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# Numerical Study of Some Intelligent Robot Systems Governed by the Fractional Differential Equations

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**ABSTRACT** We provide a high order numerical methods for solving some intelligent Robot Systems, which are governed by the fractional nonlinear variable coefficient reaction diffusion equations with delay. The convergence of this numerical algorithm is discussed via the energy method. Besides, this algorithm can also be extended to solve the reaction diffusion equation with multi-delays. Finally, we carry on the numerical analysis, and the results of the numerical tests are coincide with the theoretical results.

**INDEX TERMS** Intelligent robot systems, compact finite difference scheme, fractional variable coefficient reaction diffusion equation with delay, convergence.

## I. INTRODUCTION

In recent years, the time fractional differential equations are widely used in robot systems [1]–[7], control theory [8], [9], and signal processing [10]–[12]. Diffusion control problem of mobile actuator (robots) networks can be seen as a system governed by fractional partial differential equation [13], [14]. The optimization and estimation problem of the trajectories of the mobile robots can be found in [15]–[19]. However, this paper will consider the trajectories of the mobile robots.

As a result, the research of fractional differential equations becomes an active research direction, theoretical study can be found in [20]–[23]. As for numerical analysis, [24]–[27] considered finite element methods, [28]–[33] considered finite difference schemes, [34] and [35] constructed higher order compact difference schemes, and [36]–[38] constructed other algorithms. In fact, time delay may appear inevitably, and it should be better to consider delay in differential equation models [39]–[44]. Higher order compact difference schemes [45]–[50] and finite element methods [51]–[54] were considered to numerically solve such models with delay.

Recently, some researchers turned into the study of fractional delay partial differential equations [55]–[59]. However, the analytical solutions of fractional delay partial differential equations can be obtained in few cases. As for numerical solutions for fractional delay partial differential equations, few articles are considered on this topic, one can refer to the articles [60]–[63].

We will study the following fractional nonlinear variable coefficient reaction diffusion equation with delay,

$$\tilde{r}(x,t)\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \tilde{b}(x,t)\frac{\partial u}{\partial x} - \tilde{c}(x,t)u + \tilde{f}(u(x,t), u(x,t-s),x,t), \quad (x,t)\in(0,1)\times[0,T], \tag{1}$$

$$u(x,t) = \phi(x,t), \quad x\in[0,1],\ t\in[-s,0], \tag{2}$$

$$u(0,t) = \gamma(t), \quad u(1,t) = \beta(t), \quad t\in(0,T], \tag{3}$$

Based on the skills used in [64] and multiplying (1) by $\exp(\int_0^x \tilde{b}(s,t)ds)$, the following equivalent equation can be obtained

$$r(x,t)\frac{\partial^\alpha u}{\partial t^\alpha} = (D(x,t)u_x)_x - c(x,t)u + f(u(x,t), u(x,t-s),x,t), \quad (x,t)\in(0,1)\times[0,T], \tag{4}$$

where $r(x,t) = \tilde{r}(x,t)\exp(\int_0^x \tilde{b}(s,t)ds)$, $D(x,t) = \exp(\int_0^x \tilde{b}(s,t)ds)$, $c(x,t) = \tilde{c}(x,t)\exp(\int_0^x \tilde{b}(s,t)ds)$, $f(u(x,t),u(x,t-s),x,t) = \tilde{f}(u(x,t),u(x,t-s),x,t)\exp(\int_0^x \tilde{b}(s,t)ds)$.

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For the reason of simplicity, we firstly consider the following reaction diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = (D(x)u_x)_x + f(u(x,t), u(x,t-s), x, t), \quad (x,t) \in (0,1) \times (0,T], \tag{5}$$

with delay instead of equation (1), where $s > 0$ is a constant delay term, $0 < c_0 \leq D(x) \leq c_1$ is a sufficient smooth function. Time fractional partial derivative $\frac{\partial^\alpha u}{\partial t^\alpha}(0 < \alpha < 1)$ is defined in the Caputo sense as the following

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(x,\xi)}{\partial \xi} d\xi, \tag{6}$$

$\Gamma(\cdot)$ denotes the Gamma function.

In the following of the paper, a high order numerical technique is provided to solve (2)-(3) and (5). The convergence of this numerical algorithm is discussed in detail. Furthermore, the scheme can be extended to solve the reaction diffusion equation with multi-delays. Finally, the results of the numerical tests verify that the proposed numerical algorithm is accurate and efficient.

This paper is designed as follows. A high order numerical technique for solving the equations of (2)-(3) and (5) is constructed in the Section II. In Section III, we consider the convergence of the constructed difference scheme. In Section IV, we conduct several numerical tests to validate the theoretical results. In Section V, some conclusions of the paper are provided.

## II. THE LINEARIZED COMPACT DIFFERENCE SCHEME

In this section, we develop a high order numerical technique for the problem of (2)-(3) and (5). We assume function $f(u(x,t), u(x,t-s), x, t)$ is sufficiently smooth and satisfies

$$|f(\mu+\epsilon_1, \nu+\epsilon_2, x, t) - f(\mu, \nu, x, t)| \leq c_2|\epsilon_1| + c_3|\epsilon_2|, \tag{7}$$

where $\epsilon_1, \epsilon_2$ are arbitrary real numbers, $c_2$ and $c_3$ are positive constants.

Firstly assume two integers $M > 0$ and $N > 0$, and let $h = 1/M$, $\tau = s/n$, where $n > 0$ is a positive integer, $x_i = ih$, $t_k = k\tau$. Define $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i | 0 \leq i \leq M\}$, $\Omega_\tau = \{t_k | -n \leq k \leq N\}$, $N = [T/\tau]$, and then let $\{v_i^k | 0 \leq i \leq M, -n \leq k \leq N\}$ be the grid function space defined on $\Omega_{h\tau}$. We introduce the notations as follows

$$\begin{aligned} \delta_x v_{i+\frac{1}{2}}^k &= \frac{v_{i+1}^k - v_i^k}{h}, \ \delta_x^2 v_i^k = \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \\ \delta_x(D\delta_x v)_i^k &= (D_{i+\frac{1}{2}}\delta_x v_{i+\frac{1}{2}}^k - D_{i-\frac{1}{2}}\delta_x v_{i-\frac{1}{2}}^k)/h, \\ \delta_{0x} v_i^k &= \frac{v_{i+1}^k - v_{i-1}^k}{2h}, \\ \delta_t^\alpha v^k &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[d_0 v^k - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})v^j - d_{k-1}v^0], \end{aligned}$$

where $d_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j \geq 0$.

In addition, we denote

$$v = \frac{\partial}{\partial x}(D(x)\frac{\partial u}{\partial x}), \tag{8}$$

and define $U_i^k = u(x_i, t_k)$, $V_i^k = v(x_i, t_k)$, $0 \leq i \leq M$, $-n \leq k \leq N$ to be grid functions throughout this paper.

Regarding the time fractional derivative, we have the following Lemmas

*Lemma 1 [65]:* Suppose $0 < \alpha < 1$, $y \in C^2[0, t_k]$, it holds that

$$\begin{aligned} &|\frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{y'(s)ds}{(t_k - s)^\alpha} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[d_0 y(t_k) - \sum_{j=1}^{k-1}(d_{k-j-1} \\ &\quad - d_{k-j})y(t_j) - d_{k-1}y(t_0)]| \\ &\leq \frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})] \max_{0 \leq t \leq t_k} |y''(t)| \tau^{2-\alpha}. \end{aligned}$$

In the Lemma 1 listed above, $d_j$ satisfies the following Lemma

*Lemma 2 [66]:* Assume $0 < \alpha < 1$, then it holds that

(1) $d_j$ decreases monotonically with $j$ increases, and $0 < d_j \leq 1$;

(2) $d_0 = 1$, $\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j}) = d_0 - d_{k-1}$.

Now considering (5) at the point $(x_i, t_k)$, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = v(x_i, t_k) + f(u(x_i, t_k), u(x_i, t_k - s), x_i, t_k), \ 0 \leq i \leq M, 0 \leq k \leq N. \tag{9}$$

From Lemma 1, one have

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = \delta_t^\alpha U_i^k + r_i^k, \tag{10}$$

where

$$r_i^k = \frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})] \max_{0 \leq t \leq t_k} |\frac{\partial^2 u(x_i, t_k)}{\partial t^2}| \tau^{2-\alpha}.$$

From Taylor expansion, we then have

$$\begin{aligned} f(u(x_i, t_k), u(x_i, t_k - s), x_i, t_k) &= f(2U_i^{k-1} - U_i^{k-2}, \\ &U_i^{k-n}, x_i, t_k) + \tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \eta^k) f_\mu(\varrho_i^k, U_i^{k-n}, x_i, t_k). \end{aligned} \tag{11}$$

where $\eta^k \in (t_{k-2}, t_k)$, $\varrho_i^k$ in between $u(x_i, t_k)$ and $2U_i^{k-1} - U_i^{k-2}$.

Substituting (10) and (11) into (9), we obtain

$$\delta_t^\alpha U_i^k = V_i^k + f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) + R_{0i}^k, \tag{12}$$

where

$$R_{0i}^k = r_i^k + \tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \eta^k) f_\mu(\varrho_i^k, U_i^{k-n}, x_i, t_k).$$

Thus, (8) can be approximated by the following compact finite difference scheme

$$\mathcal{A}_h V_i^k = \delta_x(\hat{D}\delta_x U)_i^k + O(h^4), \tag{13}$$

where $\mathcal{A}_h$ is defined as [67]

$$\mathcal{A}_h = \begin{cases} u_i^k + \frac{h^2}{12}(\delta_x^2 u_i^k - \delta_{0x}(\frac{D'}{D}u)_i^k), & 1 \le i \le M-1, \\ u_i^k, & i = 0 \quad or\ M, \end{cases}$$

where in $\hat{D} = D - (h^2\tilde{D})/12$, $\tilde{D} = (D')^2/D - D''/2$, and $D'$, $D''$ denote the first and second order spatial derivatives of D(x) respectively.

Here acting the operator $\mathcal{A}_h$ on both side of (12), we obtain

$$\mathcal{A}_h\delta_t^\alpha U_i^k = \delta_x(\hat{D}\delta_x U)_i^k + \mathcal{A}_h f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) + R_i^k,\ 1 \le i \le M-1, 1 \le k \le N, \quad (14)$$

where $R_i^k = \mathcal{A}_h R_{0i}^k + O(h^4)$. It can be easily shown that

$$R_i^k \le C_R(\tau^{2-\alpha} + h^4), \quad 0 \le i \le M, 1 \le k \le N, \quad (15)$$

where $C_R > 0$ is a constant.

Equations (2) and (3) can be discretized as

$$U_i^k = \phi(x_i, t_k), \quad 0 \le i \le M, -n \le k \le 0, \quad (16)$$
$$U_0^k = \gamma(t_k), \quad U_M^k = \beta(t_k),\ 1 \le k \le N. \quad (17)$$

Using $u_i^k$ to replace $U_i^k$ in (14), (16) and (17), ignoring $R_i^k$, we have the following numerical algorithm

$$\mathcal{A}_h\delta_t^\alpha u_i^k = \delta_x(\hat{D}\delta_x u)_i^k + \mathcal{A}_h f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k), \quad 1 \le i \le M-1,\ 1 \le k \le N, \quad (18)$$
$$u_i^k = \phi(x_i, t_k), \quad 0 \le i \le M, \ -n \le k \le 0, \quad (19)$$
$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k),\ 1 \le k \le N. \quad (20)$$

For each time level, the compact difference scheme (18)-(20) is a linear traditional system with strictly diagonally dominant coefficient matrix, therefore the proposed scheme has a unique solution.

## III. THE CONVERGENCE OF THE COMPACT DIFFERENCE SCHEME

To study the convergence of the scheme proposed above, we first give some Lemmas before we provide the convergence theorem. Assuming $0 < c_4 \le \hat{D} \le c_5$, $|(\frac{D'}{D})'| \le c_6$ and $|\frac{D'}{D}| \le c_6$, where $c_4$, $c_5$ and $c_6$ are positive constants. Introduce grid function space belong to $\Omega_h$ as follows,

$$\mathcal{V}_{h,0} = \{u | u = (u_0, u_1, \cdots, u_M), u_0 = u_M = 0\}.$$

Suppose $u, v \in \mathcal{V}_{h,0}$, we have the following notations,

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i,\ \|u\| = \sqrt{(u, u)},$$
$$\|u\|_\infty = \max_{1\le i\le M-1} |u_i|,$$
$$\langle\delta_x u, \delta_x v\rangle = h\sum_{i=0}^{M-1}(\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}),$$
$$|\delta_x u|_1 = \sqrt{\langle\delta_x u, \delta_x u\rangle},$$
$$\langle\delta_x u, \delta_x v\rangle_{\hat{D}} = h\sum_{i=0}^{M-1}\hat{D}(x_{i+\frac{1}{2}})(\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}),$$
$$|\delta_x u|_{1\hat{D}} = \sqrt{\langle\delta_x u, \delta_x u\rangle_{\hat{D}}}.$$

By the definition of $|\delta_x u|_{1\hat{D}}$, one can easily obtain Lemma 3.

*Lemma 3:* For $\forall\ u \in \mathcal{V}_{h,0}$, we obtain $\sqrt{c_4}|\delta_x u|_1 \le |\delta_x u|_{1\hat{D}} \le \sqrt{c_5}|\delta_x u|_1$.

*Lemma 4 [68]:* For $\forall\ u \in \mathcal{V}_{h,0}$, we have

$$\|u\|_\infty \le \frac{1}{2}|\delta_x u|_1, \quad \|u\| \le \frac{1}{\sqrt{6}}|\delta_x u|_1.$$

To prove the convergence analysis, we introduce Lemma 5 as follows,

*Lemma 5 [68]:* Assume $\{F^k | k \ge 0\}$ to be non-negative sequence, and satisfies

$$F^{k+1} \le A + B\tau\sum_{i=1}^{k} F^l, \quad k = 0, 1, \cdots,$$

then we have

$$F^{k+1} \le Ae^{Bk\tau}, \quad k = 0, 1, 2, \cdots,$$

where $A$ and $B$ are non-negative constants.

We denote $e_i^k = U_i^k - u_i^k$, $0 \le i \le M, -n \le k \le N$. By subtracting (18)-(20) from (14), (16) and (17) respectively, we have

$$\begin{aligned} &\mathcal{A}_h\delta_t^\alpha e_i^k = \delta_x(\hat{D}\delta_x e)_i^k + \mathcal{A}_h p_i^k + R_i^k, \\ &\quad 1 \le i \le M-1, 1 \le k \le N, \\ &e_i^k = 0, \quad 0 \le i \le M, -n \le k \le 0, \\ &e_0^k = 0, \quad e_M^k = 0, \quad 1 \le k \le N, \end{aligned} \quad (21)$$

where $p_i^k = f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) - f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k)$.

*Theorem 1:* Let $u(x, t)$ be the solution of (5) and (2)-(3), and $\{u_i^k | 0 \le i \le M, -n \le k \le N\}$ be the solution of (18), (19) and (20). Then, we have

$$\|e^k\|_\infty \le C(\tau^{2-\alpha} + h^4), \quad 1 \le k \le N, \quad (22)$$

where $C$ is a positive constant independent of $h$ and $\tau$.

*Proof:* First, multiplying (21) by $h\delta_t^\alpha e_i^k$, then summing up index $i$, we obtain

$$\begin{aligned} h\sum_{i=1}^{M-1}(\mathcal{A}_h\delta_t^\alpha e_i^k)(\delta_t^\alpha e_i^k) &= h\sum_{i=1}^{M-1}(\delta_x(\hat{D}\delta_x e)_i^k)(\delta_t^\alpha e_i^k) + h\sum_{i=1}^{M-1}(\mathcal{A}_h p_i^k)(\delta_t^\alpha e_i^k) \\ &\quad + h\sum_{i=1}^{M-1}(R_i^k)(\delta_t^\alpha e_i^k), \end{aligned} \quad (23)$$

where $1 \le i \le M-1$, $1 \le k \le N$. Then we estimate equation (23) term by term. By using the discrete Green formula, we have

$$
\begin{aligned}
& h\sum_{i=1}^{M-1}(\mathcal{A}_h\delta_t^\alpha e_i^k)(\delta_t^\alpha e_i^k) \\
& = h\sum_{i=1}^{M-1}(\delta_t^\alpha e_i^k)^2 + h\frac{h^2}{12}\sum_{i=1}^{M-1} \\
& \quad (\delta_x^2\delta_t^\alpha e_i^k)(\delta_t^\alpha e_i^k) - h\frac{h^2}{12}\sum_{i=1}^{M-1}(\delta_{0x}(\frac{D'}{D}\delta_t^\alpha e)_i^k)(\delta_t^\alpha e_i^k) \\
& = \|\delta_t^\alpha e^k\|^2 - \frac{h^2}{12}h\sum_{i=1}^{M-1}(\delta_x\delta_t^\alpha e_i^k)(\delta_x\delta_t^\alpha e_i^k) \\
& \quad - \frac{h^2}{12}h\sum_{i=1}^{M-1}\frac{(\frac{D'}{D})_{i+1}\delta_t^\alpha e_{i+1}^k - (\frac{D'}{D})_{i-1}\delta_t^\alpha e_{i-1}^k}{2h}\delta_t^\alpha e_i^k \\
& = \|\delta_t^\alpha e^k\|^2 - \frac{h^2}{12}\|\delta_x\delta_t^\alpha e^k\|^2 - \frac{h^2}{12}h\sum_{i=1}^{M-1} \\
& \quad \frac{(\frac{D'}{D})_{i+1} - (\frac{D'}{D})_i}{2h}\delta_t^\alpha e_i^k\delta_t^\alpha e_{i+1}^k \\
& \geq \|\delta_t^\alpha e^k\|^2 - \frac{1}{3}\|\delta_t^\alpha e^k\|^2 - \frac{h^2}{24}c_6\|\delta_t^\alpha e^k\|^2 \\
& = (\frac{2}{3} - \frac{h^2}{24}c_6)\|\delta_t^\alpha e^k\|^2. \qquad (24)
\end{aligned}
$$

Now we denote $\lambda = \tau^\alpha\Gamma(2-\alpha)$. By the discrete Green formula, one obtain

$$
\begin{aligned}
& h\sum_{i=1}^{M-1}(\delta_x(\hat{D}\delta_x e)_i^k)(\delta_t^\alpha e_i^k) \\
& = -h\sum_{i=0}^{M-1}\hat{D}_{i+1/2}(\delta_x e_{i+1/2}^k)(\delta_t^\alpha\delta_x e_{i+1/2}^k) \\
& = -\frac{1}{\lambda}h\sum_{i=0}^{M-1}\hat{D}_{i+1/2}(\delta_x e_{i+1/2}^k)[\delta_x e_{i+1/2}^k \\
& \quad - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\delta_x e_{i+1/2}^j - d_{k-1}\delta_x e_{i+1/2}^0] \\
& = -\frac{1}{\lambda}\{|\delta_x e^k|_{1\hat{D}}^2 - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\langle\delta_x e^k, \delta_x e^j\rangle_{\hat{D}} \\
& \quad - d_{k-1}\langle\delta_x e^k, \delta_x e^0\rangle_{\hat{D}}\} \\
& \leq -\frac{1}{\lambda}\{|\delta_x e^k|_{1\hat{D}}^2 - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j}) \\
& \quad \frac{|\delta_x e^k|_{1\hat{D}}^2 + |\delta_x e^j|_{1\hat{D}}^2}{2} - d_{k-1}\frac{|\delta_x e^k|_{1\hat{D}}^2 + |\delta_x e^0|_{1\hat{D}}^2}{2}\} \\
& = -\frac{1}{\lambda}\{\frac{|\delta_x e^k|_{1\hat{D}}^2}{2} - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\frac{|\delta_x e^j|_{1\hat{D}}^2}{2}\}. \qquad (25)
\end{aligned}
$$

By the Cauchy-Schwarz inequality, one obtains

$$
\begin{aligned}
h\sum_{i=1}^{M-1}(R_i^k)(\delta_t^\alpha e_i^k) & \leq h\sum_{i=1}^{M-1}(\frac{(R_i^k)^2}{2\varepsilon} + \frac{\varepsilon}{2}(\delta_t^\alpha e_i^k)^2) \\
& = \frac{1}{2\varepsilon}\|R^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e^k\|^2. \qquad (26)
\end{aligned}
$$

Similarly, from the Cauchy-Schwarz inequality and the condition of (7), we have

$$
\begin{aligned}
& h\sum_{i=1}^{M-1}(\mathcal{A}_h p_i^k)(\delta_t^\alpha e_i^k) \\
& \quad \leq h\sum_{i=1}^{M-1}(\mathcal{A}_h(c_2|2e_i^{k-1} - e_i^{k-2}| + c_3|e_i^{k-n}|))|\delta_t^\alpha e_i^k| \\
& \quad \doteq A_1 + A_2 + A_3 + A_4 + A_5. \qquad (27)
\end{aligned}
$$

Applying $\varepsilon$-inequality, we have

$$
\begin{aligned}
A_1 & = \frac{h}{12}\sum_{i=1}^{M-1}(c_2|2e_{i+1}^{k-1} - e_{i+1}^{k-2}| + c_3|e_{i+1}^{k-n}|)|\delta_t^\alpha e_i^k| \\
& \leq \frac{h}{12}\sum_{i=1}^{M-1}[\frac{(c_2|2e_{i+1}^{k-1} - e_{i+1}^{k-2}| + c_3|e_{i+1}^{k-n}|)^2}{2\varepsilon} + \frac{\varepsilon}{2}(\delta_t^\alpha e_i^k)^2] \\
& \leq \frac{hc_2^2}{12\varepsilon}\sum_{i=1}^{M-1}(2e_{i+1}^{k-1} - e_{i+1}^{k-2})^2 + \frac{hc_3^2}{12\varepsilon}\sum_{i=1}^{M-1}(e_{i+1}^{k-n})^2 \\
& \quad + \frac{\varepsilon}{24}\|\delta_t^\alpha e^k\|^2 \\
& \leq \frac{2c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{c_3^2}{12\varepsilon}\|e^{k-n}\|^2 \\
& \quad + \frac{\varepsilon}{24}\|\delta_t^\alpha e^k\|^2. \qquad (28)
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
A_2 & = \frac{5h}{6}\sum_{i=1}^{M-1}(c_2|2e_i^{k-1} - e_i^{k-2}| + c_3|e_i^{k-n}|)|\delta_t^\alpha e_i^k| \\
& \leq \frac{20c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{5c_3^2}{6\varepsilon}\|e^{k-n}\|^2 \\
& \quad + \frac{5\varepsilon}{12}\|\delta_t^\alpha e^k\|^2, \qquad (29) \\
A_3 & = \frac{h}{12}\sum_{i=1}^{M-1}(c_2|2e_{i-1}^{k-1} - e_{i-1}^{k-2}| + c_3|e_{i-1}^{k-n}|)|\delta_t^\alpha e_i^k| \\
& \leq \frac{2c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{c_3^2}{12\varepsilon}\|e^{k-n}\|^2 \\
& \quad + \frac{\varepsilon}{24}\|\delta_t^\alpha e^k\|^2, \qquad (30) \\
A_4 & = \frac{h}{24}\sum_{i=1}^{M-1}(c_2|2e_{i-1}^{k-1} - e_{i-1}^{k-2}| + c_3|e_{i-1}^{k-n}|) \\
& \quad (h(\frac{D'}{D})_{i-1})|\delta_t^\alpha e_i^k|
\end{aligned}
$$

$$\leq \frac{c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{c_3^2}{24\varepsilon}\|e^{k-n}\|^2 + \frac{\varepsilon c_6^2 h^2}{48}\|\delta_t^\alpha e^k\|^2, \quad (31)$$

$$A_5 = -\frac{h}{24}\sum_{i=1}^{M-1}(c_2|2e_{i+1}^{k-1} - e_{i+1}^{k-2}| + c_3|e_{i+1}^{k-n}|)(h(\frac{D'}{D})_{i+1})|\delta_t^\alpha e_i^k| \leq \frac{c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{c_3^2}{24\varepsilon}\|e^{k-n}\|^2 + \frac{\varepsilon c_6^2 h^2}{48}\|\delta_t^\alpha e^k\|^2. \quad (32)$$

Substituting (28)-(32) into (27), we get

$$h\sum_{i=1}^{M-1}(\mathcal{A}_h p_i^k)(\delta_t^\alpha e_i^k) \leq \frac{26c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{13c_3^2}{12\varepsilon}\|e^{k-n}\|^2 + (\frac{\varepsilon}{2} + \frac{\varepsilon c_6^2 h^2}{24})\|\delta_t^\alpha e^k\|^2. \quad (33)$$

Inserting (24), (25), (26) and (33) into (23), we obtain

$$(\frac{2}{3} - \frac{h^2}{24}c_6)\|\delta_t^\alpha e^k\|^2 \leq -\frac{1}{\lambda}\{\frac{|\delta_x e^k|_{1\hat{D}}^2}{2} - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\frac{|\delta_x e^j|_{1\hat{D}}^2}{2}\} + \frac{26c_2^2}{3\varepsilon}(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{13c_3^2}{12\varepsilon}\|e^{k-n}\|^2 + (\frac{\varepsilon}{2} + \frac{\varepsilon c_6^2 h^2}{24})\|\delta_t^\alpha e^k\|^2 + \frac{1}{2\varepsilon}\|R^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e^k\|^2. \quad (34)$$

Multiplying (34) by $2\lambda$, and letting $\varepsilon = \frac{2/3 - c_6 h^2/24}{1 + c_6^2 h^2/24}$, where $h$ is chosen to be small enough to guarantee $\varepsilon \geq 1/3$ in this paper, thus we arrive at

$$|\delta_x e^k|_{1\hat{D}}^2 \leq \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})|\delta_x e^j|_{1\hat{D}}^2 + 52\lambda c_2^2(\|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \frac{13\lambda c_3^2}{2}\|e^{k-n}\|^2 + 3\lambda\|R^k\|^2.$$

From Lemma 3 and Lemma 4, and (15), we obtain

$$|\delta_x e^k|_1^2 \leq \frac{c_5}{c_4}\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})|\delta_x e^j|_1^2 + \frac{3\lambda}{c_4}C_R^2(\tau^{2-\alpha} + h^4)^2 + \frac{26\lambda c_2^2}{3c_4}(|\delta_x e^{k-1}|_1^2 + |\delta_x e^{k-2}|_1^2) + \frac{13\lambda c_3^2}{12c_4}|\delta_x e^{k-n}|_1^2. \quad (35)$$

By denoting

$$C_k = \frac{1}{c_4}\Gamma(2-\alpha)\max\{3C_R^2, 26c_2^2/3, 13c_3^2/12\} > 0,$$

We notice $\lambda = \tau^\alpha\Gamma(2-\alpha)$, and for $0 < \tau < 1$ we have $\tau^\alpha < 1$. Then from inequality (35), we obtain

$$|\delta_x e^k|_1^2 \leq \frac{c_5}{c_4}\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})|\delta_x e^j|_1^2 + C_k[(\tau^{2-\alpha} + h^4)^2 + |\delta_x e^{k-1}|_1^2 + |\delta_x e^{k-2}|_1^2 + |\delta_x e^{k-n}|_1^2].$$

From Lemma 2 and Lemma 5, we have

$$|\delta_x e^k|_1^2 \leq C_k \exp(3C_k + \frac{c_5}{c_4}\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j}))(\tau^{2-\alpha} + h^4)^2 = C_k \exp(3C_k + \frac{c_5}{c_4}(1 - d_{k-1}))(\tau^{2-\alpha} + h^4)^2 \leq C_1(\tau^{2-\alpha} + h^4)^2, \quad (36)$$

where $C_1 = C_k \exp(3C_k + \frac{c_5}{c_4})$. From Lemma 4, we have

$$\|e^k\|_\infty \leq \sqrt{C_1(\tau^{2-\alpha} + h^4)^2}/2 \doteq C(\tau^{2-\alpha} + h^4).$$

The proof is completed. □

*Remark 1:* Analogous to (2)-(3) and (5), we can obtain the corresponding compact difference scheme for (2)-(4) as the following scheme

$$\mathcal{A}_h(r_i^k\delta_t^\alpha u_i^k) = \delta_x(\hat{D}\delta_x u)_i^k - c_i^k u_i^k + \mathcal{A}_h f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k), \quad 1 \leq i \leq M-1, \ 1 \leq k \leq N,$$
$$u_i^k = \phi(x_i, t_k), \quad 0 \leq i \leq M, \ -n \leq k \leq 0,$$
$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k), \ 1 \leq k \leq N.$$

*Remark 2:* The scheme considered in the paper can also be extended to solve the following equations with multi-delays

$$\frac{\partial^\alpha u}{\partial t^\alpha} = (D(x)u_x)_x + f(u(x,t), u(x,t-s_1), u(x,t-s_2), \cdots, u(x,t-s_q), x, t), \quad (x,t) \in (0,1) \times (0,T],$$
$$u(x,t) = \phi(x,t), \quad x \in [0,1], \ t \in [-s, 0],$$
$$u(0,t) = \gamma(t), \quad u(1,t) = \beta(t), \ t \in (0,T],$$

where $s_i > 0, i = 1, 2, \cdots, q$, and $s = \max_{1\leq i\leq q}\{s_i\}$.

## IV. NUMERICAL TEST

In this section, two numerical tests are considered to validate the performance of the proposed scheme (18), (19) and (20). Introduce the following notation to stand by the maximum error

$$e(h,\tau) = \max_{1\leq k\leq N}\|U^k - u^k\|_\infty,$$

and we also introduce the convergence order in time and space

$$Rate_\tau = \frac{log(e(h,\tau_1)/e(h,\tau_2))}{log(\tau_1/\tau_2)},$$

**TABLE 1. Numerical errors and convergence orders in time direction with $h = 1/2000$.**

| $\tau$ | $\alpha = 0.3$ $e(h,\tau)$ | $Rate_\tau$ | $\alpha = 0.5$ $e(h,\tau)$ | $Rate_\tau$ | $\alpha = 0.8$ $e(h,\tau)$ | $Rate_\tau$ |
|---|---|---|---|---|---|---|
| 1/50 | 2.002e-005 | * | 8.928e-005 | * | 6.499e-004 | * |
| 1/100 | 6.361e-006 | 1.654 | 3.203e-005 | 1.479 | 2.844e-004 | 1.192 |
| 1/150 | 3.242e-006 | 1.662 | 1.755e-005 | 1.484 | 1.752e-004 | 1.195 |
| 1/200 | 2.007e-006 | 1.666 | 1.144e-005 | 1.487 | 1.242e-004 | 1.196 |

**TABLE 2. Numerical errors and convergence orders in spatial direction with $\tau = 1/2000$.**

| $h$ | $\alpha = 0.25$ $e(h,\tau)$ | $Rate_h$ | $\alpha = 0.5$ $e(h,\tau)$ | $Rate_h$ | $\alpha = 0.75$ $e(h,\tau)$ | $Rate_h$ |
|---|---|---|---|---|---|---|
| 1/8 | 3.082e-003 | * | 3.007e-003 | * | 2.907e-003 | * |
| 1/12 | 6.114e-004 | 3.989 | 5.978e-004 | 3.984 | 5.825e-004 | 3.965 |
| 1/16 | 1.945e-004 | 3.982 | 1.903e-004 | 3.979 | 1.882e-004 | 3.927 |
| 1/20 | 7.963e-005 | 4.001 | 7.811e-005 | 3.991 | 7.974e-005 | 3.849 |

$$Rate_h = \frac{log(e(h_1,\tau)/e(h_2,\tau))}{log(h_1/h_2)}.$$

In the $Rate_\tau$ and $Rate_h$ of converge, we require that $h$ and $\tau$ is fixed and small enough.

*Example 1:* Considering the following problem, where $D(x) = x^2 + 1$,

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial u}{\partial x}(D(x)\frac{\partial u}{\partial x}) = \frac{u(x,t-0.2)}{1+u^2(x,t-0.1)} + G(x,t), \\ \qquad (x,t) \in (0,1) \times (0,T], \\ u(x,t) = t^{2+\alpha}\cos(2\pi x), \quad x \in [0,1],\ t \in [-0.2,0], \\ u(0,t) = t^{2+\alpha}, \quad u(1,t) = t^{2+\alpha},\ t \in (0,1], \end{cases} \tag{37}$$

the exact solution of (37) is $u(x,t) = t^{2+\alpha}\cos(2\pi x)$, and

$$G(x,t) = [\frac{\Gamma(3+\alpha)}{\Gamma(3)}t^2 + 4\pi^2(x^2+1)t^{2+\alpha}]\cos(2\pi x) + 4\pi x t^{2+\alpha}\sin(2\pi x) - \frac{(t-0.2)^{2+\alpha}\cos(2\pi x)}{1+(t-0.1)^{4+2\alpha}\cos^2(2\pi x)}.$$

Now we conduct the numerical analysis by implementing our linearized compact difference scheme.

Table 1 and 2 show the numerical results of Example 1. Table 1 reports that the numerical errors in time directions for different $\alpha$. We can find that the time convergence order is $2 - \alpha$. In Table 2, when we fixed the temporal step $\tau = 1/2000$, numerical errors in the spatial direction are presented with $\alpha = 0.25, 0.5, 0.75$. The results show that the convergence orders in spatial direction is 4.

Furthermore, we make the error analysis with different $\alpha$ chosen. Figure 1 presents the error planes of different $\alpha$ respectively, where we can find that the numerical errors are negatively correlated with $\alpha$. Figure 2 gives the the absolute values of the approximation solutions of Example 1 for different $\alpha$. In conclusion that smaller solutions can be taken with bigger $\alpha$.

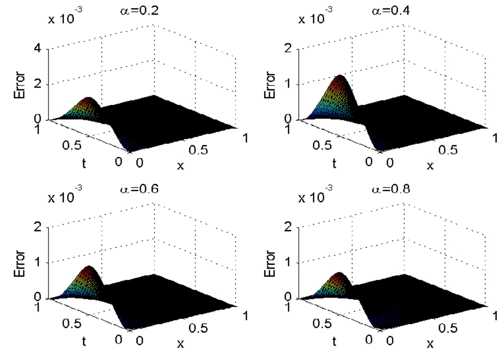


**FIGURE 1. Error planes under different $\alpha$ for Example 1, where $\tau = h = 1/100$.**

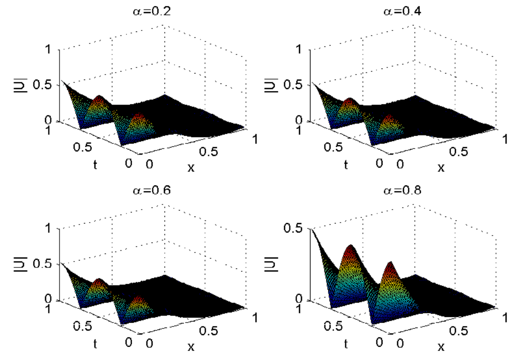


**FIGURE 2. The absolute values of numerical solutions for Example 1 under different $\alpha$.**

**TABLE 3. Numerical errors and convergence orders in time direction with $h = 1/1000$.**

| $\tau$ | $\alpha = 0.4$ $e(h,\tau)$ | $Rate_\tau$ | $\alpha = 0.6$ $e(h,\tau)$ | $Rate_\tau$ | $\alpha = 0.8$ $e(h,\tau)$ | $Rate_\tau$ |
|---|---|---|---|---|---|---|
| 1/200 | 2.811e-006 | * | 7.877e-006 | * | 2.901e-005 | * |
| 1/300 | 1.338e-006 | 1.830 | 4.366e-006 | 1.456 | 1.794e-005 | 1.185 |
| 1/400 | 8.036e-007 | 1.773 | 2.881e-006 | 1.445 | 1.276e-005 | 1.184 |
| 1/500 | 5.439e-007 | 1.749 | 2.090e-006 | 1.438 | 9.793e-006 | 1.187 |

*Example 2:* This example considers the following problem,

$$\begin{cases} e^{-x}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = u(x,t)(1-u(x,t-0.2)) \\ \qquad + G(x,t),\ (x,t) \in (0,1) \times (0,T], \\ u(x,t) = t^2\sin(2\pi x), \quad x \in [0,1],\ t \in [-0.2,0], \\ u(0,t) = 0, \quad u(1,t) = 0,\ t \in (0,1], \end{cases} \tag{38}$$

the exact solution of (38) is $u(x,t) = t^2\sin(2\pi x)$, and

$$G(x,t) = [e^{-x}\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 4\pi^2t^2 - t^2]\sin(2\pi x) - 2\pi t^2\cos(2\pi x) + t^2(t-0.2)^2\sin^2(2\pi x).$$

The problem in Example 2 can be solved by using the applying algorithm referred in Remark 1. The numerical results are shown in Table 3 and Table 4. From the results,

**TABLE 4. Numerical errors and convergence orders in spatial direction with $\tau = 1/2000$.**

| $\tau$ | $\alpha = 0.2$ | | $\alpha = 0.5$ | | $\alpha = 0.8$ | |
|---|---|---|---|---|---|---|
| | $e(h,\tau)$ | $Rate_{\mathcal{T}}$ | $e(h,\tau)$ | $Rate_{\mathcal{T}}$ | $e(h,\tau)$ | $Rate_{\mathcal{T}}$ |
| 1/8 | 2.051e-003 | * | 2.032e-003 | * | 2.016e-003 | * |
| 1/12 | 4.075e-004 | 3.986 | 4.037e-004 | 3.986 | 4.016e-004 | 3.979 |
| 1/16 | 1.303e-004 | 3.964 | 1.292e-004 | 3.962 | 1.296e-004 | 3.933 |
| 1/20 | 5.339e-005 | 3.997 | 5.299e-005 | 3.992 | 5.411e-005 | 3.913 |

we can draw a conclusion that the numerical results are in accordance with the theoretical results.

*Remark 3:* The numerical examples considered in this paper can be used to model the system of mobile robotics with wireless sensor networks. Each robot has limited sensing and communication ability. The robots can coordinate with each other to control the diffusion process by temporal and spatial feed back closed loop control [69], [70].

## V. CONCLUSION

We provide a high order numerical technique for fractional nonlinear variable coefficient reaction diffusion equation with delay. The convergence of this numerical algorithm is considered. Two numerical experiments are provided to support the theoretical results and validate the efficiency of the compact difference scheme.

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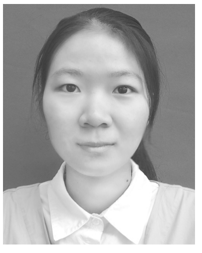

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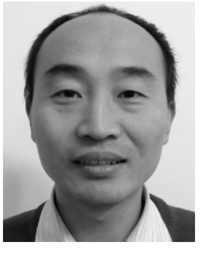

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