IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received August 24, 2019, accepted September 4, 2019, date of publication September 11, 2019, date of current version September 24, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2940541

Bipartite Consensus of Multi-Agent Systems With Intermittent Interaction

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This work was supported in part by the National Natural Science Foundation of China under Grant 61873136, Grant 61903210, Grant 61374062, and Grant 61603288, in part by the Science Foundation of Shandong Province for Distinguished Young Scholars under Grant JQ201419, and in part by the Shandong Provincial Natural Science Foundation, China, under Grant ZR201709260010.

ABSTRACT This paper studies the bipartite consensus problem of multi-agent systems with intermittent interaction under signed directed graph. It is assumed that each agent receives its states information relative to its neighbors at the sampling time and updates the control input by using the states information, and the period of each agent updates control input is equal to a positive integer multiple of its sampling period. Cooperation and competition between agents are represented by positive and negative weights of edge respectively in the signed topology. The sufficient condition for achieving bipartite consensus is obtained by Shure-Cohen stability criterion, which reveals the relationship among sampling period, update control input period and controller gain of system. Finally, simulation tests show the bipartite consensus performances of agents under intermittent protocol and signed topology.

INDEX TERMS Multi-agent systems, intermittent interaction, bipartite consensus, positive and negative weights, Shure-Cohen stability criterion.

I. INTRODUCTION

The problems of cooperative control for multi-agent systems have been concerned widely by researchers at home and abroad because of its important significance both in theory and reality. For example, the consensus [1]–[6] and controllability [7]–[16] have been widely studied for multi-agent system. As a basic problem of cooperative control, the research on consensus of multi-agent systems is always a hot topic. At present, the consensus problem has been widely applied in many fields, such as computer science and system, biology, physics, and control science. The research direction mainly focuses on fish swarm, bird swarm coordination, formation control of UAV system, target tracking of sensor network, etc.

In the study of consensus problem, it is a significant and fundamental problem to design appropriate control protocols such that agents of multi-agent systems achieve a consensus value with their neighbors, that is, the states of all agents (such as speed, location) converge to a agreement value. For the second-order multi-agent system, Liu etc first proposed a consensus algorithm in 2017, in which different agents used different absolute velocity damping gains, and necessary and sufficient conditions for system to achieve consensus were obtained [17]. For high-order multi-agent, an adaptive protocol with guaranteed performance constraints was proposed for the first time, which not only adaptively adjusts the weights among neighboring agents and the state errors among all agents, but also achieves consensus [18]. When considering the problem of energy consumption under the condition of cost budget, a dynamic output feedback consensus protocol with guaranteed cost constraints was proposed for the first time in [19].

Most of these studies were involved with complete consensus, in which protocols drove all agents to converge to the same consensus value. However, with the increase of system scale and complexity, single equilibrium point cannot meet the control requirements, so some scholars put forward the concepts of scaled consensus, group consensus and bipartite consensus. Scaled consensus refers to the states of all agents which reach assigned proportions, rather than a common value. Shang investigated the scaled consensus of switching topologies with continuous-time and discrete-time subsystems [20]. Group consensus means that agents of different subnetworks achieve different consensus value.

The associate editor coordinating the review of this manuscript and approving it for publication was Zhan Bu.

Under the assumptions of degree balance and inter-group balance, group consensus problem under Markovian switching topologies and non-linear dynamics have been solved in [21], [22], respectively. Bipartite consensus, as a special case of scaled consensus, means that the states of all agents converge to a consensus value with the same modulus, but with different sign [23]–[28]. Sign graphs are often used to represent competitive-cooperative multi-agent systems. The conditions for achieving bipartite consensus of high-order multi-agent systems with unknown disturbances were found [24], [25]. On the rate of achieving consensus, Qu *et al.* described how to get the fastest convergence rate under antagonistic interaction [26]. However, most of the existing bipartite consensus results were obtained under signed digraphs of structural balanced. Based on linear matrix inequality(LMI) approach, Tian et al. proved that the system with structural unbalance can achieve consensus by choosing appropriate parameters [28].

All references mentioned above involve isomorphic multi-agent systems with the same dynamic equation. However, in reality, there are differences in functions and structures among individuals due to the limitation of various factors, which lead to different dynamic equations for each agent. Therefore, it is necessary to study the hybrid multi-agent system [29]–[34]. Zheng, Ma and Wang proved that the hybrid multi-agent system can achieve the consensus if and only if the communication network has a directed spanning tree [29]. A game-theoretic approach was used to analyze the consensus of hybrid multi-agent systems [30]. Under signed digraphs, Liu *et al.* studied the consensus of hybrid multi-agent systems, and proved that the system has the lowest control cost under the discovered topology structure [32]. In the recent work [33], Shang framed a hybrid censoring strategies for achieving consensus of hybrid multi-agent systems with malicious nodes. The strategy further resolved resilient consensus.

In a real multi-agent system, agents communicate with each other through communication networks. Therefore, limited channel bandwidth will bring great constraints on multi-agent coordinate control. Considering the problem of network resource utilization and agent's own energy, the use of periodic sampling control can reduce the number of communications between agents, thus reducing the waste of resources [35]–[41]. Consensus of first-order multi-agent systems was considered with and without sampling delays, respectively, and some necessary and sufficient conditions were obtained in the case of fixed topology in [36]. Some necessary and sufficient conditions were provided for the second-order multi-agent system under periodic sampling in [37]. Consensus of sampled-data driven by a hybrid event-time for higher-order multi-agent systems was studied, and the robustness of the hybrid driven protocol against event-detection time delays was proved in [38]. For undirected graphs, Gao found that when the period of the zero-order holder is a positive integer multiple of the sampling period, the convergence rate is faster than that in the case when such two periods are the same [39]. In addition, Gao *et al.* considered the consensus problem of multi-agent in directed graphs under synchronous and asynchronous conditions [40]. It was proved that the consensus can be achieved when the update time interval is small enough.

Different from [39], with each agent's update control input being a positive integer multiple of the sampling period, we consider the bipartite consensus of the multi-agent system with intermittent protocol under signed digraph. In this paper, intermittent protocol to achieve bipartite consensus is proposed. And Schur-Cohen stability criterion is introduced to analyze the dynamic characteristics of an equivalent system. We find out the condition range of sampling period T when the system realizes bipartite consensus. Finally, the accuracy of the results is verified through simulation test.

The structure of this paper is as follows: In the second section, we introduce the concepts of graph theory and convergence protocol. In the third section, the main results and proofs are stated. In the fourth section, simulation experiments are presented to illustrate the accuracy of obtained results. In the last section, we summarize the work done in this paper.

The following notations and concept are used in this paper. $1_{r,s,t,\ldots} \in \mathbb{R}^{n\times 1}$ represents a column vector, where the rth, the sth and tth $\ldots$ element is -1, and the rest is 1. 1_n is the n-dimensional column vector where all elements are 1. $\mathcal{I} = \{1, 2, \cdots, n\}$ is an index set. $A \in \mathbb{R}^{m\times n}$ is a $m \times n$ matrix. Denote a complex number $\lambda_i \in \mathbb{C}$, which represents the ith eigenvalue of Laplacian matrix L. Its real part and imaginary part are, respectively, denoted by $Re(\lambda_i)$ and $Im(\lambda_i)$.

II. PRELIMINARIES

A. THEORY OF GRAPH

This paper uses the signed directed graph to represent the network topology formed among agents in a multi-agent system. We denote the signed directed graph with n vertices by $\mathcal{G}=(\mathcal{V}, E, A)$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ is the set of vertices, $E \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix of signed digraph $\mathcal{G}$, where $a_{ij} \neq 0$ for $(v_i, v_j) \in E$ and (v_i, v_j) is a directed edge of the weighted directed graph $\mathcal{G}$ from vertex v_j to vertex v_i. If the connection from v_j to v_i is cooperative, then $a_{ij} > 0$; otherwise, $a_{ij} < 0$ if the connection from v_j to v_i is antagonistic. We assume that $(v_i, v_i) \notin E$ and hence $a_{ii} = 0$. In the following discussion, we refer to $\mathcal{G}(A)$ as the corresponding signed directed graph with the weighted adjacency matrix A.

A directed path $\mathcal{P}$ of $\mathcal{G}(A)$ is a series of interrelated edges in E:

$$\mathcal{P} = \left\{\left(v_{i_1}, v_{i_2}\right), \left(v_{i_2}, v_{i_3}\right), \ldots, \left(v_{i_{p-1}}, v_{i_p}\right)\right\} \subseteq E,$$

where all vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_p}$ are different. The signed digraph $\mathcal{G}(A)$ is strongly connected if any two vertices in the digraph can be connected by a directed path. The directed graph $\mathcal{G}(A)$ is said to contain a directed spanning tree, if there exists a root vertex in the directed graph $\mathcal{G}(A)$ so that there exists a directed path from the root vertex to any other vertex. For a given signed digraph $\mathcal{G}(A)$ with the adjacency matrix A, the Laplacian matrix used in this paper is $L = C - A$, where C is a diagonal matrix, its diagonal elements are $c_{ii} = \sum_{j\in N_i} |a_{ij}|$, and define N_i as the vertex set adjacent to v_i in $\mathcal{V}$. Therefore, the elements of L are

$$L_{ik} = \begin{cases} \sum_{j\in N_i} |a_{ij}|, & k = i \\ -a_{ik}, & k \neq i. \end{cases}$$

Definition 1 (Signum Function): The notation $sgn(x)$ represents the signum function, and it is defined as follows:

$$sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Definition 2 (Structural Balance): Graph $\mathcal{G}(A)$ is structural balanced if and only if the vertex set $\mathcal{V}$ can be divided into two non-empty sets $\mathcal{V}_1$ and $\mathcal{V}_2$, and satisfy the following two conditions:

(1) $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \phi$;
(2) $a_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}_q\ (q \in \{1, 2\})$;
$a_{ij} \leq 0, \forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_r, q \neq r\ (q, r \in \{1, 2\})$.

Otherwise, the structure is unbalanced.

Definitions of degree balance and inter-group balance are given below.

Degree Balance [21]: Considering a multi-agent system $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ containing $N + M$ agents, we partition the node set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $\mathcal{V}_1 = \{v_1, \ldots, v_N\}$ and $\mathcal{V}_2 = \{v_{N+1}, \ldots, v_{N+M}\}$. The two induced subnetworks of $\mathcal{G}$ on $\mathcal{V}_1$ and $\mathcal{V}_2$ will be denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. For fixed communication topology, in graph $\mathcal{G}$, all nodes in one subnetwork share the same in-degree originating from the other subnetwork if $\sum_{j=N+1}^{N+M} a_{ij} = \alpha$ for all $i = 1, \ldots, N$; $\sum_{j=1}^{N} a_{ij} = \beta$ for all $i = N + 1, \ldots, N + M$, where α and β are constants. $\mathcal{G}_1$ is said to be in-degree balanced to $\mathcal{G}_2$ if $\alpha = 0$ for all $i = 1, \ldots, N$; $\mathcal{G}_2$ is said to be in-degree balanced to $\mathcal{G}_1$ if $\beta = 0$ for all $i = N + 1, \ldots, N + M$.

Inter-Group Balance [22]: In the weighted adjacency matrix A of $\mathcal{G}$, $\sum_{j\in\mathcal{G}_{k'}} a_{ij} \equiv 0$ for all $i \in \mathcal{G}_k$ and $k \neq k'$.

Remark 1: This paper is discussed under the condition of structural balance. An interesting research direction is the consensus of multi-agent systems with the intermittent interaction under the condition of degree balance or inter-group balance.

Definition 3 (Gauge Transformation): The gauge transformation is a linear transformation Dx for the state variable x of the multi-agent system by means of the orthogonal matrix D, where $D = diag\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, $\sigma_i \in \{\pm 1\}$ and its specific value is selected according to the need when it is used.

Denote $\mathcal{D} = \{D = diag\{\sigma_1, \sigma_2, \cdots, \sigma_n\}, \sigma_i \in \{\pm 1\}\}$ is the set of all gauge transformation matrices in $\mathbb{R}^{n\times n}$. The elements of DAD are non-negative by selecting the appropriate $D \in \mathcal{D}$, and at the same time, the non-diagonal elements of DLD are non-positive and the sum of rows is zero.

Lemma 1 [6]: For a given matrix $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ satisfying $b_{ii} \geq 0, b_{ij} \leq 0, i \neq j$ and $\sum_{j=1}^{n} b_{ij} = 0$, the matrix B has at least one zero eigenvalue and all the remaining non-zero eigenvalues have positive real parts.

Lemma 2 [27]:

(1) L and DLD are isospectral: $sp(L) = sp(DLD)$, in other words, the eigenvalues are the same.
(2) If a signed digraph $\mathcal{G}(A)$ which has a spanning tree is structurally balanced, then 0 is an eigenvalue of L, and $1_{r,s,t,\ldots} \in \mathbb{R}^{n\times 1}$ is the associated right eigenvectors.

Proof: If a signed digraph is structurally balanced, all of its vertices can be divided into two parts $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$, and $a_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}_q\ (q \in \{1, 2\})$; $a_{ij} \leq 0, \forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_r, q \neq r\ (q, r \in \{1, 2\})$. Firstly, we assume that $\mathcal{V}_2$ only contains one vertex v_r, then $a_{ir} \leq 0, a_{ri} \leq 0, i \in \mathcal{I}$. The corresponding elements of Laplacian matrix L satisfy $l_{ir} \geq 0, l_{ri} \geq 0, L1_r = 0$. Secondly, we assume that $\mathcal{V}_2$ contains two vertices v_r, v_s, then $a_{r,s} \geq 0, a_{s,r} \geq 0$, $L1_{r,s} = 0$. Similarly, $\mathcal{V}_2$ contains more than two vertices, $L1_{r,s,t\ldots} = 0$. So (2) holds.

B. SYSTEM MODEL

Consider a first-order system, which is as follows:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ represents the position of the ith agent at time t, $u_i(t)$ is a control input designed based on information acquired by the ith agent at time t. For the sake of convenience, we only consider the case of $m = 1$.

Next, we consider the control protocol of the system:

1. In the literature [39], each agent can obtain its state information relative to its neighbors at $t = t_0 + kT$, $k = 0, 1, \ldots$, with

$$y_i(t_0 + kT) = \sum_{j\in N_i} a_{ij}(x_j(t_0 + kT) - x_i(t_0 + kT)).$$

This paper, however, considers the bipartite consensus of the first-order multi-agent system under signed digraph. So, in this paper, we give the following hypothesis:

$$y_i(t_0 + kT) = -\sum_{j\in N_i} |a_{ij}|(x_i(t_0 + kT) - sgn(a_{ij})x_j(t_0 + kT)).$$

2. Each agent uses information relative to its neighbors obtained at $[(k-1)m + 1]T, \ldots, kmT$ to update control input u at $t = kmT$, $k = 0, 1, \ldots$, time, where m is a positive integer and $T > 0$. In the following analysis, we assume that $t_0 = 0$ for convenience, therefore, we consider the following control protocol:

$$u_i(t) = a_1 y_i((k-1)mT + T) + \cdots + a_m y_i(kmT) \tag{2}$$

where $a_1, \ldots, a_m$ are the parameters to be designed, and $t \in [kmT, (k+1)mT)$.

Given $u_i(t), i = 1, \ldots, n$, and any initial state $x_i(0)$ with $i, j = 1, \ldots, n$, system (1) asymptotically achieves the bipartite consensus if and only if $\lim_{t\to\infty} (|x_i(t)| - |x_j(t)|) = 0$.

III. CONSENSUS ANALYSIS

Integrating the two sides of system (1) and using the protocol (2), the following formula holds:

$$\begin{cases} x_i(kmT+T) = x_i(kmT) + Tu_i(kmT) \\ x_i(kmT+2T) = x_i(kmT) + 2Tu_i(kmT) \\ \vdots \\ x_i((k+1)mT) = x_i(kmT) + mTu_i(kmT) \end{cases}$$

Let $\xi_i(k) = [x_i((k-1)mT+T), \ldots, x_i(kmT)]^T$, $\xi(k) = \left[\xi_1(k)^T, \ldots, \xi_n(k)^T\right]^T$, we derive

$$\xi_i(k+1) = B\xi_i(k) - T(L_i \otimes C)\xi(k), k = 0, 1, \ldots$$

where

$$B = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} a_1 & \cdots & a_{m-1} & a_m \\ 2a_1 & \cdots & 2a_{m-1} & 2a_m \\ \vdots & \vdots & \vdots & \vdots \\ ma_1 & \cdots & ma_{m-1} & ma_m \end{bmatrix},$$

and L_i is the ith row of Laplacian matrix L, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{m\times m}$. Hence

$$\xi(k+1) = (I_n \otimes B - T(L \otimes C))\,\xi(k), k = 0, 1, \ldots \quad (3)$$

Let D= $diag\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, where $\sigma_r, \sigma_s, \sigma_t, \ldots$ is -1, and the rest is 1, $H = D \otimes I_m$, making gauge transformation $\tilde{\xi} = H\xi$ for ξ, system (3) can be written as follows:

$$\tilde{\xi}(k+1) = (I_n \otimes B - T(DLD \otimes C))\,\tilde{\xi}(k), \quad k = 0, 1, \ldots \quad (4)$$

Remark 2: For a strongly connected digraph $\mathcal{G}(A)$ which is structurally balanced, denote $\hat{A}$=$\frac{A+A^T}{2}$, so that the undirected graph $\mathcal{G}\left(\hat{A}\right)$ is the induced-graph of the digraph $\mathcal{G}(A)$. $\hat{L}$ of an undirected graph $\mathcal{G}\left(\hat{A}\right)$ is diagonalizable, so L of a directed graph $\mathcal{G}(A)$ is also diagonalizable. Hence, for matrix L of strongly connected, structurally balanced signed digraph $\mathcal{G}(A)$, there exists an invertible matrix P, in which the first column is listed as $1_{r,s,t,\ldots}$, such that $P^{-1}LP = diag\{0, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_i = Re(\lambda_i) + Im(\lambda_i)$.

Compared with results in [39], differences lie in the positive and negative elements of the first column of matrix U, in which all the elements of the first column in [39] are 1.

However, for matrix $L_D = DLD$, there exists an invertible matrix U, in which the first column is listed as 1_n, such that $U^{-1}L_DU = diag\{0, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_i = Re(\lambda_i) + Im(\lambda_i)$.

That system (1) asymptotically achieves bipartite consensus means $\lim_{k\to\infty}\left(|\xi_i(k)| - \left|\xi_j(k)\right|\right) = 0$. As can be seen from the gauge transformation, $\lim_{k\to\infty}\left(|\xi_i(k)| - \left|\xi_j(k)\right|\right) = 0$ is equivalent to $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0$. So, the following Lemma 3 holds.

Lemma 3: System (1) asymptotically achieves bipartite consensus under the protocol (2) if and only if consensus is asymptotically achieved for system (4).

Theorem 1: The consensus is asymptotically achieved for system (4) if and only if $B - \lambda_i TC$, $i = 2, \ldots, n$ are Schur stable.

Proof: That system (1) asymptotically achieves bipartite consensus is equivalent to $\lim_{t\to\infty}\left(|x_i(t)| - \left|x_j(t)\right|\right) = 0$. We known from Lemma 3 that $\lim_{t\to\infty}\left(|x_i(t)| - \left|x_j(t)\right|\right) = 0$ holds if and only if $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0$. Therefore, the following aim is to prove that $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0$ holds if and only if $B - \lambda_i TC$, $i = 2, \ldots, n$, are Schur stable.

Sufficiency: Prove that if $B - \lambda_i TC$, $i = 2, \ldots, n$ are Schur stable, then $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0$, $i = 2, \ldots, n$.

Denote $U = [1_n, U_1]$, and

$$\delta(k) = \left[\delta_1(k)^T, \tilde{\delta}(k)^T\right]^T = \begin{bmatrix} \delta_1(k) \\ \tilde{\delta}(k) \end{bmatrix} = \left(U^{-1} \otimes I_m\right)\tilde{\xi}(k), \quad (5)$$

where $\delta_1(k) \in \mathbb{R}^{m\times 1}$, $U_1 \in \mathbb{R}^{n\times(n-1)}$. We obtain from Eq. (5) that

$$\tilde{\xi}(k) = (U \otimes I_m)\,\delta(k) = [(1_n, U_1) \otimes I_m]\begin{bmatrix} \delta_1(k) \\ \tilde{\delta}(k) \end{bmatrix} = 1_n \otimes \delta_1(k) + (U_1 \otimes I_m)\,\tilde{\delta}(k). \quad (6)$$

By Eq. (5), we derive that

$$\delta(k+1) = \begin{bmatrix} \delta_1(k+1) \\ \tilde{\delta}(k+1) \end{bmatrix} = \left(U^{-1} \otimes I_m\right)\tilde{\xi}(k+1). \quad (7)$$

Combining Eq. (4) with Eq. (7), we derive that

$$\begin{aligned} \delta(k+1) &= \left(U^{-1} \otimes I_m\right)(I_n \otimes B - T(L_D \otimes C))\,(U \otimes I_m)\,\delta(k) \\ &= \left(\left(U^{-1} \otimes B\right) - T\left(U^{-1}L_D \otimes C\right)\right)(U \otimes I_m)\,\delta(k) \\ &= \left(I_n \otimes B - T\left(U^{-1}L_DU \otimes C\right)\right)\delta(k). \end{aligned} \quad (8)$$

Noting that

$$U^{-1}L_DU = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

so Eq. (8) is equal to

$$\begin{aligned}\delta(k+1) &= \begin{bmatrix}\delta_1(k+1)\\ \tilde{\delta}(k+1)\end{bmatrix}\\ &= \left(\begin{bmatrix}B & & & \\ & B & & \\ & & \ddots & \\ & & & B\end{bmatrix} - T\begin{bmatrix}0 & & & \\ & \lambda_2 C & & \\ & & \ddots & \\ & & & \lambda_n C\end{bmatrix}\right)\\ &\quad\times\begin{bmatrix}\delta_1(k)\\ \tilde{\delta}(k)\end{bmatrix},\end{aligned}$$

i.e.

$$\begin{cases}\delta_1(k+1) = B\delta_1(k)\\ \tilde{\delta}(k+1) = diag\{B - T\lambda_2 C, \ldots, B - T\lambda_n C\}\,\tilde{\delta}(k).\end{cases}$$

Hence, if $B - \lambda_i TC, i = 2, \ldots, n$ are Schur stable, then $\lim_{k\to\infty}\tilde{\delta}(k) = 0$, and Eq. (6) is equivalent to

$$\tilde{\xi}(k) = 1_n \otimes \delta_1(k),$$

i.e.

$$\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0, \quad i = 2, \ldots, n.$$

Necessary: Prove that if $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0, i = 2, \ldots, n$, then $B - \lambda_i TC, i = 2, \ldots, n$ are Schur stable.

Denote $U^{-1} = \left(u_2, U_2{}^T\right)^T$, where $U_2 \in \mathbb{R}^{(n-1)\times n}$. It follows from $U^{-1}U = I_n$ that

$$\begin{bmatrix}u_2{}^T\\ U_2\end{bmatrix}[1_n, U_1] = \begin{bmatrix}u_2{}^T 1_n & u_2{}^T U_1\\ U_2 1_n & U_2 U_1\end{bmatrix} = I_n,$$

we have

$$u_2{}^T 1_n = 1, \quad U_2 1_n = 0. \tag{9}$$

Since $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \tilde{\xi}_j(k)\right) = 0, i, j = 1, \ldots, n$, there exists $\phi(k) \in \mathbb{R}^{m\times 1}$ such that $\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \phi(k)\right) = 0, i = 1, \ldots, n$.

Combining Eq. (9) with $\delta(k) - \left(U^{-1} \otimes I_m\right)(1_n \otimes \phi(k))$, we have

$$\begin{aligned}&\delta(k) - \left(U^{-1} \otimes I_m\right)(1_n \otimes \phi(k))\\ &= \delta(k) - \left(U^{-1}1_n \otimes \phi_k\right)\\ &= \delta(k) - \begin{bmatrix}u_2{}^T 1_n\\ U_2 1_n\end{bmatrix} \otimes \phi(k)\\ &= \delta(k) - \begin{bmatrix}1\\ 0\end{bmatrix} \otimes \phi(k)\\ &= \begin{bmatrix}\delta_1(k) - \phi(k)\\ \tilde{\delta}(k)\end{bmatrix}.\end{aligned} \tag{10}$$

Combining Eq. (5) with $\delta(k) - \left(U^{-1} \otimes I_m\right)(1_n \otimes \phi(k))$, we have

$$\begin{aligned}&\delta(k) - \left(U^{-1} \otimes I_m\right)(1_n \otimes \phi(k))\\ &= \left(U^{-1} \otimes I_m\right)\left(\tilde{\xi}(k) - 1_n \otimes \phi(k)\right)\\ &= \begin{bmatrix}u_2{}^T \otimes I_m\\ U_2 \otimes I_m\end{bmatrix}\left(\tilde{\xi}(k) - 1_n \otimes \phi(k)\right).\end{aligned} \tag{11}$$

Combining Eq. (10) with Eq. (11), we obtain $\lim_{k\to\infty}\tilde{\delta}(k) = (U_2 \otimes I_m)\lim_{k\to\infty}\left(\tilde{\xi}(k) - 1_n \otimes \phi(k)\right)$. Because

$$\lim_{k\to\infty}\left(\tilde{\xi}_i(k) - \phi(k)\right) = 0, \quad i = 1, \ldots, n$$

so $\lim_{k\to\infty}\tilde{\delta}(k) = (U_2 \otimes I_m)\lim_{k\to\infty}\left(\tilde{\xi}(k) - 1_n \otimes \phi(k)\right) = 0$, which means that system

$$\tilde{\delta}(k+1) = diag\{B - \lambda_2 TC, \ldots, B - \lambda_n TC\}\,\tilde{\delta}(k),$$

$k = 0, 1, \ldots$ is asymptotically stable, namely $B - \lambda_i TC$, $i = 2, \ldots, n$ are all Schur stable.

Therefore, the consensus is asymptotically achieved for system (4) if and only if $B - \lambda_i TC, i = 2, \ldots, n$ are Schur stable.

Remark 3: A matrix is schur stable if all of its eigenvalues lie inside the unit disk centered at the origin.

Remark 4 (Schur-Cohen Stability Criterion): The characteristic equation of linear discrete system is

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

The number of closed-loop characteristic roots in the unit disk is equal to the number of symbolic changes of sequence $\{1, \Delta_1, \Delta_2, \cdots, \Delta_n\}$. The necessary and sufficient condition for the stability of discrete systems is that the sign of sequence $\{1, \Delta_1, \Delta_2, \cdots, \Delta_n\}$ changes n times, which is equivalent to the following formula:

$$\begin{cases}\Delta_j < 0, & j = 1, 3, 5, \cdots\\ \Delta_j > 0, & j = 2, 4, 6, \cdots\end{cases}$$

Define determinant Δ_j as follows:

$$\Delta_j = \begin{vmatrix} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_{n-j+1}\\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_{n-j+2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{j-1} & a_{j-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n\\ \bar{a}_n & 0 & \cdots & 0 & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_{j-1}\\ \bar{a}_{n-1} & \bar{a}_n & \cdots & 0 & 0 & \bar{a}_0 & \cdots & \bar{a}_{j-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \bar{a}_{n-j+1} & \bar{a}_{n-j+2} & \cdots & \bar{a}_n & 0 & 0 & \cdots & \bar{a}_0 \end{vmatrix}_{2j\times 2j}$$

where $\bar{a}$ is the conjugate complex of a.

Theorem 2: Under the assumption of strong connectivity, if signed digraph $\mathcal{G}(A)$ is structurally balanced, the multi-agent system (1) with protocol (2) achieves bipartite consensus asymptotically with

$$\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j > 0,$$

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2Re(\lambda_i)}{|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)}\right) \tag{12}$$

$$D_i = \begin{bmatrix} -(\beta_{i1}+j\beta_{i2})\,a_1 & \cdots & -(\beta_{i1}+j\beta_{i2})\,a_{m-1} & 1-(\beta_{i1}+j\beta_{i2})\,a_m \\ -(\beta_{i1}+j\beta_{i2})\,2a_1 & \cdots & -(\beta_{i1}+j\beta_{i2})\,2a_{m-1} & 1-(\beta_{i1}+j\beta_{i2})\,2a_m \\ \vdots & \vdots & \vdots & \vdots \\ -(\beta_{i1}+j\beta_{i2})\,ma_1 & \cdots & -(\beta_{i1}+j\beta_{i2})\,ma_{m-1} & 1-(\beta_{i1}+j\beta_{i2})\,ma_m \end{bmatrix}$$

$$D_i \rightarrow \begin{bmatrix} -(\beta_{i1}+j\beta_{i2})\,a_1 & \cdots & -(\beta_{i1}+j\beta_{i2})\,a_{m-1} & 1-(\beta_{i1}+j\beta_{i2})\,a_m \\ 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

or

$$\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j > 0,$$

$$0 < T < \min_{i=2,3,\ldots,n}\left(\min_{i=2,3,\ldots,n}\left(\frac{2\left(|\lambda_i|-|Im(\lambda_i)|\right)}{|\lambda_i|\,Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right),\right.$$

$$\left.\times\frac{2}{\max|\lambda_i|\left(\sum_{j=1}^{m} ma_j\right)}\right), \quad (13)$$

where λ_i is the eigenvalue of matrix L which needs to be calculated in advance.

Proof: Let $D_i = B - \lambda_i TC$, $i = 2, \ldots, n$, where $\lambda_i = p_i + jq_i$, $\beta_{i1} = p_i T$, $\beta_{i2} = q_i T$, then $D_i = B - \beta_{i1}C - j\beta_{i2}C$, and matrix D_i can be expanded to the following form, D_i, as shown at the top of this page.

Through elementary row transformations, D_i can be written as the following matrix, D_i, as shown at the top of this page. Therefore, the characteristic polynomial of D_i is

$$s^m + \beta_{i,m-1}s^{m-1} + \beta_{i,m-2}s^{m-2} = 0,$$

where

$$\beta_{i,m-1} = (\beta_{i1}+j\beta_{i2})\,(a_1 + 2a_2 + \cdots + ma_m) - 1,$$
$$\beta_{i,m-2} = (\beta_{i1}+j\beta_{i2})\left[(m-1)\,a_1 + (m-2)\,a_2 + \cdots + a_{m-1}\right].$$

Now, the key point is to find the condition that all the eigenvalues of matrix D_i are in the unit disk centered at the origin, which is the same objective as finding the condition that all the eigenvalues of matrix H_i are in the unit disk, where the characteristic polynomial of H_i is $s^2 + \beta_{i,m-1}s + \beta_{i,m-2} = 0$. By Schur-Cohen stability criterion, $s^2+\beta_{i,m-1}s+\beta_{i,m-2} = 0$ is Schur stable if and only if

$$\Delta_1 = \begin{vmatrix} \beta_{i,m-2} & 1 \\ 1 & \bar{\beta}_{i,m-2} \end{vmatrix} < 0,$$

$$\Delta_2 = \begin{vmatrix} \beta_{i,m-2} & 0 & 1 & \beta_{i,m-1} \\ \beta_{i,m-1} & \beta_{i,m-2} & 0 & 1 \\ 1 & 0 & \bar{\beta}_{i,m-2} & \bar{\beta}_{i,m-1} \\ \bar{\beta}_{i,m-1} & 1 & 0 & \bar{\beta}_{i,m-2} \end{vmatrix} > 0.$$

By $\Delta_1 < 0$, the following formula holds:

$$\left(\beta_{i1}^2 + \beta_{i2}^2\right)\left(\sum_{j=1}^{m}(m-j)\,a_j\right)^2 - 1 < 0.$$

Hence, the effective range of T is

$$0 < T < \frac{1}{\max|\lambda_i|\left|\sum_{j=1}^{m}(m-j)\,a_j\right|} \quad (14)$$

By $\Delta_2 > 0$, $\Delta_2 = \left(c^2+d^2\right)^2 - \left(2+a^2+b^2\right)\left(c^2+d^2\right) + 1 - \left(a^2+b^2\right) + 2c\left(a^2-b^2\right) + 4abd > 0$, where $a = Re\left(\beta_{i,m-1}\right)$, $b = Im\left(\beta_{i,m-1}\right)$, $c = Re\left(\beta_{i,m-2}\right)$, $d = Im\left(\beta_{i,m-2}\right)$.

$$\begin{aligned}
&\left(c^2+d^2\right)^2 \\
&= \left(T^2|\lambda_i|^2\Big(\sum_{j=1}^{m} ma_j - \sum_{j=1}^{m} ja_j\Big)^2\right)^2 \\
&= T^4|\lambda_i|^4\Big(\Big(\sum_{j=1}^{m} ma_j\Big)^2 - 2\sum_{j=1}^{m} ma_j\sum_{j=1}^{m} ja_j + \Big(\sum_{j=1}^{m} ja_j\Big)^2\Big)^2 \\
&= T^4|\lambda_i|^4\Big(\Big(\sum_{j=1}^{m} ma_j\Big)^4 + \Big(\sum_{j=1}^{m} ja_j\Big)^4 \\
&\quad + 6\Big(\sum_{j=1}^{m} ma_j\Big)^2\Big(\sum_{j=1}^{m} ja_j\Big)^2 - 4\Big(\sum_{j=1}^{m} ma_j\Big)^3\sum_{j=1}^{m} ja_j \\
&\quad - 4\Big(\sum_{j=1}^{m} ja_j\Big)^3\sum_{j=1}^{m} ma_j\Big); \\
&\quad - \left(2+a^2+b^2\right)\left(c^2+d^2\right) \\
&= -\Big(T^2|\lambda_i|^2\Big(\sum_{j=1}^{m} ja_j\Big)^2 - 2\beta_{i1}\sum_{j=1}^{m} ja_j + 3\Big) \\
&\quad \times \Big(T^2|\lambda_i|^2\Big(\sum_{j=1}^{m} ma_j - \sum_{j=1}^{m} ja_j\Big)^2\Big)
\end{aligned}$$

$$= -\left(T^2|\lambda_i|^2\left(\sum_{j=1}^{m} ja_j\right)^2 - 2\beta_{i1}\sum_{j=1}^{m} ja_j + 3\right)\left(T^2|\lambda_i|^2\right) \times\left(\left(\sum_{j=1}^{m} ma_j\right)^2 + \left(\sum_{j=1}^{m} ja_j\right)^2 - 2\sum_{j=1}^{m} ma_j\sum_{j=1}^{m} ja_j\right);$$

$$2c\left(a^2-b^2\right) = \left(2\beta_{i1}\sum_{j=1}^{m} ma_j - 2\beta_{i1}\sum_{j=1}^{m} ja_j\right)\left(\beta_{i1}^2\left(\sum_{j=1}^{m} ja_j\right)^2 - \beta_{i2}^2\left(\sum_{j=1}^{m} ja_j\right)^2 - 2\beta_{i1}\sum_{j=1}^{m} ja_j + 1\right)$$
$$= 2\beta_{i1}^3\sum_{j=1}^{m} ma_j\left(\sum_{j=1}^{m} ja_j\right)^2 - 2\beta_{i1}\beta_{i2}^2\sum_{j=1}^{m} ma_j\left(\sum_{j=1}^{m} ja_j\right)^2 - 4\beta_{i1}^2\sum_{j=1}^{m} ja_j\sum_{j=1}^{m} ma_j + 2\beta_{i1}\sum_{j=1}^{m} ma_j - 2\beta_{i1}^3\left(\sum_{j=1}^{m} ja_j\right)^3 + 2\beta_{i1}\beta_{i2}^2\left(\sum_{j=1}^{m} ja_j\right)^3 + 4\beta_{i1}^2\left(\sum_{j=1}^{m} ja_j\right)^2 - 2\beta_{i1}\sum_{j=1}^{m} ja_j;$$

$$4abd = 4\left(\beta_{i1}\sum_{j=1}^{m} ja_j - 1\right)\beta_{i2}^2\sum_{j=1}^{m} ja_j\left(\sum_{j=1}^{m} ma_j - \sum_{j=1}^{m} ja_j\right)$$
$$= 4\beta_{i1}\beta_{i2}^2\left(\sum_{j=1}^{m} ja_j\right)^2\sum_{j=1}^{m} ma_j - 4\beta_{i2}^2\sum_{j=1}^{m} ja_j\sum_{j=1}^{m} ma_j - 4\beta_{i1}\beta_{i2}^2\left(\sum_{j=1}^{m} ja_j\right)^3 + 4\beta_{i2}^2\left(\sum_{j=1}^{m} ja_j\right)^2;$$

$$2c\left(a^2-b^2\right) + 4abd = 2T^2|\lambda_i|^2\beta_{i1}\sum_{j=1}^{m} ma_j\left(\sum_{j=1}^{m} ja_j\right)^2 - 4T^2|\lambda_i|^2\sum_{j=1}^{m} ja_j\sum_{j=1}^{m} ma_j - 2T^2|\lambda_i|^2\beta_{i1}\left(\sum_{j=1}^{m} ja_j\right)^3 + 4T^2|\lambda_i|^2\left(\sum_{j=1}^{m} ja_j\right)^2 + 2\beta_{i1}\sum_{j=1}^{m} ma_j - 2\beta_{i1}\sum_{j=1}^{m} ja_j;$$

$$-\left(a^2+b^2\right) = -\left(\left(\beta_{i1}\sum_{j=1}^{m} ja_j - 1\right)^2 + \left(\beta_{i2}\sum_{j=1}^{m} ja_j\right)^2\right) = -\left(T^2|\lambda_i|^2\left(\sum_{j=1}^{m} ja_j\right)^2 - 2\beta_{i1}\sum_{j=1}^{m} ja_j + 1\right);$$

Then

$$\Delta_2 = T^4|\lambda_i|^4\left(\left(\sum_{j=1}^{m} ma_j\right)^4 + 5\left(\sum_{j=1}^{m} ma_j\right)^2\left(\sum_{j=1}^{m} ja_j\right)^2 - 4\left(\sum_{j=1}^{m} ma_j\right)^3\left(\sum_{j=1}^{m} ja_j\right) - 2\left(\sum_{j=1}^{m} ma_j\right)\times\left(\sum_{j=1}^{m} ja_j\right)^3\right) + T^2|\lambda_i|^2\left(2\beta_{i1}\left(\sum_{j=1}^{m} ma_j\right)^2 \times\left(\sum_{j=1}^{m} ja_j\right) - 3\left(\sum_{j=1}^{m} ma_j\right)^2 - 2\beta_{i1}\left(\sum_{j=1}^{m} ma_j\right) \times\left(\sum_{j=1}^{m} ja_j\right)^2 + 2\left(\sum_{j=1}^{m} ma_j\right)\left(\sum_{j=1}^{m} ja_j\right)\right) + 2\beta_{i1}\left(\sum_{j=1}^{m} ma_j\right)$$
$$= T^4|\lambda_i|^4\left(\left(\sum_{j=1}^{m} ma_j\right)^2 - 2\left(\sum_{j=1}^{m} ma_j\right)\left(\sum_{j=1}^{m} ja_j\right)\right) \times\left(\sum_{j=1}^{m}\left(m-j\right)a_j\right)^2 + 2T^2|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right) \times\left(\beta_{i1}\left(\sum_{j=1}^{m} ja_j\right) - 1\right)\left(\sum_{j=1}^{m}\left(m-j\right)a_j\right) - T^2|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)^2 + 2\beta_{i1}\left(\sum_{j=1}^{m} ma_j\right);$$

Simplifying the above formula, we obtain

$$\Delta_2 = |\lambda_i|^4\left(\left(\sum_{j=1}^{m} ma_j\right)^2 - 2\left(\sum_{j=1}^{m} ma_j\right)\left(\sum_{j=1}^{m} ja_j\right)\right) \times\left(\sum_{j=1}^{m}\left(m-j\right)a_j\right)^2 T^4 + 2|\lambda_i|^2 Re\left(\lambda_i\right)\left(\sum_{j=1}^{m} ma_j\right) \times\left(\sum_{j=1}^{m} ja_j\right)\left(\sum_{j=1}^{m}\left(m-j\right)a_j\right)T^3 - |\lambda_i|^2\left(3\left(\sum_{j=1}^{m} ma_j\right)^2 - 2\left(\sum_{j=1}^{m} ma_j\right)\left(\sum_{j=1}^{m} ja_j\right)\right)T^2 + 2Re\left(\lambda_i\right)\left(\sum_{j=1}^{m} ma_j\right)T;$$

For the convenience of calculation, we consider two cases: $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j$ and $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j$.

(a) if $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j$, then

$$\Delta_2 = -|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)^2 T^2 + 2Re\left(\lambda_i\right)\left(\sum_{j=1}^{m} ma_j\right)T.$$

Since $4Re(\lambda)^2\left(\sum_{j=1}^{m} ma_j\right)^2 > 0$, let $\Delta_2 = 0$, then $T_1 = 0$, $T_2 = \frac{2Re(\lambda_i)}{|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)}$ holds. We consider the following two cases:

(i) if $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j > 0$, $\Delta_1 < 0$ holds, then by $\Delta_2 > 0$, the range of T is

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2Re\left(\lambda_i\right)}{|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

which means that (12) holds.

(ii) if $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j < 0$, $\Delta_1 < 0$ holds, then by $\Delta_2 > 0$, the range of T is

$$\max_{i=2,3,\ldots,n}\left(\frac{2Re(\lambda_i)}{|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)}\right) < T < 0.$$

In this case, the conditions are not satisfied.

(b) if $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j$, then

$$\Delta_2 = \frac{1}{2}|\lambda_i|^2 Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)^3 T^3 - 2|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)^2 T^2 + 2Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)T$$

is a cubic equation of one variable. Let $\Delta_2 = 0$, then $T_1 = 0$, $T_{2,3} = \frac{2(|\lambda_i| \pm |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}$. We consider the following two cases:

(i) if $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j > 0$, then by $\Delta_2 > 0$, the range of T is

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| - |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

$$T > \max_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| + |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right)$$

Comparing T with that in formula (14) and taking the common part of T, we get

$$0 < T < \min_{i=2,3,\ldots,n} \times\left(\min_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| - |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right), \times \frac{2}{\max|\lambda_i|\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

hence (13) holds.

(ii) if $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j < 0$, then by $\Delta_2 > 0$, the range of T is

$$T < \min_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| + |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

$$\max_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| - |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right) < T < 0.$$

In this case, the conditions are not satisfied.

By Lemma 3 and Schur-Cohen stability criterion, system (1) asymptotically achieves bipartite consensus under the protocol (2) if and only if (12) and (13) hold.

Now our task is to design $a_1, \ldots a_m$ to ensure $\sum_{j=1}^{m} a_j > 0$, and select T to satisfy (12) and (13). In the following analysis, we assume that $a_i > 0$.

Theorem 3: Under the assumption of strong connectivity, if signed digraph $G(A)$ is structurally balanced, the multi-agent system (1) with protocol (2) can achieve bipartite consensus. Moreover, if $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j > 0$, $m = 1$, $a_1 > 0$, we take the range of sampling period as

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2Re(\lambda_i)}{|\lambda_i|^2 a_1}\right),$$

or

if $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j > 0$, $m \geq 3$, $a_j > 0$ $(j = 1, 2, \ldots m)$, we take the range of sampling period as

$$0 < T < \min_{i=2,3,\ldots,n}\left(\min_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| - |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right), \frac{2}{\max|\lambda_i|\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

then the bipartite consensus can be achieved.

Proof: In the case of $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j > 0$, this is equivalent to $m(a_1 + a_2 + \ldots + a_m) = a_1 + 2a_2 + \ldots + ma_m > 0$, i.e. $(m-1)a_1 + (m-2)a_2 + \ldots + (m-(m-1))a_{m-1} = 0$. Since $a_i > 0$, only $m = 1$ satisfies the situation. At this point,

$$\Delta_2 = -|\lambda_i|^2 {a_1}^2 T^2 + 2Re(\lambda) a_1 T.$$

The range of T from $\Delta_2 > 0$ is

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2Re(\lambda_i)}{|\lambda_i|^2 a_1}\right).$$

In the case of $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j > 0$, this is equivalent to $m(a_1 + a_2 + \ldots + a_m) = 2a_1 + 4a_2 + \ldots + 2ma_m > 0$, i.e. $(m-2)a_1 + (m-4)a_2 + \ldots + (m-2m)a_m = 0$, where $(m-2) > (m-4) > \ldots > (m-2m)$. Because $a_j > 0\,(j = 1, 2, \ldots m)$, so $(m-2) > 0$, $m \geq 3$. At this point,

$$\Delta_2 = \frac{1}{2}|\lambda_i|^2 Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)^3 T^3 - 2|\lambda_i|^2\left(\sum_{j=1}^{m} ma_j\right)^2 \times T^2 + 2Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)T.$$

By $\Delta_2 > 0$, the range of T is

$$0 < T < \min_{i=2,3,\ldots,n}\left(\frac{2(|\lambda_i| - |Im(\lambda_i)|)}{|\lambda_i| Re(\lambda_i)\left(\sum_{j=1}^{m} ma_j\right)}\right),$$

$$T > \max_{i=2,3,\ldots,n}\left(\frac{2\left(|\lambda_i| + |Im\left(\lambda_i\right)|\right)}{|\lambda_i|\, Re\left(\lambda_i\right)\left(\sum_{j=1}^{m} ma_j\right)}\right).$$

Comparing with T in formula (14) and taking the common part of T to get

$$0 < T < \min_{i=2,3,\ldots,n}\left(\min_{i=2,3,\ldots,n}\left(\frac{2\left(|\lambda_i| - |Im(\lambda_i)|\right)}{|\lambda_i|\, Re\left(\lambda_i\right)\left(\sum_{j=1}^{m} ma_j\right)}\right), \frac{2}{\max|\lambda_i|\left(\sum_{j=1}^{m} ma_j\right)}\right).$$

Theorem 15 shows that system (1) asymptotically achieves bipartite consensus if $\sum_{j=1}^{m} ma_j = \sum_{j=1}^{m} ja_j > 0$ and $m = 1$ or $\sum_{j=1}^{m} ma_j = 2\sum_{j=1}^{m} ja_j > 0$ and $m \geq 3$.

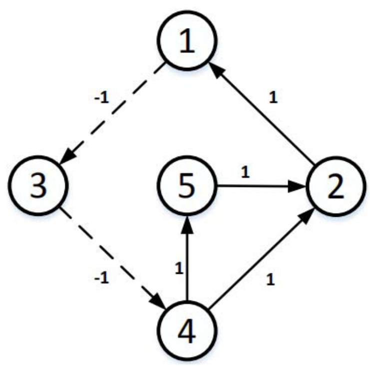

FIGURE 1. Communication topology.

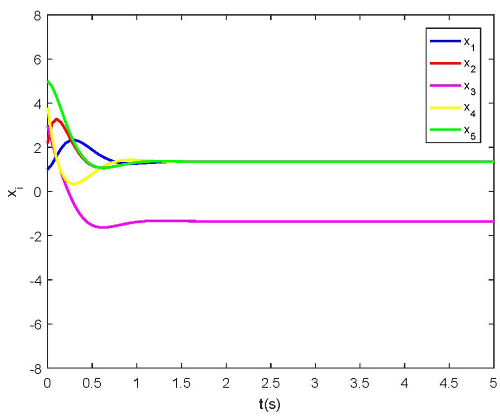

FIGURE 2. The positions of each agents of (1) using (2) when $T = 0.05, m = 1$.

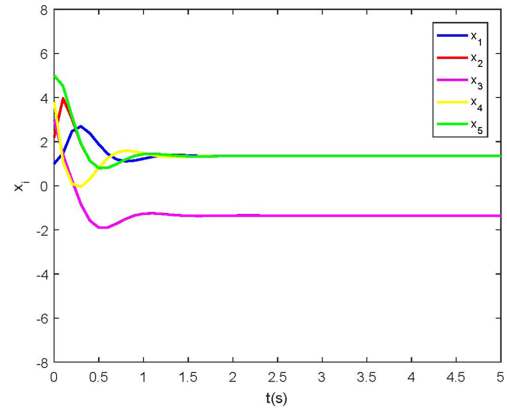

FIGURE 3. The positions of each agents of (1) using (2) when $T = 0.1, m = 1$.

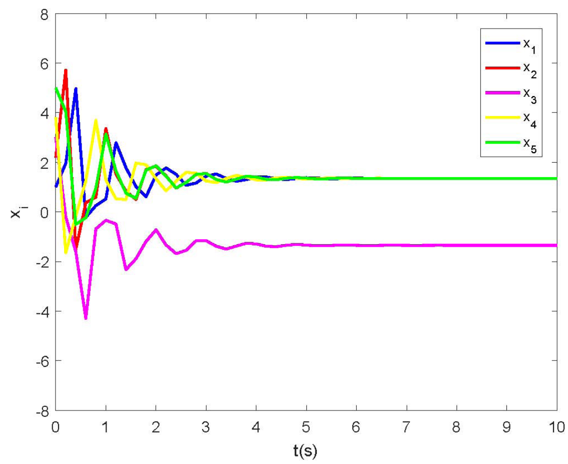

FIGURE 4. The positions of each agents of (1) using (2) when $T = 0.2, m = 1$.

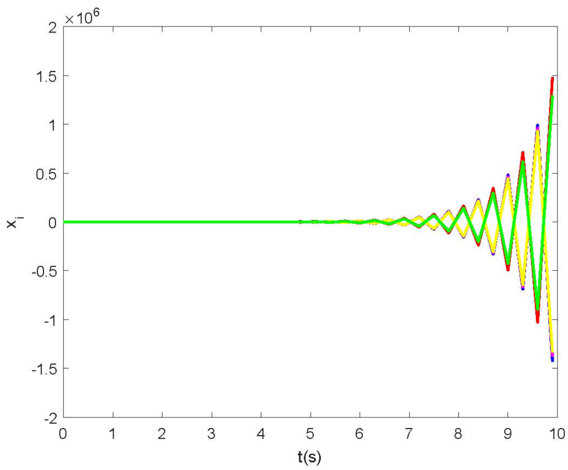

FIGURE 5. The positions of each agents of (1) using (2) when $T = 0.3, m = 1$.

IV. SIMULATIONS

Let us consider five agents. The interaction topology is shown in Figure 1 below. The cooperative relationship among agents is represented by solid lines between vertices, and the corresponding weight is 1. The competition relationship among agents is represented by dotted lines between vertices, and the corresponding weight is -1. Let $a_1 = 4, a_2 = 1, a_3 = 1$, initial states $x(0) = \begin{bmatrix} 1 & 2.2 & 3 & 3.8 & 5 \end{bmatrix}^T$.

From Theorem 3, system (1) asymptotically achieves bipartite consensus if $m = 1, T < 0.25$ or $m = 3$,

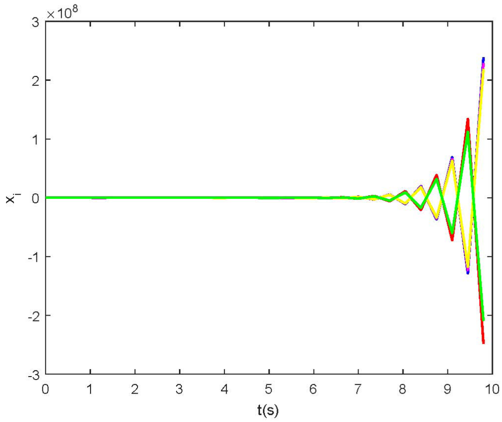

FIGURE 6. The positions of each agents of (1) using (2) when $T = 0.35, m = 1$.

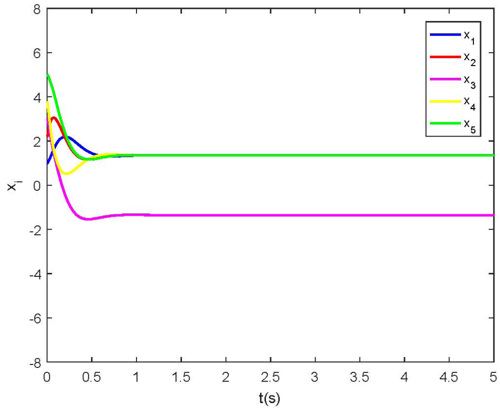

FIGURE 7. The positions of each agents of (1) using (2) when $T = 0.005, m = 3$.

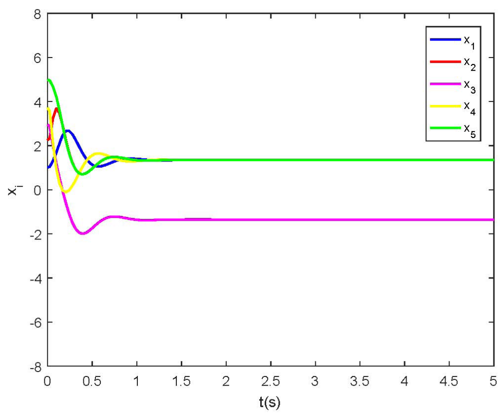

FIGURE 8. The positions of each agents of (1) using (2) when $T = 0.01, m = 3$.

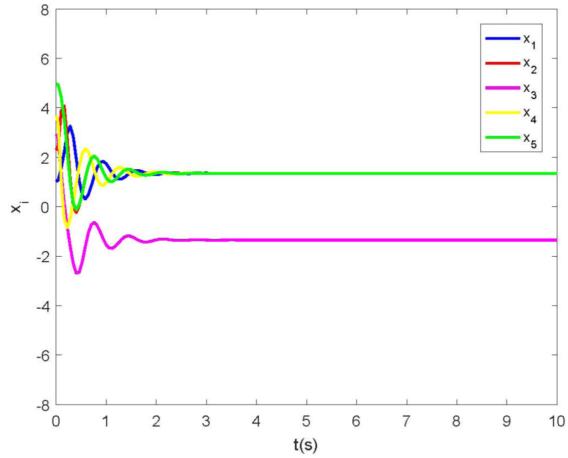

FIGURE 9. The positions of each agents of (1) using (2) when $T = 0.02, m = 3$.

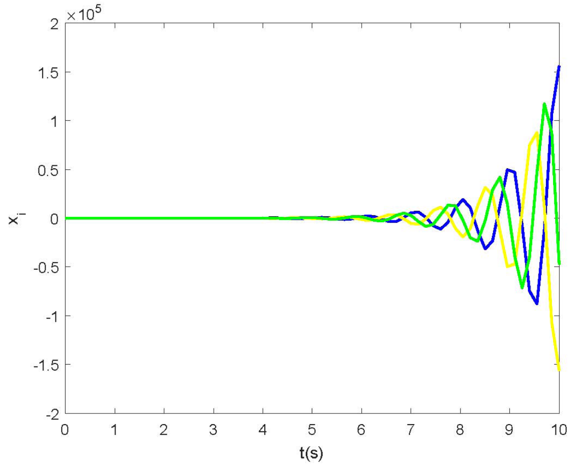

FIGURE 10. The positions of each agents of (1) using (2) when $T = 0.04, m = 3$.

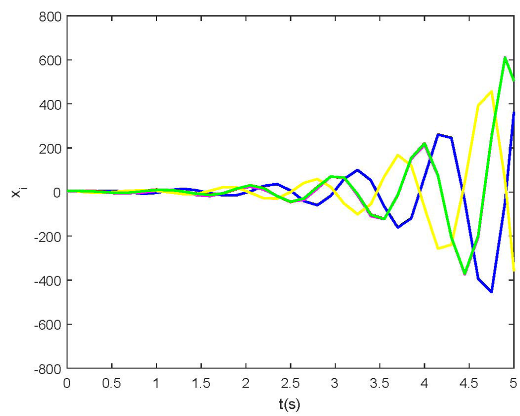

FIGURE 11. The positions of each agents of (1) using (2) when $T = 0.05, m = 3$.

$T < 0.0325$. Figure 2 to Figure 6 show, respectively, the state trajectories of the five agents when T varies from 0.05 to 0.35 with $m = 1$. Obviously, the multi-agent system can achieve bipartite consensus in the case of $T < 0.25$, while in the case of $T > 0.25$, the system cannot achieve bipartite consensus. Figures 7 to Figure 11 show, respectively, the state trajectories of the five agents when T varies from 0.005 to 0.05 with $m = 3$. In the case of $T < 0.0325$, the multi-agent system can achieve bipartite consensus, while in the case of $T > 0.0325$, the system cannot achieve bipartite consensus, which validates the accuracy of Theorem 3.

V. CONCLUSION

In this paper, the bipartite consensus problem of the first-order multi-agent system with intermittent interaction under the signed digraph in continuous-time has been studied. It is assumed that each agent can only obtain states information related to its neighbor at the sampling time, and the sampling period is different from the period of updating control input. Analysing the characteristic polynomials of the equivalent system by Schur-Cohen stability criterion and matrix theory, the sufficient conditions for the bipartite consensus of the system are obtained. The correctness of the conclusion is verified by simulation experiments. It is shown that when the period of the agent's update control input is a positive integer multiple of the sampling period, the smaller the sampling period is, the faster the convergence rate is.

In addition to solving the necessary and sufficient conditions for multi-agent systems to achieve bipartite consensus under signed directed graphs, the research direction in the future includes considering the consensus of multi-agent system with intermittent protocol under degree balance, inter-group balance, structural unbalance or designing new consensus protocol with time-varying delays.

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