

Received August 14, 2019, accepted September 11, 2019, date of publication September 23, 2019, date of current version October 10, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2943163

# Centralized Fusion Methods for Multi-Sensor System With Bounded Disturbances

QIANG SHEN<sup>ID</sup>, JIEYU LIU, XIAOGANG ZHOU, WEIWEI QIN, LIXIN WANG, AND QI WANG

Xi'an Research Institute of High-Tech, Xi'an 710025, China

Corresponding author: Weiwei Qin (johnson\_outlook@163.com)

This work was supported by the National Natural Science Foundation of China under Grant NSFC-61503390 and Grant NSFC-61503392.

**ABSTRACT** The set-membership information fusion problem is studied for general multi-sensor dynamic systems. Based on set-membership theory, three centralized state fusion estimation algorithms in the presence of bounded disturbances are proposed, namely augmented algorithm, combined measurement filtering algorithm and pseudo-sequential filtering algorithm. Theoretical discussions on the convergence and boundedness of the proposed fusion algorithms are provided and their stability is proved. The estimate accuracy, computational complexity and flexibility of these three fusion algorithms are compared through theoretical analysis and simulation. And their exchanging property of measurement update order is discussed. Results show that these algorithms are functionally equivalent in terms of the estimation accuracy and the exchangeability of the measurement update order can be guaranteed as long as the parameters satisfy certain conditions. Meanwhile the simulation results prove the role of the proposed algorithms in improving state estimation accuracy. In addition, the combined measurement filtering algorithm has the highest calculation speed due to lower dimension. But it is less flexible because the sensor measurement matrices need to satisfy some additional conditions. These conclusions are valuable in applications.

**INDEX TERMS** Centralized fusion, state estimation, ellipsoidal state bounding, multi-sensor system, set-membership estimation, unknown but bounded noise.

## I. INTRODUCTION

In recent years, the functional requirements of the large complicated systems are rapidly improving, especially the high performance estimation requirements for the system state. Due to noticeable defects in measurement accuracy, range, stability and reliability when using single sensor, the multi-sensor system and related data fusion technology have drawn more and more attention recently. And they are widely applied in many civil and defense fields, including sensor network[1], [2], target tracking[3], [4], navigation[5], [6] and big data[7]. The problem of multi-sensor estimation fusion is how to provide more useful and accurate state estimation results by fusing measurement data from multiple sensors [8], [9].

Most existing information fusion algorithms are probability-based and need accurate statistical knowledge of the noises. Many results for this kind of fusion method have been obtained (see e.g., books [10], [11]). And in recent years, it is still developing rapidly. Z. Duan and X. R. Li proposed lossless linear transformation of the raw measurements of

each sensor for distributed estimation fusion [12]. A. Fatehi and B. Huang studied the state estimation for a system with irregular rate and delayed measurements and a modified track to track Kalman filter fusion method was improved to handle this problem [13]. An unscented Kalman filter was used to combine the data coming from sensors to solve the problem of wheelchair position estimation in indoor environments with noisy measurements [14]. To realize robust weighted state fusion estimation problem for a class of time-varying multi-sensor networked systems with mixed uncertainties including uncertain-variance multiplicative and linearly correlated additive white noises, and packet dropouts, the original system was converted into one with only uncertain noise variances by augmented state method and fictitious noise technique. And finally a robust weighted state fusion Kalman estimator was proposed in [15].

Apparently, the probability-based fusion algorithms have been widely applied, and have excellent fusion estimation results under certain conditions. However, accurate statistical knowledge of process and measurement noise should be known for most probability-based algorithms and such idealized assumptions are difficult to satisfy in certain

The associate editor coordinating the review of this manuscript and approving it for publication was Bora Onat.

practical situations, and this may lead to poor performance for the state estimation [16], [17]. But in many engineering applications, it is easier to obtain the bounds of unknown noises. So the set membership (SM) algorithms offer an interesting alternative and draw increasing attention recently since the noises of their models are assumed only to be bounded [18]–[21]. SM methods have wide applications in automatic control [22], [23], simultaneous localization, mapping (SLAM) [24], faulty detection [25] and so on. For multi-sensor SM fusion in bounded setting, Becis-Aubry proposed a hierarchical SM estimation method for multi-sensor systems equipped with a local processor [26]. Afterwards the method was extended for non-linear models with potentially failing measurements [27]. Wang converted the SM information fusion problem into a semi-definite programming problem, and proposed corresponding fusion algorithms [28]. The algorithms perform well in terms of the fusion accuracy, but they have heavy computational burden. F. Farina et al. addressed the distributed estimation problem in a set membership framework [29]. And in their work two distributed algorithms were considered and such algorithms were proved to be asymptotic interpolatory estimators. In addition, the covariance intersection (CI) algorithm [30] and ellipsoidal intersection algorithm [31] has a lot of similarities with SM fusion algorithms. Now the SM-based fusion estimation methods have been successfully used in sensor networks [32]–[35] and location [36]–[38].

There are two traditional architectures for estimation fusion, centralized fusion and distributed fusion. For the centralized fusion, the raw data of each sensor is sent to the fusion center to be processed. The characteristics of the centralized fusion are high performance and complex computation, while the distributed fusion with loss of accuracy has advantages in the system survivability and computation burden [8]. For multi-sensor linear dynamic system in probabilistic setting, there exist three typical centralized fusion methods, namely augmented method, pseudo-sequential filtering method, and combined measurement filtering method. And many conclusions of the centralized point estimation fusion have been drawn [39], [40], i.e., 1) when observation matrices are identical, the augmented method and the combined measurement filtering method are functionally equivalent in terms of estimation accuracy; 2) the estimate accuracy of the pseudo-sequential filtering method and augmented method are equivalent; 3) the estimate accuracy of the three method remains unchanged when the measurement update order changes.

However, similar research on the multi-sensor fusion in bounded setting has not received enough attention and few conclusions are drawn. Even the detailed process of the three typical centralized fusion algorithms with bounded disturbances is not yet fully presented. Further, it is interesting to find out that whether the conclusions and properties 1)~3) can be maintained in bounded setting. These facts motivate us to further research the SM centralized fusion problems for

multi-sensor dynamic systems with bounded disturbances. And the following objects are focused on:

- 1) To design three centralized fusion algorithms for multi-sensor system with bounded disturbances based on ellipsoidal bounding estimation;
- 2) To analyze the properties of the proposed algorithms, including the stability, convergence, computation complexity, the exchangeability of measurement update order and the equivalence between different algorithms.

The remainder of this paper is presented as follows. The centralized fusion problem for multi-sensor system with bounded disturbances for linear model is introduced in Section II. The state evolution analysis is presented in Section III. Then three centralized SM fusion algorithms with selection of optimal parameters are derived in Section IV. Some properties of the proposed algorithm are given and proved in Section V. In Section VI, the effectiveness and properties of the algorithms are demonstrated through a numerical example. Section VII concludes this paper.

## II. PROBLEM FORMULATION

The Definition of the bounded ellipsoid  $\mathcal{E}$  of  $\mathbb{R}^n$  is given by [19].

Consider the following N-sensor dynamic linear time-varying (LTV) system with unknown but bounded noises

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{w}_{k-1} \quad (1)$$

$$\mathbf{z}_{i,k} = \mathbf{H}_{i,k}\mathbf{x}_k + \mathbf{v}_{i,k}, \quad i = 1, 2, \dots, N \quad (2)$$

where  $\mathbf{x}_k \in \mathbb{R}^n$  is the state vector at time  $k$ , and  $\mathbf{z}_{i,k} \in \mathbb{R}^{m_i}$  is the observation vector of the  $i$ th sensor.  $\mathbf{F}_{k-1}$  is the state transition matrix,  $\mathbf{G}_{k-1}$  is process noise input matrix, and  $\mathbf{H}_{i,k}$  is the observation matrix with full row rank.  $\mathbf{w}_{k-1} \in \mathbb{R}^l$  and  $\mathbf{v}_{i,k} \in \mathbb{R}^{m_i}$  are modeling process and observation noises, which are assumed to be of unknown distribution, but bounded by the following ellipsoids

$$\mathcal{W}_{k-1} = \{\mathbf{w}_{k-1} : \mathbf{w}_{k-1}^T \mathbf{Q}_{k-1}^{-1} \mathbf{w}_{k-1} \leq 1\} \quad (3)$$

$$\mathcal{V}_{i,k} = \{\mathbf{v}_{i,k} : \mathbf{v}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{v}_{i,k} \leq 1\} \quad (4)$$

where  $\mathbf{Q}_{k-1}$  and  $\mathbf{R}_{i,k}$  are known positive definite matrices. The initial state belongs to an ellipsoid given by

$$\mathcal{E}(\hat{\mathbf{x}}_0, \sigma_0 \mathbf{P}_0) = \{\mathbf{x}_0 : (\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T \mathbf{P}_0^{-1} (\mathbf{x}_0 - \hat{\mathbf{x}}_0) \leq \sigma_0\} \quad (5)$$

where  $\hat{\mathbf{x}}_0$  is the center and  $\mathbf{P}_0$  is a positive definite matrix. The variable  $\sigma_0 \in \mathbb{R}$  is a positive scalar variable. All above matrices have appropriate dimension in accordance with the state and observation vector. Then the problem of interest can now be stated as follows.

Given the model and the observations  $\mathbf{z}_{i,k}$  from different sensors, it is desired to calculate the smallest set which must contain the true but unknown state  $\mathbf{x}_k$  at time  $k$ .

## III. THE STATE EVOLUTION ANALYSIS

Under unknown but bounded noise assumptions, the centralized SM estimation fusion problem can be formulated as follows.

Assume  $\mathbf{x}_0$  belongs to the given bounding ellipsoid  $\mathcal{E}(\hat{\mathbf{x}}_0, \sigma_0 \mathbf{P}_0)$ . At time  $k-1$ , the ellipsoid containing the state  $\mathbf{x}_{k-1}$  is described by  $\mathcal{E}_{k-1} = \mathcal{E}(\hat{\mathbf{x}}_{k-1}, \sigma_{k-1} \mathbf{P}_{k-1})$ , then it can be derived from equation (1) that

$$\mathbf{x}_k \in \mathbf{F}_{k-1} \mathcal{E}_{k-1} \oplus \mathbf{G}_{k-1} \mathcal{E}(\mathbf{0}, \mathbf{Q}_{k-1}) = \{\mathbf{x} + \mathbf{w} : \mathbf{x} \in \mathbf{F}_{k-1} \mathcal{E}_{k-1} \text{ and } \mathbf{w} \in \mathbf{G}_{k-1} \mathcal{E}(\mathbf{0}, \mathbf{Q}_{k-1})\} \quad (6)$$

where  $\oplus$  stands for the Minkowski sum.

Thus the objective in prediction step is to find an ellipsoid

$$\mathcal{E}_{k|k-1} \supseteq \mathbf{F}_{k-1} \mathcal{E}_{k-1} \oplus \mathbf{G}_{k-1} \mathcal{E}(\mathbf{0}, \mathbf{Q}_{k-1}) \quad (7)$$

where  $\mathcal{E}_{k|k-1} = \mathcal{E}(\hat{\mathbf{x}}_{k|k-1}, \sigma_{k|k-1} \mathbf{P}_{k|k-1})$ .

According to the observation equation (2) and (4),  $\mathbf{x}_k$  belongs to the bounding sets

$$\mathcal{X}_k = \{\mathbf{x} : (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x})^T \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}) \leq 1, i=1, 2, \dots, N\} \quad (8)$$

Thus in the fusion update step, the objective is to find an ellipsoid  $\mathcal{E}_k = \mathcal{E}(\hat{\mathbf{x}}_k, \sigma_k \mathbf{P}_k)$  which contains the state set consistent with the time-updated ellipsoid  $\mathcal{E}_{k|k-1}$  and the current observations from multiple sensors with bounded noise, i.e., the ellipsoid  $\mathcal{E}_k \supseteq \mathcal{E}_{k|k-1} \cap \mathcal{X}_k$  is the result of the fusion update.

#### IV. THE CENTRALIZED SM ESTIMATION FUSION ALGORITHMS

In this section, we design three centralized SM fusion algorithms and each includes prediction and fusion update step.

##### A. THE AUGMENTED ALGORITHM

The core of the augmented fusion algorithm is to expand the low-dimensional measurements from multiple sensors into a single high-dimensional measurement vector. Then, this high-dimensional vector is filtered once at time  $k$  to achieve the purpose of multi-sensor fusion. The following lemma is given before the main results.

*Lemma 1:* Given two ellipsoids  $\mathcal{E}(\mathbf{a}_1, \mathbf{M}_1)$  and  $\mathcal{E}(\mathbf{a}_2, \mathbf{M}_2)$  ( $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n, \mathbf{M}_1, \mathbf{M}_2 \in \mathbb{R}^{n \times n}$ ), then  $\forall p \in (0, +\infty)$ ,  $\mathcal{E}(\mathbf{a}_s, \mathbf{M}_s) \supseteq \mathcal{E}(\mathbf{a}_1, \mathbf{M}_1) \oplus \mathcal{E}(\mathbf{a}_2, \mathbf{M}_2)$ , where  $\mathbf{a}_s = \mathbf{a}_1 + \mathbf{a}_2$  and  $\mathbf{M}_s = (1+p^{-1})\mathbf{M}_1 + (1+p)\mathbf{M}_2$ .

*Proof:* See the works of Maksarov and Norton [8].

*Theorem 1:* The recursive procedures of the augmented algorithm for combining numerous sensors with bounded disturbances are following equations.

*Prediction Step:* Given  $\mathbf{x}_{k-1} \in \mathcal{E}_{k-1} = \mathcal{E}(\hat{\mathbf{x}}_{k-1}, \sigma_{k-1} \mathbf{P}_{k-1})$  obeying (1) with  $\mathbf{w}_k \in \mathcal{E}(\mathbf{0}, \mathbf{Q}_k)$  and

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1} \quad (9)$$

$$\sigma_{k|k-1}^a = \sigma_{k-1}^a \quad (10)$$

$$\mathbf{P}_{k|k-1}^a = (1+p_k^{-1})\mathbf{F}_{k-1} \mathbf{P}_{k-1}^a \mathbf{F}_{k-1}^T + \frac{1+p_k}{\sigma_{k|k-1}^a} \mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T \quad (11)$$

then  $\forall p_k \in (0, +\infty)$ ,  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^a = \mathcal{E}(\hat{\mathbf{x}}_{k|k-1}^a, \sigma_{k|k-1}^a \mathbf{P}_{k|k-1}^a)$ .

*Fusion Update Step:* Given  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^a$  obeying (2) with  $\mathbf{v}_{i,k} \in \mathcal{E}(\mathbf{0}, \mathbf{R}_{i,k}), i \in \{1, 2, \dots, N\}$  and

$$\hat{\mathbf{x}}_k^a = \hat{\mathbf{x}}_{k|k-1}^a + q_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \quad (12)$$

$$(\mathbf{P}_k^a)^{-1} = (\mathbf{P}_{k|k-1}^a)^{-1} + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \quad (13)$$

$$\sigma_k^a = \sigma_{k|k-1}^a + q_k - q_k (\delta_k^a)^T \times (q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T + \mathbf{R}_k^a)^{-1} \delta_k^a \quad (14)$$

where

$$\delta_k^a = \mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a,$$

$$\mathbf{z}_k^a = [\mathbf{z}_{1,k}^T, \mathbf{z}_{2,k}^T, \dots, \mathbf{z}_{N,k}^T]^T,$$

$$\mathbf{H}_k^a = [\mathbf{H}_{1,k}^T, \mathbf{H}_{2,k}^T, \dots, \mathbf{H}_{N,k}^T]^T,$$

$$\mathbf{v}_k^a = [\mathbf{v}_{1,k}^T, \mathbf{v}_{2,k}^T, \dots, \mathbf{v}_{N,k}^T]^T,$$

$$(\mathbf{R}_k^a)^{-1} = \text{diag}(\alpha_{i,k} \mathbf{R}_{i,k}^{-1}),$$

with  $\alpha_{i,k} \in [0, 1], \sum_{i=1}^N \alpha_{i,k} = 1$ , then  $\forall q_k \in [0, +\infty), \mathbf{x}_k \in \mathcal{E}(\hat{\mathbf{x}}_k^a, \sigma_k^a \mathbf{P}_k^a) = \mathcal{E}_k^a \supseteq \mathcal{E}_{k|k-1}^a \cap \mathcal{X}_k$ , where  $\mathcal{X}_k$  is given by (9).

*Proof:* In prediction step, based on **Lemma 1** and through some variable substitutions, we obtain the formula (9) and

$$\sigma_{k|k-1}^a \mathbf{P}_{k|k-1}^a = (1+p_k^{-1})\mathbf{F}_{k-1} (\sigma_{k-1}^a \mathbf{P}_{k-1}^a) \mathbf{F}_{k-1}^T + (1+p_k) \mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T \quad (15)$$

Here  $\sigma_{k|k-1}^a$  is chosen to be equal to  $\sigma_{k-1}^a$ , and then (11) is derived.

In fusion update step, after the dimension expansion, the pseudo (generalized) measurement equation corresponding to all measurements received from multiple sensors can be expressed as

$$\mathbf{z}_k^a = \mathbf{H}_k^a \mathbf{x}_k + \mathbf{v}_k^a \quad (16)$$

And  $\mathbf{v}_k^a$  is assumed to be bounded by

$$\mathcal{V}_k^a = \{\mathbf{v}_k^a : (\mathbf{v}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{v}_k^a \leq 1\} \quad (17)$$

It can be derived from (4) that  $\mathbf{v}_{i,k}$  satisfies

$$\sum_{i=1}^N \alpha_{i,k} \mathbf{v}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{v}_{i,k} \leq 1, \text{ which is equivalent to } (\mathbf{v}_k^a)^T \text{diag}(\alpha_{i,k} \mathbf{R}_{i,k}^{-1}) \mathbf{v}_k^a \leq 1. \text{ Then we obtain } (\mathbf{R}_k^a)^{-1} = \text{diag}(\alpha_{i,k} \mathbf{R}_{i,k}^{-1}).$$

According to the equation (16) and (17),  $\mathbf{x}_k$  belongs to the bounding set given by

$$\begin{aligned} \mathcal{X}_k^a &= \{\mathbf{x}_k : (\mathbf{z}_k^a - \mathbf{H}_k^a \mathbf{x}_k)^T (\mathbf{R}_k^a)^{-1} (\mathbf{z}_k^a - \mathbf{H}_k^a \mathbf{x}_k) \leq 1\} \\ &= \{\mathbf{x}_k : \sum_{i=1}^N \alpha_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}_k)^T \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}_k) \leq 1\} \end{aligned} \quad (18)$$

It is obvious that  $\mathcal{X}_k^a \supseteq \mathcal{X}_k$ . Then the state  $\mathbf{x}_k$  lying in  $\mathcal{E}_k^a$  which contains  $\mathcal{E}_{k|k-1}^a \cap \mathcal{X}_k^a$  must satisfy

$$\begin{aligned}
& (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a)^T (\mathbf{P}_{k|k-1}^a)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a) \\
& + q_k (\mathbf{z}_k^a - \mathbf{H}_k^a \mathbf{x}_k)^T (\mathbf{R}_k^a)^{-1} (\mathbf{z}_k^a - \mathbf{H}_k^a \mathbf{x}_k) \\
& \leq \sigma_{k|k-1}^a + q_k
\end{aligned} \quad (19)$$

for any  $q_k \geq 0$ .

After some manipulations, (19) becomes

$$\begin{aligned}
& (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a)^T \left[ (\mathbf{P}_{k|k-1}^a)^{-1} + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \right] \\
& \times (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a) - 2q_k (\mathbf{z}_k^a - \mathbf{H}_k^a \mathbf{x}_k)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \\
& \times (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a) \\
& \leq \sigma_{k|k-1}^a + q_k - q_k (\mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a)^T (\mathbf{R}_k^a)^{-1} \\
& \times (\mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a)
\end{aligned} \quad (20)$$

Let  $\delta_k^a = \mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a$  and

$$(\mathbf{P}_k^a)^{-1} = (\mathbf{P}_{k|k-1}^a)^{-1} + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \quad (21)$$

then the equation (20) is equivalent to

$$\begin{aligned}
& (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a - q_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a)^T (\mathbf{P}_k^a)^{-1} \\
& \times (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a - q_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a) \\
& \leq \sigma_{k|k-1}^a + q_k - q_k (\delta_k^a)^T (\mathbf{R}_k^a)^{-1} (\delta_k^a) \\
& + \left[ q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \right]^T \mathbf{P}_k^a \left[ q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \right] \\
& = \sigma_{k|k-1}^a + q_k - q_k (\delta_k^a)^T \left( q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T + \mathbf{R}_k^a \right)^{-1} \delta_k^a
\end{aligned} \quad (22)$$

Thus we obtain (12) and (15). From (19), it is obvious that  $\mathcal{E}_k^a \supseteq \mathcal{E}_{k|k-1}^a \cap \mathcal{X}_k$ . The proof is completed.  $\square$

**Remark 1:** This method does not require any external condition of measurement equations for each sensor, and the measurements can be directly processed by the centralized processor. Thus it is flexible in applications. Here the flexibility means the augmented method could be used in any linear multi-sensor systems with bounded disturbances because no external condition of measurement equations for each sensor is needed.

## B. COMBINED MEASUREMENT FILTERING ALGORITHM

The combined measurement filtering algorithm always obtains the fused measurement information by weighted observations. Then, this fused measurement vector is filtered once at time  $k$  to achieve the purpose of multi-sensor fusion.

**Theorem 2:** The recursive procedures of the combined measurement filtering algorithm for combining numerous sensors with bounded disturbances are following equations.

**Prediction Step:** Given  $\mathbf{x}_{k-1} \in \mathcal{E}_{k-1}^b = \mathcal{E}(\hat{\mathbf{x}}_{k-1}^b, \sigma_{k-1}^b \mathbf{P}_{k-1}^b)$  obeying (1) with  $\mathbf{w}_k \in \mathcal{E}(\boldsymbol{\theta}, \mathbf{Q}_k)$  and

$$\hat{\mathbf{x}}_{k|k-1}^b = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1}^b \quad (23)$$

$$\sigma_{k|k-1}^b = \sigma_{k-1}^b \quad (24)$$

$$\begin{aligned}
\mathbf{P}_{k|k-1}^b &= (1 + p_k^{-1}) \mathbf{F}_{k-1} \mathbf{P}_{k-1}^b \mathbf{F}_{k-1}^T \\
& + \frac{1 + p_k}{\sigma_{k|k-1}^b} \mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T
\end{aligned} \quad (25)$$

then  $\forall p_k \in (0, +\infty)$ ,  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^b = \mathcal{E}(\hat{\mathbf{x}}_{k|k-1}^b, \sigma_{k|k-1}^b \mathbf{P}_{k|k-1}^b)$ .

**Fusion Update Step:** Given  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^b$  obeying (2) with  $v_{i,k} \in \mathcal{E}(\boldsymbol{\theta}, \mathbf{R}_{i,k})$ ,  $i \in \{1, 2, \dots, N\}$ , and assuming that  $\sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{H}_{i,k}$  is positive definite, let

$$\hat{\mathbf{x}}_k^b = \hat{\mathbf{x}}_{k|k-1}^b + q_k (1 - t_k) \mathbf{P}_k^b (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \delta_k^b \quad (26)$$

$$(\mathbf{P}_k^b)^{-1} = (\mathbf{P}_{k|k-1}^b)^{-1} + q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \mathbf{H}_k^b \quad (27)$$

$$\begin{aligned}
\sigma_k^b &= \sigma_{k|k-1}^b + q_k (1 - t_k) - q_k (\delta_k^b)^T \\
& \times \left( q_k \mathbf{H}_k^b \mathbf{P}_{k|k-1}^b (\mathbf{H}_k^b)^T + (1 - t_k)^{-1} \mathbf{R}_k^b \right)^{-1} \delta_k^b
\end{aligned} \quad (28)$$

where  $\delta_k^b = \mathbf{z}_k^b - \mathbf{H}_k^b \hat{\mathbf{x}}_{k|k-1}^b$ ,  $\mathbf{z}_k^b = \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k}$ ,

$$\mathbf{H}_k^b = \mathbf{I}, (\mathbf{R}_k^b)^{-1} = \mathbf{M}_k / (1 - t_k),$$

$$\mathbf{M}_k = \left( \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{H}_{i,k} \right),$$

$$t_k = \sum_{i=1}^N \alpha_{i,k} \mathbf{z}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k}$$

$$\begin{aligned}
& - \left( \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k} \right)^T \\
& \times \mathbf{M}_k^{-1} \left( \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k} \right)
\end{aligned}$$

with  $\alpha_{i,k} \in [0, 1]$ ,  $\sum_{i=1}^N \alpha_{i,k} = 1$ , then  $\forall q_k \in [0, +\infty)$ ,  $\mathbf{x}_k \in \mathcal{E}(\hat{\mathbf{x}}_k^b, \sigma_k^b \mathbf{P}_k^b) = \mathcal{E}_k^b \supseteq \mathcal{E}_{k|k-1}^b \cap \mathcal{X}_k$ , where  $\mathcal{X}_k$  is given by (8).

**Proof:** Proof of the prediction step is omitted because it closely follows the corresponding proofs in **Theorem 1**.

In fusion update step, let

$$\varphi_{i,k}(\mathbf{x}) = (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x})^T (\mathbf{R}_{i,k})^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}),$$

then we have

$$\mathcal{X}_k = \{\mathbf{x} : \varphi_{i,k}(\mathbf{x}) \leq 1, i = 1, 2, \dots, N\}.$$

Of course, if  $\mathbf{x} \in \mathcal{X}_k$ , then  $\varphi_k(\mathbf{x}) = \sum_{i=1}^N \alpha_{i,k} \varphi_{i,k}(\mathbf{x}) \leq 1$

for any  $\alpha_{i,k} \in [0, 1]$ ,  $\sum_{i=1}^N \alpha_{i,k} = 1$ , and it is obvious that  $\mathcal{X}_k^b = \{\mathbf{x} : \varphi_k(\mathbf{x}) \leq 1\} \supseteq \mathcal{X}_k$ .

After some transformations,  $\varphi_k(\mathbf{x})$  can be written as

$$\begin{aligned} \varphi_k(\mathbf{x}) &= \mathbf{x}^T \mathbf{M}_k \mathbf{x} - \sum_{i=1}^N \alpha_{i,k} \mathbf{z}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \mathbf{M}_k^{-1} \mathbf{M}_k \mathbf{x} \\ &\quad - \mathbf{x}^T \mathbf{M}_k \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} + \sum_{i=1}^N \alpha_{i,k} \mathbf{z}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \\ &= \left( \mathbf{x} - \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \right)^T \mathbf{M}_k \\ &\quad \times \left( \mathbf{x} - \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \right) + t_k \end{aligned} \quad (29)$$

Thus  $\varphi_k(\mathbf{x}) \leq 1$  is equivalent to

$$\begin{aligned} &\left( \mathbf{x} - \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \right)^T \frac{\mathbf{M}_k}{1 - t_k} \\ &\quad \times \left( \mathbf{x} - \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \right) \leq 1 \end{aligned}$$

Assume

$$\mathcal{X}_k^b = \left\{ \mathbf{x} : (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x})^T (\mathbf{R}_k^b)^{-1} (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x}) \leq 1 \right\},$$

then we obtain  $\mathbf{z}_k^b = \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k}$ ,  $\mathbf{H}_k^b = \mathbf{I}$  and  $(\mathbf{R}_k^b)^{-1} = \mathbf{M}_k / (1 - t_k)$ .

$\mathcal{X}_k^b$  can also be written as  $\mathcal{X}_k^b = \{ \mathbf{x} : (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x})^T \mathbf{M}_k (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x}) \leq 1 - t_k \}$ . Then the state  $\mathbf{x}_k$  lying in  $\mathcal{E}_k^b$  which contains  $\mathcal{E}_{k|k-1}^b \cap \mathcal{X}_k^b$  must satisfy

$$\begin{aligned} &(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^b)^T (\mathbf{P}_{k|k-1}^b)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^b) \\ &\quad + q_k (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x}_k)^T \mathbf{M}_k (\mathbf{z}_k^b - \mathbf{H}_k^b \mathbf{x}_k) \\ &\leq \sigma_{k|k-1}^b + q_k (1 - t_k) \end{aligned} \quad (30)$$

for any  $q_k \geq 0$ .

After some manipulations similar to (20),(21) and (22), we can obtain (26), (27) and (28). And it is obvious that  $\mathcal{E}_k^b \supseteq \mathcal{E}_{k|k-1}^b \cap \mathcal{X}_k^b$ . The proof is completed.  $\square$

**Remark 2:** It should be noted that the use of combined measurement filtering algorithm relies on the assumption that  $\sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{H}_{i,k}$  is positive definite. Thus this algorithm is less flexible in applications than the augmented algorithm.

### C. PSEUDO-SEQUENTIAL FILTERING ALGORITHM

The main idea of the pseudo-sequential filtering algorithm is that: After one-step prediction, the fusion center sequentially updates the estimate of current state by using each sensor measurement, and then a fusion estimate based on global measurement information can be obtained.

**Theorem 3:** The recursive procedures of the pseudo-sequential filtering algorithm for combining numerous sensors with bounded disturbances are following equations.

**Prediction Step:** Given  $\mathbf{x}_{k-1} \in \mathcal{E}_{k-1}^c = \mathcal{E}(\hat{\mathbf{x}}_{k-1}^c, \sigma_{k-1}^c \mathbf{P}_{k-1}^c)$  obeying (1) with  $\mathbf{w}_k \in \mathcal{E}(\mathbf{0}, \mathbf{Q}_k)$  and

$$\hat{\mathbf{x}}_{k|k-1}^c = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1}^c \quad (31)$$

$$\sigma_{k|k-1}^c = \sigma_{k-1}^c \quad (32)$$

$$\begin{aligned} \mathbf{P}_{k|k-1}^c &= (\mathbf{I} + p_k^{-1}) \mathbf{F}_{k-1} \mathbf{P}_{k-1}^c \mathbf{F}_{k-1}^T \\ &\quad + \frac{1 + p_k}{\sigma_{k|k-1}^c} \mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T \end{aligned} \quad (33)$$

then  $\forall p_k \in (0, +\infty)$ ,  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^c = \mathcal{E}(\hat{\mathbf{x}}_{k|k-1}^c, \sigma_{k|k-1}^c \mathbf{P}_{k|k-1}^c)$ .

**Fusion Update Step:** Given  $\mathbf{x}_k \in \mathcal{E}_{k|k-1}^c$  obeying (2) with  $\mathbf{v}_{i,k} \in \mathcal{E}(\mathbf{0}, \mathbf{R}_{i,k})$ ,  $i \in \{1, 2, \dots, N\}$ , set  $\hat{\mathbf{x}}_{k,0} = \hat{\mathbf{x}}_{k|k-1}^c$ ,  $\mathbf{P}_{k,0} = \mathbf{P}_{k|k-1}^c$ ,  $\sigma_{k,0} = \sigma_{k|k-1}^c$  and let

$$\hat{\mathbf{x}}_{i,k} = \hat{\mathbf{x}}_{i-1,k} + q_{i,k} \mathbf{P}_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \quad (34)$$

$$(\mathbf{P}_{i,k})^{-1} = (\mathbf{P}_{i-1,k})^{-1} + q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \quad (35)$$

$$\begin{aligned} \sigma_{i,k} &= q_{i,k} \delta_{i,k}^T \left( q_{i,k} \mathbf{H}_{i,k} \mathbf{P}_{i-1,k} \mathbf{H}_{i,k}^T + \mathbf{R}_{i,k} \right)^{-1} \delta_{i,k} \\ &\quad + \sigma_{i-1,k} - q_{i,k} \end{aligned} \quad (36)$$

where  $\delta_{i,k} = \mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{i-1,k}$ , then  $\forall q_{i,k} \in [0, +\infty)$ ,  $\mathbf{x}_k \in \mathcal{E}(\hat{\mathbf{x}}_k^c, \sigma_k^c \mathbf{P}_k^c) = \mathcal{E}_{k|k-1}^c \cap \mathcal{X}_k$ , where  $\hat{\mathbf{x}}_k^c = \hat{\mathbf{x}}_{N,k}^c$ ,  $\mathbf{P}_k^c = \mathbf{P}_{N,k}^c$ ,  $\sigma_k^c = \sigma_{N,k}^c$  and  $\mathcal{X}_k$  is given by (9).

**Proof:** Proof of the prediction step is omitted because it closely follows the corresponding proofs in **Theorem 1**.

In fusion update step, according to the observation equation (2) and (4),  $\mathbf{x}_k$  belongs to the bounding sets

$$\mathcal{X}_{i,k} = \{ \mathbf{x} : (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x})^T \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}) \leq 1 \} \quad (37)$$

It is obvious that  $\mathcal{X}_k = \cap_i \mathcal{X}_{i,k}$ . Set  $\hat{\mathbf{x}}_{k,0} = \hat{\mathbf{x}}_{k|k-1}^c$ ,  $\mathbf{P}_{k,0} = \mathbf{P}_{k|k-1}^c$ ,  $\sigma_{k,0} = \sigma_{k|k-1}^c$  and  $\mathcal{E}_{i,k} = \mathcal{E}(\hat{\mathbf{x}}_{i,k}^c, \sigma_{i,k}^c \mathbf{P}_{i,k}^c)$ , then  $\mathcal{E}_{0,k}^c = \mathcal{E}_{k|k-1}^c$ . The state  $\mathbf{x}_k$  lying in  $\mathcal{E}_{i,k}^c$  which contains  $\mathcal{E}_{i-1,k}^c \cap \mathcal{X}_{i,k}$  must satisfy

$$\begin{aligned} &(\mathbf{x} - \hat{\mathbf{x}}_{i-1,k}^c)^T (\mathbf{P}_{k,i-1}^c)^{-1} (\mathbf{x} - \hat{\mathbf{x}}_{i-1,k}^c) \\ &\quad + q_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x})^T \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \mathbf{x}) \\ &\leq \sigma_{i-1,k}^c + q_{i,k} \end{aligned} \quad (38)$$

for any  $q_{i,k} \geq 0$ .

After some manipulations similar to (20), (21) and (22), we can obtain (34), (35) and (36). Let  $\hat{\mathbf{x}}_k^c = \hat{\mathbf{x}}_{N,k}^c$ ,  $\mathbf{P}_k^c = \mathbf{P}_{N,k}^c$ ,  $\sigma_k^c = \sigma_{N,k}^c$ , then we have

$$\begin{aligned} \mathcal{E}_{N,k}^c &\supseteq \mathcal{E}_{N-1,k}^c \cap \mathcal{X}_{N,k} \\ &\supseteq (\mathcal{E}_{N-2,k}^c \cap \mathcal{X}_{N-1,k}) \cap \mathcal{X}_{N,k} \\ &= \mathcal{E}_{N-2,k}^c \cap (\mathcal{X}_{N-1,k} \cap \mathcal{X}_{N,k}) \\ &\quad \dots \\ &\supseteq \mathcal{E}_{0,k}^c \cap \left( \bigcap_{i=1}^N \mathcal{X}_{i,k} \right) \\ &= \mathcal{E}_{k|k-1}^c \cap \mathcal{X}_k \end{aligned}$$

The proof is completed.  $\square$

**Remark 3:** The pseudo-sequential filtering algorithm has no limitation on the measurement equations of each sensor. And every time the fusion center receives a measurement,

the central processor immediately processes it. Thus the advantage of the pseudo-sequential filtering algorithm is its flexibility and it can even embody its advantage when the measurements are not received at the same time instant.

#### D. SELECTION OF OPTIMAL PARAMETERS

The scalar parameters can be numerically optimized to find the smallest bounding ellipsoid [8] or ensure the stability of the estimator error [9], or set to be a reasonable constant [7]. Taking the augmented algorithm as an example, the different parameters are obtained depending on the different criteria, as below.

In prediction step, trace-minimum criterion is used, because the explicit expression of the optimal parameter can be obtained by using this criterion, thereby avoiding the solution of the nonlinear equation and improving the computational efficiency of the algorithm. According to the theory in [8], the expression of  $p_k$  is derived as follows:

$$\tilde{p}_k = \left[ \frac{\sigma_{k|k-1}^a \text{tr}(\mathbf{F}_{k-1} \mathbf{P}_{k-1}^a \mathbf{F}_{k-1}^T)}{\text{tr}(\mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T)} \right]^{1/2} \quad (39)$$

In fusion update step, contrary to those algorithms that minimize the size of the ellipsoid, here  $\sigma_k^a$  is minimized to obtain the optimal parameters due to the concern for the stability of the estimated state vector:

*Theorem 4:* Let  $\tilde{q}_k$  denote the optimal value of  $q_k$ . When  $(\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \geq 1$ , the value of  $q_k$  given in (15) that solves  $\min_{q_k \geq 0} \sigma_k^a$  satisfies

$$\begin{aligned} & (\delta_k^a)^T (\mathbf{R}_k^a + q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T)^{-1} \mathbf{R}_k^a \\ & \times (\mathbf{R}_k^a + q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T)^{-1} \delta_k^a = 1 \quad (40) \end{aligned}$$

When  $(\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a < 1$ , there is no solution for equation (40), and  $\tilde{q}_k = 0$  is the optimal value.

*Proof:* Based on (15),  $\sigma_k^a$  can be seen as a function associated with  $q_k$ , i.e.,  $\sigma_k^a = f(q_k)$ . And the object is described by

$$\tilde{q}_k = \arg \min_{q_k \geq 0} f(q_k)$$

The derivation of  $\sigma_k^a$  to  $q_k$  is

$$\begin{aligned} f'(q_k) &= 1 - (\delta_k^a)^T (\mathbf{R}_k^a + q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T)^{-1} \mathbf{R}_k^a \\ & \times (\mathbf{R}_k^a + q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T)^{-1} \delta_k^a \quad (41) \end{aligned}$$

Differentiation shows that the second derivative  $f''(q_k)$  is always non-negative, which means  $f'(q_k)$  is monotone increasing for any  $q_k \geq 0$ .

If  $f'(0) > 0$ , i.e.,  $(\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a < 1$ , we can obtain  $f'(q_k) > 0, \forall q_k \geq 0$ , which means  $f(q_k)$  is increasing for any  $q_k \geq 0$ . Thus the optimal value of  $q_k$  is 0;

If  $f'(0) \leq 0$ , i.e.,  $(\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \geq 1$ , we have  $f'(q_k) \leq 0$  when  $q_k < 0$ , and  $f'(q_k) \geq 0$  when  $q_k > 0$ . This implies the single minimum of  $f(q_k)$  is obtained when first derivation is 0. Thus (40) is obtained. The proof is completed.  $\square$

*Remark 4:*  $\tilde{q}_k = 0$  means that the observation is informative and no updating takes place, i.e.,  $\mathbf{P}_k^a = \mathbf{P}_{k|k-1}^a, \hat{\mathbf{x}}_k^a = \hat{\mathbf{x}}_{k|k-1}^a, \sigma_k^a = \sigma_{k|k-1}^a$ .

The parameters  $\alpha_{i,k}$  can be chosen at each step in an effort to reduce the effect of the faulty output  $z_{i,k}$ . The faulty output is subject to an abnormal perturbation which is not in the ellipsoid  $\mathcal{V}_{i,k}$ . Thus  $\alpha_{i,k}$  should be chosen as a decreasing function of the norm of the a priori output error vector  $\delta_{i,k}^a = z_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^a$ , e.g.,

$$\alpha_{i,k} = \frac{\left( \|\delta_{i,k}^a\|_{\mathbf{R}_{i,k}^{-1}} \right)^{-1}}{\sum_{i=1}^N \left( \|\delta_{i,k}^a\|_{\mathbf{R}_{i,k}^{-1}} \right)^{-1}} \quad (42)$$

For the other two algorithms, the different parameters in different stages are optimized as the augmented algorithm.

#### V. ALGORITHM PROPERTIES

In this section, the characteristics of the proposed algorithms in Section IV are analyzed, including the functional equivalence, the exchangeability of measurement update order and the stability.

##### A. THE FUNCTIONAL EQUIVALENCE OF THE AUGMENTED ALGORITHM AND THE COMBINED MEASUREMENT FILTERING ALGORITHM

*Theorem 5:* Considering a N-sensor dynamic system given by (1) and (2), the augmented algorithm and the combined measurement filtering algorithm are functionally equivalent in terms of estimation accuracy if the parameters  $p_k, q_k$  and  $\alpha_{i,k}$  of the two algorithms are chosen to be identical.

*Proof:* Combing (12), (13) and  $\delta_k^a = \mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a$  yields

$$\begin{aligned} & (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_k^a \\ &= (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_{k|k-1}^a + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} (\mathbf{z}_k^a - \mathbf{H}_k^a \hat{\mathbf{x}}_{k|k-1}^a) \\ &= \left( (\mathbf{P}_k^a)^{-1} - q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \right) \hat{\mathbf{x}}_{k|k-1}^a \\ & \quad + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{z}_k^a \\ &= \left( \mathbf{P}_{k|k-1}^a \right)^{-1} \hat{\mathbf{x}}_{k|k-1}^a + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{z}_k^a \quad (43) \end{aligned}$$

Then it is easily deduced from the equations of **Theorem 1** that

$$\begin{aligned} & (\mathbf{P}_k^a)^{-1} - \left( \mathbf{P}_{k|k-1}^a \right)^{-1} \\ &= q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \quad (44) \end{aligned}$$

$$\begin{aligned} & (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_k^a - \left( \mathbf{P}_{k|k-1}^a \right)^{-1} \hat{\mathbf{x}}_{k|k-1}^a \\ &= q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{z}_k^a \\ &= q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \quad (45) \end{aligned}$$

$$\begin{aligned}
 & \sigma_k^a - \sigma_{k|k-1}^a \\
 &= q_k - q_k (\delta_k^a)^T (\mathbf{R}_k^a)^{-1} (\delta_k^a) + \left[ q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \right]^T \mathbf{P}_k^a \\
 & \quad \times \left[ q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \right] \\
 &= q_k - q_k \sum_{i=1}^N \alpha_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^a)^T \\
 & \quad \times \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^a) \\
 & \quad + \left[ (\mathbf{P}_k^a)^{-1} (\hat{\mathbf{x}}_k^a - \hat{\mathbf{x}}_{k|k-1}^a) \right]^T \mathbf{P}_k^a \left[ (\mathbf{P}_k^a)^{-1} (\hat{\mathbf{x}}_k^a - \hat{\mathbf{x}}_{k|k-1}^a) \right] \\
 &= q_k \left( 1 - \sum_{i=1}^N \alpha_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^a)^T \right. \\
 & \quad \times \left. \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^a) \right) \\
 & \quad + (\hat{\mathbf{x}}_k^a - \hat{\mathbf{x}}_{k|k-1}^a)^T (\mathbf{P}_k^a)^{-1} (\hat{\mathbf{x}}_k^a - \hat{\mathbf{x}}_{k|k-1}^a) \quad (46)
 \end{aligned}$$

In a similar way, it can be derived from the equations in **Theorem 2** that

$$\begin{aligned}
 & (\mathbf{P}_k^b)^{-1} \hat{\mathbf{x}}_k^b \\
 &= (\mathbf{P}_k^b)^{-1} \hat{\mathbf{x}}_{k|k-1}^b + q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \\
 & \quad \times (\mathbf{z}_k^b - \mathbf{H}_k^b \hat{\mathbf{x}}_{k|k-1}^b) \\
 &= \left( (\mathbf{P}_k^b)^{-1} - q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \mathbf{H}_k^b \right) \hat{\mathbf{x}}_{k|k-1}^b \\
 & \quad + q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \mathbf{z}_k^b \\
 &= (\mathbf{P}_{k|k-1}^b)^{-1} \hat{\mathbf{x}}_{k|k-1}^b + q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \mathbf{z}_k^b \quad (47)
 \end{aligned}$$

Then we have

$$\begin{aligned}
 & (\mathbf{P}_k^b)^{-1} - (\mathbf{P}_{k|k-1}^b)^{-1} \\
 &= q_k (1 - t_k) (\mathbf{H}_k^b)^T (\mathbf{R}_k^b)^{-1} \mathbf{H}_k^b \\
 &= q_k \mathbf{M}_k = q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \\
 &= (\mathbf{P}_k^a)^{-1} - (\mathbf{P}_{k|k-1}^a)^{-1} \quad (48)
 \end{aligned}$$

and

$$\begin{aligned}
 & (\mathbf{P}_k^b)^{-1} \hat{\mathbf{x}}_k^b - (\mathbf{P}_{k|k-1}^b)^{-1} \hat{\mathbf{x}}_{k|k-1}^b \\
 &= q_k \mathbf{M}_k \left[ \mathbf{M}_k^{-1} \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{z}_{i,k} \right] \\
 &= q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_i^T (\mathbf{R}_i)^{-1} \mathbf{z}_i \\
 &= (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_k^a - (\mathbf{P}_{k|k-1}^a)^{-1} \hat{\mathbf{x}}_{k|k-1}^a \quad (49)
 \end{aligned}$$

Based on (28) and (29), we obtained

$$\begin{aligned}
 & \sigma_k^b - \sigma_{k|k-1}^b \\
 &= q_k \left[ 1 - t_k - (\delta_k^b)^T \mathbf{M}_k \delta_k^b \right] \\
 & \quad + (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b)^T (\mathbf{P}_k^b)^{-1} (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b) \\
 &= q_k \left[ 1 - \varphi(\hat{\mathbf{x}}_{k|k-1}^b) \right] \\
 & \quad + (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b)^T (\mathbf{P}_k^b)^{-1} (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b) \\
 &= q_k \left[ 1 - \sum_{i=1}^N \alpha_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^b)^T \right. \\
 & \quad \times \left. \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^b) \right] \\
 & \quad + (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b)^T (\mathbf{P}_k^b)^{-1} (\hat{\mathbf{x}}_k^b - \hat{\mathbf{x}}_{k|k-1}^b) \quad (50)
 \end{aligned}$$

Note that the prediction steps of the two algorithms are the same, and so are the initial conditions, i.e. when  $k = 1$ , we have

$$\sigma_{1|0}^b = \sigma_{1|0}^a = \sigma_0 \quad (51)$$

$$\hat{\mathbf{x}}_{1|0}^b = \hat{\mathbf{x}}_{1|0}^a = \hat{\mathbf{x}}_{1|0} = \mathbf{F}_0 \hat{\mathbf{x}}_0 \quad (52)$$

$$\begin{aligned}
 \mathbf{P}_{1|0}^b &= \mathbf{P}_{1|0}^a = (1 + p_1^{-1}) \mathbf{F}_0 \mathbf{P}_0 \mathbf{F}_0^T \\
 & \quad + \frac{1 + p_1}{\sigma_0} \mathbf{G}_0 \mathbf{Q}_0 \mathbf{G}_0^T \quad (53)
 \end{aligned}$$

Therefore, according to the mathematical induction, for any  $k \geq 1$ , the following equations hold:

$$(\mathbf{P}_k^b)^{-1} = (\mathbf{P}_k^a)^{-1}, \quad (\mathbf{P}_k^b)^{-1} \hat{\mathbf{x}}_k^b = (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_k^a$$

Finally, it concludes that  $\mathbf{P}_k^b = \mathbf{P}_k^a$ ,  $\hat{\mathbf{x}}_k^b = \hat{\mathbf{x}}_k^a$  and  $\sigma_k^b = \sigma_k^a$ , so there exists a functional equivalence between the two fusion algorithms obviously. The proof is completed.  $\square$

## B. THE FUNCTIONAL EQUIVALENCE OF THE AUGMENTED ALGORITHM AND THE PSEUDO-SEQUENTIAL FILTERING ALGORITHM

**Theorem 6:** Considering a N-sensor dynamic system given by (1) and (2), the pseudo-sequential filtering algorithm and the augmented algorithm are functionally equivalent in terms of estimation accuracy if the parameters  $p_k$  of the two algorithms are identical and  $q_{i,k} = q_k \alpha_{i,k}$ .

*Proof:* It is easily derived from (34)~(36) that

$$\mathbf{P}_{i,k}^{-1} (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k}) = q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \quad (54)$$

$$\mathbf{P}_{i,k}^{-1} \hat{\mathbf{x}}_{i,k} - \mathbf{P}_{i-1,k}^{-1} \hat{\mathbf{x}}_{i-1,k} = q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \quad (55)$$

$$\mathbf{P}_{i,k}^{-1} - \mathbf{P}_{i-1,k}^{-1} = q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \quad (56)$$

$$\begin{aligned}
 \sigma_{i,k} - \sigma_{i-1,k} &= q_{i,k} \left( 1 - \delta_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \right) \\
 & \quad + (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k})^T \\
 & \quad \times \mathbf{P}_{i,k}^{-1} (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k}) \quad (57)
 \end{aligned}$$

Thus

$$\begin{aligned} & (\mathbf{P}_k^c)^{-1} - (\mathbf{P}_{k|k-1}^c)^{-1} \\ &= \mathbf{P}_{N,k}^{-1} - \mathbf{P}_{0,k}^{-1} = \sum_{i=1}^N q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \end{aligned} \quad (58)$$

$$\begin{aligned} & (\mathbf{P}_k^c)^{-1} \hat{\mathbf{x}}_k^c - (\mathbf{P}_{k|k-1}^c)^{-1} \hat{\mathbf{x}}_{k|k-1}^c \\ &= \mathbf{P}_{N,k}^{-1} \hat{\mathbf{x}}_{N,k} - \mathbf{P}_{0,k}^{-1} \hat{\mathbf{x}}_{0,k} \\ &= \sum_{i=1}^N q_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \end{aligned} \quad (59)$$

$$\begin{aligned} & \sigma_k^c - \sigma_{k|k-1}^c \\ &= \sigma_{N,k} - \sigma_{0,k} \\ &= \sum_{i=1}^N q_{i,k} - \sum_{i=1}^N q_{i,k} \delta_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \\ &+ \sum_{i=1}^N (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k})^T \mathbf{P}_{i,k}^{-1} (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k}) \end{aligned} \quad (60)$$

Define  $\delta_{i,k}^c = \mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^c$ , such that

$$\delta_{i,k} = \delta_{i,k}^c - \mathbf{H}_{i,k} (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c) \quad (61)$$

Then

$$\begin{aligned} & \delta_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \\ &= 2(\hat{\mathbf{x}}_{k|k-1}^c - \hat{\mathbf{x}}_{i-1,k})^T \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} + (\delta_{i,k}^c)^T \mathbf{R}_{i,k}^{-1} \delta_{i,k}^c \\ &- (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c)^T \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c) \end{aligned} \quad (62)$$

Based on (54) and (56), we obtain

$$\begin{aligned} & \sum_{i=1}^N q_{i,k} \delta_{i,k}^T \mathbf{R}_{i,k}^{-1} \delta_{i,k} \\ &= \sum_{i=1}^N q_{i,k} (\delta_{i,k}^c)^T \mathbf{R}_{i,k}^{-1} \delta_{i,k}^c \\ &- \sum_{i=1}^N (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c)^T \mathbf{P}_{i,k}^{-1} (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{k|k-1}^c) \\ &+ \sum_{i=1}^N (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c)^T \mathbf{P}_{i-1,k}^{-1} (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c) \\ &- \sum_{i=1}^N (\hat{\mathbf{x}}_{i-1,k} - \hat{\mathbf{x}}_{k|k-1}^c)^T \mathbf{P}_{i,k}^{-1} (\hat{\mathbf{x}}_{i,k} - \hat{\mathbf{x}}_{i-1,k}) \end{aligned} \quad (63)$$

Substitute (63) into(60), then we have

$$\begin{aligned} \sigma_k^c - \sigma_{k|k-1}^c &= \sum_{i=1}^N q_{i,k} - \sum_{i=1}^N q_{i,k} (\delta_{i,k}^c)^T \mathbf{R}_{i,k}^{-1} \delta_{i,k}^c \\ &+ (\hat{\mathbf{x}}_k^c - \hat{\mathbf{x}}_{k|k-1}^c)^T (\mathbf{P}_k^c)^{-1} (\hat{\mathbf{x}}_k^c - \hat{\mathbf{x}}_{k|k-1}^c) \end{aligned} \quad (64)$$

It should be emphasized that some complicated but routine intermediate derivation process of Equations (61)~(64) is omitted due to space limitations.

Let  $q_{i,k} = q_k \alpha_{i,k}$ , it holds that

$$\begin{aligned} & (\mathbf{P}_k^c)^{-1} - (\mathbf{P}_{k|k-1}^c)^{-1} \\ &= q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{H}_{i,k} \\ &= (\mathbf{P}_k^a)^{-1} - (\mathbf{P}_{k|k-1}^a)^{-1} \end{aligned} \quad (65)$$

$$\begin{aligned} & (\mathbf{P}_k^c)^{-1} \hat{\mathbf{x}}_k^c - (\mathbf{P}_{k|k-1}^c)^{-1} \hat{\mathbf{x}}_{k|k-1}^c \\ &= q_k \sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T \mathbf{R}_{i,k}^{-1} \mathbf{z}_{i,k} \\ &= (\mathbf{P}_k^a)^{-1} \hat{\mathbf{x}}_k^a - (\mathbf{P}_{k|k-1}^a)^{-1} \hat{\mathbf{x}}_{k|k-1}^a \\ & \sigma_k^c - \sigma_{k|k-1}^c \\ &= q_k \left[ 1 - \sum_{i=1}^N \alpha_{i,k} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^c)^T \right. \\ & \quad \times \left. \mathbf{R}_{i,k}^{-1} (\mathbf{z}_{i,k} - \mathbf{H}_{i,k} \hat{\mathbf{x}}_{k|k-1}^c) \right] \\ & \quad + (\hat{\mathbf{x}}_k^c - \hat{\mathbf{x}}_{k|k-1}^c)^T (\mathbf{P}_k^c)^{-1} (\hat{\mathbf{x}}_k^c - \hat{\mathbf{x}}_{k|k-1}^c) \end{aligned} \quad (66)$$

Similar to the proof of **Theorem 5**, there exists a functional equivalence between the pseudo-sequential filtering algorithm and the augmented algorithm obviously. The proof is completed.  $\square$

### C. STABILITY AND CONVERGENCE ANALYSIS

In this section, we will prove the fusion estimator of the proposed fusion algorithms is input-to-state stable (ISS) under some conditions. The concrete meaning of ISS is explained below.

*Definition 2* [17], [20]: The system  $\mathbf{z}_k = f(\mathbf{z}_{k-1}, \mathbf{u}_{k-1})$  is ISS if it admits a continuous ISS-Lyapunov function  $\mathcal{L} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , that is, there exists  $\mathcal{K}\infty$  functions  $\mu_1$  and  $\mu_2$  such that for all  $\mathbf{z} \in \mathbb{R}^n$ ,  $\mu_1(\|\mathbf{z}\|) \leq \mathcal{L}(\mathbf{z}) \leq \mu_2(\|\mathbf{z}\|)$  and there exists a  $\mathcal{K}\infty$  function  $\mu_3$  and a  $\mathcal{K}$  function  $\chi$  such that for all  $\mathbf{z} \in \mathbb{R}^n$  and all  $\mathbf{u} \in \mathbb{R}^m$ ,  $\mathcal{L}(f(\mathbf{z}, \mathbf{u})) - \mathcal{L}(\mathbf{z}) \leq -\mu_3(\|\mathbf{z}\|) + \chi(\|\mathbf{u}\|)$

*Lemma 2:* Consider two positive definite matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\mathbf{B} \in \mathbb{R}^{n \times n}$ , and two vectors  $\mathbf{a} \in \mathbb{R}^n$  and  $\mathbf{b} \in \mathbb{R}^n$ , then

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{A} + \mathbf{B})^{-1} (\mathbf{a} + \mathbf{b}) \leq \mathbf{a}^T \mathbf{A}^{-1} \mathbf{a} + \mathbf{b}^T \mathbf{B}^{-1} \mathbf{b} \quad (68)$$

*Proof:* See the works of Shen et al. [17].

$$\hat{\mathbf{x}}_k^a = \hat{\mathbf{x}}_{k|k-1}^a + q_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a \quad (69)$$

$$\begin{aligned} (\mathbf{P}_k^a)^{-1} &= (\mathbf{P}_{k|k-1}^a)^{-1} + q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \\ \sigma_k^a &= \sigma_{k|k-1}^a + q_k - q_k (\delta_k^a)^T \\ & \quad \times \left( q_k \mathbf{H}_k^a \mathbf{P}_{k|k-1}^a (\mathbf{H}_k^a)^T + \mathbf{R}_k^a \right)^{-1} \delta_k^a \end{aligned} \quad (70)$$



**Theorem 7:** Considering a N-sensor dynamic system given by (1) and (2), if  $\mathcal{E}(\hat{\mathbf{x}}_k^a, \sigma_k^a \mathbf{P}_k^a)$  is estimated by **Theorem 1**,  $q_k = \tilde{q}_k$  and the pair  $(\mathbf{F}_k, [\mathbf{H}_{1,k}^T, \mathbf{H}_{2,k}^T, \dots, \mathbf{H}_{N,k}^T]^T)$  is uniformly observable, then  $\mathcal{L}_k = (\mathbf{x}_k - \hat{\mathbf{x}}_k^a)^T (\mathbf{P}_k^a)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_k^a)$  is an ISS-Lyapunov function and the estimation error  $\tilde{\mathbf{x}}_k^a = \mathbf{x}_k - \hat{\mathbf{x}}_k^a$  is ISS.

*Proof:* First, we have

$$\frac{\|\tilde{\mathbf{x}}_k^a\|^2}{\lambda_{\max}(\mathbf{P}_k^a)} \leq \mathcal{L}_k \leq \frac{\|\tilde{\mathbf{x}}_k^a\|^2}{\lambda_{\min}(\mathbf{P}_k^a)} \quad (71)$$

where  $\lambda(\mathbf{P}_k^a)$  is the eigenvalue of matrix  $\mathbf{P}_k^a$ .

Let  $\mathcal{L}_{k|k-1} = (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a)^T (\mathbf{P}_{k|k-1}^a)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a)$ , from(12), (13) and (15), we obtain (72), as shown at the bottom of the next page thus  $\mathcal{L}_k - \mathcal{L}_{k-1} \leq \mathcal{L}_{k|k-1} - \mathcal{L}_{k-1}$ . Due to (1) and (9), we have  $\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a = \mathbf{F}_{k-1} \tilde{\mathbf{x}}_{k-1}^a + \mathbf{G}_{k-1} \mathbf{w}_{k-1}$ . Now using (11) and **Lemma 3**,  $\mathcal{L}_{k|k-1}$  can be expressed as:

$$\begin{aligned} \mathcal{L}_{k|k-1} &= (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a)^T (\mathbf{P}_{k|k-1}^a)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a) \\ &\leq (\mathbf{F}_{k-1} \tilde{\mathbf{x}}_{k-1}^a)^T \left( (1 + p_k^{-1}) \mathbf{F}_{k-1} \mathbf{P}_{k-1}^a \mathbf{F}_{k-1}^T \right)^{-1} \\ &\quad \times (\mathbf{F}_{k-1} \tilde{\mathbf{x}}_{k-1}^a) \\ &\quad + (\mathbf{G}_{k-1} \mathbf{w}_{k-1})^T \left( \frac{1 + p_k}{\sigma_{k|k-1}^a} \mathbf{G}_{k-1} \mathbf{Q}_{k-1} \mathbf{G}_{k-1}^T \right)^{-1} \\ &\quad \times (\mathbf{G}_{k-1} \mathbf{w}_{k-1}) \\ &= \frac{p_k}{1 + p_k} \mathcal{L}_{k-1} + \frac{\sigma_{k-1}^a}{1 + p_k} \mathbf{w}_{k-1}^T \mathbf{Q}_{k-1}^{-1} \mathbf{w}_{k-1} \quad (73) \end{aligned}$$

It implies that

$$\begin{aligned} \mathcal{L}_k - \mathcal{L}_{k-1} &\leq \frac{p_k}{1 + p_k} \mathcal{L}_{k-1} - \mathcal{L}_{k-1} + \frac{\sigma_{k-1}^a}{1 + p_k} \mathbf{w}_{k-1}^T \mathbf{Q}_{k-1}^{-1} \mathbf{w}_{k-1} \\ &= -\frac{1}{1 + p_k} \mathcal{L}_{k-1} + \frac{\sigma_{k-1}^a}{1 + p_k} \mathbf{w}_{k-1}^T \mathbf{Q}_{k-1}^{-1} \mathbf{w}_{k-1} \\ &\leq -\frac{\|\tilde{\mathbf{x}}_{k-1}^a\|^2}{(1 + p_k) \lambda_{\max}(\mathbf{P}_{k-1}^a)} + \frac{\sigma_{k-1}^a \|\mathbf{w}_{k-1}\|^2}{(1 + p_k) \lambda_{\min}(\mathbf{Q}_{k-1})} \quad (74) \end{aligned}$$

(71) and (75) show that  $\mathcal{L}_k$  is an ISS-Lyapunov function and the estimation error  $\tilde{\mathbf{x}}_k^a$  is ISS for this system. The proof is completed.  $\square$

**Remark 5:** Assume no noise is present, then  $\mathcal{L}_k - \mathcal{L}_{k-1} \leq 0$  holds, hence the estimation error goes to zero.

**Remark 6:** ISS system has some important properties: 1) globally asymptotic stable;2) bounded-input bounded-state stable;3) converged-input converged-state stable. Based on **Theorem 7** and above ISS properties, the convergence and boundedness of the augmented algorithm described by **Theorem 1** could be summarized as follows:

- 1) The estimation error is bounded when the noise is nonzero, the initial error is bounded, and the pair  $(\mathbf{F}_k, [\mathbf{H}_{1,k}^T, \mathbf{H}_{2,k}^T, \dots, \mathbf{H}_{N,k}^T]^T)$  uniformly observable.
- 2) The estimation error tends to zero exponentially when the noise terms to zero.

- 3) The estimation error is converged when the noise is converged, the initial error is bounded, and the pair  $(\mathbf{F}_k, [\mathbf{H}_{1,k}^T, \mathbf{H}_{2,k}^T, \dots, \mathbf{H}_{N,k}^T]^T)$  uniformly observable.

**Remark 7:** Considering the functional equivalence of the proposed algorithms, the convergence and boundedness of the combined measurement filtering algorithm and pseudo-sequential filtering algorithm is the same as that of the augmented when the conditions in **Theorem 5** and **6** satisfy.

#### D. EXCHANGEABILITY ANALYSIS OF MEASUREMENT UPDATE ORDER IN FUSION ALGORITHMS

For the linear dynamic multi-sensor system in probabilistic setting, it has been proven that the estimate accuracy of the three methods remains unchanged when the measurement's update order exchanged. Thus it is interesting to know whether the SM fusion algorithms designed for the system with bounded disturbances can hold the property or not. Essentially, whether the fusion accuracy is influenced by the measurement update order depend on whether the choice of the parameters is affected, which can be seen from the proofs of above theorems to some extent.

**Remark 8:** For the combined measurement filtering algorithm, the information which participated in the measurement update are  $\mathbf{z}_k^b$ ,  $\mathbf{H}_k^b$  and  $\mathbf{R}_k^b$ . Since the calculation process of these vectors and matrices has nothing to do with the measurement update order, changing the order has absolutely no affect on the fusion estimate results.

**Remark 9:** For the augmented algorithm, it can be seen that the measurement information order can directly influence the values of  $\mathbf{z}_k^a$ ,  $\mathbf{H}_k^a$  and  $\mathbf{R}_k^a$ . They play an essential role in the computation of the parameter  $q_k$  if it is optimized based on a certain criteria as **Theorem 4**. Thus changing the order may alter the fusion estimate results. However, if the selection of parameters is independent of the measurement and predicted states (i.e., parameters are set to be a constant), it is obvious that exchange of the measurement update order has absolutely no effect on the fusion accuracy.

**Remark 10:** For the pseudo-sequential filtering algorithm, if one is willing to choose some scalar function of the matrix defining the ellipsoid as a criterion of size, then the optimum value of  $q_{i,k}$  must be related to the estimate  $\hat{\mathbf{x}}_{i-1,k}$  updated by the measurement of the previous sensor. Thus different measurement update order will yield different parameter sequence, and then yield different fusion results. Certainly, like the augmented algorithm, the exchange of measurement update order has absolutely no affect on the fusion accuracy if the selection of parameters is independent of the measurement and predicted states

#### E. COMPUTATIONAL COMPLEXITY ANALYSIS

The computational complexities of the proposed algorithms are determined by  $n$ ,  $m_i$  and  $N$ . To make the results clear, we set  $m_i = m$  for  $i = 1, 2, \dots, N$ . Then the computational complexities are shown in Table 1. In this table, Algorithm 1~3 refers to the augmented algorithm, the combined

**TABLE 1. The computational complexities of the proposed algorithms.**

Algorithm	Computational complexity	
	Prediction step	Fusion update step
Algorithm 1	$O(n^3)$	$O(n^3 + N^3 m^3)$
Algorithm 2	$O(n^3)$	$O(n^3 + Nm^3)$
Algorithm 3	$O(n^3)$	$O(Nn^3 + Nm^3)$

measurement filtering algorithm and the pseudo-sequential filtering algorithm, respectively, and this is also the case in the following figures and tables. Since the augmented algorithm introduces multiplication and inversion of high-dimensional matrices, it is computationally intensive. For the pseudo-sequential filtering algorithm, since the fusion center performs a filtering process for each batch of sensor measurement, the computational resources consumed by the filter will be large when sensor measurements received by the fusion center per unit time are large. It is obvious that the combined measurement filtering algorithm usually has an advantage in the computation load among above three algorithms due to low dimension.

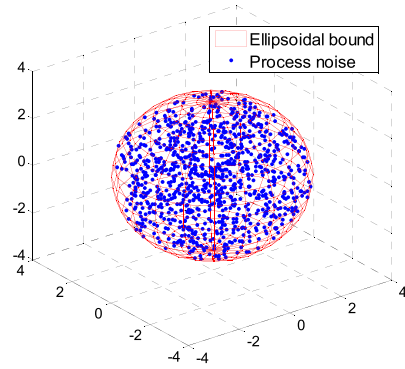
## VI. SIMULATION EXAMPLES

Monte Carlo simulations are performed to assess the performance of the proposed algorithms in this paper and verify relevant conclusions.

Consider the target tracking system with 3 sensors. The system is formulated as (1) and (2), with corresponding matrices

$$\begin{aligned}
 F_k &= \begin{bmatrix} 1 & T_0 & 0.5T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
 H_{1,k} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_{2,k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
 H_{3,k} &= [0 \quad 0 \quad 1],
 \end{aligned}$$

where  $T_0 = 0.1$  is the sampling period. In the state vector  $\mathbf{x}_k = [x_{1,k} \ x_{2,k} \ x_{3,k}]^T$  at time  $k$ ,  $x_{1,k}$ ,  $x_{2,k}$  and  $x_{3,k}$  is the position, velocity and acceleration of the moving target respectively.  $z_{i,k}$  is the detection signal to the target from the  $i$ th sensor. The parameters of the initial ellipsoid are assumed

**FIGURE 1. The distribution of process noise.**

to be  $P_0 = 100I_3, \hat{\mathbf{x}}_0 = [0 \ 0 \ 0]^T, \sigma_0 = 1$ . The center of the ellipsoid,  $\hat{\mathbf{x}}_k$ , is considered to be the point estimate at each recursive step. And the simulations are run under Matlab R2014a on Intel Core i5 PC (3.2GHz, 4G RAM).

The known matrices in (3) and (4) are given as  $R_{1,k} = \text{diag}(0.2, 0.2)$ ,  $R_{2,k} = \text{diag}(0.8, 0.6)$ ,  $R_{3,k} = 0.7$ ,  $Q_k = \text{diag}(10, 10, 10)$ . In the simulation study, the process and observation noises of sensor 1 and sensor 2 are uniformly distributed inside the ellipsoids  $\mathcal{E}(0, Q_k), \mathcal{E}(0, R_{1,k})$  and  $\mathcal{E}(0, R_{2,k})$ , respectively, as illustrated in Fig. 1 and Fig. 2. The observations noises of sensor 3 are uniformly distributed inside an interval  $[-\sqrt{0.7}, \sqrt{0.7}]$ , which can be seen as an ellipsoid with one dimension.

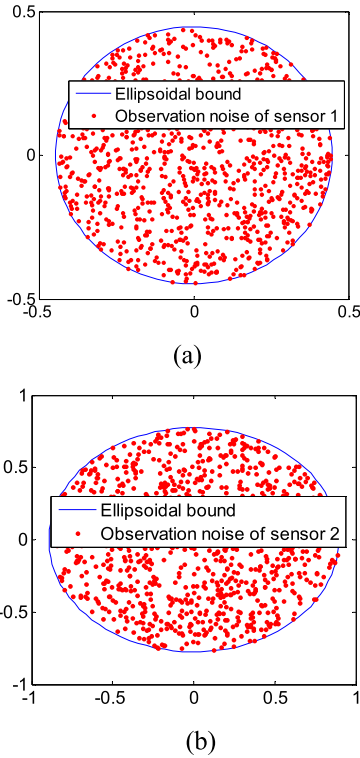
To verify the correctness of the conclusions in Section V, two cases are considered.

*Case 1:* In the first scenario, The augmented algorithm and the combined measurement filtering algorithm are performed with  $p_k = 2, q_k = 1$ , and the parameters  $\alpha_{i,k}$  are calculated according to (42). And in the pseudo-sequential filtering algorithm, we set  $q_{i,k} = q_k \alpha_{i,k}$ . In this way, the conditions in Theorem 5 and 6 are satisfied.

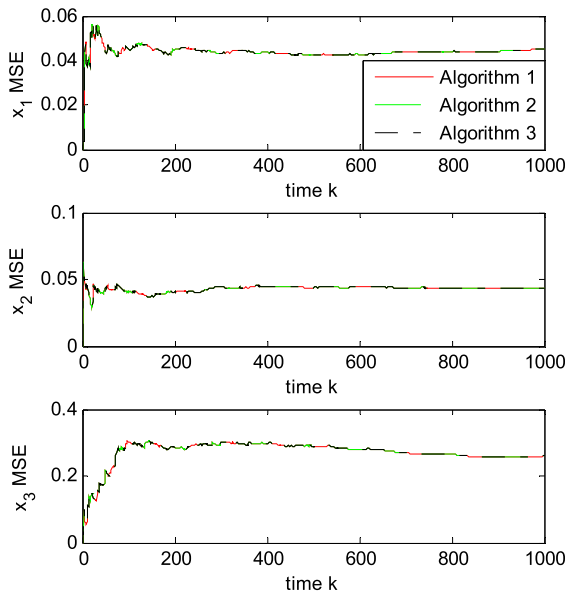
*Case 2:* In the second scenario, the parameters  $\alpha_{i,k}$  are still calculated according to (42), but all other parameters in the fusion algorithms are chosen with the methods in Section IV. Then, the conditions in Theorem 5 and 6 are not satisfied.

In each case, the SM fusion algorithms are firstly performed with measurement update order  $z_{1,k} \rightarrow z_{2,k} \rightarrow z_{3,k}$  (ascending order), then performed again with another order  $z_{3,k} \rightarrow z_{2,k} \rightarrow z_{1,k}$  (descending order).

$$\begin{aligned}
 \mathcal{L}_k &= (\mathbf{x}_k - \hat{\mathbf{x}}_k^a)^T (\mathbf{P}_k^a)^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_k^a) \\
 &= (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a - \tilde{q}_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a)^T \left( (\mathbf{P}_{k|k-1}^a)^{-1} + \tilde{q}_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \mathbf{H}_k^a \right) (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^a - q_k \mathbf{P}_k^a (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a) \\
 &= \mathcal{L}_{k|k-1} + q_k (z_k^a - \mathbf{H}_k^a \mathbf{x}_k)^T (\mathbf{R}_k^a)^{-1} (z_k^a - \mathbf{H}_k^a \mathbf{x}_k) - q_k (\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a + (q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a)^T \mathbf{P}_k^a (q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a) \\
 &\leq \mathcal{L}_{k|k-1} + q_k - q_k (\delta_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a + (q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a)^T \mathbf{P}_k^a (q_k (\mathbf{H}_k^a)^T (\mathbf{R}_k^a)^{-1} \delta_k^a) \\
 &= \mathcal{L}_{k|k-1} + \sigma_k^a - \sigma_{k|k-1}^a \\
 &\leq \mathcal{L}_{k|k-1}
 \end{aligned} \tag{72}$$

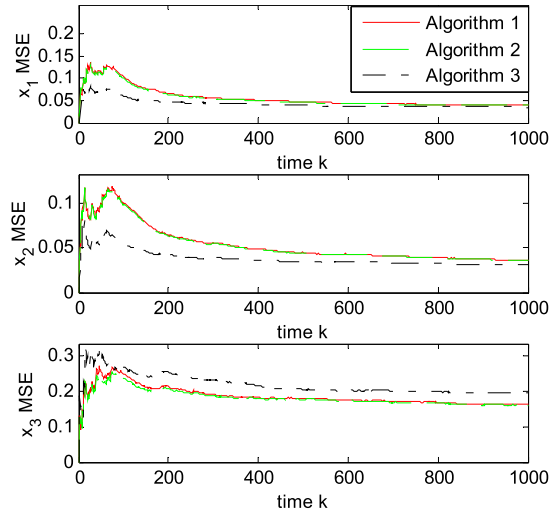


**FIGURE 2.** The distribution of observation noise. Graph (a) indicates observation noise of sensor 1. Graph (b) indicates observation noise of sensor 2.

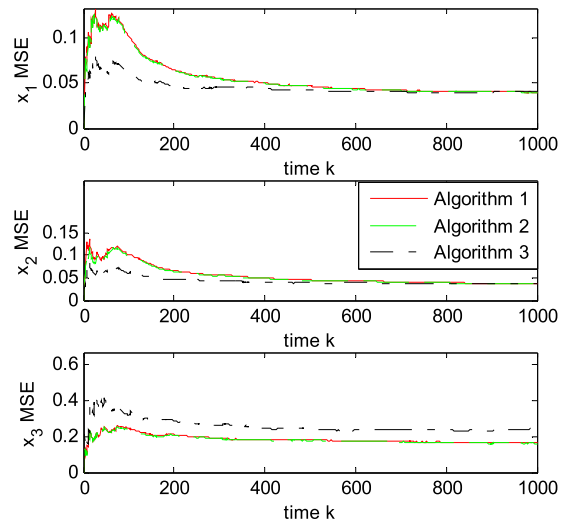


**FIGURE 3.** The MSE of the states estimated by proposed algorithms for Case 1.

The satisfying tracking performances of the state estimates for both algorithms are shown in Fig. 3~5. Furthermore, the evaluation indexes of the algorithms also include the average mean square error (MSE) in each state variable and the average volume, which are illustrated in Table 2. By contrast, the results estimated with the observations from individual sensor are given in Table 3 and Fig. 6.

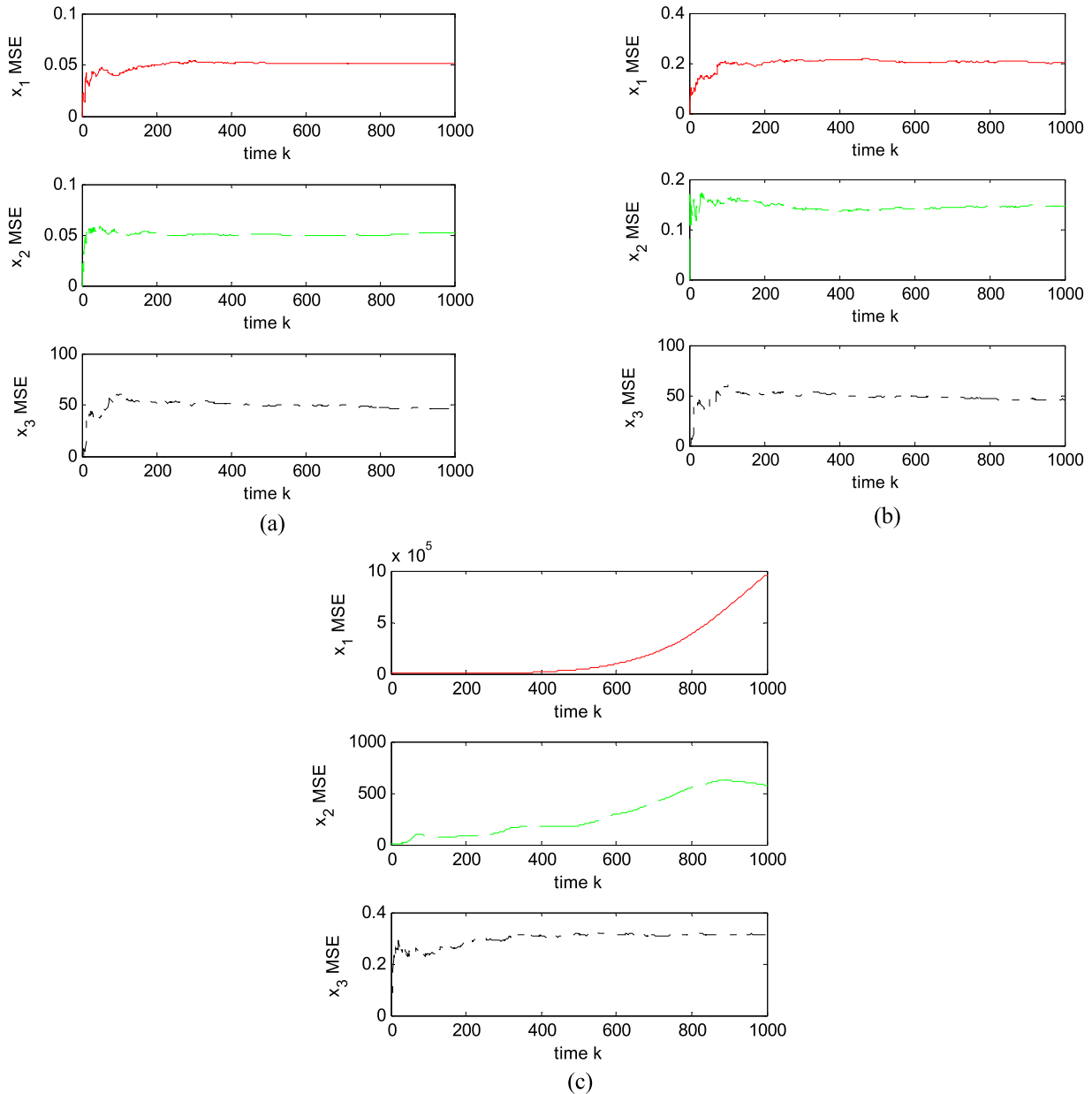


**FIGURE 4.** The MSE of the states estimated by proposed algorithms with ascending update order for Case 2.



**FIGURE 5.** The MSE of the states estimated by proposed algorithms with descending update order for Case 2.

Firstly, compared with the results estimated with the observations from single sensor, the fusion algorithms perform significantly better, which demonstrates the effectiveness of the proposed algorithms. It should be noted that the MSE of  $x_1$  increases as the time goes on, as shown in Figure 6, which means the estimation is divergent when only utilizing the measurements from sensor 3. However, the proposed centralized fusion algorithm is convergent by utilizing all available measurements from sensor 1, 2 and 3. Thus the proposed fusion algorithms could not only improve the estimation accuracy, but also improve the convergence of the estimated results. Actually, Theorem 7 can be used to explain the reason why this is happening: for this simulation example, it is easy to draw that the pair  $(F_k, H_{3,k})$  is not uniformly observable, which means the ISS condition is not satisfied. But the pair  $(F_k, [H_{1,k}^T, H_{2,k}^T, H_{3,k}^T]^T)$  is uniformly observable, thus the



**FIGURE 6.** The MSE of the states estimated with the observations from single sensor. Graph (a) indicates the MSE of the states estimated with the observations from sensor 1. Graph (b) indicates the MSE of the states estimated with the observations from sensor 2. Graph (c) indicates the MSE of the states estimated with the observations from sensor 3.

estimation error of the proposed fusion algorithms is ISS when utilizing all available measurements from all sensors. And the input-to-state stability leads to convergent estimation results.

The results in Table 2 and Fig. 3 show that the accuracy of these three algorithms is the same in the first scenario, no matter whether the measurements are updated with ascending order or descending order. This verifies the correctness of Theorem 5 and Theorem 6. It can also be seen from Table 2 and Fig. 4-5 that different algorithms have different fusion effects in the second scenario, but only the combing measurement filtering algorithm is not affected by

measurement update order. Above results verify the conclusions which are drawn in the exchangeability analysis of measurement updates. In particular, it can be seen that the pseudo-sequential filtering algorithm is significantly affected by the order of measurement. The pseudo-sequential filtering algorithm performs the best among three proposed algorithms in terms of ellipsoid volume and MSE of  $x_1$  and  $x_2$  when the measurements are updated with ascending order in the second scenario. However, when the measurements are updated with descending order, it performs the worst. This reminds us that we need to pay attention to the measurement update order in the application of the pseudo-sequential filtering algorithm.

**TABLE 2.** The average MSE of the estimation and volume of bounding ellipsoids estimated by proposed fusion algorithms.

Case	Order	Algorithms	MSE			Volume
			$\bar{e}_1^2$	$\bar{e}_2^2$	$\bar{e}_3^2$	
Case 1	ascending order	Algorithm 1	0.0449	0.0437	0.2603	206.3049
		Algorithm 2	0.0449	0.0437	0.2603	206.3049
		Algorithm 3	0.0449	0.0437	0.2603	206.3049
	descending order	Algorithm 1	0.0449	0.0437	0.2603	206.3049
		Algorithm 2	0.0449	0.0437	0.2603	206.3049
		Algorithm 3	0.0449	0.0437	0.2603	206.3049
Case 2	ascending order	Algorithm 1	0.0403	0.0361	0.1617	157.1523
		Algorithm 2	0.0399	0.0359	0.1603	148.1524
		Algorithm 3	0.0352	0.0307	0.1929	134.3245
	descending order	Algorithm 1	0.0401	0.0363	0.1611	153.2340
		Algorithm 2	0.0399	0.0359	0.1603	148.1524
		Algorithm 3	0.0406	0.0364	0.2437	244.2903

**TABLE 3.** The average MSE of the states and volume of bounding ellipsoids estimated the observations from single sensor.

Sensor	MSE			Volume
	$\bar{e}_1^2$	$\bar{e}_2^2$	$\bar{e}_3^2$	
sensor 1	0.0511	0.0520	46.7601	1.3575e+04
sensor 2	0.2012	0.1462	45.9926	1.9640e+05
sensor 3	9.6031e+05	576.4524	0.3134	Inf

And the choice of the optimal order will be studied in the future.

It seems reasonable that the proposed algorithms in the second scenario produce ellipsoids with smaller volume and give better results in terms of the estimation MSE than those in the first scenario. This shows that the selection method of the optimal parameters in the proposed algorithms is effective for accuracy improvement.

In addition, the average computational time at each recursive step of the augmented algorithm, the combined measurement filtering algorithm and the pseudo-sequential filtering algorithm is 3.81ms, 3.32ms and 5.70ms, respectively. This means the combined measurement filtering algorithm is faster than the other two algorithms.

**VII. CONCLUSION**

Based on the SM theory, three ellipsoidal outer-bounding state fusion estimation algorithms with centralized structure have been proposed, i.e., the augmented algorithm, the pseudo-sequential filtering algorithm, and the combined measurement filtering algorithm. This paper also presents both theoretical and simulation results on the comparison of these three algorithms and the exchangeability of the measurement update order. The three fusion algorithms are functionally equivalent if the parameters meet certain conditions, as in Theorem 5 and 6. In this case, the estimation accuracy of all three algorithms is not affected by the change of the

measurement update order. However, if these conditions are not satisfied, the property of the functional equivalence for the three algorithms is lost and the exchangeability can't be hold except for the combined measurement filtering algorithm. In terms of their computation speed, the combined measurement filtering algorithm is faster than the other two algorithms because of its lower dimension. By comparatively examining the formulations of the three fusion algorithms, we note that the combined measurement filtering algorithm is less flexible since it needs external conditions to perform. In the meantime, the role of the proposed algorithms in improving state estimation accuracy is verified by the simulation results.

In addition, considering operational efficiency, above conclusions are helpful for choosing appropriate algorithms in applications and the chosen criterion is as follows: If  $\sum_{i=1}^N \alpha_{i,k} \mathbf{H}_{i,k}^T (\mathbf{R}_{i,k})^{-1} \mathbf{H}_{i,k}$  is positive definite, the combined measurement filtering algorithm should be firstly considered to use. When this condition is not satisfied, we choose the augmented algorithm if the measurements of different sensors are received at the same time instant. And otherwise we could use the pseudo-sequential filtering algorithm because the observation of each sensor can be processed as soon as it arrives.

**ACKNOWLEDGMENT**

The authors would like to thank the other members of the team for their support to this research.

**REFERENCES**

- [1] Z. Jin, Y. Hu, and C. Sun, "Event-triggered information fusion for networked systems with missing measurements and correlated noises," *Neurocomputing*, vol. 332, pp. 15–28, Mar. 2019.
- [2] C. Wen, Z. Wang, G. Tao, and F. E. Alsaadi, "Event-based distributed recursive filtering for state-saturated systems with redundant channels," *Inf. Fusion*, vol. 39, pp. 96–107, Jan. 2017.
- [3] Y. Zhang, B. Song, X. Du, and M. Guizani, "Vehicle tracking using surveillance with multimodal data fusion," *IEEE Trans. Intell. Transp. Syst.*, vol. 19, no. 7, pp. 2353–2361, Jul. 2018.

- [4] S. Lee and J. McBride, "Extended object tracking via positive and negative information fusion," *IEEE Trans. Signal Process.*, vol. 67, no. 7, pp. 1812–1823, Apr. 2019.
- [5] F. Caron, E. Duflos, D. Pomorski, and P. Vanheeghe, "GPS/IMU data fusion using multisensor Kalman filtering: Introduction of contextual aspects," *Inf. Fusion*, vol. 7, no. 2, pp. 221–230, 2006.
- [6] B. Gao, G. Hu, S. Gao, Y. Zhong, and C. Gu, "Multi-sensor optimal data fusion for INS/GNSS/CNS integration based on unscented Kalman filter," *Int. J. Control Autom. Syst.*, vol. 16, no. 1, pp. 129–140, Feb. 2018.
- [7] S. Din, A. Ahmad, A. Paul, M. M. U. Rathore, and J. Gwanggil, "A cluster-based data fusion technique to analyze big data in wireless multi-sensor system," *IEEE Access*, vol. 5, pp. 5069–5083, 2017.
- [8] D. L. Hall and J. Llinas, "An introduction to multisensor data fusion," *Proc. IEEE*, vol. 85, no. 1, pp. 6–23, Jan. 1997.
- [9] Y. Zheng, "Methodologies for cross-domain data fusion: An overview," *IEEE Trans. Big Data*, vol. 1, no. 1, pp. 16–34, Mar. 2015.
- [10] Y. Zhu, *Multisensor Decision and Estimation Fusion*. Boston, MA, USA: Kluwer, 2003.
- [11] H. Fourati, *Multisensor Data Fusion: From Algorithms and Architectural Design to Applications*. Boca Raton, FL, USA: CRC Press, 2017.
- [12] Z. Duan and X. R. Li, "Lossless linear transformation of sensor data for distributed estimation fusion," *IEEE Trans. Signal Process.*, vol. 59, no. 1, pp. 362–372, Jan. 2011.
- [13] A. Fatehi and B. Huang, "Kalman filtering approach to multi-rate information fusion in the presence of irregular sampling rate and variable measurement delay," *J. Process Control*, vol. 53, pp. 15–25, May 2017.
- [14] D. Nada, M. Bousbia-Salah, and M. Bettayeb, "Multi-sensor data fusion for wheelchair position estimation with unscented Kalman Filter," *Int. J. Automat. Comput.*, vol. 15, no. 2, pp. 207–217, Apr. 2018.
- [15] C. Yang, Z. Yang, and Z. Deng, "Robust weighted state fusion Kalman estimators for networked systems with mixed uncertainties," *Inf. Fusion*, vol. 45, pp. 246–265, Jan. 2019.
- [16] Y. Zhu, "From Kalman filtering to set-valued filtering for dynamic systems with uncertainty," *Commun. Inf. Syst.*, vol. 12, no. 1, pp. 97–130, 2012.
- [17] Q. Shen, J. Liu, X. Zhou, Q. Zhao, and Q. Wang, "Low-complexity ISS state estimation approach with bounded disturbances," *Int. J. Adapt. Control Signal Process.*, vol. 32, no. 10, pp. 1473–1488, Oct. 2018.
- [18] F. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Trans. Autom. Control*, vol. 13, no. 1, pp. 22–28, Feb. 1968.
- [19] D. G. Maksirov and J. P. Norton, "State bounding with ellipsoidal set description of uncertainty," *Int. J. Control*, vol. 65, no. 5, pp. 847–866, 1996.
- [20] Y. Becis-Aubry, M. Boutayeb, and M. Darouach, "State estimation in the presence of bounded disturbances," *Automatica*, vol. 44, no. 7, pp. 1867–1873, Jul. 2008.
- [21] Y. Liu, Y. Zhao, and F. Wu, "Ellipsoidal state-bounding-based set-membership estimation for linear system with unknown-but-bounded disturbances," *IET Control Theory Appl.*, vol. 10, no. 4, pp. 431–442, 2016.
- [22] C. Novara, M. Canale, M. Milanese, and M. C. Signorile, "Set membership inversion and robust control from data of nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 24, no. 18, pp. 1226–1233, 2015.
- [23] M. H. A. Hasanien, "A set-membership affine projection algorithm-based adaptive-controlled SMES units for wind farms output power smoothing," *IEEE Trans. Sustain. Energy*, vol. 5, no. 4, pp. 1226–1233, Oct. 2014.
- [24] W. Yu, E. Zamora, and A. Soria, "Ellipsoid SLAM: A novel set membership method for simultaneous localization and mapping," *Auto. Robots*, vol. 40, no. 1, pp. 125–137, 2016.
- [25] R. M. Fernández-Cantí, J. Blesa, and V. Puig, "Set-membership identification and fault detection using a Bayesian framework," *Int. J. Syst. Sci.*, vol. 47, no. 7, pp. 1710–1724, 2014.
- [26] Y. Becis-Aubry, "Multisensor fusion for state estimation of linear models in the presence of bounded disturbances," in *Proc. Amer. Control Conf.*, Baltimore, MD, USA, Jun./Jul. 2010, pp. 6781–6782.
- [27] Y. Becis-Aubry, "Multisensor set-membership state estimation of nonlinear models with potentially failing measurements," *IFAC Proc. Volumes*, vol. 44, no. 1, pp. 12030–12035, Jan. 2011.
- [28] Z. Wang, X. Shen, and Y. Zhu, "Set-membership information fusion for multisensor nonlinear dynamic systems," in *Proc. 20th Int. Conf. Inf. Fusion (Fusion)*, Xi'an, China, Jul. 2017, pp. 1–8.
- [29] F. Farina, A. Garulli, and A. Giannitrapani, "Distributed interpolatory algorithms for set membership estimation," *IEEE Trans. Autom. Control*, vol. 64, no. 9, pp. 3817–3822, Sep. 2019.
- [30] S. Werner, M. Mohammed, Y. F. Huang, and V. Koivunen, "Decentralized set-membership adaptive estimation for clustered sensor networks," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, Las Vegas, NV, USA, Mar./Apr. 2008, pp. 3573–3576.
- [31] L. Ma, Z. Wang, H.-K. Lam, and N. Kyriakoulis, "Distributed event-based set-membership filtering for a class of nonlinear systems with sensor saturations over sensor networks," *IEEE Trans. Cybern.*, vol. 47, no. 11, pp. 3772–3783, Nov. 2017.
- [32] N. Xia, F. Yang, and Q.-L. Han, "Distributed networked set-membership filtering with ellipsoidal state estimations," *Inf. Sci.*, vol. 432, pp. 52–62, Mar. 2018.
- [33] Y. Wang and X. R. Li, "Distributed estimation fusion with unavailable cross-correlation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 48, no. 1, pp. 259–278, Jan. 2012.
- [34] J. Sijs and M. Lazar, "State fusion with unknown correlation: Ellipsoidal intersection," *Automatica*, vol. 48, pp. 1874–1878, Aug. 2012.
- [35] P. Ghofrani, T. Wang, and A. Schmeink, "A fast converging channel estimation algorithm for wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 66, no. 12, pp. 3169–3184, Jun. 2018.
- [36] V. Drevelle and P. Bonnifait, "A set-membership approach for high integrity height-aided satellite positioning," *GPS Solutions*, vol. 15, no. 4, pp. 357–368, 2011.
- [37] F. Gu, Y. He, and J. Han, "Active persistent localization of a three-dimensional moving target under set-membership uncertainty description through cooperation of multiple mobile robots," *IEEE Trans. Ind. Electron.*, vol. 62, no. 8, pp. 4958–4971, Aug. 2015.
- [38] L. C. Bento, P. Bonnifait, and U. J. Nunes, "Set-membership position estimation with GNSS pseudorange error mitigation using lane-boundary measurements," *IEEE Trans. Intell. Transp. Syst.*, vol. 20, no. 1, pp. 185–194, Jan. 2019.
- [39] Q. Gan and C. J. Harris, "Comparison of two measurement fusion methods for Kalman-filter-based multisensor data fusion," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 37, no. 1, pp. 273–279, Jan. 2001.
- [40] A. X. Yu, W. D. Hu, and W. H. Zhou, "Performance comparison of multisensor measurement fusion algorithms," *J. Nat. Univ. Def. Technol.*, vol. 25, no. 6, pp. 39–44, 2001.

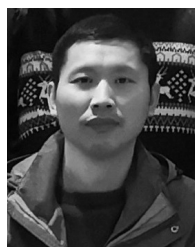


**QIANG SHEN** received the Ph.D. degree from the Xi'an Research Institute of High-Tech, Xi'an, China, in 2019, where he is currently a Lecturer. His current research interests include the set-membership theory, state estimation, information fusion and inertial navigation.



**JIEYU LIU** received the Ph.D. degree from Xi'an Jiaotong University, Xi'an, China, in 2008.

She was a Visiting Scholar with the Michigan State University, from January 2017 to July 2017. She is currently a Professor and a Ph.D. Supervisor with the Xi'an Research Institute of High-Tech. She has published three books, and about 100 articles. Her research interests include signal processing and inertial navigation.

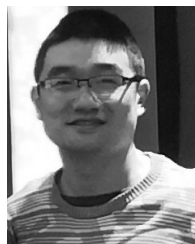


**XIAOGANG ZHOU** received the Ph.D. degree from the Xi'an Research Institute of High-Tech, Xi'an, China, in 2009, where he is currently a Lecturer.

His current research interests include inertial platform and inertial navigation.



**WEIWEI QIN** received the Ph.D. degree from the Xi'an Research Institute of High-Tech, Xi'an, China, in 2012, where he is currently an Associate Professor. His current research has been supported by the National Science Foundation of China. His current research interests include the hypersonic flight vehicle and nonlinear control.



**QI WANG** received the B.S. degree from Xi'an Research Institute of High-Tech, Xi'an, China, in 2014, where he is currently pursuing the Ph.D. degree. His current research interests include parameter identification, and state estimation and inertial navigation.

...



**LIXIN WANG** received the B.S. degree from Xi'an Jiaotong University, Xi'an, China, in 1997 and the Ph.D. degree from the Xi'an Research Institute of High-Tech, Xi'an, in 2008, where he is currently a Professor and a Ph.D. Supervisor. He has published three books, and about 200 articles. His research interests include inertial platform and inertial navigation.