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Constructing Higher Nonlinear Odd-Variable RSBFs With Optimal AI and Almost Optimal FAI

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ABSTRACT Rotation symmetric Boolean functions (RSBFs) are nowadays studied a lot because of its easy operations and good performance in cryptosystem. This paper constructs a new class of odd-variable RSBFs with optimal algebraic immunity (AI). The nonlinearity of the new function, $2^{n-1}-\binom{n-1}{k}+2^{k-4}(k-3)(k-2)$, is the highest among all existing RSBFs with optimal AI and known nonlinearity, and its algebraic degree is also almost highest. Besides, the class of functions have almost optimal fast algebraic immunity (FAI) at least for $n < 17$, which is actually the highest possible value for the designated number of variables.

INDEX TERMS Rotation symmetric Boolean function, algebraic immunity, nonlinearity, algebraic degree, fast algebraic immunity.

I. INTRODUCTION

Boolean functions play an important role in cryptosystems of stream ciphers. They are required to satisfy kinds of cryptographic properties in order to resist many attacks. In 2003, the algebraic attack was proposed by Courtois and Meier in [1]. Then algebraic immunity (AI), a new cryptographic property, was introduced [2], [3]. Boolean functions should have high AI to resist algebraic attacks. The algebraic immunity of an n-variable Boolean function f can at highest be $\mathrm{AI}(f) = \lceil \frac{n}{2} \rceil$ [3], in which case we say that the function have optimal AI. Since a tiny difference of AI may change the resistance much, functions with optimal AI have been chased, and efforts are paid to find them and their properties [5]–[16]. Later, Courtois introduced fast algebraic attacks [4]. The fast algebraic attack is possible if a nonzero function g exists such that $\deg(g)$ and $\deg(g{\cdot}f)$ are low enough. In 2011, another new cryptographic property called fast algebraic immunity (FAI) was introduced in [17], which works as a measurement of the ability of Boolean functions to resist fast algebraic attacks. Some effort have been paid to study about FAI [18]–[20], but they are still under too much limitation.

Rotation symmetric Boolean functions (RSBFs) don't change under the action of cyclic group, and lots of them have optimal AI. Up to now, lots of functions with good properties, including optimal algebraic immunity, are RSBFs [21]–[37]. In 2007, Sarkar and Maitra [22] firstly constructed a class of odd-variable RSBFs with optimal AI and nonlinearity $2^{n-1} - \binom{n-1}{\frac{n-1}{2}} + 2$. In 2009, 2011 and 2013, [24], [26], [27] presented constructions of even-variable RSBFs with optimal AI, with their nonlinearity polynomial higher than $2^{n-1} - \binom{n-1}{\frac{n-1}{2}}$. In 2014, Su and Tang [31] made use of integer partition to present new kinds of RSBFs with optimal AI and first

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exponentially higher nonlinearity, $2^{n-1}-\binom{n-1}{k}-2+2^k$ ($n=2k+1\geq 11$) and $2^{n-1}-\binom{n-1}{k}-2+3\cdot 2^{k-2}$ ($n=2k\geq 10$), both of which are later improved in [32]–[34]. However, the constructions of [31]–[33] totally ignore the fast algebraic attack. In 2017, Sun and Fu [35] presented two classes of even-variable RSBFs with optimal AI, high nonlinearity and high fast algebraic immunity. In 2019, Chen *et al.* [36] presented a class of odd-variable RSBFs with optimal AI and higher nonlinearity. These two classes of functions have almost optimal immunity for $n=11,13$ and $n=11,13,15$, respectively. In the same year, Zhang and Su [37] constructed another type of RSBFs, whose AI is optimal and nonlinearity equals to $2^{n-1}-\binom{n-1}{k}+(k-5)2^{k-1}+2k+2$ ($n=2k+1\geq 11$).

We here construct a new type of odd-variable RSBFs with the following properties: i) They are balanced with optimal AI. ii) Nonlinearity is the highest among all odd-variable RSBFs with exact known nonlinearity. iii) Algebraic degree reaches optimal upper bound of balanced Boolean functions, i.e., $n-1$. iv) The fast algebraic immunity reaches the highest possible value for the number of variables $n<17$.

In this paper, Section II provides some basic definitions and propositions, Section III constructs the odd-variable RSBFs and shows that the functions behave well on some aspects, and Section IV concludes this paper.

II. PRELIMINARIES

Let $\mathbb{F}_2^n$ be the n-dimensional vector space over the finite field $\mathbb{F}_2=\{0,1\}$. A Boolean function is a mapping from $\mathbb{F}_2^n$ to $\mathbb{F}_2$. We'll later use B_n to represent the set of the 2^{2^n} possible n-variable Boolean functions.

For a vector $x=(x_1,x_2,\cdots,x_n)\in\mathbb{F}_2^n$, supp$(x)$ is defined as $\{i|x_i=1, 1\leq i\leq n\}$, and wt$(x)$ is $|\text{supp}(x)|$. For any two vectors $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $u=(u_1,u_2,\cdots,u_n)$ in $\mathbb{F}_2^n$, we define $\alpha\preceq u$, if $\alpha_i\leq u_i$ for all $1\leq i\leq n$.

A quite usually used way to represent a Boolean function $f(x_1,x_2,\cdots,x_n)$ is the algebraic normal form (ANF), that is to say,

$$f(x_1,x_2,\cdots,x_n)=\bigoplus_{\alpha\in\mathbb{F}_2^n} c(\alpha)x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \quad (1)$$

where $c(\alpha)\in\mathbb{F}_2$ and "$\oplus$" means the addition on $\mathbb{F}_2$. By the Möbius transform,

$$c(\alpha)=\bigoplus_{x\in\mathbb{F}_2^n, x\preceq\alpha} f(x). \quad (2)$$

Definition 1: The algebraic degree of function f expressed in format (1) is defined as

$$\deg(f)=\max\{\text{wt}(\alpha)|\alpha\in\mathbb{F}_2^n, c(\alpha)=1\}.$$

A_n represents the set containing all n-variable functions whose algebraic degree is at most one.

supp(f) is denoted by $\{x|f(x)=1, x\in\mathbb{F}_2^n\}$, and wt$(f)=|\text{supp}(f)|$. A Boolean function $f\in B_n$ is balanced if wt$(f)=$ wt$(f\oplus 1)$, or equally, its Hamming weight is 2^{n-1}.

Definition 2: The Walsh spectrum of a Boolean function $f\in B_n$ is defined as

$$W_f(\omega)=\sum_{x\in\mathbb{F}_2^n}(-1)^{f(x)\oplus x\cdot\omega}, \quad \omega\in\mathbb{F}_2^n$$

where $(x_1,x_2,\cdots,x_n)\cdot(\omega_1,\omega_2,\cdots,\omega_n)=x_1\omega_1\oplus x_2\omega_2\oplus\cdots\oplus x_n\omega_n$.

Definition 3: The nonlinearity (NL) of a Boolean function f with n variables is defined as

$$\text{NL}(f)=\min_{g\in A_n}\text{wt}(f\oplus g).$$

or equally,

$$\text{NL}(f)=2^{n-1}-\frac{1}{2}\max_{\omega\in\mathbb{F}_2^n}|\text{W}_f(\omega)|. \quad (3)$$

Definition 4 [3]: The algebraic immunity of a Boolean function f, denoted by AI(f), *is*

$$\text{AI}(f)=\min\{\deg(g)|0\neq g\in B_n, f\cdot g=0 \text{ or } (f\oplus 1)\cdot g=0\}.$$

Functions without high AI will be easily attacked. But even for ones which having high AI, fast algebraic attacks is still possible if someone can find two nonzero functions g and h with low algebraic degree such that $f\cdot g=h$ [4], [38]. Fast algebraic immunity was later introduced as a measurement of the resistance of Boolean functions against fast algebraic attacks.

Definition 5 [17]: The fast algebraic immunity of a Boolean function f, denoted by FAI(f) *is defined as*

$$\text{FAI}(f)=\min\Big\{2\,\text{AI}(f), \min\{\deg(g)+\deg(f\cdot g)| 1\leq\deg(g)<\text{AI}(f)\}\Big\}.$$

We say that f has *almost optimal* fast algebraic immunity if FAI(f) is $n-1$. We won't get a balanced function with its FAI higher than n, and can only have n reached if $n=2^m+1$ for some integer m [39].

A simple function,

$$F(x)=\begin{cases}1, & \text{if wt}(x)\geq\lceil\frac{n}{2}\rceil;\\ 0, & \text{else,}\end{cases}$$

called the majority function [40], is showed to achieve the optimal AI in [6] by Dalai et al. Yet, NL$(F(x))$ is $2^{n-1}-\binom{n-1}{\lfloor\frac{n}{2}\rfloor}$, which is exactly Lobanov's lowerbound [41].

Proposition 1 [6], [22]: Let $F(x)$ be the n-variable majority function with $n=2k+1$. For $\omega\in\mathbb{F}_2^n$,

i) If wt$(\omega)=1$, *then* $\text{W}_F(\omega)=2\binom{n-1}{k}$;
ii) If wt$(\omega)=n$, *then* $\text{W}_F(\omega)=2(-1)^k\binom{n-1}{k}$;
iii) Otherwise, $|\text{W}_F(\omega)|\leq 2\left(\binom{n-3}{k-1}-\binom{n-3}{k}\right)$ *for* $n\geq 7$.

Definition 6 [23]: Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{F}_2^n$, then for any x and $0\leq h<n$, $\rho_n^h(x)$ is defined as

$$\rho_n^h(x_1,x_2,\cdots,x_n)=(x_{h+1},x_{h+2},\cdots,x_n,x_1,x_2,\cdots,x_h).$$

Definition 7 [23]: A Boolean function $f \in B_n$ satisfying that $f\left(\rho_n^k(x)\right) = f(x)$ holds for all $x \in \mathbb{F}_2^n$ and $0 \le k < n$ is called rotation symmetric Boolean function (RSBF).

Since $f\left(\rho_n^k(x)\right)$ and $f(x)$ are always equal, we can separate $\mathbb{F}_2^n$ into several orbits such that, x and y are in same orbit if there exists some k, $y = \rho_n^k(x)$.

III. CONSTRUCTION OF ODD-VARIABLE RSBFS

In this paper, we'll assume that $n = 2k+1 \ge 11$, construct a kind of balanced RSBF on n variables, and prove that $f(x)$ has optimal AI, high nonlinearity, almost optimal algebraic degree, then check that it has almost optimal FAI.

A. CONSTRUCTION

For convenience, we denote by $W_{\le i} = \{\alpha \in \mathbb{F}_2^n | \operatorname{wt}(\alpha) \le i\}$ and $W_i = \{\alpha \in \mathbb{F}_2^n | \operatorname{wt}(\alpha) = i\}$. For $1 \le i \le k-2$ and $2 \le j \le i$, we define:

$$T_{i,j} = \Big\{(1,1,\underbrace{1,\cdots,1}_{i-j},0,\underbrace{1,\cdots,1}_{j-1},1,1,\underbrace{0,\cdots,0}_{k_1},1,\underbrace{0,\cdots,0}_{k_2},1,\cdots,\underbrace{0,\cdots,0}_{k_{k-i-1}}) \in W_{k+1}\Big| k_1,k_2,\cdots,k_{k-i-1} \ge 1\Big\}.$$

It is quite obvious that $k_1 + k_2 + \cdots + k_{k-i-1} = k-1$, as by definition $\operatorname{wt}(\alpha) = k+1$ for all $\alpha \in T_{i,j}$. Therefore,

$$T = \bigcup_{i=2}^{k-2}\bigcup_{j=2}^{i} T_{i,j} \subseteq W_{k+1}.$$

We write the vectors defined in T as

$$\begin{aligned}T = \Big\{&\alpha_{2,2,1},\alpha_{2,2,2},\cdots,\alpha_{2,2,|T_{2,2}|},\\ &\alpha_{3,2,1},\alpha_{3,2,2},\cdots,\alpha_{3,2,|T_{3,2}|},\\ &\alpha_{3,3,1},\alpha_{3,3,2},\cdots,\alpha_{3,3,|T_{3,3}|},\\ &\cdots,\\ &\alpha_{k-2,2,1},\alpha_{k-2,2,2},\cdots,\alpha_{k-2,2,|T_{k-2,2}|},\\ &\alpha_{k-2,3,1},\alpha_{k-2,3,2},\cdots,\alpha_{k-2,3,|T_{k-2,3}|},\\ &\cdots,\\ &\alpha_{k-2,k-2,1},\alpha_{k-2,k-2,2},\cdots,\alpha_{k-2,k-2,|T_{k-2,k-2}|}\Big\},\end{aligned}$$

where $\alpha_{i,j,s}$ means the sth smallest vector according to the lexicographic order in $T_{i,j}$.

We can find $u_{i,j,s}$ for each $\alpha_{i,j,s}$ as

$$\begin{aligned}U_{i,j} = \Big\{&u_{i,j,s} = \alpha_{i,j,s} \oplus (\underbrace{0,\cdots,0}_{i-j+3},\underbrace{1,\cdots,1}_{j-1},\underbrace{0,\cdots,0}_{2k-1-i})\Big|\alpha_{i,j,s} \in T_{i,j}\Big\}\\ &\subseteq W_{k-j+2}\end{aligned}$$

and similarly

$$\begin{aligned}U = \bigcup_{i=2}^{k-2}\bigcup_{j=2}^{i} U_{i,j} = \Big\{&u_{2,2,1},u_{2,2,2},\cdots,u_{2,2,|U_{2,1}|},\\ &u_{3,2,1},u_{3,2,2},\cdots,u_{3,2,|U_{3,2}|},\\ &u_{3,3,1},u_{3,3,2},\cdots,u_{3,3,|U_{3,3}|},\\ &\cdots,\\ &u_{k-2,2,1},u_{k-2,2,2},\cdots,u_{k-2,2,|T_{k-2,2}|},\\ &u_{k-2,3,1},u_{k-2,3,2},\cdots,u_{k-2,3,|T_{k-2,3}|},\\ &\cdots,\\ &u_{k-2,k-2,1},u_{k-2,k-2,2},\cdots,u_{k-2,k-2,|T_{k-2,k-2}|}\Big\},\end{aligned}$$

which is a subset of $W_{\le k}$. It's a direct result that $\left|T_{i,j}\right| = \left|U_{i,j}\right|$ for every possible i and j, and $|T| = |U|$.

Example 1: For $k = 5$, i.e., $n = 11$, we have

$$\begin{aligned}T_{2,2} &= \{(1,1,0,1,1,1,0,0,0,1,0),\\ &\quad (1,1,0,1,1,1,0,0,1,0,0),\\ &\quad (1,1,0,1,1,1,0,1,0,0,0)\},\\ T_{3,2} &= \{(1,1,1,0,1,1,1,0,0,0,0)\},\\ T_{3,3} &= \{(1,1,0,1,1,1,1,0,0,0,0)\},\end{aligned}$$

and

$$\begin{aligned}U_{2,2} &= \{(1,1,0,0,1,1,0,0,0,1,0),\\ &\quad (1,1,0,0,1,1,0,0,1,0,0),\\ &\quad (1,1,0,0,1,1,0,1,0,0,0)\},\\ U_{3,2} &= \{(1,1,1,0,0,1,1,0,0,0,0)\},\\ U_{3,3} &= \{(1,1,0,0,0,1,1,0,0,0,0)\}.\end{aligned}$$

Example 2: For $k = 6$, i.e., $n = 13$, we have

$$\begin{aligned}T_{2,2} &= \{(1,1,0,1,1,1,0,0,0,1,0,1,0),\\ &\quad (1,1,0,1,1,1,0,0,1,0,0,1,0),\\ &\quad (1,1,0,1,1,1,0,0,1,0,1,0,0),\\ &\quad (1,1,0,1,1,1,0,1,0,0,0,1,0),\\ &\quad (1,1,0,1,1,1,0,1,0,0,1,0,0),\\ &\quad (1,1,0,1,1,1,0,1,0,1,0,0,0)\},\\ T_{3,2} &= \{(1,1,1,0,1,1,1,0,0,0,0,1,0),\\ &\quad (1,1,1,0,1,1,1,0,0,0,1,0,0),\\ &\quad (1,1,1,0,1,1,1,0,0,1,0,0,0),\\ &\quad (1,1,1,0,1,1,1,0,1,0,0,0,0)\},\\ T_{3,3} &= \{(1,1,0,1,1,1,1,0,0,0,0,1,0),\\ &\quad (1,1,0,1,1,1,1,0,0,0,1,0,0),\\ &\quad (1,1,0,1,1,1,1,0,0,1,0,0,0),\\ &\quad (1,1,0,1,1,1,1,0,1,0,0,0,0)\},\\ T_{4,2} &= \{(1,1,1,1,0,1,1,1,0,0,0,0,0)\},\\ T_{4,3} &= \{(1,1,1,0,1,1,1,1,0,0,0,0,0)\},\\ T_{4,4} &= \{(1,1,0,1,1,1,1,1,0,0,0,0,0)\}.\end{aligned}$$

We can see that, when k increases from 5 to 6, the amount of vectors is more than tripled.

Now, define

$$\tilde{P} = \left\{\rho_n^l(x)\middle|x \in P, 0 \le l < n\right\}$$

for P being any subset of $\mathbb{F}_2^n$, and we can have:

Construction 1: Let $F(x)$ as the majority function, then

$$f(x) = \begin{cases} F(x) \oplus 1, & x \in \tilde{T} \cup \tilde{U}; \\ F(x), & \text{otherwise.} \end{cases} \quad (4)$$

Obviously f is a balanced RSBF.

B. ALGEBRAIC IMMUNITY

Define "$(x_1, x_2, x_3, \cdots, x_h) < (y_1, y_2, y_3, \cdots, y_h)$" as "$x_1 < y_1$ or $x_1 = y_1$ and $(x_2, x_3, \cdots, x_h) < (y_2, y_3, \cdots, y_h)$", and that $() \not< ()$, then we have:

Lemma 1: For $\alpha_{i,j,s} \in T$, $u_{i,j,s}, u_{i',j',s'} \in U$, the following properties hold.

i) $\rho_n^l(u_{i,j,s}) \preceq \rho_n^l(\alpha_{i,j,s})$, *for* $0 \le l < n$.
ii) $\rho_n^l(\alpha_{i,j,s}) \neq \alpha_{i,j,s}$, $\rho_n^l(u_{i,j,s}) \neq u_{i,j,s}$, *for* $1 \le l < n$.
iii) $u_{i,j,s} \npreceq \rho_n^l(\alpha_{i,j,s})$, *for* $1 \le l < n$.
iv) $u_{i',j',s'} \npreceq \rho_n^l(\alpha_{i,j,s})$, *for* $(i',j',s') < (i,j,s)$ *and* $0 \le l < n$.

Proof: For convenience, we write

$$\alpha_{i,j,s} = (1, 1, \underbrace{1, \cdots, 1}_{i-j}, 0, \underbrace{1, \cdots, 1}_{j-1}, 1, 1, \underbrace{0, \cdots, 0}_{k_1}, 1, \underbrace{0, \cdots, 0}_{k_2}, 1, \cdots, \underbrace{0, \cdots, 0}_{k_{k-i-1}})$$

and

$$u_{i,j,s} = (1, 1, \underbrace{1, \cdots, 1}_{i-j}, 0, \underbrace{0, \cdots, 0}_{j-1}, 1, 1, \underbrace{0, \cdots, 0}_{k_1}, 1, \underbrace{0, \cdots, 0}_{k_2}, 1, \cdots, \underbrace{0, \cdots, 0}_{k_{k-i-1}}).$$

Here, we define $(x_1, x_2, \cdots, x_n)_p^q$ as $(x_p, x_{p+1}, \cdots, x_q)$.

i) holds by the definitions of $T_{i,j}$ and $U_{i,j}$.

If $1 \le l < n$, then $(u_{i,j,s})_0^1 = (u_{i,j,s})_{3+i}^{4+i} = (1, 1)$, but the existances of $(1, 1)$ only appear in the range of the bits. Therefore, $u_{i,j,s} \npreceq \rho_n^l(\alpha_{i,j,s})$ for $1 \le l < n$, and then ii) and iii) hold.

If $i' < i$, and $u_{i',j',s'} \preceq \rho_n^l(\alpha_{i,j,s})$, by the necessarity of existance of the two $(1, 1)$'s we know $u_{i',j',s'} \oplus (\underbrace{0, \cdots, 0}_{i-j+2}, \underbrace{1, \cdots, 1}_{j}, \underbrace{0, \cdots, 0}_{2k-1-i}) \preceq \alpha_{i,j,s} \oplus (\underbrace{0, \cdots, 0}_{i-j+2}, 1, \underbrace{0, \cdots, 0}_{2k-i+j-2})$. Since the two vectors have same weight, they should be equal, which is obviously impossible.

If $i' = i$ then we use the same method when proving ii) and iii) and get that $l = 0$. In this case, if $j' < j$, then $(u_{i',j',s'})_{j+2}^{j+2} > (\alpha_{i,j,s})_{j+2}^{j+2}$; otherwise, they use different partitions, so $u_{i',j',s'} \npreceq \rho_n^l(\alpha_{i,j,s})$, still. This completes the proof of iv). ■

Lemma 2 [8]: Define $F(x)$ as the majority function. Let $T = \{\alpha_1, \cdots, \alpha_l\} \subseteq W^{\le k}$ and $U = \{u_1, \cdots, u_l\} \subseteq W^{k+1}$, for some integer l. If the vectors in T and U satisfy

$$P1. \quad \alpha_i \preceq u_i \quad \text{for } 1 \le i \le l,$$

and

$$P2. \quad \alpha_i \npreceq u_j \quad \text{for } 1 \le i < j \le l,$$

then

$$f_1(x) = \begin{cases} F(x) \oplus 1, & x \in T \cup U \\ F(x), & \text{otherwise} \end{cases}$$

has optimal AI.

Theorem 1: The Boolean function $f(x)$ in Construction 1 has optimal AI.

Proof: Since $\rho_n^l(u_{i,j,s}) \preceq \rho_n^l(\alpha_{i,j,s})$ for $0 \le l < n$, $\rho_n^l(u_{i,j,s}) \npreceq \rho_n^m(\alpha_{i,j,s})$ for $0 \le l, m < n$ and $m \neq n$, and that $\rho_n^l(u_{i',j',s'}) \npreceq \rho_n^m(\alpha_{i,j,s})$ for $(i',j',s') < (i,j,s)$ according to Lemma 1, we can renumber the elements $\rho_n^l(\alpha_{i,j,s})$ in $\tilde{T}$ in order of (i, j, s, l), and the same operation can apply on $\tilde{U}$. In this way the conditions in Lemma 2 satisfy, and the proof is completed. ■

C. NONLINEARITY

Theorem 2: The nonlinearity of $f(x)$ in (4) is

$$\mathrm{NL}(f) = 2^{n-1} - \binom{n-1}{k} + 2^{k-4}(k-3)(k-2),$$

where $n = 2k + 1 \ge 11$.

Proof: For $\omega \in \mathbb{F}_2^n$, we'll first calculate the maximum of Walsh transform on ω.

Case 1: If $\mathrm{wt}(\omega) = 1$, then

$$\begin{aligned} &\sum_{x \in \tilde{T}} \Big((-1)^{f(x) \oplus \omega \cdot x} - (-1)^{F(x) \oplus \omega \cdot x}\Big) \\ &= \sum_{x \in T} \Big(-\mathrm{wt}(x) - \big(n - \mathrm{wt}(x)\big)\Big) - \Big(\mathrm{wt}(x) - \big(n - \mathrm{wt}(x)\big)\Big) \\ &= \sum_{x \in T} \big(2n - 4\,\mathrm{wt}(x)\big) \end{aligned}$$

and

$$\begin{aligned} &\sum_{x \in \tilde{U}} \Big((-1)^{f(x) \oplus \omega \cdot x} - (-1)^{F(x) \oplus \omega \cdot x}\Big) \\ &= \sum_{x \in U} \Big(\mathrm{wt}(x) - \big(n - \mathrm{wt}(x)\big)\Big) - \Big(-\mathrm{wt}(x) + \big(n - \mathrm{wt}(x)\big)\Big) \\ &= \sum_{x \in U} \big(-2n + 4\,\mathrm{wt}(x)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} W_f(\omega) &= W_F(\omega) + \sum_{x \in \tilde{T}} \Big((-1)^{f(x) \oplus \omega \cdot x} - (-1)^{F(x) \oplus \omega \cdot x}\Big) \\ &\quad + \sum_{x \in \tilde{U}} \Big((-1)^{f(x) \oplus \omega \cdot x} - (-1)^{F(x) \oplus \omega \cdot x}\Big) \\ &= 2\binom{n-1}{k} + 4 \sum_{\{(i,j,s) \mid \alpha_{i,j,s} \in T\}} \big(-\mathrm{wt}(\alpha_{i,j,s}) + \mathrm{wt}(u_{i,j,s})\big) \\ &= 2\binom{n-1}{k} - 4 \sum_{i=2}^{k-2} \sum_{j=1}^{i-1} (j-1) \binom{k-2}{k-i-2} \\ &= 2\binom{n-1}{k} - 2^{k-3}(k-3)(k-2). \end{aligned}$$

Case 2: If $\mathrm{wt}(\omega) = n$, then

$$\sum_{x\in\tilde{T}}\left((-1)^{f(x)\oplus\omega\cdot x} - (-1)^{F(x)\oplus\omega\cdot x}\right) = \sum_{x\in T} n(-1)^{f(x)+\mathrm{wt}(x)} - n(-1)^{F(x)+\mathrm{wt}(x)} = \sum_{x\in T} 2n(-1)^{\mathrm{wt}(x)}$$

and

$$\sum_{x\in\tilde{U}}\left((-1)^{f(x)\oplus\omega\cdot x} - (-1)^{F(x)\oplus\omega\cdot x}\right) = \sum_{x\in U} n(-1)^{f(x)+\mathrm{wt}(x)} - n(-1)^{F(x)+\mathrm{wt}(x)} = \sum_{x\in U} -2n(-1)^{\mathrm{wt}(x)}.$$

Thus,

$$\begin{aligned} W_f(\omega) &= 2(-1)^k\binom{n-1}{k} + \sum_{x\in T} 2n(-1)^{\mathrm{wt}(x)} + \sum_{x\in U} 2n(-1)^{\mathrm{wt}(x)} \\ &= 2(-1)^k\binom{n-1}{k} + 2n\sum_{i,j,s}\left((-1)^{\mathrm{wt}(\alpha_{i,j,s})} - (-1)^{\mathrm{wt}(u_{i,j,s})}\right) \\ &= 2(-1)^k\binom{n-1}{k} - 4(-1)^k n\sum_{i=2}^{k-2}\sum_{t=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{k-2}{k-i-2}. \end{aligned}$$

Since

$$n\sum_{i=2}^{k-2}\sum_{t=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{k-2}{k-i-2} > \sum_{i=2}^{k-2}\sum_{j=1}^{i-1}(j-1)\binom{k-2}{k-i-2}$$

as

$$\sum_{t=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{k-2}{k-i-2} \le \frac{1}{2}\sum_{j=1}^{i-1}\binom{k-2}{k-i-2}$$

and $2(j-1) < n$, and

$$4n\sum_{i=2}^{k-2}\sum_{t=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{k-2}{k-i-2} < 2\binom{n-1}{k},$$

the absolute of $W_f(\omega)$ is obviously larger when $\omega = 1$ than $\omega = n$.

Case 3: If $2 \le \mathrm{wt}(\omega) < n$, then $|W_f(\omega)| \le 2(\binom{n-3}{k-1} - \binom{n-3}{k}) + 4n|T|$, which is also smaller than the Walsh when $\omega = 1$.

Hence, the maximum absolute of Walsh transform appears when $\mathrm{wt}(\omega) = 1$, in which case we can know from (3) that the nonlinearity of f is $2^{n-1} - \binom{n-1}{k} + 2^{k-4}(k-3)(k-2)$. ■

TABLE 1. Comparisons of nonlinearities among the odd-variable RSBFs.

n	$F(x)$	Construction in [31]	Construction in [37][1]	$f(x)$ in (4)
11	772	802	784	784
13	3172	3234	3218	3218
15	12952	13078	13096	13110
17	52666	52920	53068	53146
19	213524	214034	214568	214866
21	863820	864842	866402	867402
23	3488872	3490918	3495040	3498086
25	14073060	14077154	14087422	14096098

1. The value is calculated according to the given nonlinearity formula.

D. ALGEBRAIC DEGREE

Lemma 3 [6]: For the n-variable majority function F,

$$\deg(F) = 2^{\lfloor\log_2 n\rfloor}.$$

Lemma 4: Let f be the function defined in (4), and F be the majority function, then

$$\deg(f\oplus F) < n-1.$$

Proof: Let's define $\eta_{n-1} = (\underbrace{1,\cdots,1}_{n-1},0)$. Since f is balanced, by (2), $\deg(f) < n$; Because of the rotated symmetrical of f, $\deg(f\oplus F) = n-1$ iff $\bigoplus_{t\preceq\eta_{n-1}} f(t)$, which has same parity to $\sum_{x\in T\cup U}(n-\mathrm{wt}(x))$, and same to $\frac{1}{4}W_f(1,\underbrace{0,\cdots,0}_{n-1})$. Since it's always even, $\deg(f\oplus F) < n-1$. ■

Theorem 3: For Boolean function $f(x)$ defined in (4), if $k = 2^m$, then $\deg(f) = n-1$, and vise versa.

Proof: Since $\deg(f\oplus F) < n-1$, we know that $\deg(f) = n-1$ iff $\deg(F) = n-1$. By Lemma 4, it only happens when $n = 2^{m+1}+1$, i.e., $k = 2^m$. ■

Therefore, in most situations, $\deg(f\oplus F) < n-1$. The exception happens when $k = 2^m$ for some integer m. That's quite small amount.

To improve the degree, we define:

Construction 2:

$$f'(x) = \begin{cases} f(x)\oplus 1, & \text{if } \rho_n^l(x) \in \{\alpha_{k-2,2,1}, u_{k-2,2,1}\}; \\ f(x), & \text{else.} \end{cases}$$

Because it inverts $\bigoplus_{t\preceq\eta_{n-1}} f(t)$, with the same methods used before, we can get:

Theorem 4: For function f' defined in Construction 2, where $k \ne 2^m$, f' has optimal AI, $\deg(f') = n-1$, and $\mathrm{NL}(f') = 2^{n-1} - \binom{n-1}{k} + 2^{k-4}(k-3)(k-2) - 2.$

E. FAST ATTACK IMMUNITY

Currently exact FAI is still only available for majority function on some special n [18], [19], and for our function, we're only able to analyze for small n. With the computer program in [42] we can get that, for odd $n < 17$, $\mathrm{FAI}(f') = \mathrm{FAI}(f) = n-1$. As proved in [39], this is the highest possible value for RSBFs with optimal AI.

IV. CONCLUSION

In this paper, we construct a new type of balanced odd-variable RSBFs with optimal AI, and show the exact value of nonlinearity of our construction, which is higher than the known ones before. And such functions also have highest possible algebraic degree. In addition, for odd $n < 17$, the function has almost optimal FAI. However, the value of FAI for higher n still need some work, which is a significant open research area.

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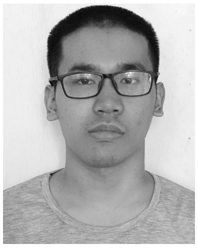

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