delta(2i) = \Delta(2i'-1)$, then we get:
$i(2k+1) + (i-1)k + 1 = (i'-1)(2k+1) + (i'-1)k + 1$
$(i'-i)(3k+1) = -2k-1 \equiv 2k+1(\bmod n)$
$2(i'-i)(k+1) \equiv 0(\bmod n)$
Since k is odd therefore, $g.c.d.(2k+2, n) = 2$ and $i' - i \equiv 0(\bmod \frac{n}{2})$. But this is not possible.
Now, to show Δ' is a permutation

$$\begin{aligned} \Delta'(2i-1) &= (i-1)(2k+1) + (i-1)(k-2) + 1(\bmod n) \\ \Delta'(2i) &= i(2k+1) + (i-1)(k-2) + 1(\bmod n) \end{aligned}$$

Since $g.c.d(n, k) = 1$ and $3k-1 \equiv -k-3(\bmod n)$,
$(3k-1)(i-i') \equiv (k+3)(i'-i) \not\equiv 0(\bmod n)$ if $0 < i-i' < \frac{n}{2}$.
This implies that $\Delta(2i-1) \neq \Delta'(2i'-1)$ and $\Delta'(2i) \neq \Delta'(2i'-1)$ if $i \neq i'$.
However, if $\Delta'(2i) = \Delta'(2i'-1)$, then we get
$i(2k+1) + (i-1)(k-2) + 1 = (i'-1)(2k+1) + (i'-1)(k-2) + 1$
$i(2k+1+k-2) = i'(2k+1+k-2) - (2k+1)$
$(i-i')(3k-1) = -2k-1 \equiv 2k+1(\bmod n)$
$2(3k-1)(i-i') \equiv 0(\bmod n)$
$2(k+3)(i'-i) \equiv 0(\bmod n)$
Since k is odd and $g.c.d.(n, k) = 1$ it follows that $g.c.d.(2k+6, n) = 2$ and $i - i' \equiv 0(\bmod \frac{n}{2})$. But this contradicts the fact that $0 < i - i' < \frac{n}{2}$.
The span of RL F is equal to

$$\begin{aligned} &F_1 + F_2 + F_3 +, \ldots, F_{n-2} + F_{n-1} + F' + F'_1 \\ &\quad + F'_2 + F'_3 +, \ldots, F'_{n-2} + F'_{n-1} \\ &= [(F_1 + F_3 + F_5 +, \ldots, +F_{n-1})] \\ &\quad + [(F_2 + F_4 + F_6 +, \ldots, +F_{n-2})] + F' \\ &\quad + [(F'_1 + F'_3 + F'_5 +, \ldots, +F'_{n-1})] \\ &\quad + [(F'_2 + F'_4 + F'_6 +, \ldots, +F'_{n-2})] \end{aligned}$$

$$= \frac{n}{2}(1) + \frac{n-2}{2}(\frac{k+1}{2} + 1) + \frac{k+1}{2} + 2$$
$$+ \frac{n}{2}(1) + \frac{n-2}{2}(\frac{k+1}{2} + 2)$$
$$= \frac{4k^2 + 25k + 9}{2}$$

IV. A LOWER BOUND FOR $P(N, 2)$, $N = 4K$

In this section, the lower bound for radio number of $P(n, 2)$, where $n = 4k$ is determined. Here,

$$V(G) = V(P(n, 2)) = \{\alpha_i, \beta_i : i = 1, 2, ...n\}$$

and an edge set $E(G) = \{\alpha_i\alpha_{i+1}, \beta_i\beta_{i+2}, \alpha_i\beta_i|$ with indices taken modulo $n\}$.

Note that, $diam(P(n, 2)) = \frac{n}{4} + 2 = k + 2$ when $n = 4k$.

Lemma 6: Let $P(n, 2)$ be the family of generalized Petersen graphs, $n = 4k$.

i. *For every vertex α_1 that lies on exterior cycle there are only 3 vertices α_{2k}, α_{2k+1} and α_{2k+2} at a distance d of $P(n, 2)$.*
ii. *For every vertex β_1 lies on the interior cycle there are only 2 vertices β_{2k}, β_{2k+2} at a distance d of $P(n, 2)$.*

Lemma 7: Let α, β, γ are any 3 vertices that lies on exterior cycle of $P(n, 2)$, $n = 4k$ then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d + 3.$$

Lemma 8: If α, β, γ are any 3 vertices that lies on interior cycles of $P(n, 2)$, $n = 4k$ then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d + 1.$$

Lemma 9: Let α, β, γ are any 3 vertices in $P(n, 2)$, for $n = 4k$ such that 2 of them lies on exterior cycle and 1 of them lies on interior cycle, then

$$d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha) \leq 2d.$$

Lemma 10: Let F be RL of $P(n, 2)$, for $n = 4k$ and $k \geq 5$. Then we have

i. *Let $\{\mu_i : 1 \leq i \leq n\}$ is vertex set lies on exterior cycle and $F(\mu_i) < F(\mu_j)$ whenever $i < j$.*
Then $|F(\mu_{i+2}) - F(\mu_i)| \geq \phi(n)$, where

$$\phi(n) = \begin{cases} \frac{k}{2} + 1, & \text{for even } k; \\ \frac{k+1}{2} + 1, & \text{for odd } k. \end{cases}$$

ii. *Let $\{\omega_i : 1 \leq i \leq n\}$ is vertex set of interior cycles and $F(\omega_i) < F(\omega_j)$ whenever $i < j$.*
Then $|F(\omega_{i+2}) - F(\omega_i)| \geq \psi(n)$, where

$$\psi(n) = \begin{cases} \frac{k}{2} + 2, & \text{for even } k; \\ \frac{k+1}{2} + 2, & \text{for odd } k. \end{cases}$$

Proof:

i. Consider $\{\mu_i, \mu_{i+1}, \mu_{i+2}\}$ are any 3 vertices lies on exterior cycle of $P(n, 2)$ with $n = 4k$. Using the radio condition to every pair of vertex set $\{\mu_i, \mu_{i+1}, \mu_{i+2}\}$ and taking sum of three inequalities.
$|F(\mu_{i+1}) - F(\mu_i)| \geq diam(G) - d(\mu_{i+1}, \mu_i) + 1$
$|F(\mu_{i+2}) - F(\mu_{i+1})| \geq diam(G) - d(\mu_{i+2}, \mu_{i+1}) + 1$
$|F(\mu_{i+2}) - F(\mu_i)| \geq diam(G) - d(\mu_{i+2}, \mu_i) + 1$
$|F(\mu_{i+1}) - F(\mu_i)| + |F(\mu_{i+2}) - F(\mu_{i+1})| + |F(\mu_{i+2}) - F(\mu_i)| \geq 3diam(G) + 3 - d(\mu_{i+1}, \mu_i) - d(\mu_{i+2}, \mu_{i+1}) - d(\mu_{i+2}, \mu_i)$
We can omit the absolute sign, because $F(\mu_i) < F(\mu_{i+1}) < f(\mu_{i+2})$ and utilizing Lemma 7, we obtaine
$2[F(\mu_{i+2}) - F(\mu_i)] \geq 3 + 3diam(G) - (2d + 3) = d$
$[F(\mu_{i+2}) - F(\mu_i)] \geq \frac{d}{2} = \frac{k+2}{2} = \frac{k}{2} + 1$
Thus

$$\phi(n) = \begin{cases} \frac{k}{2} + 1, & \text{for even } k; \\ \frac{k+1}{2} + 1, & \text{for odd } k. \end{cases}$$

ii. Now suppose $\{\omega_i, \omega_{i+1}, \omega_{i+2}\}$ are any 3 vertices of interior cycle of $P(n, 2)$ with $n = 4k$. Using radio condition to everypair in the above manner and utilizing Lemma , we obtain
$2[F(\omega_{i+2}) - F(\omega_i)] \geq 3 + 3diam(G) - (2d + 1) = d + 2$
$[F(\mu_{i+2}) - F(\mu_i)] \geq \frac{d+2}{2} = \frac{k+4}{2} = \frac{k}{2} + 2$
Thus

$$\psi(n) = \begin{cases} \frac{k}{2} + 2, & \text{for even } k; \\ \frac{k+1}{2} + 2, & \text{for odd } k. \end{cases}$$

■

Theorem 4: For $P(n, 2)$ with $n = 4k$ and $k \geq 5$ we have

$$rn(P(n, 2)) \geq \begin{cases} \frac{4k^2 + 11k}{2}, & \text{for even } k; \\ \frac{4k^2 + 15k - 1}{2}, & \text{for odd } k. \end{cases}$$

Proof: A generalized Petersen graph has $2n$ vertices. Let us divide the set of vertices into two subsets $\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ and $\{\beta_1, \beta_2, \beta_3, ..., \beta_n\}$. Suppose F is a distance labeling for $P(n, 2)$. We order the vertices of $P(n, 2)$ on the outer cycle by $\mu_1, \mu_2, \mu_3, ..., \mu_n$ with $F(\mu_i) < F(\mu_{i+1})$ and the vertices on the inner cycles by $\omega_1, \omega_2, \omega_3, ..., \omega_n$ with

$F(\omega_i) < F(\omega_{i+1})$. Denote the $diam(P(n, 2))$ by d, then $d = k + 2$.

For $i = 1, 2, 3, ...n - 1$, set $d_i = d(\mu_i, \mu_{i+1})$ and $F_i = F(\mu_{i+1}) - F(\mu_i)$

Then $F_i \geq d - d_i + 1$ for all i.

By Lemma 10(i), the span of a distance labeling F of $P(n, 2)$ for the vertices on the outer cycle is

$$F(\mu_n) = \sum_{i=1}^{n-1} F_i = F_1 + F_2 + F_3 + ... + F_{n-2} + F_{n-1}$$
$$= [F(\mu_2) - F(\mu_1)] + [F(\mu_3) - F(\mu_2)] + ...$$
$$+ [F(\mu_{n-1}) - F(\mu_{n-2})] + [F(\mu_n) - F(\mu_{n-1})]$$
$$= (F_1 + F_2) + (F_3 + F_4) + (F_4 + F_5) + ...$$

$$+(F_{n-3}+F_{n-2})+F_{n-1}$$
$$=\sum_{i=1}^{\frac{n-2}{2}}(F_{2i-1}+F_{2i})+F_{n-1}$$
$$\geq \frac{n-2}{2}\phi(n)+1$$
$$F(\mu_n)\geq \begin{cases} \frac{n-2}{2}.(\frac{k}{2}+1)+1, & \text{for even } k; \\ \frac{n-2}{2}.(\frac{k+1}{2}+1)+1, & \text{for odd } k. \end{cases}$$
$$F(\mu_n)\geq \begin{cases} \frac{2k^2+3k}{2}, & \text{for even } k; \\ \frac{2k^2+5k-1}{2}, & \text{for odd } k. \end{cases}$$

Applying Lemma IV and Lemma 10(ii) to the vertices μ_{n-1}, μ_n, ω_1 such that

$F(\mu_{n-1}) < F(\mu_n) < F(\omega_1)$, then we have:

$$|F(\omega_1)-F(\mu_{n-1})| \geq \begin{cases} \frac{k}{2}+2, & \text{for even } k; \\ \frac{k+1}{2}+2, & \text{for odd } k. \end{cases}$$
$$F(\omega_1) \geq \begin{cases} F(\mu_{n-1})+\frac{k}{2}+2=k^2+2k+1, & \text{for even } k; \\ F(\mu_{n-1})+\frac{k+1}{2}+2=k^2+3k+1, & \text{for odd } k. \end{cases}$$

By Lemma 10(ii), the span of distance labeling f of $P(n, 2)$ for the vertices on the inner cycles is

$$F(\omega_n)-F(\omega_1)=\sum_{i=1}^{n-1}F_i=(F_1+F_2)+(F_2+F_3)$$
$$+\ldots+(F_{n-3}+F_{n-2})+F_{n-1}$$
$$F(\omega_n)-F(\omega_1)=\sum_{i=1}^{\frac{n-2}{2}}F_{2i-1}+F_{2i})+F_{n-1}$$
$$\geq \frac{n-2}{2}\phi(n)+1$$
$$F(\omega_n)-F(\omega_1)\geq \begin{cases} \frac{n-2}{2}.(\frac{k}{2}+2)+1, & \text{for even } k; \\ \frac{n-2}{2}.(\frac{k+1}{2}+2)+1, & \text{for odd } k. \end{cases}$$
$$F(\omega_n)\geq \begin{cases} \frac{2k^2+7k-2}{2}+F(\omega_1), & \text{for even } k; \\ \frac{2k^2+9k-3}{2}+F(\omega_1), & \text{for odd } k. \end{cases}$$
$$F(\omega_n)\geq \begin{cases} \frac{4k^2+11k}{2}, & \text{for even } k; \\ \frac{4k^2+15k-1}{2}, & \text{for odd } k. \end{cases}$$

Hence

$$rn(P(n,2))\geq \begin{cases} \frac{4k^2+11k}{2}, & \text{for even } k; \\ \frac{4k^2+15k-1}{2}, & \text{for odd } k. \end{cases}$$

■

V. CONCLUSION

The radio range is the part of the electromagnetic range with frequencies from 3 Hz to 30000 GHz (3 THz). Electromagnetic waves in this recurrence extend, called radio waves, are amazingly generally utilized in current innovation, especially in media transmission. To forestall interference between various users, RL is brisk alter in this course on the grounds that the level of interference. Very few graphs have been proved to have RL and achieve the radio number. In this paper, we have investigated the values of radio number for Peterson graphs [28]–[30]. Graph labeling has many applications in coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management.

RESEARCH QUESTIONS

It is an important problem to determine Radio labeling and radio number for different families of graphs. Radio number of only few families of graph is known. The interesting researchers can compute the radio number of the families of graphs studied in [37]–[41].

REFERENCES

[1] E. W. Weisstein. *Labeled Graph*. MathWorld. Accessed: Sep. 17, 2019. [Online]. Available: http://mathworld.wolfram.com/LabeledGraph.html

[2] J. A. Gallian, "A dynamic survey of graph labelings," *Electron. J. Combinatorics*, vol. 16, pp. 1–219, Jan. 2009.

[3] A. Rosa, "On certain valuations of the vertices of a graph," in *Proc. Theory Graphs Int. Symp.*, Rome, Italy, Jul. 1966, pp. 349–355.

[4] M. A. Umar, N. Ali, A. Tabassum, and B. R. Ali, "Book graphs are cycle antimagic," *Open J. Math. Sci.*, vol. 3, no. 1, pp. 184–190, 2019.

[5] M. A. Umar, "Cyclic-antimagic construction of ladders," *Eng. Appl. Sci. Lett.*, vol. 2, no. 2, pp. 43–47, 2019.

[6] M. A. Umar, M. A. Javed, M. Hussain, and B. R. Ali, "Super(a,d)-C_4-antimagicness of book graphs," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 115–121, 2018.

[7] H. Nagesh and M. Mahesh Kumar, "Block digraph of a directed graph," *Open J. Math. Sci.*, vol. 2, no. 3, pp. 202–208, Jun. 2018.

[8] F. Aslam, Z. Zahid, and S. Zafar, "3-total edge mean cordial labeling of some standard graphs," *Open J. Math. Sci.*, vol. 3, no. 1, pp. 129–138, 2019.

[9] W. Gao, "Remarks on fractional locally harmonious coloring," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 301–306, 2018.

[10] N. Ali, M. A. Umar, A. Tabassum, and A. Raheem, "SUPER(a, d)-C_3-antimagicness of a corona graph," *Open J. Math. sci.*, vol. 2, no. 1, pp. 371–378, 2018.

[11] W. K. Hale, "Frequency assignment: Theory and applications," *Proc. IEEE*, vol. 68, no. 12, pp. 1497–1514, Dec. 1980.

[12] F. S. Roberts, "T-colorings of graphs: Recent results and open problems," *Discrete Math.*, vol. 93, nos. 2–3, pp. 229–245, 1991.

[13] J. R. Griggs and R. K. Yeh, "Labelling graphs with a condition at distance 2," *SIAM J. Discrete Math.*, vol. 5, no. 4, pp. 586–595, 1992.

[14] R. K. Yeh, "A survey on labeling graphs with a condition at distance two," *Discrete Math.*, vol. 306, no. 12, pp. 1217–1231, 2006.

[15] D. Sakai, "Labeling chordal graphs: Distance two condition," *SIAM J. Discrete Math.*, vol. 7, no. 1, pp. 133–140, 1994.

[16] G. J. Chang and D. Kuo, "The $L(2, 1)$-labeling problem on graphs," *SIAM J. Discrete Math.*, vol. 9, no. 2, pp. 309–316, 1996.

[17] S. Vaidya, P. Vihol, N. Dani, and D. Bantva, "L(2,1)-labeling in the context of some graph operations," *J. Math. Res.*, vol. 2, no. 3, pp. 109–119, 2010.

[18] S. K. Vaidya and D. D. Bantva, "Labeling cacti with a condition at distance two," *Le Matematiche*, vol. 66, no. 1, pp. 29–36, 2011.

[19] G. Chartrand, D. Erwin, P. Zhang, and F. Harary, "Radio labelings of graphs," *Bull. Inst. Combinatorics Appl.*, vol. 33, pp. 77–85, Sep. 2001.

[20] D. D.-F. Liu and X. Zhu, "Multilevel distance labelings for paths and cycles," *SIAM J. Discrete Math.*, vol. 19, no. 3, pp. 610–621, 2005.

[21] D. D.-F. Liu and M. Xie, "Radio number for square of cycles," *Congressus Numerantium*, vol. 169, pp. 105–125, Jan. 2004.
[22] D. D.-F. Liu and M. Xie, "Radio number for square paths," *Ars Combinatoria-Waterloo Winnipeg*, vol. 90, pp. 307–319, Jan. 2009.
[23] D. D.-F. Liu, "Radio number for trees," *Discrete Math.*, vol. 308, no. 7, pp. 1153–1164, Apr. 2008.
[24] D. D.-F. Liu and X. Zhu, "Multilevel distance labeling for paths and cycles," *SIAM J. Discrete Math.*, vol. 19, no. 3, pp. 610–620, 2009.
[25] H. Wang, X. Xu, Y. Yang, B. Zhang, M. Luo, and G. Wang, "Radio number of ladder graphs," *Int. J. Comput. Math.*, vol. 88, no. 10, pp. 2026–2034, 2011.
[26] I. Kousar, "Dimension theory of graphs," Ph.D. dissertation, FAST NUCES Lahore Campus, Lahore, Pakistan, 2012.
[27] P. Martinez, J. Ortiz, M. Tomova, and C. Wyels, "Radio numbers for generalized prism graphs," *Discussiones Mathematicae Graph Theory*, vol. 31, no. 1, pp. 45–62, 2011.
[28] Z. Shao, H. Jiang, P. Wu, S. Wang, J. Žerovnik, X. Zhang, and J.-B. Liu, "On 2-rainbow domination of generalized Petersen graphs," *Discrete Appl. Math.*, vol. 257, pp. 370–384, Mar. 2019.
[29] H. Jiang, P. Wu, Z. Shao, Y. Rao, and J.-B. Liu, "The double roman domination numbers of generalized Petersen graphs P(n, 2)," *Mathematics*, vol. 6, p. 206, Oct. 2018. doi: 10.3390/math6100206.
[30] E. Zhu, Z. Li, Z. Shao, J. Xu, and C. Liu, "Acyclic 3-coloring of generalized Petersen graphs," *J. Combinat. Optim.*, vol. 31, no. 2, pp. 902–911, Feb. 2016.
[31] R. Gera and P. Stănică, "The spectrum of generalized Petersen graphs," *Australas. J. Combinatorics*, vol. 49, pp. 39–45, Sep. 2010.
[32] M. Alaeiyan and H. Karami, "Perfect 2-colorings of the generalized Petersen graph," *Proc.-Math. Sci.*, vol. 126, no. 3, pp. 289–294, 2016.
[33] J. S. Kuhl, "The Tutte polynomial and the generalized Petersen graph," *Australas. J. Combinatorics*, vol. 40, pp. 87–97, 2008. [Online]. Available: https://ajc.maths.uq.edu.au/pdf/40/ajc_v40_p087.pdf
[34] Z. Shao, S. M. Sheikholeslami, P. Wu, and J.-B. Liu, "The metric dimension of some generalized Petersen graphs," *Discrete Dyn. Nature Soc.*, vol. 2018, Aug. 2018, Art. no. 4531958.
[35] R. Frucht, J. E. Graver, and M. E. Watkins, "The groups of the generalized Petersen graphs," *Math. Proc. Cambridge Philos. Soc.*, vol. 70, no. 2, pp. 211–218, Sep. 1971.
[36] K. Bannai, "Hamiltonian cycles in generalized Petersen graphs," *J. Combinat. Theory, B*, vol. 24, no. 2, pp. 181–188, 1978.
[37] Z. Shao, A. U. R. Virk, M. S. Javed, M. A. Rehman, and M. R. Farahani, "Degree based graph invariants for the molecular graph of Bismuth Tri-Iodide," *Eng. Appl. Sci. Lett.*, vol. 2, no. 1, pp. 1–11, Dec. 2019.
[38] M. Imran, A. Asghar, and A. Q. Baig, "On graph invariants of oxide network," *Eng. Appl. Sci. Lett.*, vol. 1, no. 1, pp. 23–28, 2018.
[39] W. Gao, M. Asif, and W. Nazeer, "The study of honey comb derived network via topological indices," *Open J. Math. Anal.*, vol. 2, no. 2, pp. 10–26, 2018.
[40] S. Noreen and A. Mahmood, "Zagreb polynomials and redefined Zagreb indices for the line graph of carbon nanocones," *Open J. Math. Anal.*, vol. 2, no. 1, pp. 66–73, 2018.
[41] H. Siddiqui and M. R. Farahani, "Forgotten polynomial and forgotten index of certain interconnection networks," *Open J. Math. Anal.*, vol. 1, no. 1, pp. 44–59, 2017.

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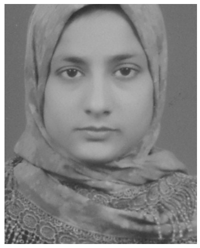

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