Received August 6, 2019, accepted September 2, 2019, date of publication September 24, 2019, date of current version October 9, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2943348

Multi-Attribute Group Decision Making Method Based on EDAS Under Picture Fuzzy Environment

XIA LI[1], YANBING JU[1], DAWEI JU[2], WENKAI ZHANG[3], PEIWU DONG[1], AND AIHUA WANG[4]
[1]School of Management and Economics, Beijing Institute of Technology, Beijing 100081, China
[2]Beijing Engineering Research Center of Post Intelligent Equipment, Postal Scientific Research and Planning Academy, Beijing 100096, China
[3]School of Economics and Management, China University of Geosciences, Beijing 100083, China
[4]Graduate School of Education, Peking University, Beijing 100871, China

Corresponding author: Yanbing Ju (juyb@bit.edu.cn)

This work was supported in part by the Natural Science Foundation of China under Grant 71873015 and Grant 71401012, in part by the Program for New Century Excellent Talents in the University of China under Grant NCET-13-0037, and in part by the Humanities and Social Sciences Foundation of the Ministry of Education of China under Grant 14YJA630019.

ABSTRACT Multi-attribute group decision making (MAGDM) is one of the most important research hotspots in the field of decision sciences. Many practical problems are often characterized by MAGDM. The aim of this paper is to develop a new approach for MAGDM problems, in which the attribute values take the form of picture fuzzy information, and the information about the weights of attributes and decision makers is unknown. Firstly, some picture fuzzy interaction operators are presented, such as the picture fuzzy weighted interaction averaging (PFWIA) operator, picture fuzzy ordered weighted interaction averaging (PFOWIA) operator and picture fuzzy hybrid ordered weighted interaction averaging (PFHOWIA) operator. In the meantime, some desirable properties of these operators are discussed in detail. Secondly, to get reasonable decision result, we propose a method to determine the weights of decision makers under picture fuzzy setting based on the idea of the Dice similarity measure. Thirdly, for the situations where the information about the attribute weights is partly known, we establish an optimization model to determine the attribute weights on the basis of the maximizing deviation method. Fourthly, we propose a new method to solve MAGDM problems by extending the traditional Evaluation based on Distance from Average Solution (EDAS) method. Finally, an illustrative example is given to demonstrate the calculation process of the proposed method, and the method is verified by comparing the evaluation result with that of two existing methods.

INDEX TERMS Evaluation based on distance from average solution (EDAS), multi-attribute group decision making (MAGDM), picture fuzzy numbers (PFNs), picture fuzzy interaction aggregation operators, maximizing deviation method, dice similarity measure.

I. INTRODUCTION

Multi-attribute group decision making (MAGDM) is an important branch of decision theory, which has been widely used in many fields [1]–[17]. In many cases, decision makers are puzzled when giving a reasonable result, because the decision process involves identifying multiple criteria and evaluating multiple alternatives. Moreover, in the practical decision process, due to the ambiguity as well as intangibility arising from human qualitative judgments, experts' opinions could involve more types of answers: yes, abstain, no and refusal, which cannot be accurately expressed by crisp values, and even cannot be described by the fuzzy set theory [18] or intuitionistic fuzzy set theory [19]. Recently, Cuong and Kreinovich [20] proposed the picture fuzzy set (PFS) method and investigated some basic operations and properties of PFS. The picture fuzzy set method is characterized by three functions expressing the degree of positive membership, the degree of neutral membership and the degree of negative membership. Since its appearance, picture fuzzy set has received more and more attention from researchers. Singh [21] proposed correlation coefficients for picture fuzzy sets which consider the degree of positive membership, degree of neutral membership, degree of negative membership and the degree of refusal membership, and applied the correlation coefficients to clustering analysis under picture fuzzy environment. Son [22] presented a novel distributed picture fuzzy clustering method on picture fuzzy sets and

The associate editor coordinating the review of this manuscript and approving it for publication was Alba Amato.

developed two novel hybrid forecast methods based on picture fuzzy clustering method. Thong and Son [23] proposed automatic picture fuzzy clustering method for determining the most suitable number of clusters through combining Particle Swarm Optimization (PSO) algorithm and fuzzy C-means under picture fuzzy environment. Wei [24] defined the cross entropy of picture fuzzy sets and utilized the picture fuzzy weighted cross entropy between the feasible alternatives and the ideal alternative to select the most desirable alternative(s). Wei *et al.* [25] established the projection model with picture fuzzy information to measure the similarity degrees between the feasible alternatives and the ideal one. Son [26] defined a generalized picture distance measure and integrated it to a novel hierarchical picture fuzzy clustering method. Wei [27] developed some picture fuzzy aggregation operators and applied these operators to solve multi-attribute decision making problems. Thong and Son [28] presented an effective hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems and applied this model to deal with medical diagnosis problem. Nie *et al.* [29] advanced a shareholder voting method for proxy advisory firm selection based on 2-tuple linguistic picture preference relation. Wang *et al.* [30] introduced some new operational rules for picture fuzzy numbers (PFNs) and proposed geometric aggregation operators based on the operational laws and applied these operators to deal with MAGDM problems. Ashraf *et al.* [31] proposed some aggregation operators for PFNs and applied these operators to solve MAGDM problem.

Information aggregation operators have attracted wide attention of researchers and have become an important research topic in the research fields of MAGDM problem. Over the past few years, to aggregate the individual preference information into a collective one or obtain the overall evaluation value of each alternative on all attributes, various aggregation operators have been proposed, such as ordered weighted averaging (OWA) operator [32], generalized OWA aggregation operators [33], generalized hybrid aggregation operators [34], and so on. For picture fuzzy information, some aggregation operators have been proposed by some researchers, such as fuzzy logic operators [35], picture fuzzy aggregation operators and picture 2-tuple linguistic aggregation operators [27], [36]. However, there is still short of appropriate aggregation operators to integrate picture fuzzy information. Therefore, an important objective of this paper is to develop some new aggregation operators to integrate picture fuzzy information and then to provide an effective tool to deal with MAGDM problem.

For the traditional multi-attribute decision making (MADM) methods, such as TOPSIS and VIKOR [37], the best alternative is got by computing the distance from ideal and nadir solutions. The desirable alternative has lower distance from ideal solution and higher distance from nadir solution in these MADM methods. Evaluation based on Distance from Average Solution (EDAS), originally proposed by Ghorabaee *et al.* [38], is a novel MADM method. It is very useful for us to deal with MAGDM problem with conflict parameters. Ghorabaee *et al.* [39] extended the EDAS method to supplier selection. Peng and Liu [40] proposed an algorithm to solve single-valued neutrosophic soft decision making problem by EDAS. Ecer [41] used fuzzy AHP to calculate the priority weights of each criteria and employed the EDAS to achieve the final ranking of third-party logistics providers. As far as we know, however, the study of the MAGDM problem based on EDAS method have not been applied under picture fuzzy environment in the existing academic literature. Hence, it is an interesting research topic to extend the traditional EDAS method to solve MAGDM problem under picture fuzzy environment.

Motivated by the advantages of EADS method and picture fuzzy set, this paper extends the traditional EDAS method to solve MAGDM problem with unknown weights of attributes and decision makers under picture fuzzy environment. First, we develop some operators to aggregate picture fuzzy information. Then, a method is proposed to determine the weights of decision makers and an optimization model is established to determine the weights of attributes. Finally, a novel approach is developed to solve MAGDM problem under picture fuzzy environment based on the idea of the EADS method. To do so, the remainder of the paper is organized as follows. Section II briefly reviews some related basic concepts. Section III proposes some aggregation operators, including picture fuzzy weighted interaction averaging (PFWIA) operator, picture fuzzy ordered weighted interaction averaging (PFOWIA) operator and picture fuzzy hybrid ordered weighted interaction averaging (PFHOWIA) operator. In Section IV, a new method is given to determine the weights of decision makers based on the idea of the Dice similarity measure, an optimization model is established to determine the weights of attributes on the basis of the maximizing deviation method, and further the EDAS method is extended to rank alternatives. Section V provides a numerical example and comparison analysis between the proposed method in this paper and other existing methods. The paper is concluded in Section VI.

II. PRELIMINARIES

A. PICTURE FUZZY SETS

Cuong and Kreinovich [20] introduced the concept of picture fuzzy set, which is a generalization of the intuitionistic fuzzy set.

Definition 1 [20]: A picture fuzzy set (PFS) A on the universe X is an object of the form

$$A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x) > | x \in X\} \tag{1}$$

where $\mu_A(x) \in [0, 1]$ is the degree of positive membership of A, $\eta_A(x) \in [0, 1]$ is the degree of neutral membership of A, $\nu_A(x) \in [0, 1]$ is the degree of negative membership of A, and $\mu_A(x), \eta_A(x), \nu_A(x)$ satisfy the following condition: $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X$. For $x \in X$, $\pi_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called the degree of refusal membership of x in A. For convenience, we call

$a_a = < \mu_a, \eta_a, \nu_a >$ a picture fuzzy number (PFN), where $\mu_a \in [0, 1], \eta_a \in [0, 1], \nu_a \in [0, 1], 0 \le \mu_a + \eta_a + \nu_a \le 1$.

Definition 2 [20]: Suppose that A and B be two PFSs on universe X, the inclusion, union, intersection and complement operations are defined as follows:

$$(1) A \subseteq B, \text{ if } \mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x) \text{ and } \nu_A(x) \le \nu_B(x), \forall x \in X;$$
$$(2) A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) \,|x \in X\};$$
$$(3) A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) \,|x \in X\};$$
$$(4) \tilde{A} = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,|x \in X\}.$$

Definition 3 [20]: Let $a = < u_a, \eta_a, \nu_a >$ be a picture fuzzy number, and then the score function $S(a)$ and accuracy function $H(a)$ of the PFN are defined as follows:

$$S(a) = (\mu_a + 1 - \eta_a + 1 - \nu_a)/3 \tag{2}$$
$$H(a) = \mu_a - \nu_a \tag{3}$$

The larger the score value of $S(a)$, the greater the PFN a. And the larger the score value of $H(a)$, the greater the PFN a.

Definition 4 [20]: Let $a_i = < \mu_{a_i}, \eta_{a_i}, \nu_{a_i} >$ and $b_i = < \mu_{b_i}, \eta_{b_i}, \nu_{b_i} >, (i = 1, 2, \ldots, n)$ are two picture fuzzy sets, then the distance between the two picture fuzzy sets a and b is defined as follows:

$$d(a, b) = \left(\frac{1}{n}\sum_{i=1}^{n}((\mu_{a_i} - \mu_{b_i})^p + (\eta_{a_i} - \eta_{b_i})^p + (\nu_{a_i} - \nu_{b_i})^p)\right)^{1/p} \tag{4}$$

In this paper, we take $p = 2$ to calculate the distance between picture fuzzy sets.

B. THE EDAS METHOD

Evaluation based on Distance from Average Solution (EDAS) is a novel MADM method, which is suitable for solving MADM problem with conflicting parameters. Let $A = \{A_1, A_2, \ldots, A_m\} (m \ge 2)$ be a finite set of alternatives, $C = \{C_1, C_2, \ldots, C_n\} (n \ge 2)$ be a finite set of attributes, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the attributes, where $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. The detailed steps of the traditional EDAS method are given as follows [38]:

Step 1: Select suitable attributes that describe alternatives.

Step 2: Construct the decision making matrix X, shown as follows:

$$X = (x_{ij})_{m\times n}$$

where x_{ij} is the evaluation information of the alternative A_i with respect the attribute C_j.

Step 3: Determine the average solution with respect to each attribute, which is shown as follows:

$$AV_j = \frac{1}{n}\sum\nolimits_{i=1}^{n} x_{ij}, \quad j = 1, 2, \ldots n$$

Step 4: Construct the positive distance matrix $PDA = (PDA_{ij})_{m\times n}$ from the average solution and the negative distance matrix $NDA = (NDA_{ij})_{m\times n}$ from the average solution according to the type of attribute. If the jth attribute is beneficial,

$$PDA_{ij} = \frac{\max(0, (x_{ij} - AV_j))}{AV_j}, \quad NDA_{ij} = \frac{\max(0, (AV_j - x_{ij}))}{AV_j},$$

and if the jth attribute is cost,

$$PDA_{ij} = \frac{\max(0, (AV_j - x_{ij}))}{AV_j}, \quad NDA_{ij} = \frac{\max(0, (x_{ij} - AV_j))}{AV_j},$$

where PDA_{ij} and NDA_{ij} denote the positive and negative distance of the ith alternative from average solution in terms of the jth attribute, respectively.

Step 5: Determine the weight sum of PDA and NDA for all alternatives, which is shown as follows:

$$SP_i = \sum_{j=1}^{n} w_j PDA_{ij}, \quad SN_i = \sum_{j=1}^{n} w_j NDA_{ij}, \ i = 1, 2, \ldots, m,$$

where w_j is the weight of the jth attribute.

Step 6: Normalize the values of SP_i and SN_i for all alternatives.

$$NSP_i = \frac{SP_i}{\max\limits_{1\le i\le m}\{SP_i\}}, \quad NSN_i = 1 - \frac{SN_i}{\max\limits_{1\le i\le m}\{SN_i\}}, \quad i = 1, 2, \ldots, m.$$

Step 7: Calculate the appraisal score for all alternatives.

$$AS_i = \frac{1}{2}(NSP_i + NSN_i), \quad i = 1, 2, \ldots, m. \tag{5}$$

Step 8: Rank all the alternatives according to the decreasing values of appraisal score, and select the most desirable alternative(s).

III. THE INTERACTION OPERATIONAL LAWS AND PICTURE FUZZY INTERACTION AGGREGATION OPERATORS

A. THE INTERACTION OPERATIONAL LAWS OF PICTURE FUZZY NUMBERS

In this section, we use some novel operational laws on picture fuzzy numbers to develop aggregation operators. The existing operational laws of picture fuzzy information and aggregation operators in [27] suffer from serious drawbacks. The aggregation results usually conflicts with the constraint that the sum of the three degree must not exceed 1. At the same time, it is found that the operational laws and geometric aggregation operators on picture fuzzy sets in [27] are not suitable to be used in the special circumstances. For example, if $a_i = < \mu_i, \eta_i, \nu_i >, (i = 1, 2, \ldots, n)$ are a collection of picture fuzzy numbers, where $a_k = < \mu_k, \eta_k, \nu_k >$ is one the element of $a_i (i = 1, 2, \ldots, n)$ with $\mu_k = 0$. Then we have $u_{PFGA}(a_1, a_2, \ldots, a_n) = 0$ by using the operation laws in [27]. It is obvious that the elements $a_i (i \ne k)$ have no effects on the aggregation result.

Inspired by the interaction operational laws in [42], [43] and the view of probability, we adopt the operational laws of picture fuzzy numbers in [44], [45]. The feature of the operational laws in [44], [45] is that the interactions are taken into consideration among positive membership, neutral membership, negative membership and refusal membership values of picture fuzzy numbers.

Definition 5 [44], [45]: Let $a =< u_a, \eta_a, v_a >$ and $b =< u_b, \eta_b, v_b >$ be two PFNs, and $\lambda > 0$, then

$$(1) a \oplus b =< 1-(1-u_a)(1-u_b), (1-u_a)(1-u_b)-(1-u_a-\eta_a)(1-u_b-\eta_b), (1-u_a-\eta_a)(1-u_b-\eta_b)-(1-u_a-\eta_a-v_a)(1-u_b-\eta_b-v_b) > .$$

$$(2) a \otimes b = < (1-v_a-\eta_a)(1-v_b-\eta_b)-(1-v_a-\eta_a-u_a)(1-v_b-\eta_b-u_b), (1-v_a)(1-v_b)-(1-v_a-\eta_a)(1-v_b-\eta_b), 1-(1-v_a)(1-v_b) > .$$

$$(3) \lambda a =< 1-(1-\mu_a)^\lambda, (1-\mu_a)^\lambda-(1-\mu_a-\eta_a)^\lambda, (1-\mu_a-\eta_a)^\lambda-(1-\mu_a-\eta_a-v_a)^\lambda > .$$

$$(4) (a)^\lambda =< (1-v_a-\eta_a)^\lambda-(1-v_a-\eta_a-\mu_a)^\lambda, (1-v_a)^\lambda-(1-v_a-\eta_a)^\lambda, 1-(1-v_a)^\lambda > .$$

Example 1: Let $a =< 0.3, 0.2, 0.4 >$ and $b =< 0.4, 0.2, 0.3 >$ be two PFNs and $\lambda = 0.5$. Now we calculate the picture numbers by using the operational laws in Definition 5, then we obtain as follows:

$$(1) a \oplus b =< 1-(1-0.3)(1-0.4), (1-0.3)(1-0.4)-(1-0.3-0.2)(1-0.4-0.2), (1-0.3-0.2)(1-0.4-0.2)-(1-0.3-0.2-0.4)(1-0.4-0.2-0.3) > = < 0.58, 0.22, 0.19 > .$$

$$(2) a \otimes b = < (1-0.4-0.2)(1-0.3-0.2)-(1-0.4-0.2-0.3)(1-0.3-0.2-0.4), (1-0.4)(1-0.3)-(1-0.4-0.2)(1-0.3-0.2), 1-(1-0.4)(1-0.3) > = < 0.19, 0.22, 0.58 > .$$

$$(3) \lambda a =< 1-(1-0.3)^{0.5}, (1-0.3)^{0.5}-(1-0.3-0.2)^{0.5}, (1-0.3-0.2)^{0.5}-(1-0.3-0.2-0.4)^{0.5} > = < 0.16, 0.13, 0.39 > .$$

$$(4) (a)^\lambda =< (1-0.4-0.2)^{0.5}-(1-0.4-0.2-0.3)^{0.5}, (1-0.4)^{0.5}-(1-0.4-0.2)^{0.5}, 1-(1-0.4)^{0.5} > = < 0.32, 0.14, 0.23 > .$$

Theorem 1: Let $a =< u_a, \eta_a, v_a >, b =< u_b, \eta_b, v_b >$ and $c =< u_c, \eta_c, v_c >$ be three PFNs, then

(1) $a \oplus b = b \oplus a$;
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(3) $\lambda_1 a \oplus \lambda_2 a = (\lambda_1 + \lambda_2)a$;
(4) $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.

Proof: According to Definition 5, we can easily get (1), so the proof of (1) is omitted here. In what follows, we just give the proofs of (2), (3) and (4).

(2) According to the operational laws in Definition 5, we obtain

$$a \oplus (b \oplus c) =< 1-(1-u_a)(1-u_{b\oplus c}), (1-u_a)(1-u_{b\oplus c})-(1-u_a-\eta_{b\oplus c})(1-u_b-\eta_{b\oplus c}), (1-u_a-\eta_a)(1-u_{b\oplus c}-\eta_{b\oplus c})- (1-u_a-\eta_a-v_a)(1-u_{b\oplus c}-\eta_{b\oplus c}-v_{b\oplus c}) >,$$

where $u_{b\oplus c} = 1-(1-u_a)(1-u_b)$, $\eta_{b\oplus c} = (1-u_a)(1-u_b)-(1-u_a-\eta_a)(1-u_b-\eta_b)$, $v_{b\oplus c} = (1-u_a-\eta_a)(1-u_b-\eta_b)-(1-u_a-\eta_a-v_a)(1-u_b-\eta_b-v_b)$.

Then, we have

$$a \oplus (b \oplus c) =< 1-(1-u_a)(1-u_b)(1-u_c), (1-u_a)(1-u_b)(1-u_c)-(1-u_a-\eta_a)(1-u_b-\eta_b)(1-u_c-\eta_c), (1-u_a-\eta_a)(1-u_b-\eta_b)(1-u_c-\eta_c)- (1-u_a-\eta_a-v_a)(1-u_b-\eta_b-v_b)(1-u_c-\eta_c-v_c) > .$$

Similarly, we have

$$(a \oplus b) \oplus c =< 1-(1-u_a)(1-u_b)(1-u_c), (1-u_a)(1-u_b)(1-u_c)-(1-u_a-\eta_a)(1-u_b-\eta_b)(1-u_c-\eta_c), (1-u_a-\eta_a)(1-u_b-\eta_b)(1-u_c-\eta_c)- (1-u_a-\eta_a-v_a)(1-u_b-\eta_b-v_b)(1-u_c-\eta_c-v_c) > .$$

Therefore, the expression $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ holds.

(3) According to the operational laws in Definition 5, we obtain

$$\lambda_1 a = < 1-(1-\mu_a)^{\lambda_1}, (1-\mu_a)^{\lambda_1}-(1-\mu_a-\eta_a)^{\lambda_1}, (1-\mu_a-\eta_a)^{\lambda_1}-(1-\mu_a-\eta_a-v_a)^{\lambda_1} >,$$
$$\lambda_2 a = < 1-(1-\mu_a)^{\lambda_2}, (1-\mu_a)^{\lambda_2}-(1-\mu_a-\eta_a)^{\lambda_2}, (1-\mu_a-\eta_a)^{\lambda_2}-(1-\mu_a-\eta_a-v_a)^{\lambda_2} >$$

and

$$(\lambda_1+\lambda_2)a = < 1-(1-\mu_a)^{(\lambda_1+\lambda_2)}, (1-\mu_a)^{(\lambda_1+\lambda_2)}-(1-\mu_a-\eta_a)^{(\lambda_1+\lambda_2)}, (1-\mu_a-\eta_a)^{(\lambda_1+\lambda_2)}-(1-\mu_a-\eta_a-v_a)^{(\lambda_1+\lambda_2)} > .$$

Further, we have the following expression:

$$\lambda_1 a \oplus \lambda_2 a = < 1-(1-\mu_a)^{(\lambda_1+\lambda_2)}, (1-\mu_a)^{(\lambda_1+\lambda_2)}-(1-\mu_a-\eta_a)^{(\lambda_1+\lambda_2)}, (1-\mu_a-\eta_a)^{(\lambda_1+\lambda_2)}-(1-\mu_a-\eta_a-v_a)^{(\lambda_1+\lambda_2)} > .$$

Therefore, the expression $\lambda_1 a \oplus \lambda_2 a = (\lambda_1+\lambda_2)a$ holds.

(4) According to the operational laws in Definition 5, we obtain we obtain

$$\lambda a = < 1-(1-\mu_a)^\lambda, (1-\mu_a)^\lambda-(1-\mu_a-\eta_a)^\lambda, (1-\mu_a-\eta_a)^\lambda-(1-\mu_a-\eta_a-\nu_a)^\lambda >,$$
$$\lambda b = < 1-(1-\mu_b)^\lambda, (1-\mu_b)^\lambda-(1-\mu_b-\eta_b)^\lambda, (1-\mu_b-\eta_b)^\lambda-(1-\mu_b-\eta_b-\nu_b)^\lambda >,$$

and

$$a\oplus b =< 1-(1-u_a)(1-u_b), (1-u_a)(1-u_b)-(1-u_a-\eta_a)(1-u_b-\eta_b), (1-u_a-\eta_a)(1-u_b-\eta_b)-(1-u_a-\eta_a-\nu_a)(1-u_b-\eta_b-\nu_b) > .$$

Further, we can get the following aggregation result according the operational laws in Definition 5:

$$\lambda(a\oplus b) = < 1-(1-u_a)^\lambda(1-u_b)^\lambda, (1-u_a)^\lambda(1-u_b)^\lambda-(1-u_a-\eta_a)^\lambda(1-u_b-\eta_b)^\lambda, (1-u_a-\eta_a)^\lambda(1-u_b-\eta_b)^\lambda - (1-u_a-\eta_a-\nu_a)^\lambda(1-u_b-\eta_b-\nu_b)^\lambda > = \lambda a\oplus\lambda b.$$

B. PICTURE FUZZY WEIGHTED INTERACTION AVERAGING OPERATOR

Definition 7 : Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of PFNs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, then the picture fuzzy weighted interaction averaging (PFWIA) operator is defined as:

$$PFWIA(a_1, a_2, \ldots, a_n) = \mathop{\oplus}_{i=1}^{n}(w_i a_i). \quad (6)$$

According to the operational laws of PFNs described in Definition 5, we can derive the following theorem:

Theorem 2: Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of PFNs, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $a_i (i = 1, 2, \ldots, n)$, such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated result by using *PFWIA* operator is shown as follows:

$$PFWIA(a_1, a_2, \ldots, a_n) = \mathop{\oplus}_{i=1}^{n}(w_i a_i) = < 1-\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}, \prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}, \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} > . \quad (7)$$

Proof: In what follows, we prove that Equation (7) holds by using mathematical induction on n.

When $n = 1$, we have

$$w_1 a_1 = < 1-(1-\mu_{a_1})^{w_1}, (1-\mu_{a_1})^{w_1}-(1-\mu_{a_1}-\eta_{a_1})^{w_1}, (1-\mu_{a_1}-\eta_{a_1})^{w_1}-(1-\mu_{a_1}-\eta_{a_1}-\nu_{a_1})^{w_1} > .$$

Thus, Equation (7) holds for $n = 1$. If Equation (7) holds for $n = k$. Then, when $n = k + 1$, by inductive assumption and the operational laws of PFNs in Definition 5, we have

$$\mathop{\oplus}_{i=1}^{k+1}(w_i a_i) = \mathop{\oplus}_{i=1}^{k}(w_i a_i)\oplus(w_{k+1}a_{k+1})$$
$$= < 1-\prod_{i=1}^{k}(1-\mu_{a_1})^{w_i}, \prod_{i=1}^{k}(1-\mu_{a_1})^{w_i}-\prod_{i=1}^{k}(1-\mu_{a_1}-\eta_{a_1})^{w_i}, \prod_{i=1}^{k}(1-\mu_{a_1}-\eta_{a_1})^{w_i}-\prod_{i=1}^{k}(1-\mu_{a_1}-\eta_{a_1}-\nu_{a_1})^{w_i} > \oplus$$
$$< 1-(1-\mu_{a_{k+1}})^{w_{k+1}}, (1-\mu_{a_{k+1}})^{w_{k+1}}-(1-\mu_{a_{k+1}}-\eta_{a_{k+1}})^{w_{k+1}}, (1-\mu_{a_{k+1}}-\eta_{a_{k+1}})^{w_{k+1}}-(1-\mu_{a_{k+1}}-\eta_{a_{k+1}}-\nu_{a_{k+1}})^{w_{k+1}} >$$
$$= < 1-\prod_{i=1}^{k+1}(1-\mu_{a_i})^{w_i}, \prod_{i=1}^{k+1}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{k+1}(1-\mu_{a_i}-\eta_{a_i})^{w_i}, \prod_{i=1}^{k+1}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{k+1}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} >$$

Then, Equation (7) holds for $n = k + 1$.

Therefore, by using mathematical induction on n, Equation (7) holds for all positive natural numbers n.

Theorem 3: Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of PFNs, then the aggregated result by using *PFWIA* operator is also a picture fuzzy number, that is $PFWIA(a_1, a_2, \ldots, a_n)$ is also a PFN.

Proof: Since $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} >\in PFNs$, we have $0 \leq \mu_{a_i}, \eta_{a_i}, \nu_{a_i} \leq 1$ and $0 \leq \mu_{a_i}+\eta_{a_i}+\nu_{a_i} \leq 1$, then by Definition 5, we know that the following expressions hold:

$$0 \leq 1-\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i} \leq 1,$$
$$0 \leq \prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} \leq 1,$$
$$0 \leq \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \leq 1,$$

and

$$\left(1-\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}\right)+\left(\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}\right) + \left(\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i}\right) = 1-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \in [0, 1].$$

Thus, we know that the aggregated result $PFWIA(a_1, a_2, \ldots, a_n)$ of the $PFWIA$ operator is also a PFN.

Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then the $PFWIA$ operator reduces to the picture fuzzy interaction arithmetic averaging ($PFIAA$) operator:

$$\begin{aligned} &PFIAA(a_1, a_2, \ldots, a_n) \\ &= < 1 - \prod_{i=1}^{n}(1-\mu_{a_i})^{1/n}, \\ &\quad \prod_{i=1}^{n}(1-\mu_{a_i})^{1/n} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{1/n}, \\ &\quad \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{1/n} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{1/n} > . \end{aligned} \tag{8}$$

Based on above analysis, we can further discuss some important properties of the $PFWIA$ operator.

Theorem 4 (Idempotency): Let $a_i = (\mu_{a_i}, \eta_{a_i}, \nu_{a_i})(i = 1, 2, \ldots, n)$ be a set of picture fuzzy numbers, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $a_i(i = 1, 2, \ldots, n)$, such that $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. If all $a_i(i = 1, 2, \ldots, n)$ are equal and $a_i = (\mu_{a_0}, \eta_{a_0}, \nu_{a_0}) = a_0$ for all $i = 1, 2, \ldots, n$, then

$$PFWIA(a_1, a_2, \ldots, a_n) = a_0 \tag{9}$$

Proof: Since $a_i = a_0 =< \mu_{a_0}, \eta_{a_0}, \nu_{a_0} >$ for all $i = 1, 2, \ldots, n$, $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, according to Theorem 2, we have

$$\begin{aligned} &PFWIA(a_1, a_2, \ldots, a_n) \\ &= \bigoplus_{i=1}^{n}(w_i a_i) \\ &= < 1 - \prod_{i=1}^{n}(1-\mu_{a_0})^{w_i}, \prod_{i=1}^{n}(1-\mu_{a_0})^{w_i} \\ &\quad - \prod_{i=1}^{n}(1-\mu_{a_0}-\eta_{a_0})^{w_i}, \\ &\quad \prod_{i=1}^{n}(1-\mu_{a_0}-\eta_{a_0})^{w_i} - \prod_{i=1}^{n}(1-\mu_{a_0}-\eta_{a_0}-\nu_{a_0})^{w_i} > \\ &= < 1 - (1-\mu_{a_0})^{\sum_{i=1}^{n} w_i}, (1-\mu_{a_0})^{\sum_{i=1}^{n} w_i} \\ &\quad - (1-\mu_{a_0}-\eta_{a_0})^{\sum_{i=1}^{n} w_i}, \\ &\quad (1-\mu_{a_0}-\eta_{a_0})^{\sum_{i=1}^{n} w_i} - (1-\mu_{a_0}-\eta_{a_0}-\nu_{a_0})^{\sum_{i=1}^{n} w_i} > \\ &= < \mu_{a_0}, \eta_{a_0}, \nu_{a_0} > \\ &= a_0 \end{aligned}$$

Theorem 5 (Boundary): Let $a_i = (\mu_{a_i}, \eta_{a_i}, \nu_{a_i})(i = 1, 2, \ldots, n)$ be a set of picture fuzzy numbers, if

$$\begin{aligned} a^+ = &< \max_{1\leq i\leq m}(\mu_{a_i}), \max\{0, (\min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) - \max_{1\leq i\leq m}(\mu_{a_i}))\}, \\ &\max\{0, (\min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}) - \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))\} > \end{aligned}$$

and

$$\begin{aligned} a^- = &< \min_{1\leq i\leq m}(\mu_{a_i}), \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) - \min_{1\leq i\leq m}(\mu_{a_i}), \\ &\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}) - \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) > . \end{aligned}$$

Then we have

$$a^- \leq PFWIA(a_1, a_2, \ldots, a_n) \leq a^+ \tag{10}$$

Proof: According to the given condition, we can get the following two inequalities:

$$\max_{1\leq i\leq m}(\mu_{a_i}) = 1 - (1 - \max_{1\leq i\leq m}(\mu_{a_i}))^{\sum_{i=1}^{n} w_i} \geq 1 - \prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}$$

and

$$1 - \prod_{i=1}^{n}(1-\mu_{a_i})^{w_i} \geq 1 - (1 - \min_{1\leq i\leq m}(\mu_{a_i}))^{\sum_{i=1}^{n} w_i} = \min_{1\leq i\leq m}(\mu_{a_i}).$$

Similarly, we can get the following two inequalities:

$$\begin{aligned} &\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) - \min_{1\leq i\leq m}(\mu_{a_i}) \\ &= (1 - \min_{1\leq i\leq m}(\mu_{a_i}))^{\sum_{i=1}^{n} w_i} - (1 - \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &\geq \prod_{i=1}^{n}(1-\mu_{a_i})^{w_i} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} \end{aligned}$$

and

$$\begin{aligned} &\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} \\ &\geq (1 - \max_{1\leq i\leq m}(\mu_{a_i}))^{\sum_{i=1}^{n} w_i} - (1 - \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &= \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) - \max_{1\leq i\leq m}(\mu_{a_i}). \end{aligned}$$

Further, we know that the following two inequalities hold:

$$\begin{aligned} &\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}) - \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}) \\ &\quad = (1 - \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &\qquad - (1 - \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &\quad \geq \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \end{aligned}$$

and

$$\begin{aligned} &\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} - \prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \\ &\quad \geq (1 - \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &\qquad - (1 - \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}))^{\sum_{i=1}^{n} w_i} \\ &\quad = \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i}) - \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}). \end{aligned}$$

According to Theorem 3, we know that the aggregated result of the PFWIA operator is also PFN. Thus, we know the following expressions hold:

$$\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}\geq 0,$$

$$\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i}\geq 0.$$

Therefore, we can get:

$$\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} \geq \max\left\{0,\ \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i})-\max_{1\leq i\leq m}(\mu_{a_i})\right\}$$

and

$$\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \geq \max\left\{0,\ \min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i})-\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i})\right\}.$$

According to Definition 3, we can prove the conclusion: $a^{-}\leq PFWIA(a_1,a_2,\ldots,a_n)\leq a^{+}$.

Theorem 6 (Monotonicity): If $a_i=<u_{a_i},\eta_{a_i},\nu_{a_i}>$ $(i=1,2,\ldots,n)$ and $b_i=<u_{b_i},\eta_{b_i},\nu_{b_i}>$ $(i=1,2,\ldots,n)$ be two sets of picture fuzzy numbers, and $\mu_{a_i}\leq\mu_{b_i}$, $\mu_{a_i}+\eta_{a_i}\leq\mu_{b_i}+\eta_{b_i}$, $\mu_{a_i}+\eta_{a_i}+\nu_{a_i}\leq\mu_{b_i}+\eta_{b_i}+\nu_{b_i}$ for all $i=1,2,\ldots,n$, then we have

$$PFWIA(a_1,a_2,\ldots,a_n)\leq PFWIA(b_1,b_2,\ldots,b_n) \quad (11)$$

Proof: Since $\mu_{a_i}\leq\mu_{b_i}$, we have the following inequality: $1-\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}\leq 1-\prod_{i=1}^{n}(1-\mu_{b_i})^{w_i}$.

Similarly, since $\mu_{a_i}\leq\mu_{b_i}$ and $\mu_{a_i}+\eta_{a_i}\leq\mu_{b_i}+\eta_{b_i}$, we can get the following inequality:

$$\prod_{i=1}^{n}(1-\mu_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i} \geq \prod_{i=1}^{n}(1-\mu_{b_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{b_i}-\eta_{b_i})^{w_i}.$$

Further, since $\mu_{a_i}+\eta_{a_i}\leq\mu_{b_i}+\eta_{b_i}$ and $\mu_{a_i}+\eta_{a_i}+\nu_{a_i}\leq\mu_{b_i}+\eta_{b_i}+\nu_{b_i}$, we have the following inequality:

$$\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{a_i}-\eta_{a_i}-\nu_{a_i})^{w_i} \geq \prod_{i=1}^{n}(1-\mu_{b_i}-\eta_{b_i})^{w_i}-\prod_{i=1}^{n}(1-\mu_{b_i}-\eta_{b_i}-\nu_{b_i})^{w_i}.$$

Therefore, according to Definition 3, we know that the following inequality holds.

$$PFWIA(a_1,a_2,\ldots,a_n)\leq PFWIA(b_1,b_2,\ldots,b_n).$$

C. A PICTURE FUZZY ORDERED WEIGHTED INTERACTION AVERAGING OPERATOR

Definition 7: Let $a_i=<\mu_{a_i},\eta_{a_i},\nu_{a_i}>$ $(i=1,2,\ldots,n)$ be a collection of picture fuzzy numbers, $a_{\sigma(i)}$ be the ith largest of them, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the aggregation-associated vector with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the picture fuzzy ordered weighted interaction averaging (*PFOWIA*) operator is defined as:

$$PFOWIA(a_1,a_2,\ldots,a_n)=\mathop{\oplus}_{i=1}^{n}(\omega_i a_{\sigma(i)}). \quad (12)$$

Theorem 7: Let $a_i(i=1,2,\ldots,n)$ be a collection of picture fuzzy numbers, $a_{\sigma(i)}$ be the ith largest of them, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the aggregation-associated vector with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the aggregated result by using *PFOWIA* operator is shown as follows:

$$\begin{aligned} &PFOWIA(a_1,a_2,\ldots,a_n)\\ &=\mathop{\oplus}_{i=1}^{n}(\omega_i a_{\sigma(i)})\\ &=\Big\langle 1-\prod_{i=1}^{n}(1-\mu_{a_{\sigma(i)}})^{\omega_i},\\ &\quad\prod_{i=1}^{n}(1-\mu_{a_{\sigma(i)}})^{\omega_i}-\prod_{i=1}^{n}(1-\mu_{a_{\sigma(i)}}-\eta_{a_{\sigma(i)}})^{\omega_i},\\ &\quad\prod_{i=1}^{n}(1-\mu_{a_{\sigma(i)}}-\eta_{a_{\sigma(i)}})^{\omega_i}-\prod_{i=1}^{n}(1-\mu_{a_{\sigma(i)}}-\eta_{a_{\sigma(i)}}-\nu_{a_{\sigma(i)}})^{\omega_i}\Big\rangle \end{aligned} \quad (13)$$

The proof of this theorem is similar to Theorem 2.

Theorem 8: Let $a_i=<\mu_{a_i},\eta_{a_i},\nu_{a_i}>$ $(i=1,2,\ldots,n)$ be a collection of picture fuzzy numbers, $a_{\sigma(i)}$ be the ith largest of them, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the aggregation-associated vector with $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then the aggregated result by using *PFOWIA* operator is also a picture fuzzy number.

The proof of this theorem is similar to Theorem 3.

Theorem 9 (Idempotency): Let $a_i=<\mu_{a_i},\eta_{a_i},\nu_{a_i}>$ $(i=1,2,\ldots,n)$ be a collection of picture fuzzy numbers, if all $a_i(i=1,2,\ldots,n)$ are equal and $a_i=(\mu_{a_0},\eta_{a_0},\nu_{a_0})=a_0$ for all $i=1,2,\ldots,n$, then

$$PFOWIA(a_1,a_2,\ldots,a_n)=a_0 \quad (14)$$

The proof of this theorem is similar to that of Theorem 4.

Theorem 10 (Boundary): Let $a_i=<\mu_{a_i},\eta_{a_i},\nu_{a_i}>$ $(i=1,2,\ldots,n)$ be a collection of picture fuzzy numbers, and a^{+} are a^{-} denoted as

$$\begin{aligned} a^{+}=&<\max_{1\leq i\leq m}(\mu_{a_i}),\max\{0,(\min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i})-\max_{1\leq i\leq m}(\mu_{a_i}))\},\\ &\max\{0,(\min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i})-\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}))\}>,\\ a^{-}=&<\min_{1\leq i\leq m}(\mu_{a_i}),\ \max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i})-\min_{1\leq i\leq m}(\mu_{a_i}),\\ &\max_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i}+\nu_{a_i})-\min_{1\leq i\leq m}(\mu_{a_i}+\eta_{a_i})>. \end{aligned}$$

Then, we have

$$a^{+} \leq PFOIWA(a_1, a_2, \ldots, a_n) \leq a^{-} \tag{15}$$

The proof of this theorem is similar to that of Theorem 5.

D. PICTURE FUZZY HYBRID ORDERED WEIGHTED INTERACTION AVERAGING OPERATOR

Definition 8: Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of picture fuzzy numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them with $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$, and n be the balancing coefficient which plays a role of balance, then based on the aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$, the picture fuzzy hybrid ordered weighted interaction averaging (PFHOWIA) operator is defined as follows:

$$PFHOWIA(a_1, a_2, \ldots, a_n) = \mathop{\oplus}_{i=1}^{n} (\omega_i b_{\sigma(i)}) \tag{16}$$

where $b_{\sigma(i)}$ is the ith largest of the picture fuzzy weighted arguments $b_i = nw_i a_i (i = 1, 2, \ldots, n)$.

Theorem 11: Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of picture fuzzy numbers, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them with $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$, n be the balancing coefficient which plays a role of balance and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$, then the aggregated result by using *PFHOWIA* operator is shown as follows:

$$\begin{aligned}
&PFHOWIA(a_1, a_2, \ldots, a_n) \\
&= \mathop{\oplus}_{i=1}^{n} (\omega_i b_{\sigma(i)}) \\
&= < 1 - \prod_{i=1}^{n} (1 - \tilde{\mu}_{b_{\sigma(i)}})^{\omega_i}, \prod_{i=1}^{n} (1 - \tilde{\mu}_{b_{\sigma(i)}})^{\omega_i} \\
&\quad - \prod_{i=1}^{n} (1 - \tilde{\mu}_{b_{\sigma(i)}} - \tilde{\eta}_{b_{\sigma(i)}})^{\omega_i}, \\
&\quad \prod_{i=1}^{n} (1 - \tilde{\mu}_{b_{\sigma(i)}} - \tilde{\eta}_{b_{\sigma(i)}})^{\omega_i} - \prod_{i=1}^{n} (1 - \tilde{\mu}_{b_{\sigma(i)}} - \tilde{\eta}_{b_{\sigma(i)}} - \tilde{\nu}_{b_{\sigma(i)}})^{\omega_i} >
\end{aligned}$$

where $b_{\sigma(i)}$ is the ith largest of the picture fuzzy weighted arguments $b_i = nw_i a_i (i = 1, 2, \ldots, n)$.

The proof is similar to that of Theorem 2.

Theorem 12: Let $a_i =< \mu_{a_i}, \eta_{a_i}, \nu_{a_i} > (i = 1, 2, \ldots, n)$ be a collection of picture fuzzy numbers, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them with $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$, n be the balancing coefficient which plays a role of balance and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$, then the aggregated result $PFHOWIA(a_1, a_2, \ldots, a_n)$ by using the *PFHOWIA* operator is also a picture fuzzy number.

The proof of this theorem is similar to Theorem 3.

IV. AN EXTENDED EDAS METHOD FOR MAGDM PROMLEM UNDER PICTURE FUZZY ENVIRONMENT

A. DESCRIPTION OF THE MAGDM PROMLEM UNDER PICTURE FUZZY ENVIRONMENT

For MAGDM problem, let $A = \{A_1, A_2, \ldots, A_m\} (m \geq 2)$ be a finite set of feasible alternatives among which experts have to select, $C = \{C_1, C_2, \ldots, C_n\} (n \geq 2)$ be a finite set of attributes with which alternative performance is assessed, and $DM = \{DM_1, DM_2, \ldots, DM_t\} (t \geq 2)$ be a set of decision makers. Suppose that $R^k = (r_{ij}^k)_{m \times n}$ is a picture fuzzy decision matrix provided by the kth decision maker, in which r_{ij}^k is the assessment on alternative $A_i \in A$ with respect to the attribute $C_i \in C$ determined by the kth decision maker. Subsequently, we will develop a new method for solving the MAGDM problem with picture fuzzy information, in which the information about decision makers' weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T$ is completely unknown and the information about attribute weight vector $w = (w_1, w_2, \ldots, w_n)^T$ is incompletely known.

B. DETERMINE THE WEIGHTS OF DECISION MAKERS

In real life, decision maker often come from different departments, and each decision maker has his unique characteristics with regard to knowledge, skills and experience. The process of determining decision makers' weight plays an important role in obtaining the reasonable result for the MAGDM problem [14]. In the following, we develop a new method to determine the objective weights of decision makers under picture fuzzy environment based on the idea of Dice similarity measure. Dice similarity measure, initially proposed by Dice [46], is a similarity measure between two vectors. The basic principle of Dice similarity measure is that the more similar their geometrical shape is, the larger the Dice similarity measure of the comparing data is. Since its appearance, it has attracted more and more attention from researchers. Based on the basic principle of Dice similarity measure in [46], Wei and Gao [47] proposed the following concept of generalized Dice similarity measures for picture fuzzy sets.

Definition 9 [47]: Let $a_j =< \mu_{a_j}, \eta_{a_j}, \nu_{a_j} >$ and $b_j =< \mu_{b_j}, \eta_{b_j}, \nu_{b_j} >$ be two PFNs. then the Dice similarity measure between a_j and b_j is defined as:

$$\begin{aligned}
&D_{PFS}(a_j, b_j) \\
&= \frac{2(a_j \cdot b_j)}{|a_j|^2 + |b_j|^2} \\
&= \frac{2\sum_{j=1}^{n} (\mu_{a_j}\mu_{bj} + \eta_{a_j}\eta_{bj} + \nu_{a_j}\nu_{bj} + \pi_{a_j}\pi_{bj})}{\sum_{j=1}^{n} (\mu_{aj}^2 + \eta_{aj}^2 + \nu_{aj}^2 + \pi_{aj}^2) + \sum_{j=1}^{n} (\mu_{bj}^2 + \eta_{bj}^2 + \nu_{bj}^2 + \pi_{bj}^2)}
\end{aligned} \tag{17}$$

In the following, we adopt the generalized Dice similarity measures for picture fuzzy sets to determine the weights of decision makers:

Step 1: Obtain the individual picture fuzzy decision matrix $R^k = (r_{ij}^k)_{m\times n}$.

Step 2: Compute the relative Dice similarity measure matrices $E^{kl} = (e_{ij}^{kl})_{m\times n}, (k = 1, 2, \ldots, t, l = 1, 2, \ldots, t, k \neq l)$,

where $e_{ij}^{kl} = D_{PFS}(r_{ij}^k, r_{ij}^l)$ can be calculated by Equation (17).

Step 3 : Determine the comprehensive similarity measure matrix of the decision makers:

$$E^k = (e_{ij}^k)_{m\times n} = \sum\nolimits_{l=1, l\neq k}^{t} E^{kl}, \quad k = 1, 2, \ldots, t$$

.

Step 4 : Calculate the weight of decision makers

$$\lambda_k = \sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}^k / \sum_{k=1}^{t}\sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}^k, \quad k = 1, 2, \ldots, t. \quad (18)$$

Obviously, $0 \leq \lambda_k \leq 1$ and $\sum_{k=1}^{t} \lambda_k = 1$.

C. DETERMINE THE WEIGHTS OF ATTRIBUTES

For the situations where the information about the attribute weights is partly known, we establish a linear programming model to determine the attribute weights according to the maximizing deviation method. Let w_j be the weight of the attribute $C_j \in C$, which satisfies the normalization conditions $w_j \in [0, 1], (j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. Let Γ_0 denote the set of all the weight vectors, and

$$\Gamma_0 = \left\{(w_1, w_2, \ldots w_n) \left| w_j \geq 0, j = 1, 2, \ldots, n, \sum\nolimits_{j=1}^{n} w_j = 1\right.\right\}. \quad (19)$$

The incomplete information on the attribute weights provided by the decision-maker can usually be constructed using several basic ranking forms. For a decision making problem that contains incomplete weight information, we consider the five basic ranking forms for the incomplete information on the attribute weights [48].

- A weak ranking:

$$\Gamma_1 = \{(w_1, w_2, \ldots w_n \in \Gamma_0 | w_{j1} \geq w_{j2}, \textit{for all } j_1 \in r_1 \textit{ and } j_2 \in \Lambda_1\}. \quad (20)$$

where r_1 and Λ_1 are two disjoint subsets of the subscript index set $N = \{1, 2, \ldots, n\}$ for all attributes.

- A strict ranking:

$$\Gamma_2 = \{(w_1, w_2, \ldots w_n) \in \Gamma_0 | w_{j1} - w_{j2} \geq \delta_{j1j2}, \textit{for all } j_1 \in r_2 \textit{ and } j_2 \in \Lambda_2\}. \quad (21)$$

where δ_{j1j2} is a constant that satisfies the condition $\delta_{j1j2} > 0$, r_2 and Λ_2 are two disjoint subsets of the subscript index set $N = \{1, 2, \ldots, n\}$ for all attributes.

- A ranking of difference:

$$\Gamma_3 = \{(w_1, w_2, \ldots w_n) \in \Gamma_0 | w_{j1} - w_{j2} \geq w_{j2} - w_{j3}, \textit{for all} j_1 \in r_3, j_2 \in \Lambda_3 \textit{and} j_3 \in \Omega_3\}.$$

where r_3, Λ_3 and Ω_3 are three disjoint subsets of the subscript index set $N = \{1, 2, \ldots, n\}$ for all attributes.

- A interval bound:

$$\Gamma_4 = \{(w_1, w_2, \ldots w_n) \in \Gamma_0 | \delta_{j1} + \varepsilon_{j1} \geq w_{j1} \geq \delta_{j1}, \textit{for all } j_1 \in r_4\}. \quad (22)$$

where $\delta_{j1} \geq 0$ and $\varepsilon_{j1} \geq 0$ are constants that satisfy the condition $0 \leq \delta_{j1} \leq \delta_{j1} + \varepsilon_{j1} \leq 1$, and r_4 is the subset of the subscript index set $N = \{1, 2, \ldots, n\}$.

- A ratio bound

$$\Gamma_5 = \{(w_1, w_2, \ldots w_n) \in \Gamma_0 | w_{j1} \geq \delta_{j2} w_{j2}, \textit{for all } j_1 \in r_5 \textit{ and } j_2 \in \Lambda_5\}. \quad (23)$$

where δ_{j2} is a constant that satisfies the condition $0 \leq \delta_{j2}^l \leq 1$, and r_5 and Λ_5 are two disjoint subsets of the subscript index set $N = \{1, 2, \ldots, n\}$ for all attributes.

Let Γ denote a set of the known information on the attribute weights, and

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5. \quad (24)$$

The maximizing deviation method, initially proposed by Wang [49], is a useful tool to determine the objective weights of attributes in solving MADM problems. The basic principle of this method is given as follows: if the assessment values of each alternative have little differences under a given attribute, it shows that the attribute plays a small important role in the priority procedure, so it should be assigned a small weight; Otherwise, if an attribute makes the assessment values among all the alternatives have obvious differences, then the attribute should be assigned a big weight [49]. Based on the above analysis, a linear programming model to determine the objective weights of attributes is constructed as follows.

For the given attribute $C_j \in C$, the deviation of alternative A_i to all the other alternatives can be defined as:

$$G_{ij}(w) = \sum_{k=1}^{m} d(r_{ij}, r_{kj}) w_j \quad (25)$$

where $d(r_{ij}, r_{kj})$ can be calculated by Equation (4).

Since each alternative is non-inferior and there exists no preference relation on all alternatives, then we construct the following expression:

$$G_j(w) = \sum_{i=1}^{m}\sum_{k=1}^{m} d(r_{ij}, r_{kj}) w_j \quad (26)$$

where $G_j(w)$ represents the deviation value of all alternatives to other alternatives with regard to the given attribute $C_j \in C$.

Further, based on the idea of the maximizing deviation method, we construct the following linear programming model (M-1) to calculate the weight vector of the attributes:

$$\max G = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(r_{ij}, r_{kj})w_j$$

$$(M-1)\ s.t. \begin{cases} w_j \in \Gamma, \\ \sum_{j=1}^{n} w_j = 1, \\ w_j \geq 0, \quad j = 1, 2, \ldots, n \end{cases} \tag{27}$$

Solving the linear programming model (M-1), we can obtain the optimal solution $w = (w_1, w_2, \ldots, w_n)^T$, which can be used as the objective weight vector of attributes.

D. PROPOSED METHOD FOR MAGDM PROBLEM BASED ON THE IDEA OF EDAS

In the following, a new method is proposed to solve multi-attribute group decision making problems in which the weights of decision makers and attributes are unknown based the traditional idea of EDAS method. The procedure of the extended EDAS method for MAGDM problem is described as follows.

Step 1 : Obtain the picture fuzzy decision matrix $R^k = (r_{ij}^k)_{m\times n}, (k = 1, 2, \ldots, t)$ given by the decision makers, r_{ij}^k is the evaluation information of the ith alternative with respect to the jth attribute given by kth decision maker.

Step 2 : Determine the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T$ of decision makers by Equation (18).

Step 3 : Calculate the overall picture fuzzy decision matrix $R = (r_{ij})_{m\times n}$ by the PFIWA operator in Equation (7), where

$$r_{ij} = \overset{t}{\underset{k=1}{\oplus}} (\lambda_k r_{ij}^k), \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \tag{28}$$

Step 4: Determine the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ of attributes by solving the programming $M - 1$ based on the overall picture fuzzy decision matrix $R = (r_{ij})_{m\times n}$.

Step 5: According to the traditional EDAS method, the average picture fuzzy evaluation $\tilde{a}_j =< \mu_{\tilde{a}_j}, \eta_{\tilde{a}_j}, \nu_{\tilde{a}_j} >$ of jth attribute can be determined by the PFIAA operator in Equation (8), which is shown as follows:

$$\tilde{r}_j = \overset{m}{\underset{i=1}{\oplus}} (\frac{1}{m} r_{ij}), \quad j = 1, 2, \ldots, n \tag{29}$$

Step 6 : Construct the positive distance matrix $PDA = (PDA_{ij})_{m\times n}$ and the negative distance matrix $NDA = (NDA_{ij})_{m\times n}$. To simplify the calculation process, in what follows, we adopt Equations (30) and (31) to determine the positive distance PDA_{ij} from average solution and the negative distance NDA_{ij} from average solution.

$$PDA_{ij} = \begin{cases} s(r_{ij}) - s(\tilde{r}_j), \quad if\ r_{ij} \geq \tilde{r}_j, \\ \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \\ 0, \quad if\ r_{ij} < \tilde{r}_j, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \end{cases} \tag{30}$$

$$NDA_{ij} = \begin{cases} s(\tilde{r}_j) - s(r_{ij}), \quad if\ r_{ij} < \tilde{r}_j, \ i = 1, 2, \ldots, m, \\ \quad j = 1, 2, \ldots, n \\ 0, \quad if\ r_{ij} \geq \tilde{r}_j, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \end{cases} \tag{31}$$

where $s(r_{ij})$ and $s(\tilde{r}_j)$ can be calculated by Equation (2).

Step 7 : Obtain weighted summation of the positive and negative distances from average matrix

$$SP_i = \sum\nolimits_{j=1}^{n} w_j PDA_{ij}, \quad i = 1, 2, \ldots, m \tag{32}$$

$$SN_i = \sum\nolimits_{j=1}^{n} w_j NDA_{ij}, \quad i = 1, 2, \ldots, m \tag{33}$$

Step 8 : Calculate the normalized values of SP_i and SN_i for all alternatives.

$$NSP_i = \frac{SP_i}{\max\limits_{1\leq i\leq m} (SP_i)}, \quad i = 1, 2, \ldots, m \tag{34}$$

$$NSN_i = 1 - \frac{SN_i}{\max\limits_{1\leq i\leq m} (SN_i)}, \quad i = 1, 2, \ldots, m \tag{35}$$

Step 9 : Determine the appraisal scores of all feasible alternatives, which are shown as follows:

$$AS_i = \frac{1}{2}(NSP_i + NSN_i), \quad i = 1, 2, \ldots, m \tag{36}$$

Obviously, we have $0 \leq AS_i \leq 1$.

Step 10: Rank all feasible alternatives according to the decreasing values of appraisal score AS_i and select the most desirable alternative(s) with the highest AS_i.

V. NUMERICAL EXAMPLE AND COMPARISON ANALYSIS

A. NUMERICAL EXAMPLE

This example is adopted from reference [14]. It is described as follows: an emergency management center (EMC) wants to select an optimal emergency alternative from four feasible emergency alternatives $A_i (i = 1, 2, , 3, 4)$. A committee of three experts, $DM_k (k = 1, 2, 3)$, are invited to evaluate the four feasible alternatives and select the most suitable emergency alternative. The weights of the three experts are completely unknown. The attributes for the evaluation of emergency alternative are as follows: (1) C_1 is the emergency forecasting capability; (2) C_2 is the emergency process capability; (3) C_3 is the emergency support capability; (4) C_4 is the after-disaster process capability. The weights of the attribute are partially known, which are described as follows:

$$\Lambda = \{(w_1, w_2, w_3, w_4) \in \Gamma_0 \,|, 0.1 \leq w_1 \leq 0.2\ ,$$
$$0.2 \leq w_2 \leq 0.3, 0.3 \leq w_3 \leq 0.4, 0.1 \leq w_4 \leq 0.2\}\,.$$

In the following, we utilize the proposed approach in Section IV to solve the emergency alternative selection problem.

Step 1: Obtain the picture fuzzy decision matrix $R^k = (r_{ij}^k)_{4\times 4} (k = 1, 2, 3)$ given by the decision makers, r_{ij}^k is the evaluation information of ith alternative with respect to the jth attribute given by kth decision maker. The picture fuzzy decision matrices are shown in Tables 1-3.

TABLE 1. Picture fuzzy decision matrix R_1 given by DM_1.

	C_1	C_2	C_3	C_4
A_1	<0.4,0.2,0.3>	<0.4,0.2,0.3>	<0.3,0.2,0.3>	<0.5,0.1,0.3>
A_2	<0.5,0.3,0.2>	<0.3,0.3,0.3>	<0.4,0.2,0.3>	<0.3,0.3,0.3>
A_3	<0.2,0.2,0.4>	<0.2,0.2,0.5>	<0.3,0.2,0.4>	<0.2,0.3,0.4>
A_4	<0.1,0.3,0.4>	<0.3,0.2,0.4>	<0.2,0.3,0.4>	<0.2,0.3,0.3>

TABLE 2. Picture fuzzy decision matrix R_2 given by DM_2.

	C_1	C_2	C_3	C_4
A_1	<0.2,0.3,0.3>	<0.3,0.2,0.4>	<0.2,0.4,0.3>	<0.4,0.2,0.3>
A_2	<0.5,0.2,0.3>	<0.4,0.2,0.3>	<0.4,0.2,0.3>	<0.3,0.3,0.2>
A_3	<0.4,0.2,0.3>	<0.4,0.3,0.1>	<0.5,0.3,0.1>	<0.4,0.3,0.2>
A_4	<0.1,0.2,0.5>	<0.2,0.4,0.3>	<0.2,0.3,0.3>	<0.3,0.1,0.4>

TABLE 3. Picture fuzzy decision matrix given by DM_3.

	C_1	C_2	C_3	C_4
A_1	<0.3,0.2,0.3>	<0.2,0.2,0.5>	<0.2,0.2,0.4>	<0.2,0.3,0.4>
A_2	<0.1,0.3,0.4>	<0.2,0.2,0.5>	<0.2,0.3,0.4>	<0.1,0.3,0.4>
A_3	<0.4,0.2,0.3>	<0.3,0.2,0.3>	<0.4,0.2,0.3>	<0.4,0.3,0.2>
A_4	<0.5,0.2,0.1>	<0.4,0.2,0.3>	<0.3,0.2,0.3>	<0.3,0.3,0.2>

TABLE 4. The collective decision matrix R.

	C_1	C_2	C_3	C_4
A_1	<0.30,0.23,0.31>	<0.31,0.20,0.39>	<0.24,0.27,0.33>	<0.38,0.23,0.39>
A_2	<0.39,0.28,0.33>	<0.31,0.24,0.36>	<0.34,0.23,0.33>	<0.24,0.27,0.31>
A_3	<0.34,0.20,0.33>	<0.30,0.25,0.29>	<0.41,0.25,0.24>	<0.34,0.27,0.27>
A_4	<0.26,0.24,0.30>	<0.26,0.27,0.33>	<0.23,0.27,0.34>	<0.27,0.34,0.31>

Step 2: Determine the optimal weight vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$ of decision makers by Equation (18).

$$\lambda = (0.34, 0.33, 0.33)^T.$$

Step 3: Calculate the collective picture fuzzy decision matrix $R = (r_{ij})_{4\times 4}$ by the PFWIA operator in Equation (7), which is shown in Table 4.

Step 4: Determine the weights of attributes. For the collective picture fuzzy decision matrix $R = (r_{ij})_{4\times 4}$ in Table 4, we utilize the model (M-1) in Section IV to establish the following single objective linear programming model:

$$\max G = 0.2899w_1 + 0.2324w_2 + 0.4285w_3 + 0.3631w_4$$
$$s.t. \begin{cases} 0.1 \le w_1 \le 0.2, \\ 0.2 \le w_2 \le 0.3, \\ 0.3 \le w_3 \le 0.4, \\ 0.1 \le w_4 \le 0.2, \\ \sum_{j=1}^{4} w_j = 1, \\ w_j \ge 0, \quad j = 1, 2, 3, 4. \end{cases}$$

Solving this model using software Lingo 9.0, we obtain the weight vector of attributes:

$$W = (0.2, 0.2, 0.4, 0.2)^T.$$

Step 5 : Calculate the average picture fuzzy evaluation $\tilde{r}_j =< \mu_{\tilde{r}_j}, \eta_{\tilde{r}_j}, \nu_{\tilde{r}_j} >$ of jth attribute according to the PFIAA operator in Equation (8), which is shown as follows:

$$\tilde{r}_1 = < 0.33, 0.22, 0.31 >, \quad \tilde{r}_2 =< 0.30, 0.24, 0.34 >,$$
$$\tilde{r}_3 = < 0.31, 0.26, 0.31 >, \quad \tilde{r}_4 =< 0.31, 0.25, 0.31 > .$$

Step 6 : Calculate the positive distance from average and the negative distance from average matrices by Equations (30) and (31).

$$PDA = \begin{bmatrix} 0 & 0 & 0 & 0.0344 \\ 0.0368 & 0 & 0.0128 & 0 \\ 0.0012 & 0.0154 & 0.0564 & 0.0164 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$NDA = \begin{bmatrix} 0.0110 & 0.0038 & 0.0378 & 0 \\ 0 & 0.0038 & 0 & 0.0317 \\ 0 & 0 & 0 & 0 \\ 0.0279 & 0.0045 & 0.0382 & 0.0204 \end{bmatrix}$$

Step 7 : Determine the weighted summation of the positive and negative distances of all alternatives by Equations (32) and (33).

$$SP_1 = 0.0069, \quad SP_2 = 0.0125, \; SP_3 = 0.0291, \; SP_4 = 0;$$
$$SN_1 = 0.0181, \quad SN_2 = 0.0071, \; SN_3 = 0, \; SN_4 = 0.0258.$$

Step 8 : Calculate the normalized values of SP_i and SN_i for all alternatives.

$$NSP_1 = 0.2361, \quad NSP_2 = 0.4283, \; NSP_3 = 1, \; NSP_4 = 0;$$
$$NSN_1 = 0.2944, \quad NSN_2 = 0.7251, \; NSN_3 = 1, \; NSN_4 = 0.$$

Step 9: Calculated the appraisal scores of all feasible alternatives, which are shown as follows:

$$AS_1 = 0.2678, AS_2 = 0.5767, AS_3 = 1, AS_4 = 0.$$

Step 10: Rank all feasible alternatives according to the decreasing values of appraisal score:$A_3 \succ A_2 \succ A_1 \succ A_4$. Then, A_3 is the best alternative.

B. COMPARISON ANALYSIS

In this section, to illustrate the practicality and effectiveness of the proposed method, we compare the proposed extended EDAS method with the projection method in [25] and picture fuzzy aggregation operators in [50] for the collective picture fuzzy data in Table 4. Wei *et al.* [25] established the projection model of picture fuzzy information to measure the similarity degrees between each alternative and the ideal one in order to select the most desirable one(s). The weighted cosine of the included angle between the feasible alternative and the ideal one is defined as:

Definition 9: Suppose there are m feasible alternative $A_i = (r_{i1}, r_{i2}, \ldots, r_{in})(i = 1, 2, \ldots, m)$ among which experts have to choose. The positive ideal alternative is denoted as $A^+ = (r_1^+, r_2^+, \ldots, r_n^+)$, then the projection of each alternative on the positive ideal alternative is defined as follows:

$$Prj_{A^+}(A_i) = \frac{1}{|A^+|} \sum_{j=1}^{n} w_j^2(\mu_{ij}\mu_j^+ + \eta_{ij}\eta_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+) \tag{37}$$

where

$$r_j^+ = (\mu_j^+, \eta_j^+, \nu_j^+, \pi_j^+), \quad j = 1, 2, \ldots, n,$$
$$\mu_j^+ = \max_{1\le i\le m} (u_{ij}), \eta_j^+ = \min_{1\le i\le m} (\eta_{ij}),$$
$$\nu_j^+ = \min_{1\le i\le m} (\nu_{ij}), \pi_j^+ = 1 - \mu_j^+ - \eta_j^+ - \nu_j^+,$$

and

$$\left|A^+\right| = \sqrt{\sum_{j=1}^n ((\mu_j^+)^2 + (\eta_j^+)^2 + (\nu_j^+)^2 + (\pi_j^+)^2)}.$$

For the collective the picture fuzzy decision matrix R in Table 4, we can obtain the positive ideal alternative A^+, which is shown as follows:

$$A^+ = \{< 0.39, 0.19, 0.13 >, < 0.31, 0.20, 0.20 >, < 0.41, 0.23, 0.12 >, < 0.38, 0.19, 0.16 >\}$$

Further, we can calculate the projection of all alternatives $A_i (i = 1, 2, 3, 4)$ on the positive ideal evaluation A^+ by using Equation (37), the results is shown as follows:

$$Prj_{A^+}(A_1) = 0.5289, \quad Prj_{A^+}(A_2) = 0.5320,$$
$$Prj_{A^+}(A_3) = 0.5329, \quad Prj_{A^+}(A_4) = 0.5226.$$

Rank all the feasible alternatives according the values $Prj_{A^+}(A_i), (i = 1, 2, 3, 4)$:$A_3 \succ A_2 \succ A_1 \succ A_4$. Then, A_3 is the best alternative. Obviously, the ranks of the four emergency alternatives by the projection method and the proposed method are exactly the same.

To illustrate the validity of the proposed method, we further compare the ranking result of the proposed method in this paper with that of another existing method.

To aggregate picture fuzzy information, Grag [50] proposed some picture aggregation operators, namely picture fuzzy weighted averaging (PFWA) operator, picture fuzzy ordered weighted averaging (PFOWA) operator and picture fuzzy hybrid ordered weighted averaging (PFHOWA) operator. For the collective the picture fuzzy decision matrix R in Table 4, we can calculate the overall evaluation value of all alternatives by using the following picture fuzzy weighted averaging (PFWA) operator in [50]:

$$PFWA(a_1, a_2, \ldots, a_n) = \mathop{\oplus}_{j=1}^{n} w_j a_j = < h^{-1}(\sum_{j=1}^n w_j h(u_j)), g^{-1}(\sum_{j=1}^n w_j g(u_j)), g^{-1}(\sum_{j=1}^n w_j g(u_j)) > \tag{38}$$

The overall results of alternative $A_i (i = 1, 2, 3, 4)$ are shown as follows:

$$r_{A_1} = < 0.30, 0.22, 0.34 >, \quad r_{A_2} =< 0.32, 0.23, 0.33 >,$$
$$r_{A_3} = < 0.35, 0.24, 0.28 >, \quad r_{A_4} =< 0.27, 0.26, 0.32 > .$$

The score values of alternatives are given as follows:

$$S(A_1) = 0.5776, \quad S(A_2) = 0.5886,$$
$$S(A_3) = 0.6099, \quad S(A_4) = 0.5615.$$

TABLE 5. The results of different methods.

Methods	Values of alternatives	Ranking
Proposed method in this paper	$AS_1 = 0.2678$,	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$AS_2 = 0.5767$,	
	$AS_3 = 1$,	
	$AS_4 = 0$	
Projection method [25]	$Prj_{A^+}(A_1) = 0.5289$,	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$Prj_{A^+}(A_2) = 0.5320$,	
	$Prj_{A^+}(A_3) = 0.5329$,	
	$Prj_{A^+}(A_4) = 0.5226$	
Using PFWA operator in [50]	$S(A_1) = 0.5776$,	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$S(A_2) = 0.5886$,	
	$S(A_3) = 0.6099$,	
	$S(A_4) = 0.5615$	

Then we have $A_3 \succ A_2 \succ A_1 \succ A_4$, and A_3 is the best alternative. The score values $S(A_i)$ of alternatives $A_i (i = 1, 2, 3, 4)$ and ranking results are shown in Table 5.

As we can see from Table 5, the ranks of the four emergency alternatives by the existing two methods and the proposed method are exactly the same, and the evaluation results show that A_3 is the best emergency alternative, and A_4 is the worst. The comparison results demonstrate the effectiveness and reliability of the proposed approach. Moreover, by considering both the positive distance from the average solution and negative distance from the average solution at the same time under picture fuzzy environment, the method proposed in this paper can accurately reflect the reality.

VI. CONCLUSION

Many practical problems are often characterized by MAGDM. Because of lack of knowledge or data, and the decision makers' limited expertise about the problem domain, the attribute values provided by decision makers often take the form of picture fuzzy information. In this paper, a new method is proposed to solve MAGDM under picture fuzzy environment, in which the information about the weights of attributes is partly known and the weights of decision makers is completely unknown. The proposed method involves four main steps: (1) Some picture fuzzy interaction operators are presented, such as the PFWIA operator, the PFOWIA operator and the PFHOWIA operator. Simultaneously, some desirable properties of these operators are discussed in detail. (2) Determine the weights of decision makers under picture fuzzy setting based on the idea of the Dice similarity measure. (3) Establish an optimization model to determine the attribute weights on the basis of the maximizing deviation method. (4) Extend the traditional EDAS method to picture fuzzy environment and rank all alternatives. At the end of this paper, we give an example of practical application of the developed method to select the most desirable emergency alternative, and compare the proposed method with two existing methods. The comparison results demonstrate the effectiveness and practicality of the proposed approach.

The prominent characteristics of the developed method are that it can provide more reasonable and robust ranking results. Above all, it is simpler and more convenient to use in practical applications and it can be performed on computer easily. In future research, we will focus on the applications of the proposed method and extend it to picture fuzzy linguistic environment.

REFERENCES

[1] S.-M. Chen and J.-A. Hong, "Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets and the TOPSIS method," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 44, no. 12, pp. 1665–1673, Dec. 2014.

[2] P. Liu, J. Liu, and S.-M. Chen, "Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making," *J. Oper. Res. Soc.*, vol. 69, no. 1, pp. 1–24, Jan. 2018.

[3] F.-Y. Meng, Q.-X. An, C.-Q. Tan, and X.-H. Chen, "An approach for group decision making with interval fuzzy preference relations based on additive consistency and consensus analysis," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 2069–2082, Aug. 2017.

[4] Y. Xing, R. Zhang, M. Xia, and J. Wang, "Generalized point aggregation operators for dual hesitant fuzzy information," *J. Intell. Fuzzy Syst.*, vol. 33, no. 1, pp. 515–527, Jun. 2017.

[5] Z. Xu, "Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators," *Knowl.-Based Syst.*, vol. 24, no. 6, pp. 749–760, 2011.

[6] Z. Zhang, C. Guo, and L. Martínez, "Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 11, pp. 3063–3076, Nov. 2017.

[7] Y. Ju and S. Yang, "Approaches for multi-attribute group decision making based on intuitionistic trapezoid fuzzy linguistic power aggregation operators," *J. Intell. Fuzzy Syst.*, vol. 27, no. 2, pp. 987–1000, Jan. 2014.

[8] Y. Ju, X. Liu, and D. Ju, "Some new intuitionistic linguistic aggregation operators based on Maclaurin symmetric mean and their applications to multiple attribute group decision making," *Soft Comput.*, vol. 20, no. 11, pp. 4521–4548, 2016.

[9] P. Liu, S.-M. Chen, and P. Wang, "Multiple-attribute group decision-making based on q-rung orthopair fuzzy power maclaurin symmetric mean operators," *IEEE Trans. Syst., Man, Cybern., Syst.*, to be published. doi: 10.1109/TSMC.2018.2852948.

[10] P. Liu and J. Liu, "Some q-rung orthopai fuzzy Bonferroni mean operators and their application to multi-attribute group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 2, pp. 315–347, 2018.

[11] W. Yang and Y. Pang, "New pythagorean fuzzy interaction Maclaurin symmetric mean operators and their application in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 39241–39260, 2018.

[12] P. Liu, Q. Khan, T. Mahmood, and N. Hassan, "T-spherical fuzzy power Muirhead mean operator based on novel operational laws and their application in multi-attribute group decision making," *IEEE Access*, vol. 7, pp. 22613–22632, 2019.

[13] G. Wei, H. Garg, H. Gao, and C. Wei, "Interval-valued pythagorean fuzzy maclaurin symmetric mean operators in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 67866–67884, 2018.

[14] Y. Ju, "A new method for multiple criteria group decision making with incomplete weight information under linguistic environment," *Appl. Math. Model.*, vol. 38, nos. 21–22, pp. 5256–5268, Nov. 2014.

[15] P. Wu, J. Zhu, L. Zhou, and H. Chen, "Local feedback mechanism based on consistency-derived for consensus building in group decision making with hesitant fuzzy linguistic preference relations," *Comput. Ind. Eng.*, vol. 137, Nov. 2019, Art. no. 106001. doi: 10.1016/j.cie.2019.106001.

[16] P. Wu, L. Zhou, H. Chen, and Z. Tao, "Additive consistency of hesitant fuzzy linguistic preference relation with a new expansion principle for hesitant fuzzy linguistic term sets," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 4, pp. 716–730, Apr. 2019.

[17] H. Gao, Y. Ju, W. Zhang, and D. Ju, "Multi-attribute decision-making method based on interval-valued q-rung orthopair fuzzy archimedean muirhead mean operators," *IEEE Access*, vol. 7, pp. 74300–74315, 2019.

[18] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965.

[19] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, pp. 87–96, Aug. 1986.

[20] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets—A new concept for computational intelligence problems," in *Proc. 3rd World Congr. Inf. Commun. Technol. (WICT)*, Dec. 2013, pp. 1–6.

[21] P. Singh, "Correlation coefficients for picture fuzzy sets," *J. Intell. Fuzzy Syst.*, vol. 28, no. 2, pp. 591–604, 2015.

[22] L. H. Son, "DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets," *Expert Syst. Appl.*, vol. 42, no. 1, pp. 51–66, 2015.

[23] P. H. Thong and L. H. Son, "A novel automatic picture fuzzy clustering method based on particle swarm optimization and picture composite cardinality," *Knowl.-Based Syst.*, vol. 109, pp. 48–60, Oct. 2016.

[24] G. Wei, "Picture fuzzy cross-entropy for multiple attribute decision making problems," *J. Bus. Econ. Manage.*, vol. 17, no. 4, pp. 491–502, 2016.

[25] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Projection models for multiple attribute decision making with picture fuzzy information," *Int. J. Mach. Learn. Cybern.*, vol. 9, no. 4, pp. 713–719, 2018.

[26] L. H. Son, "Generalized picture distance measure and applications to picture fuzzy clustering," *Appl. Soft Comput.*, vol. 46, pp. 284–295, Sep. 2016.

[27] G. Wei, "Picture fuzzy aggregation operators and their application to multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 2, pp. 713–724, 2017.

[28] N. T. Thong and L. H. Son, "HIFCF: An effective hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis," *Expert Syst. Appl.*, vol. 42, no. 7, pp. 3682–3701, 2015.

[29] R.-X. Nie, J.-Q. Wang, and L. Li, "A shareholder voting method for proxy advisory firm selection based on 2-tuple linguistic picture preference relation," *Appl. Soft Comput.*, vol. 60, pp. 520–539, Nov. 2017.

[30] C. Wang, X. Zhou, H. Tu, and S. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making," *Italian J. Pure Appl. Math.*, vol. 37, pp. 477–492, Jan. 2017.

[31] S. Ashraf, T. Mahmood, S. Abdullah, and Q. Khan, "Different approaches to multi-criteria group decision making problems for picture fuzzy environment," *Bull. Brazilian Math. Soc., New Ser.*, vol. 50, no. 2, pp. 373–397, 2019. doi: 10.1007/s00574-018-0103-y.

[32] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Trans. Syst., Man, Cybern.*, vol. 18, no. 1, pp. 183–190, Jan./Feb. 1988.

[33] R. R. Yager, "Generalized OWA aggregation operators," *Fuzzy Optim. Decision Making*, vol. 3, no. 1, pp. 93–107, Mar. 2004.

[34] J. M. Merigo and M. Casanovas, "Fuzzy generalized hybrid aggregation operators and its application in fuzzy decision making," *Int. J. Fuzzy Syst.*, vol. 12, no. 1, pp. 15–24, Mar. 2010.

[35] B. C. Cuong and P. Van Hai, "Some fuzzy logic operators for picture fuzzy sets," in *Proc. 7th Int. Conf. Knowl. Syst. Eng. (KSE)*, Oct. 2015, pp. 132–137.

[36] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Picture 2-tuple linguistic aggregation operators in multiple attribute decision making," *Soft Comput.*, vol. 22, no. 2, pp. 989–1002, 2018.

[37] S. Opricovic and G.-H. Tzeng, "Compromise solution by MCDM methods: A comparative analysis of VIKOR and TOPSIS," *Eur. J. Oper. Res.*, vol. 156, pp. 445–455, Jul. 2004.

[38] M. K. Ghorabaee, E. K. Zavadskas, L. Olfat, and Z. Turskis, "Multi-criteria inventory classification using a new method of evaluation based on distance from average solution (EDAS)," *Informatica*, vol. 26, no. 3, pp. 435–451, Mar. 2015.

[39] M. K. Ghorabaee, E. K. Zavadskas, M. Amiri, and Z. Turskis, "Extended EDAS method for fuzzy multi-criteria decision-making: An application to supplier selection," *Int. J. Comput. Commun.*, vol. 11, no. 3, pp. 358–371, 2016.

[40] X. Peng and L. Chong, "Algorithms for neutrosophic soft decision making based on EDAS, new similarity measure and level soft set," *J. Intell. Fuzzy Syst.*, vol. 32, no. 1, pp. 955–968, 2017.

[41] F. Ecer, ''Third-party logistics (3PLs) provider selection via Fuzzy AHP and EDAS integrated model,'' *Technol. Econ. Develop. Economy*, vol. 24, no. 2, pp. 615–634, 2018.
[42] Y. He, H. Chen, L. Zhou, J. Liu, and Z. Tao, ''Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making,'' *Inf. Sci.*, vol. 259, pp. 142–159, Feb. 2014.
[43] Y. He, H. Chen, L. Zhou, B. Han, Q. Zhao, and J. Liu, ''Generalized intuitionistic fuzzy geometric interaction operators and their application to decision making,'' *Expert Syst. Appl.*, vol. 41, no. 5, pp. 2484–2495, Apr. 2014.
[44] Y. Ju, D. Ju, and A. Wang, ''A note on 'picture 2-tuple linguistic aggregation operators in multiple attribute decision making,''' *Soft. Comput.*, pp. 1–5, Jun. 2019. doi: 10.1007/s00500-019-04162-5.
[45] Y. Ju, D. Ju, E. D. R. S. Gonzalez, M. Giannakis, and A. Wang, ''Study of site selection of electric vehicle charging station based on extended GRP method under picture fuzzy environment,'' *Comput. Ind. Eng.*, vol. 135, pp. 1271–1285, Sep. 2019.
[46] L. R. Dice, ''Measures of the amount of ecologic association between species,'' *Ecology*, vol. 26, no. 3, pp. 297–302, 1945.
[47] G. Wei and H. Gao, ''The generalized Dice similarity measures for picture fuzzy sets and their applications,'' *Information*, vol. 29, pp. 107–124, Sep. 2018.
[48] K. S. Park, S. H. Kim, and W. C. Yoon, ''Establishing strict dominance between alternatives with special type of incomplete information,'' *Eur. J. Oper. Res.*, vol. 96, no. 2, pp. 398–406, Jan. 1997.
[49] Y. Wang, ''Using the method of maximizing deviation to make decision for multiindices,'' *J. Syst. Eng. Electron.*, vol. 8, no. 3, pp. 21–26, Sep. 1997.
[50] H. Garg, ''Some picture fuzzy aggregation operators and their applications to multicriteria decision-making,'' *Arabian J. Sci. Eng.*, vol. 42, no. 12, pp. 5275–5290, 2017.

XIA LI received the B.S. and M.S. degrees in transportation planning and management from Jilin University, Changchun, China, in 2001 and 2004, respectively, and the Ph.D. degree in urban transport planning and management from Beijing Jiaotong University, Beijing, China, in 2010. In 2008, she was a Research Assistant with the Transport Research Center of Technical University of Madrid. From 2010 to 2013, she was a Postdoctoral Fellow with the School of Mechanical Engineering, Beijing Institute of Technology. Since 2013, she has been an Assistant Professor with the Department of Management Science and Logistics, School of Management and Economics, Beijing Institute of Technology. Her research interests include urban transport planning and management, discrete system modeling and simulation, and urban transport environment.

YANBING JU received the Ph.D. degree in management science and engineering from the Beijing University of Aeronautics and Astronautics (Beihang University), China, in 2003. He is currently a Professor in management science and engineering with the Beijing Institute of Technology. He has published more than 80 articles in journals and conference proceeding, including journals, such as *Applied Mathematical Modelling*, *Computers and Industrial Engineering*, *Soft Computing*, *Expert Systems with Applications*, the *International Journal of Intelligent Systems*, the *Journal of Intelligent and Fuzzy Systems*, and *Natural Hazards*. His current research interests include decision making under uncertainty, discrete event system simulation and optimization, and emergency management.

DAWEI JU received the B.A. degree in communication science from the South China University of Technology, in 2009, and the M.B.A. and Ph.D. degrees in management science and engineering from the Beijing Institute of Technology, in 2014 and 2018, respectively. He is currently an Engineer with Postal Scientific Research and Planning Academy. He is the author of several articles in SCI journals, such as *Computers and Industrial Engineering*, *Soft Computing*, and the *Journal of Intelligent and Fuzzy Systems*. His research interests include fuzzy multi-criteria group decision making, system evaluation method, and emergency management.

WENKAI ZHANG received the B.S. degree in management from the Shandong University of Science and Technology, Qingdao, China, in 2012, and the Ph.D. degree in management science and engineering from the Beijing Institute of Technology, Beijing, China, in 2018. He is currently a Lecturer with the School of Economics and Management, China University of Geosciences (Beijing), Beijing. His current research interests include preference modeling, decision analysis, and group decision making.

PEIWU DONG received the Ph.D. degree from the Department of Management, Harbin Institute of Technology, China, in 2001. He is currently a Professor and an Assistant Dean with the School of Management and Economics, Beijing Institute of Technology, and the Deputy Director Member of Business Administration Teaching Guidance Committee for the Ministry of Education, China. He has published more than 60 articles in journals and conference proceeding. His current research interests include decision theory and method, complex system modeling, management system engineering, investment financing, and risk management.

AIHUA WANG received the B.S. degree in mathematics and applied mathematics from Northwest University, Xi'an, China, in 1991, the M.S. degree in computer science and technology from Harbin Engineering University, Harbin, China, in 1998, and the Ph.D. degree in computer science and technology from Peking University, Beijing, China, in 2002, where she is currently an Associate Professor in education technology. She has published more than 60 articles in journals and conference proceeding, including journals, such as *Applied Mathematical Modelling*, *Computers and Industrial Engineering*, *Soft Computing*, *Expert Systems with Applications*, the *International Journal of Intelligent Systems*, the *Journal of Intelligent and Fuzzy Systems*, and *Natural Hazards*. Her current research interests include education technology theory and method, decision making theory and method, and discrete event system simulation and modeling.