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Cyclic Super Magic Labelings for Toroidal and Klein-Bottle Fullerenes

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ABSTRACT A simple graph $G = (V, E)$ admits a cycle-covering if every edge in E belongs at least to one subgraph of G isomorphic to a given cycle C. The graph G is C-magic if there exists a total labeling $f : V \cup E \rightarrow \{1, 2, 3, ..., |V| + |E|\}$ such that for every subgraph $H' = (V', E')$ of G isomorphic to C, $\sum_{V \in V'} f(V) + \sum_{E \in E'} f(E)$ is constant, when $f(V) = 1, 2, 3, ..., |V|$. Then G is said to be C-supermagic. In the present paper, we investigate the cyclic-supermagic behavior of toroidal and Klein-bottle graph.

INDEX TERMS Toroidal fullerene, Klein-bottle fullerene, toroidal polyhex, Klein-bottle polyhex, cyclic supermagic labeling.

I. INTRODUCTION AND DEFINITIONS

Fullerenes, the third type of carbon, have ended up essential atoms in science and innovation. In [1], [6], a massive literature is discussed about fullerenes along with their applications in science and innovation. Fullerenes are atoms made altogether out of carbon that were found in 1985 at Rice University. A large number of the viable uses of fullerenes take after straightforwardly from their uncommon properties. Significant chemical, physical and optical properties have been discussed in [7], which makes fullerenes key segments for the eventual fate of nano-electromechanical frameworks. The most significant properties are displayed as its high electron affinity and oxidation of the atom. Fullerenes are amazingly solid atoms, ready to oppose extraordinary weights. In optical properties fullerenes have indicated specific guarantees in optical restricting and power subordinate refractive list. The uses of fullerene are utilizations in solar cell, hydrogen gas storage devices, harden metals and alloys, interdigitated capacitors (IDCs), treatment of AIDS and also in magnetic resonance imaging (MRI).

Let $G = (V(G), E(G))$ be a finite simple graph, where $V(G)$ denote vertex set and $E(G)$ denote edge set of G. The cardinality of vertices is denoted by $|V(G)|$ and cardinality of edges denoted by $|E(G)|$ respectively.

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Let $K_1, K_2, \ldots, K_t$ be a collection of subgraphs of G then G has $(K_1, K_2, \ldots, K_t)$-*edge-covering*, if every edge of $E(G)$ belongs to a subgraph K_i, $1 \leq i \leq t$. If each subgraph K_i is isomorphic to some graph K, then G admits K-*covering*.

Consider G has K-covering. A bijective function f from the set $V(G) \cup E(G)$ into the set $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called total labeling of G. The weight of a subgraph K' of G under f is the sum of vertices labels and edges labels correspond to K'. The total labeling f is a K-*magic labeling* of G if the weight of each subgraph isomorphic to K is equal to a given constant C. The graph G is K-*magic* if G has K-*magic labeling*. Moreover, if we used smallest possible labels for vertices of G, that is if $f(v) \in \{1, 2, 3, \ldots, |V(G)|\}$, then G is said to be H-*supermagic*.

In 2005 Gutierrez and Llado [18] first introduced the idea of H-supermagic labeling. In [18] they showed that for some values of n the complete bipartite graph $K_{n,m}$ are $K_{1,n}$-supermagic. They also showed that for some n the path P_n and the cycle C_n are P_h-supermagic. Later in 2007 Llado and Moragas [23] described some C_n-super-magic graphs. Jeyanthi and Selvagopal [27], [29] discussed the H-super magic strength of graphs and some C_4-super magic graphs moreover some results of C_3-super magic graphs can be found in [28]. In [24] the author showed that if a graph is C-supermagic then disjoint union of G denoted by mG for $2 \leq m$ is also C-supermagic. In [26] the authors proved that the generalized splitting graph is cyclic supermagic.

A complete survey of cyclic super magic graphs can be seen in [8]. For more details about labeling and its applications, we refer [9]–[17]

II. GRAPHICAL IDENTIFICATIONS OF TOROIDAL AND KLEIN-BOTTLE FULLERENES

Fullerene's extension was considered by Deza et al. to different surfaces and determined that only four surfaces i.e
(1)Torus
(2)Sphere
(3)Klein-bottle
(4)Projective plane
are compatible. In contrast to spherical or globular fullerenes, the two fullerenes toroidal and Klein-bottle have been seen as combined groups of perfect hexagons on their faces having no pentagons.

A cubic bipartite graph that can be drawn on the surface of torus is called the toroidal fullerene such that the each face of it is a hexagon. In [22], [30], remarkable work have been done in the context of full identification of toroidal fullerenes by employing diversified procedure. Some results about k-resonance of toroidal fullerenes and Klein-bottle polyhex can be found in [31]–[33]. 2-extendability of toroidal and Klein-bottle polyhexes is explained in [34]. The extrema fullerene graph with maximum Clar number is investigated in [35]. Numan *et al.* [4] explored the labeling (1,1,1) for toroidal fullerenes. Butt *et al.* [3] proposed the labeling (1,1,1) for Klein-bottle fullerenes. Moreover the total edge irregular strength of toroidal fullerene is investigated in [2].

In [19] and [20] the chemical relevance were studied for some features of toroidal polyhexes. For example, for the counting of isomers of toroidal polyhexes a systematic coding and classification scheme was given, the count of spanning trees and the calculation of the spectrum were discussed.

If we have a fundamental parallelogram then toroidal and Klein-bottle polyhexes can be constructed by Klein-bottle and toroidal identifications in appropriate orientation. Consider a regular hexagonal lattice L. Let cut a quadrilateral section P_n^m where m is the number of hexagon on the bottom or top side and n is the number of hexagons on lateral sides, see Figures [1] and [2]. We will use toroidal and Kelin bottle identification on P_n^m that is first we identify the lateral sides to form cylinder and then identify the top and bottom sides of the constructed cylinder in the same direction and in the opposite direction to form toroidal fullerenes $\mathbb{H}_n^m$ and Klein-bottle fullerene $\mathbb{K}_n^m$ respectively.

We consider the toroidal polyhex $\mathbb{H}_n^m$, where the vertex set $V(\mathbb{H}_n^m)$, edge set $E(\mathbb{H}_n^m)$ and face set $F(\mathbb{H}_n^m)$ are defined as follows:

$$V(\mathbb{H}_n^m) = \{u_i^j, v_i^j : 0 \le i \le m-1, 0 \le j \le n-1\},$$
$$E(\mathbb{H}_n^m) = \{u_i^j v_i^{j-1}, v_i^j u_i^{j+1} : 0 \le i \le m-1, \text{for } j \text{ even}\} \cup\{u_i^j v_{i+1}^{j-1}, v_i^j u_{i+1}^{j+1} : 0 \le i \le m-1, \text{for } j \text{ odd}\} \cup\{u_i^j v_i^j : 0 \le i \le m-1, 0 \le j \le n-1\}.$$

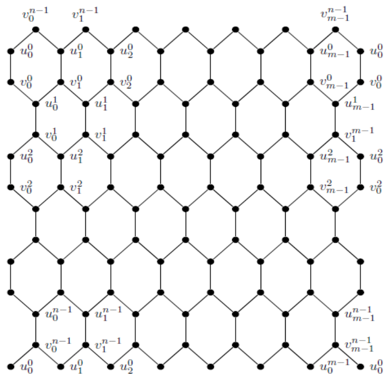


FIGURE 1. Toroidal fullerenes $\mathbb{H}_n^m$ identification.

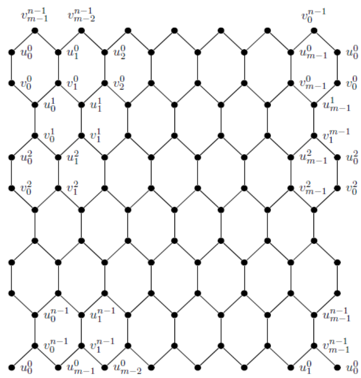


FIGURE 2. Klein-bottle fullerenes $\mathbb{K}_n^m$ identification.

Moreover the Klein-bottle polyhex $\mathbb{K}_n^m$ having the vertex set $V(\mathbb{K}_n^m)$ and edge set $E(\mathbb{K}_n^m)$ define as:

$$V(\mathbb{K}_n^m) = \{u_i^j, v_i^j : 0 \le i \le m-1, 0 \le j \le n-1\},$$
$$E(\mathbb{K}_n^m) = \{u_i^j v_i^{j-1}, v_i^j u_i^{j+1} : 0 \le i \le m-1, 2 \le j \le n-1, \text{for } j \text{ even}\} \cup\{u_i^j v_{i+1}^{j-1}, v_i^j u_{i+1}^{j+1} : 0 \le i \le m-1, 2 \le j \le n-1, \text{for } j \text{ odd}\} \cup\{u_i^j v_i^j : 0 \le i \le m-1, 0 \le j \le n-1\} \cup\{u_i^0 v_{m-1-i}^{n-1}, u_{i+1}^0 v_{m-1-i}^{n-1}, v_i^{n-1} u_{m-1-i}^0, v_i^{n-1} u_{m-i}^0 : 0 \le i \le m-1\}.$$

The index i is taken modulo m and the index j is taken modulo n. The cardinality of $|V(\mathbb{H}_n^m)| = |V(\mathbb{K}_n^m)| = 2mn$ and $|E(\mathbb{H}_n^m)| = |E(\mathbb{K}_n^m)| = 3mn$.

In the present paper we construct the cyclic super magic labeling for toroidal, Klein-bottle fullerenes, disjoint union and subdivision of toroidal and Klein-bottle fullerenes. We will show toroidal, Klein-bottle fullerenes and disjoint union of toroidal and Klein-bottle are C_6-supermagic. Moreover the subdivision of toroidal fullerene, Klein-bottle fullerene and any graph homeomorphic to toroidal fullerene or Klein-bottle fullerene are cyclic-supermagic.

III. METHODOLOGY

First, we embedded the fullerenes graph on the surface of torus and Klein-bottle by using torus and Klein-bottle identification which are called toroidal and Klein-bottle fullerenes graphs. Secondly, we partition the vertex and edge sets of toroidal and Klein-bottle fullerenes in different categories. Thirdly, we use number theory for set partition to label the vertices and edges of toroidal and Klein-bottle fullerenes and then we use Langrange interpolartion to construct labeling formulas for our constructed labeling scheme. At the end we use super cyclic magic graph definition and calculus to derive our results for toroidal and Klein-bottle fullerenes.

IV. CYCLIC SUPER MAGIC LABELING OF TORODIAL AND KLEIN-BOTTLE POLYHEXES

Toroidal polyhex can be covered by cycles having six sides. Let z_i^j, $0 \le i \le m-1$, $0 \le j \le n-1$, be the 6-sided cycles covering $\mathbb{H}_m^n$ define as follows:

$$z_i^j = \begin{cases} u_i^j v_{i+1}^{j-1} u_{i+1}^j v_{i+1}^j u_{i+1}^{j+1} v_i^j, & \text{if } j \text{ is odd} \\ u_i^j v_i^{j-1} u_{i+1}^j v_{i+1}^j u_i^{j+1} v_i^j, & \text{if } j \text{ is even.} \end{cases}$$

Let g be a total labeling of $\mathbb{H}_n^m$. The weights of the cycles z_i^j, for $0 \le i \le m-1$, as follows:

For j odd, $1 \le j \le n-1$

$$\begin{aligned} wt(z_i^j) = {} & g(u_i^j) + g(u_i^j v_{i+1}^{j-1}) + g(v_{i+1}^{j-1}) + g(v_{i+1}^{j-1} u_{i+1}^j) \\ & + g(u_{i+1}^j) + g(u_{i+1}^j v_{i+1}^j) + g(v_{i+1}^j) \\ & + g(v_{i+1}^j u_{i+1}^{j+1}) + g(u_{i+1}^{j+1}) + g(u_{i+1}^{j+1} v_i^j) \\ & + g(v_i^j) + g(v_i^j u_i^j) + g(z_i^j) \end{aligned} \tag{1}$$

and for j even, $0 \le j \le n-2$

$$\begin{aligned} wt(z_i^j) = {} & g(u_i^j) + g(u_i^j v_i^{j-1}) + g(v_i^{j-1}) + g(v_i^{j-1} u_{i+1}^j) \\ & + g(u_{i+1}^j) + g(u_{i+1}^j v_{i+1}^j) + g(v_{i+1}^j) \\ & + g(v_{i+1}^j u_i^{j+1}) + g(u_i^{j+1}) + g(u_i^{j+1} v_i^j) + g(v_i^j) \\ & + g(v_i^j u_i^j) + g(z_i^j). \end{aligned} \tag{2}$$

Theorem 1: For n even, $n \ge 4$ and $m \ge 2$, the toroidal polyhex $\mathbb{H}_n^m$ admits C_6-supermagic labeling.

Proof: For $0 \le i \le m-1$, we define a labeling g that assigns values from 1 up to $5mn$ to the vertices and edges of $\mathbb{H}_n^m$ in the following way

$$\begin{aligned} g(u_i^j) &= mj + i + 1, \quad \text{for } 1 \le j \le n-1 \\ g(v_i^{n-1}) &= 2mn - i \\ g(v_i^j) &= (2n - j - 1)m - i, \quad \text{for } 0 \le j \le n-2 \\ g(u_i^j v_i^j) &= (5n - j - 1)m + i + 1, \quad \text{for } 0 \le j \le n-1 \\ g(u_i^{j+1} v_i^j) &= \tfrac{(7n+j+2)m-2i}{2}, \quad \text{for } 0 \le j \le n-2, \text{ if } j \text{ is even} \\ g(u_i^j v_i^{j-1}) &= \tfrac{(5n+j)m+2+2i}{2}, \quad \text{for } 2 \le j \le n-2, \text{ if } j \text{ is even} \\ g(u_i^j v_{i+1}^{j-1}) &= \tfrac{(6n+j+1)m-2i}{2}, \quad \text{for } 1 \le j \le n-1, \text{ if } j \text{ is odd.} \end{aligned}$$

We label the remaining vertices and edges as follows;

$$g(u_i^0) = \begin{cases} 1, & \text{for } i = 0 \\ m - i + 1, & \text{for } 1 \le i \le m-1 \end{cases}$$

$$g(u_i^0 v_i^{n-1}) = \tfrac{5mn}{2} + 1 + i, \quad \text{for } 0 \le i \le m-1$$

$$g(v_i^{n-1} u_{i+1}^0) = \begin{cases} 2mn + m - 1 - i, & \text{for } 0 \le i \le m-2 \\ 2mn + m, & \text{for } i = m-1 \end{cases}$$

$$g(u_{i+1}^{j+1} v_i^j) = \begin{cases} \tfrac{(4n+j+3)m-2i-2}{2}, \\ \quad \text{for } 0 \le i \le m-2,\ 1 \le j \le n-3, \text{ if } j \text{ is odd} \\ \tfrac{(4n+j+3)m}{2}, \\ \quad \text{for } i = m-1,\ 1 \le j \le n-3, \text{ if } j \text{ is odd.} \end{cases}$$

It is easy to verify that the labeling g used each integer from the set $\{1, 2, \dots, 5mn\}$ exactly once and the vertices are labeled with the numbers $\{1, 2, \dots, 2mn\}$. By using (1), (2) and the total lableing g defined above the weights of 6-sided cycles z_i^j attain the values

$$wt_g(z_i^j) = 33mn + 6, \quad \text{for } 0 \le i \le m-1,\ 0 \le j \le n-1.$$

As all the cycles weights are equal to $33mn + 6$ we get that g is a C_6-supermagic labeling of $\mathbb{H}_n^m$. ■

Like toroidal polyhex $\mathbb{H}_n^m$ Klein-bottle polyhex $\mathbb{K}_n^m$ is also covered by 6-sided cycles.

Let z_i^j, $0 \le i \le m-1$ be 6-sided cycle covering $\mathbb{K}_n^m$ define as:

$$z_i^j = \begin{cases} u_i^j v_{i+1}^{j-1} u_{i+1}^j v_{i+1}^j u_{i+1}^{j+1} v_i^j, & \text{if } j \text{ is odd}, j \ne n-1 \\ u_i^j v_i^{j-1} u_{i+1}^j v_{i+1}^j u_i^{j+1} v_i^j, & \text{if } j \text{ is even}, j \ne 0 \\ u_0^{n-1} v_{i+1}^{n-2} u_{i+1}^{n-1} v_{i+1}^{n-1} u_{m-1-i}^0 v_i^{n-1}, & \text{if } j = n-1 \\ u_i^0 v_{m-1-i}^{n-1} u_{i+1}^0 v_{i+1}^0 u_i^1 v_i^0, & \text{if } j = 0. \end{cases}$$

Let h be a total labeling of $\mathbb{K}_n^m$. The weights of the cycles z_i^j, for $0 \le i \le m-1$, as follows:

For j odd, $1 \le j \le n-2$

$$\begin{aligned} wt(z_i^j) = {} & h(u_i^j) + h(u_i^j v_{i+1}^{j-1}) + h(v_{i+1}^{j-1}) + h(v_{i+1}^{j-1} u_{i+1}^j) \\ & + h(u_{i+1}^j) + h(u_{i+1}^j v_{i+1}^j) + h(v_{i+1}^j) \\ & + h(v_{i+1}^j u_{i+1}^{j+1}) + h(u_{i+1}^{j+1}) + h(u_{i+1}^{j+1} v_i^j) \\ & + h(v_i^j) + h(v_i^j u_i^j) \end{aligned} \tag{3}$$

and $j = n-1$

$$\begin{aligned} wt(z_i^{n-1}) = {} & h(u_i^{n-1}) + h(u_i^{n-1} v_{i+1}^{n-2}) + h(v_{i+1}^{n-2}) \\ & + h(v_{i+1}^{n-2} u_{i+1}^{n-1}) + h(u_{i+1}^{n-1}) + h(u_{i+1}^{n-1} v_{i+1}^{n-1}) \\ & + h(v_{i+1}^{n-1}) + h(v_{i+1}^{n-1} u_{m-1-i}^0) + h(u_{m-1-i}^0) \\ & + h(u_{m-1-i}^0 v_i^{n-1}) + h(v_i^{n-1}) + h(v_i^{n-1} u_i^{n-1}). \end{aligned} \tag{4}$$

For j even, $2 \le j \le n-2$

$$\begin{aligned} wt(z_i^j) = {} & h(u_i^j) + h(u_i^j v_i^{j-1}) + h(v_i^{j-1}) \\ & + h(v_i^{j-1} u_{i+1}^j) + h(u_{i+1}^j) + h(u_{i+1}^j v_{i+1}^j) \end{aligned}$$

$$+h(v_{i+1}^{j}) + h(v_{i+1}^{j}u_i^{j+1}) + h(u_i^{j+1})$$
$$+h(u_i^{j+1}v_i^{j}) + h(v_i^{j}) + h(v_i^{j}u_i^{j}) \qquad (5)$$

and for $j = 0$

$$\begin{aligned} wt(z_i^0) = & h(u_i^0) + h(u_i^0 v_{m-1-i}^{n-1}) + h(v_{m-1-i}^{n-1}) \\ & +h(v_{m-1-i}^{n-1}u_{i+1}^0) + h(u_{i+1}^0) + h(u_{i+1}^0 v_{i+1}^0) \\ & +h(v_{i+1}^0) + h(v_{i+1}^0 u_i^1) + h(u_i^1) \\ & +h(u_i^1 v_i^0) + h(v_i^0) + h(v_i^0 u_i^0). \end{aligned} \qquad (6)$$

Theorem 2: For n even, $n \geq 4$ and $m \geq 2$, the Klein-bottle polyhex $\mathbb{K}_n^m$ has C_6-cyclic supermagic labeling

Proof: Let h be a total labeling defined from the set $V(\mathbb{K}_n^m) \cup E(\mathbb{K}_n^m)$ to $\{1, 2, \ldots, 5mn\}$ in the following way:

$$h(u_i^j) = \begin{cases} mj + i + 1, & \text{for } 0 \leq i \leq m-1, 0 \leq j \leq n-2 \\ mn - i, & \text{for } 0 \leq i \leq m-1, j = n-1 \end{cases}$$

$$h(v_i^{n-1}) = (2n-1)m + i + 1, \quad \text{for } 0 \leq i \leq m-1$$

$$h(v_i^j) = \begin{cases} (2n-j-1)m - i, & \text{for } 0 \leq i \leq m-1, \\ & 0 \leq j \leq n-2, j \neq n-3 \\ (n+2)m - 1 - i, & \text{for } 0 \leq i \leq m-2, \\ & j = n-3 \\ (n+2)m, & \text{for } i = m-1, j = n-3 \end{cases}$$

$$h(u_i^0 v_{m-i}^{n-1}) = \begin{cases} (3n+1)m - i, & \text{for } 1 \leq i \leq m-1 \\ (3n+1)m + 1, & \text{for } i = 0 \end{cases}$$

$$h(u_{i+1}^0 v_{m-1-i}^{n-1}) = \begin{cases} 2mn + (m-1) - i, & \text{for } 0 \leq i \leq m-2 \\ 2mn + m, & \text{for } i = m-1 \end{cases}$$

$$h(u_i^j v_i^j) = \begin{cases} (5n-j-1)m + i + 1, & \text{for } 0 \leq i \leq m-1, \\ & 0 \leq j \leq n-1, \\ & j \neq n-3 \\ (4n+2)m + 2 + i, & \text{for } 0 \leq i \leq m-2, \\ & j = n-3 \\ (4n+2)m + 1, & \text{for } i = m-1, \\ & j = n-3 \end{cases}$$

$$h(u_i^{j+1} v_i^j) = \begin{cases} \frac{(5n+j)m+2i+2}{2}, & 0 \leq i \leq m-1, \\ & 0 \leq j \leq n-2, \text{ if } j \text{ is even} \end{cases}$$

$$h(u_i^j v_i^{j-1}) = \begin{cases} \frac{(6n+j+2)m-2i}{2}, & \text{for } 0 \leq i \leq m-1, \\ & 2 \leq j \leq n-2, \text{ if } j \text{ is even} \end{cases}$$

$$h(u_{i+1}^{j+1} v_i^j) = \begin{cases} \frac{(4n+j+3)m-2i-2}{2}, & \text{for } 0 \leq i \leq m-2, \\ & 1 \leq j \leq n-3 \text{ if } j \text{ is odd} \\ \frac{(4n+j+3)m}{2}, & \text{for } i = m-1, \\ & 1 \leq j \leq n-3 \text{ if } j \text{ is odd} \end{cases}$$

$$h(u_i^j v_{i+1}^{j-1}) = \begin{cases} \frac{(7n+j+1)m-2i}{2}, & \text{for } 0 \leq i \leq m-1, \\ & 1 \leq j \leq n-3 \text{ if } j \text{ is odd} \\ \frac{(8n-2)m+4+2i}{2}, & \text{for } 0 \leq i \leq m-2, j = n-1 \\ \frac{(8n-2)m+2}{2}, & \text{for } i = m-1, j = n-1. \end{cases}$$

It is easy to verify that the labeling h used each integer from the set $\{1, 2, \ldots, 5mn\}$ exactly once and the vertices are labeled with the numbers $\{1, 2, \ldots, 2mn\}$. By using (3)-(6) and total labeling h defined above, the weight of 6-sided cycles is

$$wt_g(z_i^j) = 27mn + 6, \quad \text{for } 0 \leq i \leq m-1, 0 \leq j \leq n-1.$$

Since weight of all 6-sided cycles is $27mn + 6$, which showed that the Klein-bottle polyhex $\mathbb{K}_n^m$ is C_6-supermagic. ■

V. CYCLIC SUPER MAGIC LABELING OF SOME GRAPHS DERIVED FROM TORODIAL AND KLEIN-BOTTLE POLYHEXES

If we have a graph G then r isomorphic copies of G denoted by rG that is disjoint union of r copies of G, where $r \geq 2$. M. Asif *et al.* [24] proved that if a graph G is cyclic-supermagic then the disjoint union rG of G is also cyclic-supermagic for $r \geq 2$. From this result we conclude the following result for disjoint union of torodial and Klein-bottle polyhexes.

Theorem 3: For n even, $n \geq 4$, $m \geq 2$ and $r \geq 2$, the disjoint union $r\mathbb{H}_n^m$, $r\mathbb{K}_n^m$ of toroidal polyhex $\mathbb{H}_n^m$ and Klein-bottle polyhex $\mathbb{K}_n^m$ admits C_6-supermagic labelings.

Consider a graph G then a subdivision of G is denoted by $S(G)$ is obtained from G by inserting new vertices on edges of G more precisely $S(G)$ can obtained from G if we replace edges of G by paths. If G had H-covering then $S(G)$ has $S(H)$-covering where $S(H)$ denote subdivision of H. Numan *et al.* [36] showed that if a graph G is H-supermagic then $S(G)$ is $S(H)$-supermagic. From the result of [36] we can obtain the following theorem.

Theorem 4: For n even, $n \geq 4$ and $m \geq 2$, the subdivided graphs $S(\mathbb{H}_n^m)$, $S(\mathbb{K}_n^m)$ of toroidal polyhex $\mathbb{H}_n^m$ and Klein-bottle polyhex $\mathbb{K}_n^m$ admits $S(C_6)$-supermagic labelings.

Next we extend our study to homeomorphic graphs. Two graphs G_1 and G_2 are homeomorphic to each other if they obtained from same graph G by using the operation of subdivision.

Theorem 5: Let G_1 and G_2 are two simple graphs obtained from the simple graph G. If G had cyclic-supermagic labeling then G_1 and G_2 also admits cyclic-super magic labeling.

The proof of the above theorem is a consequence of the result in [36]. Since G_1 and G_2 can obtained from subdivision of G and the graph G is cyclic-supermagic therefore by using results of [36] both G_1 and G_2 are cyclic-supermagic. From Theorem 5 we can obtain that any graph G homeomorphic to the graph $\mathbb{H}_n^m$ or $\mathbb{K}_n^m$ is cyclic-supermagic.

VI. CONCLUSION

In this paper first we show that the toroidal polyhex $\mathbb{H}_n^m$ and Klein-bottle polyhex $\mathbb{K}_n^m$ for every $m \geq 2$ and even $n \geq 4$ is C_6-supermagic. Secondly, we show that for every $m \geq 2$, $r \geq 2$ and even $n \geq 4$ the disjoint union $r(\mathbb{H}_n^m 0$ of toroidal polyhex and disjoint union $r(\mathbb{K}_n^m)$ Klein-bottle polyhex are C_6-supermagic. Finally, we show that any graph homeomorphic to $\mathbb{H}_n^m$ or $\mathbb{K}_n^m$ is cyclic-supermagic. We conclude our paper with the following open problem.

VII. OPEN PROBLEM

Is the Klein-bottle polyhex $\mathbb{K}_n^m$ is C_6-supermagic for other cases of Klein-bottle? Also it is interesting problem to find out C_6-supermagic labeling for the families of graphs studied in [37]–[41].

REFERENCES

[1] A. W. Jensen, S. R. Wilson, and D. I. Schuster, "Biological applications of fullerenes," in *Bioorganic & Medicinal Chemistry*. New York, NY, USA: Elsevier, 1996.

[2] M. Bača, M. Lascsáková, and M. K. Siddiqui, "Total edge irregularity strength of toroidal fullerene," *Math. Comput. Sci.*, vol. 7, no. 4, pp. 487–492, 2014.

[3] S. I. Butt, M. Numan, and S. Qaisar, "Labelings of type (1, 1, 1) for Klein bottle fullerenes," *J. Math. Chem.*, vol. 54, no. 2, pp. 428–441, 2016.

[4] M. Bača, M. Numan, and A. Shabbir, "Labelings of type (1,1,1) for toroidal fullerenes," *Turkish J. Math.*, vol. 37, no. 6, pp. 899–907, 2013.

[5] M. Deza, P. W. Fowler, A. Rassat, and K. M. Rogers, "Fullerenes as tilings of surfaces," *J. Chem. Inf. Comput. Sci.*, vol. 40, no. 3, pp. 550–558, 2000.

[6] M. S. Dresselhaus, G. Dresselhaus, and P. C. Eklund, *Science of Fullerenes and Carbon Nanotubes*. Boston, MA, USA: Academic, 1996.

[7] *Fullerenes and Their Applications in Science and Technology*, Eulises Ulloa 4138296, EEE-5425, Introduction to Nanotechnology Spring, 2013.

[8] J. A. Gallian, "A dynamic survey of graph labeling," *Electron. J. Combinat.*, vol. 17, 2018.

[9] M. A. Umar, N. Ali, A. Tabassum, and B. R. Ali, "Book graphs are cycle antimagic," *Open J. Math. Sci.*, vol. 3, no. 1, pp. 184–190, 2019.

[10] M. A. Umar, "Cyclic-antimagic construction of ladders," *Eng. Appl. Sci. Lett.*, vol. 2, no. 2, pp. 43–47, 2019.

[11] L. Chen, H. Jiang, Z. Shao, and M. Ivanović, "Dominator and total dominator colorings in vague graphs," *Eng. Appl. Sci. Lett.*, vol. 2, no. 2, pp. 10–17, 2019.

[12] F. Aslam, Z. Zahid, and S. Zafar, "3-total edge mean cordial labeling of some standard graphs," *Open J. Math. Sci.*, vol. 3, no. 1, pp. 129–138, 2019.

[13] N. Ali, M. A. Umar, A. Tabassum, and A. Raheem, "Super (a,d)-C_3-antimagicness of a corona graph," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 371–378, 2018.

[14] M. M. Kumar and H. M. Nagesh, "Directed pathos total digraph of an arborescence," *Eng. Appl. Sci. Lett.*, vol. 1, no. 1, pp. 29–42, 2018.

[15] W. Gao, M. Asif, and W. Nazeer, "The study of honey comb derived network via topological indices," *Open J. Math. Anal.*, vol. 2, no. 2, pp. 10–26, 2018.

[16] H. Nagesh and M. M. Kumar, "Block digraph of a directed graph," *Open J. Math. Sci.*, vol. 2, no. 3, pp. 202–208, Jun. 2018.

[17] G. V. Rajasekharaiah and U. P. Murthy, "Secure domination in Lict graphs," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 134–145, 2018.

[18] A. Gutiérrez and A. Lladó, "Magic coverings," *J. Combinat. Math. Combinat. Comput.*, vol. 55, pp. 43–56, 2005.

[19] E. C. Kirby, R. B. Mallion, and P. Pollak, "Toroidal poloyhexes fullerene," *J. Chem. Soc. Faraday Trans.*, vol. 89, pp. 1945–1953, 1993.

[20] E. C. Kirby and P. Pollak, "How to enumerate the connectional isomers of a toroidal polyhex fullerene," *J. Chem. Inf. Comput. Sci*, vol. 38, no. 1, pp. 66–70, 1998.

[21] D. J. Klein, "Elemental benzenoids," *J. Chem. Inf. Comput. Sci.*, vol. 34, no. 2, pp. 453–459, 1994.

[22] D. J. Klein and H. Zhu, "Resonance in elemental benzenoids," *Discrete Appl. Math.*, vol. 67, pp. 157–173, May 1996.

[23] A. Lladó and J. Moragas, "Cycle-magic graphs," *Discrete Math.*, vol. 307, no. 23, pp. 2925–2933, 2007.

[24] M. Asif, G. Ali, M. Numan, and A. Semanicova-Fenovcikova, "Cycle-supermagic labeling for some families of graphs," *Utilitas Math.*, vol. 103, pp. 51–59, 2017.

[25] T. K. Maryati, E. T. Baskoro, and A. N. M. Salman, "P_h-supermagic labelings of some trees," *J. Combinat. Math. Combinat. Comput.*, vol. 65, pp. 197–204, 2008.

[26] P. Jeyanthi and N. T. Muthuraja, "Some cycle-super magic graphs," *Int. J. Math. Soft Comput.*, vol. 4, no. 2, pp. 137–144, 2014.

[27] P. Jeyanthi and P. Selvagopal, "H-super magic strength of some graphs," *Tokyo J. Math.*, vol. 33, no. 2, pp. 499–507, 2010.

[28] P. Jeyanthi, P. Selvagopal, and S. S. Sundaram, "Some C_3-supermagic graphs," *Utilitas Math.*, vol. 89, pp. 357–366, 2012.

[29] P. Jeyanthi and P. Selvagopal, "Some C_4-supermagic graphs," *Ars Combinat.*, vol. 111, pp. 129–136, 2013.

[30] T. G. Schmalz, W. A. Seitz, D. J. Klein, and G. E. Hite, "Elemental carbon cages," *J. Amer. Chem. Soc.*, vol. 110, no. 4, pp. 1113–1127, 1988.

[31] W. C. Shiu, P. C. B. Lam, and H. Zhang, "k-resonance in toroidal polyhexes," *J. Math. Chem.*, vol. 38, no. 4, pp. 451–466, 2005.

[32] W. C. Shiu and H. Zhang, "A complete characterization for k-resonant Klein-bottle polyhexes," *J. Math. Chem.*, vol. 43, no. 1, pp. 45–59, 2008.

[33] D. Ye, Z. Qi, and H. Zhang, "On k-resonant fullerene graphs," *SIAM J. Discrete Math.*, vol. 23, no. 2, pp. 1023–1044, 2009.

[34] D. Ye and H. Zhang, "2-extendability of toroidal polyhexes and Klein bottle polyhexes," *Discrete Appl. Math.*, vol. 157, no. 2, pp. 292–299, 2009.

[35] D. Ye and H. Zhang, "Extremal fullerene graphs with the maximum Clar number," *Discrete Appl. Math.*, vol. 157, no. 14, pp. 3152–3173, 2009.

[36] M. Numan, G. Alib, M. Asif, and A. Semanicová-Fenovcíkovác, "Cycle-supermagic labelling of some classes of plane graphs," *ScienceAsia*, vol. 44, no. 2, pp. 129–134, 2018.

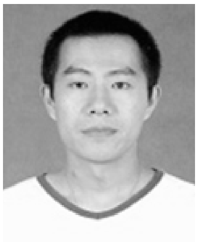
[37] H. Siddiqui and M. R. Farahani, "Forgotten polynomial and forgotten index of certain interconnection networks," *Open J. Math. Anal.*, vol. 1, no. 1, pp. 45–60, 2017.

[38] S. Noreen and A. Mahmood, "Zagreb polynomials and redefined Zagreb indices for the line graph of carbon nanocones," *Open J. Math. Anal.*, vol. 2, no. 1, pp. 66–73, 2018.

[39] Z. Tang, L. Liang, and W. Gao, "Wiener polarity index of quasi-tree molecular structures," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 73–83, 2018.

[40] M. Imran, A. Asghar, and A. Q. Baig, "On graph invariants of oxide network," *Eng. Appl. Sci. Lett.*, vol. 1, no. 1, pp. 23–28, 2018.

[41] W. Gao, A. Asghar, and W. Nazeer, "Computing degree-based topological indices of Jahangir graph," *Eng. Appl. Sci. Lett.*, vol. 1, no. 1, pp. 16–22, 2018.



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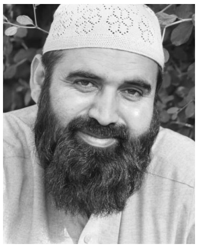

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