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Systematic and Unified Stochastic Tool to Determine the Multidimensional Joint Statistics of Arbitrary Partial Products of Ordered Random Variables

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ABSTRACT In this paper, we introduce a systematic and unified stochastic tool to determine the joint statistics of partial products of ordered random variables (RVs). With the proposed approach, we can systematically obtain the desired joint statistics of any partial products of ordered statistics in terms of the Mellin transform and the probability density function in a unified way. Our approach can be applied when all the K-ordered RVs are involved, even for more complicated cases, for example, when only the K_s ($K_s < K$) best RVs are also considered. As an example of their application, these results can be applied to the performance analysis of various wireless communication systems including wireless optical communication systems. For an applied example, we present the closed-form expressions for the exponential RV special case. We would like to emphasize that with the derived results based on our proposed stochastic tool, computational complexity and execution time can be reduced compared to the computational complexity and execution time based on an original multiple-fold integral expression of the conventional Mellin transform based approach which has been applied in cases of the product of RVs.

INDEX TERMS Joint PDF, partial products, Mellin transform (MT), order statistics, probability density function (PDF), exponential random variables, information combining.

I. INTRODUCTION

Order statistics that deal with ordered random variables (RVs) and distributions of those functions are an important sub-field of statistical theory and have been applied in a wide-range of fields [1]. Over the years, it has been applied to a vast variety of areas of statistical theory and practice [1], including signal and image processing as well as quality control, life-testing, and so on [2]–[4]. In particular, order statistics have been increasingly emerging in communications engineering and advanced signal processing fields [5]. For example, in [6], to determine the joint statistics of partial sums of ordered RVs, a moment generating function (MGF) based unified analytical framework was applied. With this analytical framework, when both all the ordered RVs are involved and only the best RVs are considered, we can obtain the target joint PDF of arbitrary partial sums of RVs as the closed-form expressions.

As one of the applied problems in statistics, the distributions that deal with the product of RVs have been raised and studied [7]–[9]. However, most of these studies have been limited to independent cases. Therefore, approaches applied to these independent case studies and related results can not be applied to dependent random variable cases such as ordered RV cases. In [10], a new characterization that involves a distributional relation of products of order RVs is introduced but this study is limited to simple/fundamental cases (i.e., the product of two independent order RVs). Recently, in [11], the distribution of products of dependent RVs is studied but the results and the approach applied to derive them are only valid for simple cases taking into account the product of two ordered RVs. It directly means that these results and approaches can not be applied to more

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complicated/general cases. These problem has not been tackled earlier, yet.

Many studies of the applications of Mellin transform to statistical distribution theory have been considered in [12]–[14], including the methods in problems relating to exact distributions of multivariate statistics in [15], [16]. In particular, [7], [9], [11], [13], [17] studied how to obtain the probability density function (PDF) of the product of RVs (e.g., the product of two independent RVs or two dependent RVs) based on the Mellin transform has been made. Similar to Laplace and Fourier transforms, the Mellin transform is one of integral transforms. In terms of probability theory, the Mellin transform of a random variable is related to its PDF and according to the properties of the Mellin transform, the Mellin transform of a product of RVs is the product of the Mellin transforms of the individual RVs [7], [9]. The definition and the related properties of the Mellin transforms allow us to derive the joint statistics of the product of ordered RVs in a systematic and unified manner with the help of the MGF-based unified framework.

In this paper, we introduce a Mellin-transform-based systematic stochastic tool to determine the multidimensional joint statistics of arbitrary partial products of ordered RVs in a unified way. More specifically, by combining the MGF-based framework introduced in [6] with the Mellin transform, we introduce a method for systematically obtaining the desired joint statistics of an arbitrary partial product of ordered RVs in a unified way. As a special case based on this, we provide some selected new closed-form results for exponential RV case. To our best knowledge, this systematic and unified stochastic tool and the related closed-form results have never been investigated. These results are not easily obtained by direct integration calculations of the series that represent the probability densities due to high computational complexity and long execution time. However, with our results, computational complexity and execution time can be reduced because it is in the form of a summation expression rather than a multiple-fold integral expression. The above mentioned points are very important in implementing the proposed tool on computer-based systems or in calculating the numerical result. Further, we would like to emphasize that such generalization may lay a foundation for other researchers to build a more rich theory of order statistics in the general order statistics theory.

II. EXAMPLES OF APPLICATION SYSTEM MODELS

In this section, as feasible application examples in which our systematic and unified stochastic tool to determine the multidimensional joint statistics can be applied, we consider the following two application examples.

A. EXAMPLE SCENARIO 1: INFORMATION ACCUMULATION

Wireless optical communications are recently considered as a promising technology that provides high-speed, improved-capacity, cost-effective, secure and easy-to-deploy wireless networks. In an optical wireless communication system, the coexistence of multiple wavelength division multiplexing (WDM) channels on a single optical channel allows simultaneous transmission of high-speed signals having multiple information, thereby expanding network capacity. Conventional optical fiber communication systems and optical wireless communication systems use similar system components. In a conventional fiber optic communication system, all signals received at the receiver are always valid, but in a wireless optical communication system, all multi-beams generated by WDM may not always be valid or may not have an acceptable signal-to-noise ratio (SNR) due to atmospheric attenuation [18]–[23]. Therefore, considering only the best valid signal, not all signals, the receiver can combine multiple information received only on the best valid signals and reduce unnecessary complexity in terms of hardware components and signal processing.

In another example, the accumulation of information for relay transmission, in particular, the use of rateless codes, can be seen as a method to achieve the information-theoretical capacity of a channel with multiple relays [24]–[26]. More specifically, ideal rateless codes and decoders at the receiver can distinguish information streams from different relay nodes, and mutual information of signals transmitted by the relay nodes can be accumulated. For example in [26], if channels between source (S) and destination (D) and between relay (R) and D have different propagation delays, then D can apply a Rake receiver for (maximal-ratio) combining signals from S and Rs. Accumulation of information is implemented when relays with distinct forwarding times use different sub-channels/codes. In this case, due to the different spreading codes, D can distinguish signals from S and R, and the S-D and R-D links combine information as they use different generating vectors. In this case, an information packet is reliably decodable at the receiver once the instantaneous accumulated information is more than a certain specified threshold R_{th}, i.e., $\sum_i Log\left(1+SNR_i\right) > R_{th}$.

Based on these examples, to reduce the complexity without any significant loss in terms of performance, we can consider that the receiver combines the received information from the best K_s signals among total K ($> K_s$) signals. Then, the combined rate can be formulated as

$$\sum_{i=1}^{K_s} Log\left(1+SNR_i\right), \tag{1}$$

where $SNR_1 \geq SNR_2 \geq \cdots \geq SNR_{K_s} \geq \cdots \geq SNR_K$. Here, if we consider the high SNR regime assumption, then we can rewrite as

$$\sum_{i=1}^{K_s} Log\left(1+SNR_i\right) \approx \sum_{i=1}^{K_s} Log\left(SNR_i\right). \tag{2}$$

Then, we can rewrite it as the product form

$$\sum_{i=1}^{K_s} Log\left(1+SNR_i\right) \approx \sum_{i=1}^{K_s} Log\left(SNR_i\right) = Log\left(\prod_{i=1}^{K_s} SNR_i\right) = Log\left(\gamma_{product}\right), \tag{3}$$

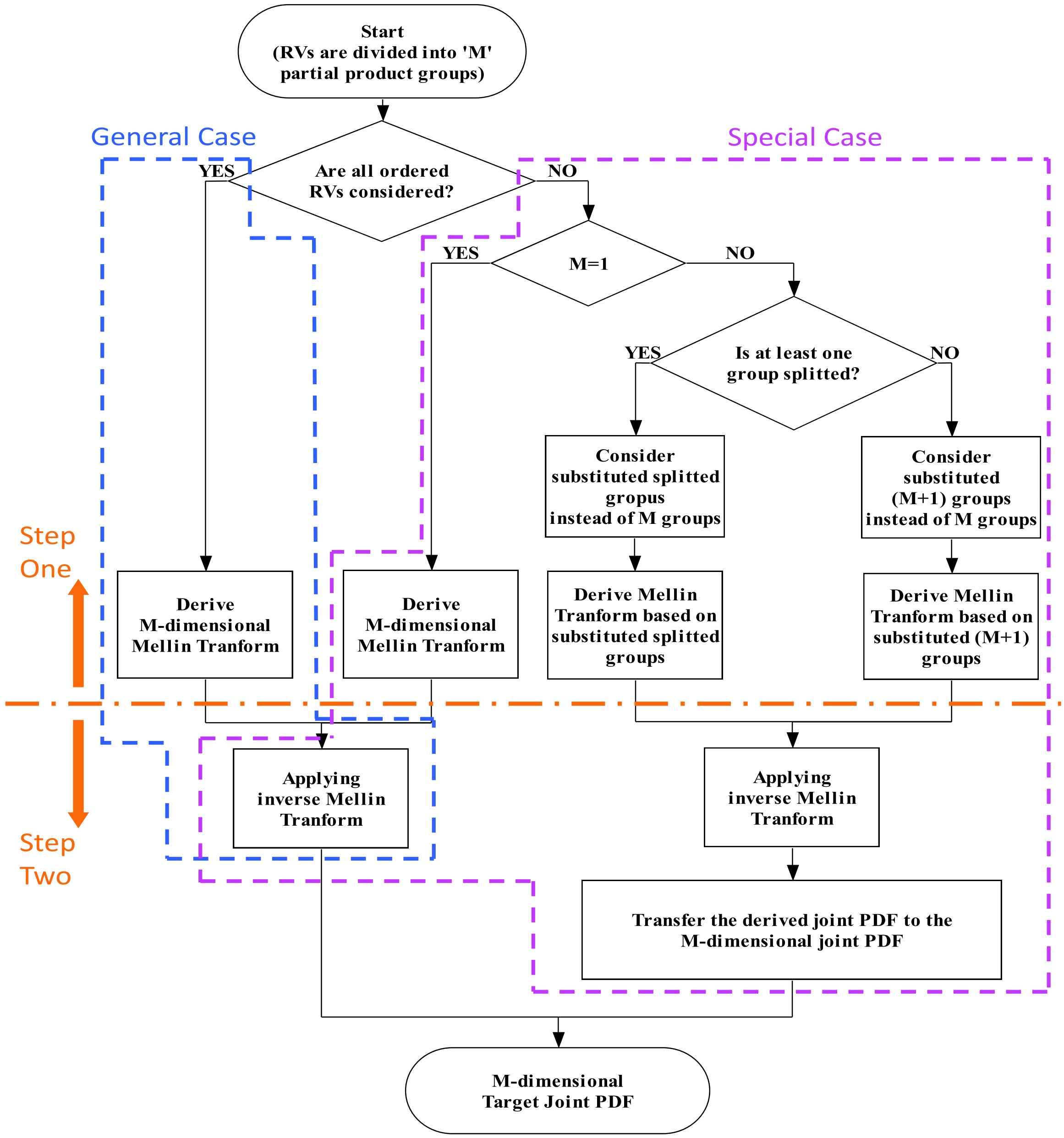

FIGURE 1. Process flowchart for deriving joint statistics of partial products of ordered RVs.

where $\gamma_{product} = \prod_{i=1}^{K_s} SNR_i$. In order to find an analytical result (e.g., average combined rate, $\int_0^\infty Log(\gamma) f_{\gamma_{product}}(\gamma)\, d\gamma$) based on this system model assumptions, we need to derive the PDF of partial product of ordered random variables, $f_{\gamma_{product}}(\gamma)$, from the best one to the K_s-th ordered one among K ordered random variables.

B. EXAMPLE SCENARIO 2: OUTAGE CAPACITY OF MULTI-CARRIER SYSTEMS

Due to robustness to frequency-selective fading and the consequent inter-symbol interference (ISI), many current wireless communication systems are being deployed using multi-carrier modulation. In addition, where channel state information is available at the transmitter, adaptive modulation can be used to allow multi-carrier systems to achieve better system spectral efficiency. To quantify this gain in spectral efficiency, it is important to calculate the outage capacity of these systems, which corresponds to the probability that the aggregate rate over all carriers fails to exceed a predetermined rate threshold [27], [28].

For considering the multi-carrier system, assume that each carrier is affected by Rayleigh fading, such that its instantaneous SNR of the l-th carrier, γ_l, follows an exponential distribution. Here, it is clear that the logarithmic sum of some numbers is equal to the logarithm of the product of these numbers. As such, the instantaneous sum capacity of the multi-carrier system is given by

$$C_s = \sum_l Log_2 (1 + \gamma_l) = Log_2 \left(\prod_l (1 + \gamma_l) \right). \quad (4)$$

Here, if we consider i) the high SNR regime assumption and ii) only the best K_s carriers/tones among the total K ($> K_s$) multi-carrier instead of considering all the K carriers, then the instantaneous sum capacity of the multi-carrier system can be obtained as

$$C_s = \sum_{l=1}^{K_s} Log_2 (1 + \gamma_l) \approx Log_2 \left(\gamma_{product} \right), \quad (5)$$

where $\gamma_{product} = \prod_{l=1}^{K_s} \gamma_l$ for $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{K_s} \geq \cdots \geq \gamma_K$. With (5), our objective is to find the outage capacity of this system. It is defined as the probability that the instantaneous sum capacity, C_s falls below a certain specified threshold C_{th}, i.e.,

$$C_{out} = \Pr [0 \leq C_s < C_{th}] = \int_0^{C_{th}} f_{C_s} (\gamma) \, d\gamma. \quad (6)$$

If we let $\gamma_{product} = \prod_{l=1}^{K_s} \gamma_l$, then we can rewrite it as the function of PDF of partial product of ordered random variables as

$$C_{out} = \int_1^{2^{C_{th}}} f_{\gamma_{product}} (\gamma) \, d\gamma. \quad (7)$$

Then, we also need to derive the PDF of partial product of ordered random variables, $f_{\gamma_{product}} (\gamma)$, from the best one to the K_s-th ordered one among K ordered random variables. In what follows, we show how to determine the key joint statistics systematically with our proposed unified stochastic tool followed by some selected examples.

III. THE MAIN FRAMEWORK

We assume that $\infty \geq \gamma_{1:K} \geq \gamma_{2:K} \geq \gamma_{3:K} \cdots \geq \gamma_{K:K} \geq 0$ are the ordered RVs which are obtained by arranging K non-negative i.i.d. RVs, $\{\gamma_i\}_{i=1}^{K}$, in decreasing order of magnitude. Here, we adopt the analytical framework proposed in [6] to systematically derive the target multidimensional joint PDF of arbitrary partial products of ordered RVs involving either all K or the first K_s ($K_s < K$) ordered RVs. Specifically, we adopt i) the interchange of multiple integrals of ordered RVs and ii) a general two-step approach when all the ordered RVs are considered including extra steps when these conditions do not hold, especially in the flowchart given in Fig. 1. More specifically, as shown in the general case in Fig. 1, after deriving the analytical expressions of multidimensional joint Mellin transform in step one, in step two, proceed to apply inverse Mellin transform to obtain the joint PDF. Note that additional integration may be required to obtain the desired joint PDF.

For the special case[1], following the special steps systematically as shown in Fig. 1, we first divide these RVs into smaller products. Then, after applying a general two-step approach based on newly divided group of RVs, we transform the result obtained through a general two-step approach to a lower dimensional desired joint PDF with finite integration. Because the unified MGF-based approach and results were limited to the cases when the joint statistics of partial sums of ordered RVs were desired, although new useful closed-form results on ordered statistics have been provided, systematically obtaining the multidimensional joint statistics in a unified manner still remains a challenge.

In this paper, the analytical framework is slightly modified making it suitable for multidimensional joint statistics of partial products of ordered RVs. Specifically, in order to systematically obtain the multidimensional joint statistic results in closed-form, a systematic and unified stochastic framework based on the Mellin transform and related fundamental/essential common core functions is provided. The main challenge is to establish a unified method for systematically deriving the target multidimensional joint PDFs. In this case, we can systematically obtain the joint Mellin transform function by applying the Mellin transform instead of joint MGF function. Note that the original expression of joint Mellin transform involves a multi-fold integral expression as shown in the following example

$$\begin{aligned} \mathcal{M}(s_1, s_2) = & \int_0^\infty d\gamma_{1:K} (\gamma_{1:K})^{s_2-1} f(\gamma_{1:K}) \\ & \cdots \int_0^{\gamma_{m-2:K}} d\gamma_{m-1:K} (\gamma_{m-1:K})^{s_2-1} f(\gamma_{m-1:K}) \\ & \times \int_0^{\gamma_{m-1:K}} d\gamma_{m:K} (\gamma_{m:K})^{s_1-1} f(\gamma_{m:K}) \\ & \times \int_0^{\gamma_{m:K}} d\gamma_{m+1:K} (\gamma_{m+1:K})^{s_2-1} f(\gamma_{m+1:K}) \\ & \cdots \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s_2-1} f(\gamma_{K:K}). \end{aligned} \quad (8)$$

Therefore, as we can see from (8), even if we apply the conventional Mellin transform-based approach which has been applied in cases of the product of RVs, we still need to compute this multi-fold integral expression.

Here, by adopting the interchange of multiple integral techniques in [6], (8) can be re-arranged

$$\mathcal{M}(s_1, s_2) = \int_0^\infty d\gamma_{m:K} (\gamma_{m:K})^{s_1-1} f(\gamma_{m:K})$$

[1]When the RVs separated by the other RVs, i.e., the RVs involved in at least one partial product is not continuous

$$\times \int_{0}^{\gamma_{m:K}} d\gamma_{m+1:K} (\gamma_{m+1:K})^{s_2-1} f(\gamma_{m+1:K})$$
$$\cdots \int_{0}^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s_2-1} f(\gamma_{K:K})$$
$$\times \int_{\gamma_{m:K}}^{\infty} d\gamma_{m-1:K} (\gamma_{m-1:K})^{s_2-1} f(\gamma_{m-1:K})$$
$$\cdots \int_{\gamma_{2:K}}^{\infty} d\gamma_{1:K} (\gamma_{1:K})^{s_2-1} f(\gamma_{1:K}). \tag{9}$$

With this re-arranged result, the joint Mellin transform based expression, especially both multi-fold integral expressions from $\gamma_{m+1:K}$ to $\gamma_{K:K}$ and from $\gamma_{m-1:K}$ to $\gamma_{1:K}$, can be made as compact as possible with the help of common core functions derived in the following sections.

Then, with the Mellin transform-based expression as the joint compact form, we can derive joint PDF results of selected partial products through the inverse Mellin transform. In most cases we are interested, the joint Mellin transform function involves basic functions, for which the inverse Mellin transform can be analytically computed. In the worst case, the results of the final non-closed form needs to be computed numerically using conventional standard mathematical packages, such as Matlab and Mathematica. Note that with original multi-fold integral expressions (e.g., the K-fold integrals given in (9)), it is difficult to estimate them accurately as K increases even with the conventional mathematical tools due to high computational complexity. Then, by applying three common core functions of a special fading case for obtaining the joint Mellin transform expression in a compact form, the desired multidimensional joint PDF can be obtained in a closed-form expression through the inverse Mellin transform. Here, we basically adopt a general two-step approach in Sec. II [6] as shown in Fig. 1 similar to the case of joint statistics of partial sums of ordered RVs, especially when all K ordered RVs are considered. When only the best K_s ($K_s < K$) ordered RVs are involved in the partial sums), similarly, to obtain a valid-joint Mellin transform function, we adopt extra steps. Specifically, the last (i.e., K_s-th) ordered-RV $\gamma_{K_s:K}$ is considered separately. In the next sections, our focus is to get the joint Mellin transform functions in a compact expression, which can be greatly simplified with the application of the following common core functions and relations.

IV. COMMON CORE FUNCTIONS

A. FOR GENERAL DISTRIBUTION CASES

This section introduces some common core functions and their properties. They are used to simplify the derivation of the Mellin transform-based joint result in a simple and unified form with the help of:

i) A mixture of a CDF and a Mellin transform
$$cm(\gamma, s) = \int_0^{\gamma} dx x^{s-1} f(x), \tag{10}$$
where $f(x)$ represents the distribution function of the RV of interest. In (10), $cm(\gamma, 1) = c(\gamma)$ leads to the CDF and $cm(\infty, s)$ is the Mellin transform, where the variable γ is real while s is complex.

ii) A mixture of an exceedance distribution function (EDF) and a Mellin transform
$$em(\gamma, s) = \int_{\gamma}^{\infty} dx x^{s-1} f(x). \tag{11}$$
In (11), $em(\gamma, 1) = e(\gamma)$ represents the EDF and $em(0, s)$ leads the Mellin transform.

iii) An interval Mellin transform
$$\mu m(\gamma_a, \gamma_b, s) = \int_{\gamma_a}^{\gamma_b} dx x^{s-1} f(x). \tag{12}$$
In (12), $\mu m(0, \infty, s)$ represents the Mellin transform.

The functions defined in (10), (11), and (12) have the following relationships:
$$cm(\gamma, s) = cm(\infty, s) - em(\gamma, s) \overset{\text{or}}{=} em(0, s) - em(\gamma, s), \tag{13}$$
$$em(\gamma, s) = em(0, s) - cm(\gamma, s) \overset{\text{or}}{=} cm(\infty, s) - cm(\gamma, s), \tag{14}$$
$$im(\gamma_a, \gamma_b, s) = cm(\gamma_b, s) - cm(\gamma_a, s) \overset{\text{or}}{=} em(\gamma_a, s) - em(\gamma_b, s). \tag{15}$$

B. FOR SPECIAL CASES: i.i.d. EXPONENTIAL RVs

Here, we specialize in the closed-form results of the common core functions for an i.i.d. Exponential RV case with a common PDF, $f(\gamma) = \frac{1}{\bar{\gamma}} \exp\left(-\frac{\gamma}{\bar{\gamma}}\right)$; i) a mixture of a CDF and a Mellin transform, ii) a mixture of an EDF and a Mellin transform, and iii) an interval Mellin transform, including the n-th power of these functions for arbitrary n as

i) Closed-form results for a mixture of a CDF and a Mellin transform

For exponential case, $cm(\gamma, s)$ can be written as
$$cm(\gamma, s) = \int_0^{\gamma} dx x^{s-1} \frac{1}{\bar{\gamma}} \exp\left(-\frac{x}{\bar{\gamma}}\right). \tag{16}$$
Then, with the help of [29, (3.381.1)], the closed-form expression of (16) can be obtained as
$$cm(\gamma, s) = \left(\frac{1}{\bar{\gamma}}\right)^{-s+1} \gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right), \tag{17}$$
or with the help of [29, (3.381.3) and (3.381.4)], it can also be written as the form of the gamma function
$$cm(\gamma, s) = \left(\frac{1}{\bar{\gamma}}\right)^{-s+1} \left[\Gamma(s) - \Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right], \tag{18}$$
where $\Gamma(\cdot)$ and $\gamma(\cdot, \cdot)$ denote the gamma function and the incomplete gamma function [29, (8.310.1) and (8.350.2)], respectively.

Then, with (17) and (18), the n-th power of $cm(\gamma, s)$ for arbitrary n can be written as

$$[cm(\gamma, s)]^n = \left(\frac{1}{\bar{\gamma}}\right)^{-n(s-1)} \left[\gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right]^n \overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^{-n(s-1)} \left[\Gamma(s) - \Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right]^n. \quad (19)$$

Note that by applying the binomial theorem, (19) can be rewritten as the following sum of the form

$$[cm(\gamma, s)]^n = \left(\frac{1}{\bar{\gamma}}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s} \times [\Gamma(s)]^l \left[\Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right]^{n-l}. \quad (20)$$

ii) Closed-form results for a mixture of a EDF and a Mellin transform
Similar to case i) (i.e., $cm(\gamma, s)$), with the help of [29, (3.381.3)], we can obtain the closed-form expression of $em(\gamma, s)$ for exponential case as

$$em(\gamma, s) = \int_0^{\gamma} dx x^{s-1} \frac{1}{\bar{\gamma}} \exp\left(-\frac{x}{\bar{\gamma}}\right) = \left(\frac{1}{\bar{\gamma}}\right)^{-s+1} \Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right). \quad (21)$$

With (21), the closed-form expression for n-th power can be written as

$$[em(\gamma, s)]^n = \left(\frac{1}{\bar{\gamma}}\right)^{-n(s-1)} \left[\Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right]^n \overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^n \left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s} \left[\Gamma\left(s, \frac{\gamma}{\bar{\gamma}}\right)\right]^n. \quad (22)$$

iii) Closed-form results for an interval Mellin transform
In this case, by applying (15) with closed-form results of $cm(\gamma, s)$ and $em(\gamma, s)$ cases, we can obtain the closed-form results of $im(\gamma_a, \gamma_b, s)$ and its n-th power form for exponential case, respectively, as

$$im(\gamma_a, \gamma_b, s) = \int_{\gamma_a}^{\gamma_b} dx x^{s-1} \frac{1}{\bar{\gamma}} \exp\left(-\frac{x}{\bar{\gamma}}\right) \overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^{-s+1} \left[\Gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right) - \Gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right)\right] \overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^{-s+1} \left[\gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right) - \gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right)\right], \quad (23)$$

and

$$[im(\gamma_a, \gamma_b, s)]^n = \left(\frac{1}{\bar{\gamma}}\right)^{-n(s-1)} \left[\Gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right) - \Gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right)\right]^n \overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s} \times \left[\Gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right)\right]^l \left[\Gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right)\right]^{n-l}. \quad (24)$$

V. EXAMPLES

A. EXAMPLE 1: TWO-DIMENSIONAL JOINT PDF OF THE LARGEST RV AND THE PRODUCT OF REMAINING RVs

In this subsection, we discuss one simple example to derive a two-dimensional joint PDF of the largest RV and the product of remaining RVs. Let $z_1 = \gamma_{1:K}$, $z_2 = \prod_{n=2}^{K} \gamma_{n:K}$, and $Z = [z_1, z_2]$. Since the original RVs $\{\gamma_n\}$ are i.i.d. with common PDF $f(x)$, the K-dimensional joint PDF of the ordered RVs $\{\gamma_{n:K}\}$ is simply for $\gamma_{1:K} \geq \gamma_{2:K} \geq \cdots \geq \gamma_{K:K}$

$$g(\gamma_{1:K}, \gamma_{2:K}, \cdots, \gamma_{K:K}) = K! f(\gamma_{1:K}) f(\gamma_{2:K}) \cdots f(\gamma_{K:K}). \quad (25)$$

In this case, K ordered RVs can be viewed as

$$\underbrace{\gamma_{1:K}}_{z_1}, \underbrace{\gamma_{2:K}, \gamma_{3:K}, \cdots, \gamma_{K-1:K}, \gamma_{K:K}}_{z_2}. \quad (26)$$

In (26), all the K ordered RVs are considered. Therefore, as shown in Fig. 1, we directly derive the 2-dimensional joint PDF by adopting a general two-step approach. More specifically, following the step one in general case systematically as shown in Fig. 1, we can first formulate the original 2-dimensional Mellin Transform as the following K-fold integral expression

$$\mathcal{M}(s_1, s_2) = K! \int_0^{\infty} d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} f(\gamma_{1:K}) \times \int_0^{\gamma_{1:K}} d\gamma_{2:K} (\gamma_{2:K})^{s_2-1} f(\gamma_{2:K}) \cdots \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s_2-1} f(\gamma_{K:K}). \quad (27)$$

With (27), to facilitate the inverse Mellin Transform calculation, this 2-dimensional Mellin Transform should be made as compact as possible. Here, we can simplify the $(K-1)$-fold integral expression in (27) from $\gamma_{2:K}$ to $\gamma_{K:K}$ with the help of Appendix I (especially, by applying (57)) as the following $(K-1)$-th power of $cm(\cdot, \cdot)$ function

$$\int_0^{\gamma_{1:K}} d\gamma_{2:K} (\gamma_{2:K})^{s_2-1} f(\gamma_{2:K}) \cdots \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s_2-1} f(\gamma_{K:K}) = \frac{1}{(K-1)!} [cm(\gamma_{1:K}, s_2)]^{K-1}. \quad (28)$$

Thus, with (28), (27) can be simplified as the 1-fold integral expression

$$\mathcal{M}(s_1, s_2) = \frac{K!}{(K-1)!} \int_0^{\infty} d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} f(\gamma_{1:K}) [cm(\gamma_{1:K}, s_2)]^{K-1}. \quad (29)$$

With (29), for an i.i.d. exponential RV case, by applying (19), (28) can be rewritten as

$$\mathcal{M}(s_1, s_2) = \frac{K!}{(K-1)!} \left(\frac{1}{\bar{\gamma}}\right)^K \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^{K-1-j}$$

$$\times \left\{ \int_0^\infty d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} \exp\left(-\frac{\gamma_{1:K}}{\bar{\gamma}}\right) \left(\frac{1}{\bar{\gamma}}\right)^{-(K-1)s_2} \times [\Gamma(s_2)]^j \left[\Gamma\left(s_2, \frac{\gamma_{1:K}}{\bar{\gamma}}\right)\right]^{K-1-j} \right\}. \quad (30)$$

Then, with the compact form of (27) in (30), following the step two in general case in Fig. 1, we apply the inverse Mellin transform to obtain the target PDF expression as

$$\begin{aligned} f(z_1, z_2) &= \mathcal{M}_{s_1,s_2}^{-1} \{\mathcal{M}(s_1, s_2)\} \\ &= \frac{K!}{(K-1)!} \left(\frac{1}{\bar{\gamma}}\right)^K \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^{K-1-j} \\ &\quad \times \mathcal{M}_{s_1}^{-1} \left\{ \int_0^\infty d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} \exp\left(-\frac{\gamma_{1:K}}{\bar{\gamma}}\right) \right. \\ &\quad \left. \times \mathcal{M}_{s_2}^{-1} \left\{ \left(\frac{1}{\bar{\gamma}}\right)^{-(K-1)s_2} [\Gamma(s_2)]^j \left[\Gamma\left(s_2, \frac{\gamma_{1:K}}{\bar{\gamma}}\right)\right]^{K-1-j} \right\} \right\}. \end{aligned} \quad (31)$$

In (31), with the help of Appendix II, $m = K - 1$, $l = 0$, $p = 0$, and $q = K - 1$ for the inverse Mellin transform of s_2. Thus, the inverse Mellin transform result of s_2 is obtained in the form of the generalized incomplete Fox's H functions which is defined in Appendix II

$$\begin{aligned} &\mathcal{M}_{s_2}^{-1} \left\{ \left(\frac{1}{\bar{\gamma}}\right)^{-(K-1)s_2} [\Gamma(s_2)]^j \left[\Gamma\left(s_2, \frac{\gamma_{1:K}}{\bar{\gamma}}\right)\right]^{K-1-j} \right\} \\ &= \mathrm{UH}_{0,K-1}^{K-1,0} \left[\frac{z_2}{(\bar{\gamma})^{K-1}} \middle| \begin{matrix} - - - - - - - - \\ \underbrace{(0,1,0)\cdots(0,1,0)}_{\#=j} \underbrace{\left(0,1,\frac{\gamma_{1:K}}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{\gamma_{1:K}}{\bar{\gamma}}\right)}_{\#=K-1-j} \end{matrix} \right]. \end{aligned} \quad (32)$$

Here, if we let

$$\begin{aligned} f(\gamma_{1:K})|_{\gamma_{1:K}=z_1} &= \exp\left(\frac{-\gamma_{1:K}}{\bar{\gamma}}\right) \\ &\quad \times \mathrm{UH}_{0,K-1}^{K-1,0} \left[\frac{z_2}{(\bar{\gamma})^{K-1}} \middle| \begin{matrix} - - - - - - - \\ (0,1,0)\cdots(0,1,0)\left(0,1,\frac{\gamma_{1:K}}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{\gamma_{1:K}}{\bar{\gamma}}\right) \end{matrix} \right] \Bigg|_{\gamma_{1:K}=z_1}, \end{aligned} \quad (33)$$

with the following inverse Mellin transform property; $\mathcal{M}_s^{-1}\{\int_0^\infty dx(x)^{s-1}f(x)\} = f(x)$, then the inverse Mellin transform result of s_1 can be obtained as the following simplified form

$$\begin{aligned} &\mathcal{M}_{s_1}^{-1} \left\{ \int_0^\infty d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} f(\gamma_{1:K}) \right\} = f(\gamma_{1:K})|_{\gamma_{1:K}=z_1} \\ &= \exp\left(\frac{-z_1}{\bar{\gamma}}\right) \\ &\quad \times \mathrm{UH}_{0,K-1}^{K-1,0} \left[\frac{z_2}{(\bar{\gamma})^{K-1}} \middle| \begin{matrix} - - - - - - - \\ (0,1,0)\cdots(0,1,0)\left(0,1,\frac{z_1}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{z_1}{\bar{\gamma}}\right) \end{matrix} \right]. \end{aligned} \quad (34)$$

Subsequently, we can finally obtain the closed-form result of the target two-dimensional joint PDF as

$$\begin{aligned} f(z_1, z_2) &= \frac{K!}{(K-1)!} \left(\frac{1}{\bar{\gamma}}\right)^K \sum_{j=0}^{K-1} \binom{K-1}{j} (-1)^{K-1-j} \exp\left(-\frac{z_1}{\bar{\gamma}}\right) \\ &\quad \times \mathrm{UH}_{0,K-1}^{K-1,0} \left[\frac{z_2}{(\bar{\gamma})^{K-1}} \middle| \begin{matrix} - - - - - - - - \\ (0,1,0)\cdots(0,1,0)\left(0,1,\frac{z_1}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{z_1}{\bar{\gamma}}\right) \end{matrix} \right]. \end{aligned} \quad (35)$$

Here, we would like to emphasize that our derived statistical result is much simpler (i.e., low computational complexity) than the original multiple-fold integral form (not the multiple products of multiple one-fold integral form) based on the conventional Mellin transform-based approaches. More specifically, if this closed-form result is not available, the numerical estimation of K-fold integrals in (27) is required. In this section, we only showed the derivation process of the simple case which is the derivation of two-dimensional joint PDF of the largest RV and product of remaining $(K-1)$ RVs among K ordered RVs. However, similarly, by following the Fig. 1 with the help of common core functions, we can obtain the desired joint statistics of any partial products of ordered RVs.

B. EXAMPLE 2: JOINT PDF OF THE PRODUCT OF THE BEST K_S ORDERED RVs AMONG THE TOTAL K ($\geq K_S$) ORDERED RVs

In this subsection, we discuss the application example to derive a joint PDF of the product of the best K_S ordered RVs among the total K ($\geq K_S$) ordered RVs which was considered as the key statistics in Sec. II-A and Sec. II-B. According to Fig. 1, we first let $Z_1 = \prod_{n=1}^{K_S-1} \gamma_{n:K}$, $Z_2 = \gamma_{K_S:K}$, and $Z = [z_1, z_2]$, where $\gamma_{1:K} \geq \gamma_{2:K} \geq \cdots \geq \gamma_{K:K}$. Since the original RVs $\{\gamma_n\}$ are i.i.d. with common PDF $f(x)$, the K-dimensional joint PDF of the ordered RVs $\gamma_{n:K}$ is simply

$$\begin{aligned} g(\gamma_{1:K}, \gamma_{2:K}, \cdots, \gamma_{K:K}) &= K! f(\gamma_{1:K}) f(\gamma_{2:K}) \cdots f(\gamma_{K_S:K}) \left[F(\gamma_{K_S:K})\right]^{K-K_S} \\ &= K! f(\gamma_{1:K}) f(\gamma_{2:K}) \cdots f(\gamma_{K_S:K}) \left[cm(\gamma_{K_S:K}, 1)\right]^{K-K_S}, \end{aligned} \quad (36)$$

With (36), based on Fig. 1, we first need to derive the 2-dimensional Mellin Transform and it can be formulated as the following multi-fold integration form

$$\begin{aligned} \mathcal{M}(s_1, s_2) &= K! \int_0^\infty d\gamma_{1:K} (\gamma_{1:K})^{s_1-1} f(\gamma_{1:K}) \\ &\quad \times \int_0^{\gamma_{1:K}} d\gamma_{2:K} (\gamma_{2:K})^{s_1-1} f(\gamma_{2:K}) \\ &\quad \cdots \int_0^{\gamma_{K_S-2:K}} d\gamma_{K_S-1:K} (\gamma_{K_S-1:K})^{s_1-1} f(\gamma_{K_S-1:K}) \\ &\quad \times \int_0^{\gamma_{K_S-1:K}} d\gamma_{K_S:K} (\gamma_{K_S:K})^{s_2-1} f(\gamma_{K_S:K}) \\ &\quad \times \left[cm(\gamma_{K_S:K}, 1)\right]^{K-K_S}, \end{aligned} \quad (37)$$

Then, by applying the interchange of multiple integrals, we can rewrite (37) as

$$\begin{aligned}\mathcal{M}(s_1,s_2) &= K!\int_0^{\infty} d\gamma_{K_S:K}\left(\gamma_{K_S:K}\right)^{s_2-1} f\left(\gamma_{K_S:K}\right)\left[cm\left(\gamma_{K_S:K},1\right)\right]^{K-K_S} \\ &\quad\times \int_{\gamma_{K_S:K}}^{\infty} d\gamma_{K_S-1:K}\left(\gamma_{K_S-1:K}\right)^{s_1-1} f\left(\gamma_{K_S-1:K}\right) \\ &\quad\cdots\int_{\gamma_{2:K}}^{\infty} d\gamma_{1:K}\left(\gamma_{1:K}\right)^{s_1-1} f\left(\gamma_{1:K}\right). \end{aligned} \tag{38}$$

In (38), the multi-fold integral expression can be simplified as the following multiple-product of a mixture of a EDF and a Mellin transform

$$\begin{aligned}&\int_{\gamma_{K_S:K}}^{\infty} d\gamma_{K_S-1:K}\left(\gamma_{K_S-1:K}\right)^{s_1-1} f\left(\gamma_{K_S-1:K}\right) \\ &\qquad\cdots\int_{\gamma_{2:K}}^{\infty} d\gamma_{1:K}\left(\gamma_{1:K}\right)^{s_1-1} f\left(\gamma_{1:K}\right) \\ &\quad= \frac{1}{\left(K_S-1\right)!}\left[em\left(\gamma_{K_S:K},s_1\right)\right]^{K_S-1}. \end{aligned} \tag{39}$$

Thus, (37) can be finally simplified as the following single integral expression

$$\begin{aligned}\mathcal{M}(s_1,s_2) = \frac{K!}{\left(K_S-1\right)!}&\int_0^{\infty} d\gamma_{K_S:K}\left(\gamma_{K_S:K}\right)^{s_2-1} f\left(\gamma_{K_S:K}\right) \\ &\times\left[cm\left(\gamma_{K_S:K},1\right)\right]^{K-K_S}\left[em\left(\gamma_{K_S:K},s_1\right)\right]^{K_S-1}, \end{aligned} \tag{40}$$

where, in (40) for i.i.d Rayleigh fading,

$$\begin{aligned}&\left[em\left(\gamma_{K_S:K},s_1\right)\right]^{K_S-1} \\ &\quad= \left(\frac{1}{\bar{\gamma}}\right)^{-(K_S-1)(s_1-1)}\left[\Gamma\left(s_1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K_S-1} \\ &\quad\overset{\text{or}}{=} \left(\frac{1}{\bar{\gamma}}\right)^{(K_S-1)}\left(\frac{1}{\bar{\gamma}}\right)^{-(K_S-1)\cdot s_1}\left[\Gamma\left(s_1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{(K_S-1)}, \end{aligned} \tag{41}$$

$$\begin{aligned}&\left[cm\left(\gamma_{K_S:K},1\right)\right]^{K-K_S} \\ &\quad= \left[\Gamma(1)-\Gamma\left(1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K-K_S} \\ &\quad= \left[1-\exp\left(-\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K-K_S} \\ &\quad= \sum_{j=0}^{K-K_S}\binom{K-K_S}{j}(-1)^j\left(\exp\left(-\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right)^j. \end{aligned} \tag{42}$$

Then, substituting (41) and (42) in (40), (40) can be finally rewritten as

$$\begin{aligned}&\mathcal{M}(s_1,s_2) \\ &\quad= \frac{K!}{\left(K_S-1\right)!}\sum_{j=0}^{K-K_S}\binom{K-K_S}{j}(-1)^j\left(\frac{1}{\bar{\gamma}}\right)^{(K_S-1)+1} \\ &\qquad\times\int_0^{\infty} d\gamma_{K_S:K}\left(\gamma_{K_S:K}\right)^{s_2-1}\left(\exp\left(-\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right)^{j+1} \\ &\qquad\times\left(\frac{1}{\bar{\gamma}}\right)^{-(K_S-1)\cdot s_1}\left[\Gamma\left(s_1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K_S-1}. \end{aligned} \tag{43}$$

After obtaining this joint Mellin transform result in a compact form, we can derive a target joint PDF through inverse Mellin transform. Note that this result involves a single one-dimensional integration, which can be easily and accurately evaluated numerically using standard mathematical packages such as Mathematica and Matlab while the original Mellin transform expression in (37) involves K_s-fold integrations.

With this 2-dimensional Mellin transform result in (43), in the next step, we now need to apply the inverse Mellin transform to obtain the 2-dimensional joint PDF result as

$$\begin{aligned}&f(z_1,z_2) = \mathcal{M}_{s_1,s_2}^{-1}\left\{\mathcal{M}(s_1,s_2)\right\} \\ &\quad= \frac{K!}{\left(K_S-1\right)!}\sum_{j=0}^{K-K_S}\binom{K-K_S}{j}(-1)^j\left(\frac{1}{\bar{\gamma}}\right)^{(K_S-1)+1} \\ &\qquad\times\mathcal{M}_{s_2}^{-1}\left\{\int_0^{\infty} d\gamma_{K_S:K}\left(\gamma_{K_S:K}\right)^{s_2-1}\left(\exp\left(-\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right)^{j+1}\right. \\ &\qquad\times\left.\mathcal{M}_{s_1}^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-(K_S-1)\cdot s_1}\left[\Gamma\left(s_1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K_S-1}\right\}\right\}. \end{aligned} \tag{44}$$

In (44), based on [27], the inverse Mellin transform with s_1 becomes as the function of generalized incomplete Fox's H function

$$\begin{aligned}&\mathcal{M}_{s_1}^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-(K_S-1)\cdot s_1}\left[\Gamma\left(s_1,\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right]^{K_S-1}\right\} \\ &\quad= \mathrm{UH}_{0,K_S-1}^{K_S-1,0}\left[\left(\frac{1}{\bar{\gamma}}\right)^{K_S-1} z_1\,\middle|\, \underbrace{\overline{\left(0,1,\tfrac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\cdots\left(0,1,\tfrac{\gamma_{K_S:K}}{\bar{\gamma}}\right)}}_{\#=K_S-1}\right]. \end{aligned} \tag{45}$$

Then, substituting (45) in (44), (44) can be rewritten as

$$\begin{aligned}&f(z_1,z_2) = \mathcal{M}_{s_1,s_2}^{-1}\left\{\mathcal{M}(s_1,s_2)\right\} \\ &\quad= \frac{K!}{\left(K_S-1\right)!}\sum_{j=0}^{K-K_S}\binom{K-K_S}{j}(-1)^j\left(\frac{1}{\bar{\gamma}}\right)^{(K_S-1)+1} \\ &\qquad\times\mathcal{M}_{s_2}^{-1}\left\{\int_0^{\infty} d\gamma_{K_S:K}\left(\gamma_{K_S:K}\right)^{s_2-1}\left(\exp\left(-\frac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\right)^{j+1}\right. \\ &\qquad\times\left.\mathrm{UH}_{0,K_S-1}^{K_S-1,0}\left[\left(\frac{1}{\bar{\gamma}}\right)^{K_S-1} z_1\,\middle|\, \underbrace{\overline{\left(0,1,\tfrac{\gamma_{K_S:K}}{\bar{\gamma}}\right)\cdots\left(0,1,\tfrac{\gamma_{K_S:K}}{\bar{\gamma}}\right)}}_{\#=K_S-1}\right]\right\}. \end{aligned} \tag{46}$$

With this simplified result in (46), by applying the following inverse Mellin transform property

$$\mathcal{M}_s^{-1}\left\{\int_0^{\infty} dx(x)^{s-1} f(x)\right\} = f(x), \tag{47}$$

we can obtain the 2-dimensional joint PDF as

$$f(z_1, z_2) = \frac{K!}{(K_S-1)!} \sum_{j=0}^{K-K_S} \binom{K-K_S}{j} (-1)^j \times \left(\frac{1}{\overline{\gamma}}\right)^{(K_S-1)+1} \left(\exp\left(-\frac{z_2}{\overline{\gamma}}\right)\right)^{j+1} \times \mathrm{UH}_{0,K_S-1}^{K_S-1,0}\left[\left(\frac{1}{\overline{\gamma}}\right)^{K_S-1} z_1 \left| \begin{array}{c} \text{-------} \\ \underbrace{\left(0, 1, \frac{z_2}{\overline{\gamma}}\right) \cdots \left(0, 1, \frac{z_2}{\overline{\gamma}}\right)}_{\# = K_S-1} \end{array} \right.\right]. \tag{48}$$

Here, let $Z' = Z_1 \cdot Z_2$ where $Z_1 = \prod_{n=1}^{K_S-1} \gamma_{n:K}$, $Z_2 = \gamma_{K_S:K}$, then $Z_1 = \frac{Z'}{Z_2}$ and $Z_2^{K_S} < Z'$. Therefore, with the help of integration by part, we can finally obtain the following one-dimensional target PDF by taking a single-integration while the original approach based on the conventional Mellin transform needs to consider K_s-fold integrations in (37)

$$f_{Z'}(z) = \int_0^{z^{\frac{1}{K_S}}} f\left(\frac{z}{z_2}, z_2\right) dz_2. \overset{\text{or}}{=} \int_{z^{\frac{K_S-1}{K_S}}}^{\infty} f\left(z_1, \frac{z}{z_1}\right) dz_1. \tag{49}$$

As a validation of our analytical result, we compare in Fig. 2 the analytical PDF in (49) with an empirical PDF obtained by Monte-Carlo simulation. Note that some selected result shows that simulation results match our analytical results well.

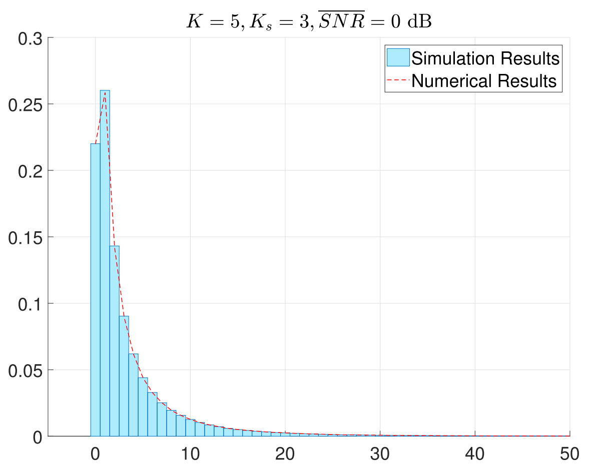

FIGURE 2. PDF comparison between the analytical and the simulation results of PDF of $Z' = \prod_{n=1}^{K_S} \gamma_{n:K}$ for $K = 5$, $K_S = 3$, $\overline{SNR} = 0$ dB, and i.i.d. Rayleigh fading condition.

VI. CONCLUSION AND DISCUSSION

In this work, we newly introduced the Mellin transform based systematic stochastic tool to determine the desired multidimensional joint statistics of arbitrary partial products of ordered RVs in a unified way. Specially according to Fig. 1 step-by step with the unified stochastic framework, we can systematically obtain the desired joint statistics. As an example of applying our proposed method, we provided some selected closed-form results on ordered statistics of partial products of ordered exponential RVs as a special case. To our best knowledge, this systematic and unified stochastic tool and related closed-form results have never appeared in the literature.

Note that if the results of the closed-form are not available in the joint statistics, especially the order statistics, the numerical computation of multi-fold integral expressions (e.g., the K-fold integrals given in (27) are required based on the conventional Mellin transform approaches. However, even with conventional mathematical computing tools, estimating them accurately as K increases is difficult because these multi-fold integral expressions require very high complexity, in contrast to simply multiplying multiple one-fold integrals in terms of computation. In particular, when K is large, it is almost impossible to estimate the analytical results. However, as a result of the closed-form derived from this paper, probabilistic analysis is numerically feasible using existing mathematical calculation tools. This is because it is a form of summing, not an multiple-fold integral expression. More specifically, with this multiple-fold integral expression, it is almost impossible to obtain numerical results physically due to estimation difficulties (i.e., high computational complexity and long execution time).

APPENDICES

APPENDIX I: SIMPLIFIED EXPRESSIONS OF MULTI-DIMENSIONAL INTEGRAL EXPRESSIONS

In this section, we apply an approach similar to that used in [6] to provide three simplified expressions of multi-fold integral expressions

a) The simplified expression of the multi-fold integral form with the upper limit, J_m:
In this case, we consider the following multi-fold integral with the upper limit, J_m, as

$$J_m = \int_0^{\gamma_{m-1:K}} d\gamma_{m:K} (\gamma_{m:K})^{s-1} f(\gamma_{m:K}) \times \int_0^{\gamma_{m:K}} d\gamma_{m+1:K} (\gamma_{m+1:K})^{s-1} f(\gamma_{m+1:K}) \cdots \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}). \tag{50}$$

First, for a single integral term in (50), we can rewrite the following single integral term as the function of $cm(\cdot,\cdot)$ as

$$\int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}) = \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} cm'(\gamma_{K:K}, s) = cm(\gamma_{K:K}, s)\Big|_0^{\gamma_{K-1:K}}$$

$$= cm(\gamma_{K-1:K}, s) - cm(0, s)$$
$$= cm(\gamma_{K-1:K}, s). \quad (51)$$

With the simplified result in (51), a following double integral form can be rewritten as the function of $cm(\cdot, \cdot)$ as

$$\int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} (\gamma_{K-1:K})^{s-1} f(\gamma_{K-1:K}) \times \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}) = \int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} cm'(\gamma_{K-1:K}, s)\, cm(\gamma_{K-1:K}, s). \quad (52)$$

Then, with (52), integration by parts gives

$$\int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} cm'(\gamma_{K-1:K}, s)\, cm(\gamma_{K-1:K}, s) = [cm(\gamma_{K-1:K}, s)]^2 \Big|_0^{\gamma_{K-2:K}} - \int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} cm(\gamma_{K-1:K}, s)\, cm'(\gamma_{K-1:K}, s). \quad (53)$$

Thus, by re-arranging them, the double integral form in (52) can be rewritten as the form of the second power of $cm(\gamma, s)$ function

$$\int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} (\gamma_{K-1:K})^{s-1} f(\gamma_{K-1:K}) \times \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}) = \frac{1}{2}[cm(\gamma_{K-2:K}, s)]^2. \quad (54)$$

Similarly, by applying results and approaches used in (51) and (54), we can rewrite the triple integral form as

$$\int_0^{\gamma_{K-3:K}} d\gamma_{K-2:K} (\gamma_{K-2:K})^{s-1} f(\gamma_{K-2:K}) \times \int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} (\gamma_{K-1:K})^{s-1} f(\gamma_{K-1:K}) \times \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K})$$
$$= \int_0^{\gamma_{K-3:K}} d\gamma_{K-2:K} cm'(\gamma_{K-2:K}, s) \frac{1}{2}[cm(\gamma_{K-2:K}, s)]^2$$
$$= \frac{1}{2}[cm(\gamma_{K-2:K}, s)]^3 \Big|_0^{\gamma_{K-3:K}} - \int_0^{\gamma_{K-3:K}} d\gamma_{K-2:K} [cm(\gamma_{K-2:K}, s)]^2 cm'(\gamma_{K-2:K}, s). \quad (55)$$

Then, re-arranging these results gives the following third power of $cm(\gamma, s)$ function

$$\int_0^{\gamma_{K-3:K}} d\gamma_{K-2:K} (\gamma_{K-2:K})^{s-1} f(\gamma_{K-2:K}) \times \int_0^{\gamma_{K-2:K}} d\gamma_{K-1:K} (\gamma_{K-1:K})^{s-1} f(\gamma_{K-1:K}) \times \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}) = \frac{1}{2 \cdot 3}[cm(\gamma_{K-3:K}, s)]^3. \quad (56)$$

For general case, by generalizing the approaches used in (51), (54), and (56), we can finally obtain the m-fold integral form as the following multiple product from of $cm(\cdot, \cdot)$ function as

$$J_m = \int_0^{\gamma_{m-1:K}} d\gamma_{m:K} (\gamma_{m:K})^{s-1} f(\gamma_{m:K}) \times \int_0^{\gamma_{m:K}} d\gamma_{m+1:K} (\gamma_{m+1:K})^{s-1} f(\gamma_{m+1:K}) \cdots \int_0^{\gamma_{K-1:K}} d\gamma_{K:K} (\gamma_{K:K})^{s-1} f(\gamma_{K:K}) = \frac{1}{(K-m+1)!}[cm(\gamma_{m-1:K}, s)]^{K-m+1}. \quad (57)$$

b) The simplified expression of the multi-fold integral with the lower limit, J'_m:
Using similar manipulations to the ones used in (57), we have the following simplified expression of the multi-fold integral with the lower limit, J'_m, as

$$J'_m = \int_{\gamma_{m+1:K}}^{\infty} d\gamma_{m:K} (\gamma_{m:K})^{s-1} f(\gamma_{m:K}) \times \int_{\gamma_{m:K}}^{\infty} d\gamma_{m-1:K} (\gamma_{m-1:K})^{s-1} f(\gamma_{m-1:K}) \cdots \int_{\gamma_{2:K}}^{\infty} d\gamma_{1:K} (\gamma_{1:K})^{s-1} f(\gamma_{1:K}) = \frac{1}{m!}[em(\gamma_{m+1:K}, s)]^m. \quad (58)$$

c) The simplified expression of the multi-fold integral with the interval, $J''_{a,b}$ (for $a < b$): Similar to two previous cases in (57) and (58), we also have the following simplified expression of the multi-fold integral with the interval, $J''_{a,b}$, as

$$J''_{a,b} = \int_{\gamma_{b:K}}^{\gamma_{a:K}} d\gamma_{b-1:K} (\gamma_{b-1:K})^{s-1} f(\gamma_{b-1:K}) \times \int_{\gamma_{b-1:K}}^{\gamma_{a:K}} d\gamma_{b-2:K} (\gamma_{b-2:K} 2 : K)^{s-1} f(\gamma_{b-2:K}) \cdots \int_{\gamma_{a+2:K}}^{\gamma_{a:K}} d\gamma_{a+1:K} (\gamma_{a+1:K})^{s-1} f(\gamma_{a+1:K}) = \frac{1}{(b-a-1)!}[im(\gamma_{b:K}, \gamma_{a:K}, s)]^{b-a-1}. \quad (59)$$

APPENDIX II: INVERSE MELLIN TRANSFORM

We can take several approaches to get the PDF expressions from Mellin transform expressions. For example, we consider the existing inverse Mellin transformation table and properties in [30]. However, applying known tables and attributes directly to our cases is difficult. Residual theorem [31] may be suitable for our case, but too complicated and not for uniform solutions. Please refer to Appendix III. As an alternative, we can apply the generalized upper incomplete Fox's H function in [27], [32], which is a suitable approach for uniform solutions, where the generalized upper incomplete Fox's H function is defined as

$$\begin{aligned}&\mathrm{UH}_{p,q}^{m,l}(c\cdot\gamma)\\&=\frac{1}{2\pi j}\\&\times\oint\frac{\prod_{i=1}^{m}\Gamma(b_i+\beta_i\cdot s,B_i)\prod_{k=1}^{l}\Gamma(1-a_k-\alpha_k\cdot s,A_k)}{\prod_{k=l+1}^{p}\Gamma(a_k+\alpha_k\cdot s,A_k)\prod_{i=m+1}^{q}\Gamma(1-b_i-\beta_i\cdot s,B_i)}(c\cdot\gamma)^{-s}ds\\&\triangleq\mathrm{UH}_{p,q}^{m,l}\left[c\cdot\gamma\left|\begin{matrix}(a_1,\alpha_1,A_1)\cdots(a_p,\alpha_p,A_p)\\(b_1,\beta_1,B_1)\cdots(b_q,\beta_q,B_q)\end{matrix}\right.\right],\end{aligned}\tag{60}$$

where

- m, l, p, and q are integers such that $0 \le m \le q$ and $0 \le l \le p$.
- $a_k, b_i \in \mathbb{C}$ and $a_k, A_k, \beta_i, B_i, c \in \mathbb{R}^+$ with $1 \le k \le p$ and $1 \le i \le q$.
- $\prod_{k=n}^{n-1}\Gamma(b_k+\beta_k\cdot s,B_k)=\prod_{k=n}^{n-1}\Gamma(1-b_k-\beta_k\cdot s,B_k)$ $=\prod_{k=n}^{n-1}\Gamma(a_k+\alpha_k\cdot s,A_k)=\prod_{k=n}^{n-1}\Gamma(1-a_k-\alpha_k\cdot s,A_k)$ $=1.$

Note that

$$\begin{aligned}\gamma(\alpha,x;b)&=\gamma(\alpha,x;b;1),\\ \gamma(\alpha,x)&=\gamma(\alpha,x;0)=\gamma(\alpha,x;0;1),\\ \Gamma(\alpha,x;b)&=\Gamma(\alpha,x;b;1),\\ \Gamma(\alpha,x)&=\Gamma(\alpha,x;0)=\Gamma(\alpha,x;0;1),\\ \Gamma(\alpha)&=\Gamma(s,0).\end{aligned}$$

A. EXAMPLE 1) THE INVERSE MELLIN TRANSFORM OF $\frac{K_N}{C_N}\prod_{i=1}^{N}\Gamma(s,\alpha_i\beta_i)\left(\frac{1}{C_N}\right)^{-s}$

In this case, the inverse Mellin transform can be written as the following residue theorem formation

$$\begin{aligned}&\frac{1}{2\pi j}\oint_C\frac{K_N}{C_N}\prod_{i=1}^{N}\Gamma(s,\alpha_i\beta_i)\left(\frac{1}{C_N}\right)^{-s}(x)^{-s}ds\\&\quad=\frac{K_N}{C_N}\cdot\frac{1}{2\pi j}\oint_C\prod_{i=1}^{N}\Gamma(s,\alpha_i\beta_i)\left(\frac{x}{C_N}\right)^{-s}ds.\end{aligned}\tag{61}$$

Here, in (60), $m = n$ and $l = 0$. Therefore, p should become 0 and q should become N. As a result, the closed-form result of (61) with $\mathrm{UH}_{p,q}^{m,l}(\cdot)$ can be obtained

$$\begin{aligned}&\frac{1}{2\pi j}\oint_C\frac{K_N}{C_N}\prod_{i=1}^{N}\Gamma(s,\alpha_i\beta_i)\left(\frac{x}{C_N}\right)^{-s}ds\\&=\frac{K_N}{C_N}\cdot\mathrm{UH}_{0,N}^{N,0}\left[\frac{x}{C_N}\left|\begin{matrix}-------\\(0,1,\alpha_1\beta_1)(0,1,\alpha_2\beta_2)\cdots(0,1,\alpha_N\beta_N)\end{matrix}\right.\right].\end{aligned}\tag{62}$$

B. EXAMPLE 2) THE INVERSE MELLIN TRANSFORM OF $\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}[\Gamma(s)]^k\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^{n-k}$

Similarly, in this case, the inverse Mellin transform can be written as

$$\begin{aligned}&\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}[\Gamma(s)]^k\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^{n-k}\right\}\\&\quad=\frac{1}{2\pi j}\oint_C[\Gamma(s)]^k\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^{n-k}\left(\left(\frac{1}{\bar{\gamma}}\right)^n x\right)^{-s}ds.\end{aligned}\tag{63}$$

Then, in (60), $m = n$ and $l = 0$, which leads to $p = 0$ and $q = n$. Thus, the inverse Mellin transform closed-form result of (63) can be obtained as

$$\begin{aligned}&\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}[\Gamma(s)]^k\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^{n-k}\right\}\\&=\mathrm{UH}_{0,n}^{n,0}\left[\frac{x}{(\bar{\gamma})^n}\left|\begin{matrix}-\;-\;-\;-\\\underbrace{(0,1,0)\cdots(0,1,0)}_{k}\underbrace{\left(0,1,\frac{\gamma}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{\gamma}{\bar{\gamma}}\right)}_{n-k}\end{matrix}\right.\right].\end{aligned}\tag{64}$$

C. EXAMPLE 3) THE INVERSE MELLIN TRANSFORM OF $\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^n$

In the following inverse Mellin transform,

$$\begin{aligned}&\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^n\right\}\\&\qquad=\frac{1}{2\pi j}\oint_C\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^n\left(\left(\frac{1}{\bar{\gamma}}\right)^n x\right)^{-s}ds.\end{aligned}\tag{65}$$

the inverse Mellin transform becomes similar to example 2) when $m = n$ and $l = 0$ (i.e., $p = 0$ and $q = n$) in (60). Thus, the closed-form result of the inverse Mellin transform (65) can be obtained as

$$\begin{aligned}&\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s,\frac{\gamma}{\bar{\gamma}}\right)\right]^n\right\}\\&\quad=\mathrm{UH}_{0,n}^{n,0}\left[\frac{x}{(\bar{\gamma})^n}\left|\begin{matrix}-\;-\;-\;-\\\underbrace{\left(0,1,\frac{\gamma}{\bar{\gamma}}\right)\cdots\left(0,1,\frac{\gamma}{\bar{\gamma}}\right)}_{n}\end{matrix}\right.\right].\end{aligned}\tag{66}$$

D. EXAMPLE 4) THE INVERSE MELLIN TRANSFORM OF $\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s,\frac{\gamma_a}{\bar{\gamma}}\right)\right]^k\left[\Gamma\left(s,\frac{\gamma_b}{\bar{\gamma}}\right)\right]^{n-k}$

Similarly, the inverse Mellin transform expression can be written as the following residue theorem format

$$\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s,\frac{\gamma_a}{\bar{\gamma}}\right)\right]^k\left[\Gamma\left(s,\frac{\gamma_b}{\bar{\gamma}}\right)\right]^{n-k}\right\}$$

$$= \frac{1}{2\pi j}\oint_C \left[\Gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right)\right]^k \left[\Gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right)\right]^{n-k} \left(\left(\frac{1}{\bar{\gamma}}\right)^n x\right)^{-s} ds. \tag{67}$$

Similar to previous cases, $m = n$ and $l = 0$ in (60). As a result, with $p = 0$ and $q = n$, the inverse Mellin transform closed-form result of (67) can be obtained as

$$\mathcal{M}_s^{-1}\left\{\left(\frac{1}{\bar{\gamma}}\right)^{-n\cdot s}\left[\Gamma\left(s, \frac{\gamma_a}{\bar{\gamma}}\right)\right]^k \left[\Gamma\left(s, \frac{\gamma_b}{\bar{\gamma}}\right)\right]^{n-k}\right\} = \mathrm{UH}_{0,n}^{n,0}\left[\frac{x}{(\bar{\gamma})^n}\left| \underbrace{\left(0, 1, \frac{\gamma_a}{\bar{\gamma}}\right)\cdots\left(0, 1, \frac{\gamma_a}{\bar{\gamma}}\right)}_{k}\underbrace{\left(0, 1, \frac{\gamma_b}{\bar{\gamma}}\right)\cdots\left(0, 1, \frac{\gamma_b}{\bar{\gamma}}\right)}_{n-k}\right.\right]. \tag{68}$$

APPENDIX III: INVERSE MELLIN TRANSFORM APPROACH BASED ON THE RESIDUE THEOREM

In this section, we show an example of a residue theorem based derivation process to demonstration of the advantages (i.e., simplicity and unification) of our proposed unified stochastic tool compared to the conventional residue theorem based derivation process.

E. THE INVERSE MELLIN TRANSFORM OF $\Gamma(s)$

If we let $F(s) = \Gamma(s)$, then based on the residue theorem, the inverse Mellin transform of $F(s)$ can be written as

$$f(x) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\, x^{-s} ds \quad \text{for } c > 0. \tag{69}$$

Here, $F(s)\, x^{-s}$ has poles at $s = 0, -1, -2, \cdots$. Therefore, (69) can be rewritten as

$$f(x) = \frac{1}{2\pi j}\cdot 2\pi j \cdot \operatorname*{Res}_{s=0,-1,-2\cdots} F(s)\, x^{-s}, \tag{70}$$

where each residue can be obtained as

$$\text{at } s=0:\ sF(s)\,x^{-s}\big|_{s=0} = F(s+1)x^{-s}\big|_{s=0} = \Gamma(1) = 1$$

$$\text{at } s=-1:\ (s+1)\,F(s)\,x^{-s}\big|_{s=-1} = \left.\frac{F(s+2)}{s}x^{-s}\right|_{s=-1} = -\Gamma(1)\,x^1 = -x$$

$$\text{at } s=-2:\ (s+2)F(s)\,x^{-s}\big|_{s=-2} = \left.\frac{F(s+3)}{s(s+1)}x^{-s}\right|_{s=-2} = \frac{\Gamma(1)}{(-2)(-1)}x^2 = \frac{x^2}{2}$$

and so on. Therefore, (70) can be written as

$$f(x) = \operatorname*{Res}_{s=0,-1,-2\cdots} F(s)\,x^{-s} = 1+(-x)+\frac{x^2}{2}+\frac{-x^3}{6}\cdots = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \exp(-x). \tag{71}$$

F. THE INVERSE MELLIN TRANSFORM OF $[\Gamma(s)]^n$

Similar to the previous case based on the residue theorem, the inverse Mellin transform of $F(s) = [\Gamma(s)]^n$ can be written as

$$f_n(x) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} [\Gamma(s)]^n x^{-s} ds \quad \text{for } c > 0. \tag{72}$$

Here, $F(s)\,x^{-s}$ has poles at $s = 0, -1, -2, \cdots$. Therefore, based on the residue theorem, the residue values are as

$$\begin{aligned}\operatorname*{Res}_{s=0,-1,-2\cdots} F(s)\,x^{-s} &= \frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s-s_0)^n[\Gamma(s)]^n x^{-s}\Big|_{s=s_0}, \\ &\quad \text{for } s_0 = 0,-1,-2,\cdots \\ &\overset{\text{or}}{=} \frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s+s_0)^n[\Gamma(s)]^n x^{-s}\Big|_{s=-s_0}, \\ &\quad \text{for } s_0 = 0, 1, 2, \cdots.\end{aligned} \tag{73}$$

Therefore, (72) can be rewritten as the summation form of residues

$$f_n(x) = \sum_{j=0}^{\infty}\frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s+j)^n[\Gamma(s)]^n x^{-s}\Big|_{s=-j} \tag{74}$$

In (74), if we let $G(s) = x^{-s}(s+j)^n[\Gamma(s)]^n$, then we still need to derive the n-th derivative term of $G(s)$. Here, if we let $\frac{G^{(n)}(s)}{G(s)} = Z_n(s)$, then for mathematical convenience, $Z_n(s)$ can be expressed in terms of $A(s), A'(s), \cdots, A^{(n)}(s)$, which are regardless of n. For the first derivative term of $G(s)$, it can be written as

$$G'(s) = G(s)\,Z_1(s) = G(s)\,A(s), \tag{75}$$

and in (75), $Z_1(s) = A(s)$. Similarly, the second and the third derivative term of $G(s)$ can be written as

$$G^{(2)}(s) = G(s)Z_2(s) = G'(s)Z_1(s) + G(s)Z_1{}'(s), \tag{76}$$

and

$$G^{(3)}(s) = G(s)Z_3(s) = G'(s)Z_2(s) + G(s)Z_2{}'(s), \tag{77}$$

where (76) and (77) can be rewritten in terms of $A(s)$, $A'(s), \cdots, A^{(n)}(s)$, respectively, as

$$G^{(2)}(s) = G(s)\left\{(A(s))^2 + A'(s)\right\}, \tag{78}$$

and

$$\begin{aligned}G^{(3)}(s) &= G(s)\left\{A(s)Z_2(s) + Z_2{}'(s)\right\} \\ &= G(s)\left\{(A(s))^3 + 3A(s)A'(s) + A^{(2)}(s)\right\}.\end{aligned} \tag{79}$$

Here, $Z_2(s)$ and $Z_3(s)$ can be written with low-order derivative terms as

$$Z_2(s) = (A(s))^2 + A'(s) = A(s)\,Z_1(s) + Z_1{}'(s), \tag{80}$$

$$\begin{aligned}Z_3(s) &= (A(s))^3 + 3A(s)\,A'(s) + A^{(2)}(s) \\ &= A(s)\,Z_2(s) + Z_2{}'(s).\end{aligned} \tag{81}$$

As a result, by generalizing the above special cases, we can obtain the following generalized result as the recursive expressions

$$G^{(n+1)}(s) = G(s)Z_{n+1}(s) = G(s)\left\{A(s)Z_n(s)+Z_n{}'(s)\right\}, \quad (82)$$

where

$$Z_{n+1}(s)=A(s)Z_n(s)+Z_n{}'(s) \quad \text{for } Z_1(s)=A(s). \quad (83)$$

With (82) and (83), we can now derive the $(n-1)$-th derivative term of $G(s)$ at $s=-j$, $G^{(n-1)}(s)\big|_{s=-j}$. Here, $G(s)$ can be rewritten as

$$G(s) = x^{-s}\left\{(s+j)\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)(s+j)}\right\}^n = x^{-s}\left\{\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right\}^n, \quad (84)$$

where $G(s)|_{s=-j} = x^j\left\{(-1)^j\frac{1}{j!}\right\}^n = \frac{x^j(-1)^{j\cdot n}}{(j!)^n}$. Then, we need to derive $Z_n(s)\left(=\frac{G^{(n)}(s)}{G(s)}\right)$ in (82) as $A(s)$ functions. For $Z_1(s)$, $Z_1(s)$ is $\frac{G'(s)}{G(s)}$ and it also can be written as $\frac{d}{ds}\ln G(s)$ where

$$\ln G(s) = -s\ln x+n\ln\left(\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right) \overset{\text{or}}{=} -s\ln x+n\left\{\ln(\Gamma(s+j+1)) - \sum_{k=0}^{j-1}\ln(s+k)\right\}. \quad (85)$$

Thus, with (85), $\frac{G'(s)}{G(s)}\left(=\frac{d}{ds}\ln G(s)\right)$ can be obtained as

$$\frac{G'(s)}{G(s)} = -\ln x + n\left\{\psi(s+j+1) - \sum_{k=0}^{j-1}\frac{1}{s+k}\right\}, \quad (86)$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the psi (Digamma) function. Therefore, based on (75), $A(s)$ becomes $\frac{d}{ds}\ln G(s)$ and it can be written as

$$A(s)|_{s=-j} = -\ln y + n\left\{\psi(1) - \sum_{k=0}^{j-1}\frac{1}{k-j}\right\} = -\ln y + n\left\{\psi(1) + \sum_{k=1}^{j}\frac{1}{k}\right\}. \quad (87)$$

Here, for integer j, $\psi(j+1) = -C + \sum_{k=1}^{j}\frac{1}{k}$, where $\psi(1) = -C$ (Euler constant). Therefore, (87) can be rewritten as

$$A(s)|_{s=-j} = -\ln y + n\psi(j+1). \quad (88)$$

Similarly, for $A(s)$ family function case, we can obtain the following results

$$A'(s)|_{s=-j} = n\left\{\psi'(1) + \sum_{k=0}^{j-1}\frac{1}{(s+k)^2}\Bigg|_{s=-j}\right\} = n\left\{\psi'(1) + \sum_{k=1}^{j}\frac{1}{(k)^2}\right\}, \quad (89)$$

and

$$A^{(2)}(s)\Big|_{s=-j} = n\left\{\psi^{(2)}(1) - \sum_{k=0}^{j-1}\frac{2}{(s+k)^3}\Bigg|_{s=-j}\right\} = n\left\{\psi^{(2)}(1) + 2\sum_{k=1}^{j}\frac{1}{(k)^3}\right\}. \quad (90)$$

Thus, generalizing the above special cases, we can obtain the following generalized result as

$$A^{(l)}(s)\Big|_{s=-j} = n\left\{\psi^{(l)}(1) + l\sum_{k=1}^{j}\frac{1}{(k)^{l+1}}\right\}. \quad (91)$$

Here, (74) can be written as the function $G(s)$

$$f_n(x) = \sum_{j=0}^{\infty}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}G(s)|_{s=-j}, \quad (92)$$

where

$$G(s) = x^{-s}\left\{\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right\}^n, \quad (93)$$

$$G(s)|_{s=-j} = \frac{x^j(-1)^{j\cdot n}}{(j!)^n}, \quad (94)$$

and the generalized derivative result becomes

$$G^{(n-1)}(s)\Big|_{s=-j} = G(s)Z_{n-1}(s)|_{s=-j} = \frac{(-1)^{j\cdot n}}{(j!)^n}Z_{n-1}(s)|_{s=-j}. \quad (95)$$

As results, for arbitrary integer n, the inverse Mellin transform of $[\Gamma(s)]^n$ can be rewritten in terms of $Z(s)$ family functions as

$$f_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{\infty}\frac{x^j(-1)^{j\cdot n}}{(j!)^n}Z_{n-1}(s)|_{s=-j}, \quad (96)$$

where $Z_{n-1}(s)|_{s=-j}$ can be obtained with (83) and (91).

With (96), if we let $L(x,n,j) = -\ln x + n\{\psi(1) + \sum_{k=1}^{j}\frac{1}{k}\}$ and $P_l(j) = \psi^{(l)}(1) + l!\sum_{k=1}^{j}\frac{1}{k^{l+1}}$, with the help of (83), then $f_2(x)$ can be written as

$$f_2(x) = \sum_{j=0}^{\infty}\frac{x^j}{(j!)^2}Z_1(s)|_{s=-j}, \quad (97)$$

where

$$Z_1(s)|_{s=-j} = A(-j) = -\ln x + 2\left\{\psi(1) + \sum_{k=1}^{j}\frac{1}{k}\right\} = L(x,2,j). \quad (98)$$

Similarly, $f_3(x)$ and $f_4(x)$ can be obtained, respectively as

$$f_3(x)=\frac{1}{2!}\sum_{j=0}^{\infty}\frac{x^j(-1)^j}{(j!)^3}\left[\{L(x,3,j)\}^2+3P_1(j)\right], \tag{99}$$

and

$$f_4(x)=\frac{1}{3!}\sum_{j=0}^{\infty}\frac{x^j(-1)^j}{(j!)^4}\Big[\{L(x,4,j)\}^3 +12L(x,4,j)P_1(j)+4P_2(j)\Big]. \tag{100}$$

Based on the above derivation process, we can see that it is difficult to systematically derive the results of inverse Mellin transform in a unified way for all cases; this is particularly the case when considering the application of residual theory in this paper.

Note that for the inverse Mellin transform of $[\gamma(s,a)]^n$, similar to previous cases, if we let $F(s)=[\gamma(s,a)]^n$, then the inverse Mellin transform of $F(s)$ can be written as

$$f_n(x)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}[\gamma(s,a)]^n x^{-s}ds, \tag{101}$$

where $[\gamma(s,a)]^n$ has the poles of order n at $s=0,-1,-2,\cdots$ as

$$\frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s-s_0)^n x^{-s}[\gamma(s,a)]^n\Big|_{s=s_0},$$
$$\text{for } s_0=0,-1,-2,\cdots$$
$$\overset{\text{or}}{=}\frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s+s_0)^n x^{-s}[\gamma(s,a)]^n\Big|_{s=-s_0},$$
$$\text{for } s_0=0,1,2,\cdots.$$

As a result, the inverse Mellin transform result can be obtained as

$$f_n(x)=\sum_{s_0=0}^{\infty}\frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{ds^{n-1}}(s+s_0)^n x^{-s}[\gamma(s,a)]^n\Big|_{s=-s_0}. \tag{102}$$

However, similar to previous cases, with this approach, it is also difficult to systematically derive the desired result in a unified way.

REFERENCES

[1] H. A. David, *Order Statistics*. New York, NY, USA: Wiley, 1981.
[2] N. Balakrishnan and C. R. Rao, *Handbook Statistics: Order Statistics: Applications*, 2nd ed. Amsterdam, The Netherlands: Elsevier, 1998.
[3] A. Goldsmith, *Wireless Communications*. New York, NY, USA: Cambridge Univ. Press, 2005.
[4] J. Gambini, M. E. Mejail, J. Jacobo-Berlles, and A. C. Frery, "Accuracy of edge detection methods with local information in speckled imagery," *Statist. Comput.*, vol. 18, no. 1, pp. 15–26, Mar. 2008.
[5] M. S. Alouini and H. C. Yang, *Order Statistics in Wireless Communications*, 1st ed. New York, NY, USA: Cambridge Univ. Press, 2011.
[6] S. S. Nam, M.-S. Alouini, and H.-C. Yang, "An MGF-based unified framework to determine the joint statistics of partial sums of ordered random variables," *IEEE Trans. Inf. Theory*, vol. 56, no. 8, pp. 5655–5672, Nov. 2010.
[7] M. D. Springer, *The Algebra Random Variables*. New York, NY, USA: Wiley, 1979.
[8] D. F. Andrews, "Asymptotic expansions of moments and cumulants," *Statist. Comput.*, vol. 11, no. 1, pp. 7–16, Jan. 2001.
[9] P. Galambos and I. Simonelli, *Products of Random Variables: Applications to Problems of Physics and to Arithmetical Functions*. New York, NY, USA: Marcel Dekker Inc., 2004.
[10] G. Arslan, "A characterization based on product of order statistics," Oct. 2012, *arXiv:1110.2879*. [Online]. Available: https://arxiv.org/abs/1110.2879
[11] K. Subrahmaniam, "On some applications of Mellin transforms to statistics: Dependent random variables," *SIAM J. Appl. Math.*, vol. 19, no. 4, pp. 658–662, Dec. 1970.
[12] V. M. Zolotarev, "Mellin-stieltjes transforms in probability theory," *Theory Probab. Appl.*, vol. 2, no. 4, pp. 433–460, 1957.
[13] M. D. Springer and W. E. Thompson, "The distribution of products of independent random variables," *SIAM J. Appl. Math.*, vol. 14, no. 3, pp. 511–526, 1966.
[14] Z. A. Lomnicki, "On the distribution of products of random variables," *J. Roy. Statist. Soc. B*, vol. 29, no. 3, pp. 513–524, 1967.
[15] P. C. Consul, "On the distribution of VOTAW's likelihood ratio criterion L for testing the bipolarity of a covariance matrix," *Math. Nachrichten*, vol. 36, nos. 1–2, pp. 1–13, 1968.
[16] P. C. Consul, "On the exact distributions of votaw's criteria for testing compound symmetry of a covariance matrix," *Ann. Math. Statist.*, vol. 340, no. 3, pp. 836–843, 1969.
[17] C. Fox, "Some applications of mellin transforms to the theory of bivariate statistical distributions," *Math. Proc. Cambridge Philos. Soc.*, vol. 53, no. 3, pp. 620–628, Jul. 1957.
[18] S. Hitam, S. Suhaimi, A. Noor, S. Anas, and R. Sahbudin, "Performance analysis on 16-channels wavelength division multiplexing in free space optical transmission under tropical regions environment," *J. Comput. Sci.*, vol. 8, no. 1, pp. 145–148, 2012.
[19] C.-Y. Lin, Y.-P. Lin, H.-H. Lu, C.-Y. Chen, T.-W. Jhang, and M.-C. Chen, "Optical free-space wavelength-division-multiplexing transport system," *Opt. Lett.*, vol. 39, no. 2, pp. 315–318, Jan. 2014.
[20] S. Robinson and R. Pavithra, "Investigation on multi-beam hybrid WDM for free space optical communication system," *Int. J. Photon. Opt. Technol.*, vol. 2, no. 2, pp. 24–28, Jun. 2016.
[21] P. V. Trinh, N. T. Dang, and A. T. Pham, "Optical amplify-and-forward multihop WDM/FSO for all-optical access networks," in *Proc. Int. Symp. CSNDSP*, Manchester, U.K., Jul. 2014, pp. 1106–1111.
[22] F. Rashidia, J. He, and L. Chena, "Spectrum slicing WDM for FSO communication systems under the heavy rain weather," *Opt. Commun.*, vol. 387, pp. 296–302, Mar. 2017.
[23] A. O. Aladeloba, M. S. Woolfson, and A. J. Phillips, "WDM FSO network with turbulence-accentuated interchannel crosstalk," *J. Opt. Commun. Netw.*, vol. 5, no. 6, pp. 641–651, Jun. 2013.
[24] A. F. Molisch, N. B. Mehta, J. S. Yedidia, and J. Zhang, "Performance of fountain codes in collaborative relay networks," *IEEE Trans. Wireless Commun.*, vol. 6, no. 11, pp. 4108–4119, Nov. 2007.
[25] A. Ravanshid, L. Lampe, and J. Huber, "Signal combining for relay transmission with rateless codes," in *Proc. IEEE ISIT*, Seoul, South Korea, Jun./Jul. 2009, pp. 508–512.
[26] S. S. Nam, M.-S. Alouini, and S. Choi, "Modified dynamic decode-and-forward relaying protocol for type II relay in LTE-Advanced and beyond," *PLoS ONE*, vol. 11, no. 11, Nov. 2016, Art. no. e0167457.
[27] F. Yilmaz and M.-S. Alouini, "Product of shifted exponential variates and outage capacity of multicarrier systems," in *Proc. Eur. Wireless Conf. (EW)*, May 2009, pp. 282–286.
[28] F. Yilmaz and M.-S. Alouini, "Outage capacity of multicarrier systems," in *Proc. IEEE Int. Conf. Telecommun. (ICT)*, Apr. 2010, pp. 260–265.
[29] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego, CA, USA: Academic, 2000.
[30] A. D. Poularikas, *Handbook of Formulas and Tables for Signal Processing*. Boca Raton, FL, USA: CRC Press, 1998.
[31] E. Kreyszig, *Advanced Engineering Mathematics*, 8th ed. New York, NY, USA: Wiley, 1998.
[32] A. Chelli, A. Hadjtaieb, and M.-S. Alouini, "Performance analysis of hybrid ARQ with incremental redundancy over amplify-and-forward dual-hop relay channels," 2014, *arXiv:1404.2131*. [Online]. Available: https://arxiv.org/abs/1404.2131