Inequalities for a Unified Integral Operator and Associated Results in Fractional Calculus

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ABSTRACT Integral operators are useful in real analysis, mathematical analysis, functional analysis and other subjects of mathematical approach. The goal of this paper is to study a unified integral operator via convexity. By using convexity and conditions of unified integral operators, bounds of these operators are obtained. Furthermore consequences of these results are discussed for fractional and conformable integral operators.

INDEX TERMS Convex function, Mittag-Leffler function, integral operator, fractional integral operator, conformable integral operator.

I. INTRODUCTION AND PRELIMINARY RESULTS

A function f satisfying the following inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad (1)$$

where $\lambda \in [0, 1]$, $x, y \in C$ and C is convex set, is called convex function on C. A function satisfying (1) in reverse order is called concave function. For properties and characterizations of convex functions, see [1].

Definition 1 [2]: Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu > 0$ are defined by:

$$^{\mu}_g I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x) - g(t))^{\mu-1} g'(t) f(t) dt, \quad x > a \quad (2)$$

and

$$^{\mu}_g I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t) - g(x))^{\mu-1} g'(t) f(t) dt, \quad x < b, \quad (3)$$

where $\Gamma(.)$ is the Gamma function.

A k-fractional analogue of above definition is given as follows:

Definition 2 [3]: Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu, k > 0$ are defined by:

$$^{\mu}_g I_{a^+}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x) - g(t))^{\frac{\mu}{k}-1} g'(t) f(t) dt, \quad x > a \quad (4)$$

and

$$^{\mu}_g I_{b^-}^k f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t) - g(x))^{\frac{\mu}{k}-1} g'(t) f(t) dt, \quad x < b, \quad (5)$$

where $\Gamma_k(.)$ is the k-Gamma function.

A generalized fractional integral with kernel an extended generalized Mittag-Leffler function is defined as follows:

Definition 3 [6]: Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ and

$\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ are defined by:

$$\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p)$$
$$= \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^\mu;p)f(t)dt, \quad (6)$$

and

$$\left(\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x;p)$$
$$= \int_x^b (t-x)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu;p)f(t)dt, \quad (7)$$

where

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^\infty \frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}} \quad (8)$$

is the extended generalized Mittag-Leffler function.

Recently Farid defined a new unified integral operator from which the fractional as well as conformable integral operators can be derived at once:

Definition 4 [7]: Let $f, g : [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a,b]$, and g be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then for $x \in [a,b]$ the left and right integral operators are defined by:

$$(_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x;p) = \int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}$$
$$\times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(t))^\mu;p)g'(t)f(t)dt \quad (9)$$

and

$$(_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f)(x;p) = \int_x^b \frac{\phi(g(t)-g(x))}{g(t)-g(x)}$$
$$\times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(t)-g(x))^\mu;p)g'(t)f(t)dt. \quad (10)$$

In [7] it is proved that the operators defined in (9) and (10) are bounded, further they are linear hence these are continuous operators.

Theorem 5 [7]: Under the assumptions of Definition 4, the following bounds hold for integral operators (9) and (10):

$$\left|(_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x;p)\right| \leq K \|f\|_{[a,b]} \quad (11)$$

and

$$\left|(_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f)(x;p)\right| \leq K \|f\|_{[a,b]}. \quad (12)$$

Hence

$$\left|(_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x;p) + (_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f)(x;p)\right| \leq 2K \|f\|_{[a,b]}, \quad (13)$$

where S is the sum of absolute terms of (8) and $K = S |\phi(g(b)-g(a))|$.

Integral operators defined in (9) and (10) are unified in the sense that for specific settings of functions ϕ, g and particular values of involved parameters in Mittag-Leffler function they contain two kinds of general fractional integral operators (4), (5), and (6), (7). These integral operators and their consequences are narrated in the following two remarks.

Remark 6: (i) Let $\phi(x) = \frac{x^{\beta/k}\Gamma(\beta)}{k\Gamma_k(\beta)}$, $k > 0$, $\beta > k$ and $p = \omega = 0$, in unified integral operators (9) and (10). Then generalized Riemann-Liouville fractional integral operators (4) and (5) are obtained.

(ii) For $k = 1$, (4) and (5) fractional integrals coincide with (2) and (3) fractional integrals, which further produce the following fractional and conformable integrals:

(iii) By taking g as identity function, (4) and (5) fractional integrals coincide with k-fractional Riemann-Liouville integrals defined by Mubeen et al. in [15].

(iv) For $k = 1$, along with g as identity function, (4) and (5) fractional integrals coincide with Riemann-Liouville fractional integrals [2].

(v) For $k = 1$ and $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$, (4) and (5) produce fractional integrals defined by Chen et al. in [10].

(vi) For $k = 1$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, (4) and (5) produce generalized conformable integrals defined by Khan et al. in [13].

(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (4) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (5), then conformable (k,s)-fractional integrals will be obtained as defined by Habib et al. in [11].

(ix) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then conformable integrals will be obtained as defined by Sarikaya et al. in [16].

(x) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (4) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (5) with $k = 1$, then conformable integrals will be obtained as defined by Jarad et al. in [12].

Remark 7: Let $\phi(x) = x^\beta$ and $g(x) = x$, $\beta > 0$, in unified integral operators (9) and (10). Then fractional integral operators (6) and (7) are obtained, which along with different settings of $p, k, \delta, l, c, \gamma$ in generalized Mittag-Leffler function give the following integral operators:

1. By setting $p = 0$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Salim-Faraj in [5].

2. By setting $l = \delta = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Rahman et al. in [18].

3. By setting $p = 0$ and $l = \delta = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Srivastava-Tomovski in [14].

4. By setting $p = 0$ and $l = \delta = k = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Prabhakar in [19].

5. By setting $p = \omega = 0$, fractional integral operators (6) and (7) are reduced to the left-sided and right-sided Riemann-Liouville fractional integrals.

For detailed study of recent generalized, fractional and conformable integral operators one can consult [3]–[6], [8], [10]–[14], [16], [17] and references therein.

The purpose of this research is the study of all above integral operators via convex functions. We are succeeded to

obtain bounds of integral operators defined in (9) and (10). These results provide formulas for bounds of all fractional and conformable integrals comprised in Remark1 and Remark 7. The paper is organized as follows:

In Section II, upper bounds of unified fractional integral operators (9) and (10) are established by using the involved conditions and convex functions. Further by imposing an additional condition of symmetry two sided Hadamard type bounds are obtained. Moreover by using convexity of $|f'|$ and applying integral operator on convolution of two functions some interesting bounds are studied. It is important to note that all these results hold for fractional and conformable integral operators comprised in Remark 6 and Remark 7. Also some fractional differential equations are solved in Section III.

II. MAIN RESULTS

Bounds of integral operators (9), (10) and their sum are obtained in the following theorem.

Theorem 8: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a positive convex function, $0 < a < b$ and $g : [a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on $[a, b]$ and $\alpha, l, \gamma, c \in \mathbb{C}, p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in [a, b]$ we have

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \right) (x; p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$$
$$\times (\phi(g(x)-g(a))) (f(x)+f(a)) \quad (14)$$

and

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \right) (x; p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b) - g(x))^\mu; p)$$
$$\times (\phi(g(b)-g(x))) (f(x)+f(b)) \quad (15)$$

hence

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \right) (x; p) + \left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \right) (x; p)$$
$$\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p) (\phi(g(x) - g(a)))$$
$$(f(x) + f(a)) + E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b) - g(x))^\mu; p)$$
$$(\phi(g(b) - g(x))) (f(x) + f(b)) . \quad (16)$$

Proof 9: As g is increasing, therefore for $t \in [a, x]$, $x \in (a, b)$, $g(x) - g(t) \leq g(x) - g(a)$. The function $\frac{\phi}{x}$ is increasing, therefore one can obtain:

$$\frac{\phi(g(x) - g(t))}{g(x) - g(t)} \leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)}. \quad (17)$$

Now by multiplying with $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p)g'(t)$ the following inequality is yielded:

$$\frac{\phi(g(x) - g(t))}{g(x) - g(t)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p)$$
$$\leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p). \quad (18)$$

Also $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p)$ is series of positive terms, therefore $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$ so the following inequality holds:

$$\frac{\phi(g(x) - g(t))}{g(x) - g(t)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^\mu; p)$$
$$\leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p). \quad (19)$$

Using convexity of f on $[a, x]$ for $x \in (a, b)$ we have

$$f(t) \leq \frac{x - t}{x - a}f(a) + \frac{t - a}{x - a}f(x). \quad (20)$$

Multiplying (19) and (20), then integrating with respect to t over $[a, x]$ we have

$$\int_a^x \frac{\phi(g(x) - g(t))}{g(x) - g(t)} g'(t)f(t)$$
$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}(\omega(g(x) - g(t))^\mu; p)dt$$
$$\leq \frac{f(a)}{x - a} \frac{\phi(g(x) - g(a))}{g(x) - g(a)} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$$
$$\times \int_a^x (x - t)g'(t)dt$$
$$+ \frac{f(x)}{x - a} \frac{\phi(g(x) - g(a))}{g(x) - g(a)} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$$
$$\times \int_a^x (t - a)g'(t)dt. \quad (21)$$

By using (9) of Definition 4, and integrating by parts we get

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \right) (x; p)$$
$$\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$$
$$\times \left(\frac{\phi(g(x) - g(a))}{g(x) - g(a)} \right) \left(\frac{f(a)}{x - a} \left(g(a)(a - x) + \int_a^x g(t)dt \right) \right.$$
$$+ \frac{f(x)}{x - a} \left((x - a)g(x) - \int_a^x g(t)dt \right) \right)$$

which further simplifies as follows:

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \right) (x; p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p)$$
$$\times (\phi(g(x)-g(a))) (f(x)+f(a)) . \quad (22)$$

Now on the other hand for $t \in (x, b]$, $x \in (a, b)$ the following inequality holds true:

$$\frac{\phi(g(t) - g(x))}{g(t) - g(x)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t) - g(x))^\mu; p)$$
$$\leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t) - g(x))^\mu; p). \quad (23)$$

Also $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t) - g(x))^\mu; p)$ is series of positive terms, therefore $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t) - g(x))^\mu; p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b) - g(x))^\mu; p)$, so the following inequality is valid:

$$\frac{\phi(g(t) - g(x))}{g(t) - g(x)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t) - g(x))^\mu; p)$$
$$\leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^\mu; p). \quad (24)$$

The following inequality also holds for convex function f:

$$f(t) \le \frac{t-x}{b-x}f(b) + \frac{b-t}{b-x}f(x). \qquad (25)$$

Multiplying (24) and (25), then integrating with respect to t over $(x, b]$ and adopting the same pattern of simplification as we did for (21), the following inequality is obtained:

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;p)$$

$$\le E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^\mu; p)$$

$$\times \left(\frac{\phi(g(b)-g(x))}{g(b)-g(x)}\right)\left(\frac{f(b)}{b-x}\left(g(b)(b-x) - \int_x^b g(t)dt\right)\right.$$

$$\left. + \frac{f(x)}{b-x}\left((x-b)g(x) + \int_x^b g(t)dt\right)\right)$$

which further simplifies as follows:

$$\left({}_g F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;p) \le E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^\mu; p)$$

$$\times (\phi(g(b)-g(x)))\,(f(x)+f(b)). \qquad (26)$$

By adding (22) and (26), (16) can be achieved.

Henceforth we give consequences of above theorem for fractional calculus and conformable integral operators defined in [2], [5], [9]–[13], [15], [16].

Proposition 10: Let $\phi(t) = t^\alpha$ and $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators (2) and (3) defined in [2], as follows:

$$\left({}_g F^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) := {}_g^\alpha I_{a^+} f(x)$$

and

$$\left({}_g F^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) := {}_g^\alpha I_{b^-} f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \ge 1$, therefore they satisfy the following bound:

$$({}_g^\alpha I_{a^+} f)(x) + ({}_g^\alpha I_{b^-} f)(x)$$

$$\le \frac{1}{\Gamma(\alpha)}\left((g(x)-g(a))^\alpha\,(f(x)+f(a))\right.$$

$$\left. + (g(b)-g(x))^\alpha(f(x)+f(b))\right).$$

Proposition 11: Let $g(x) = I(x) = x$ and $p = \omega = 0$. Then (9) and (10) produce integral operators defined in [17] as follows:

$$\Gamma(\alpha)\left({}_I F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) := ({}_{a^+} I_\phi f)(x)$$

$$= \int_a^x \frac{\phi(x-t)}{(x-t)} f(t)dt$$

and

$$\Gamma(\alpha)\left({}_I F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) := ({}_{b^-} I_\phi f)(x)$$

$$= \int_x^b \frac{\phi(t-x)}{(t-x)} f(t)dt.$$

Further they satisfy the following bound:

$$({}_{a^+} I_\phi f)(x) + ({}_{b^-} I_\phi f)(x)$$

$$\le \phi(x-a)(f(x)+f(a)) + \phi(b-x)(f(x)+f(b)).$$

Corollary 12: If we take $\phi(t) = \frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators (4) and (5) defined in [3] as follows:

$$\left({}_g F^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) := {}_g^\alpha I^k_{a^+} f(x)$$

and

$$\left({}_g F^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) := {}_g^\alpha I^k_{b^-} f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \ge k$, therefore they satisfy the following bound:

$$({}_g^\alpha I^k_{a^+} f)(x) + ({}_g^\alpha I^k_{b^-} f)(x) \le \frac{1}{k\Gamma_k(\alpha)}((g(x)-g(a))^{\frac{\alpha}{k}}(f(x)+f(a))$$

$$+ (g(b)-g(x))^{\frac{\alpha}{k}}(f(b)+f(x))).$$

Corollary 13: If we take $\phi(t) = t^\alpha$ and $g(x) = I(x) = x$ with $p = \omega = 0$. Then (9) and (10) produce left and right Riemann-Liouville fractional integrals [2] as follows:

$$\left({}_I F^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) := {}^\alpha I_{a^+} f(x)$$

and

$$\left({}_I F^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) := {}^\alpha I_{b^-} f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \ge k$ therefore, they satisfy the following bound:

$$({}^\alpha I_{a^+} f)(x) + ({}^\alpha I_{b^-} f)(x)$$

$$\le \frac{1}{\Gamma(\alpha)}((x-a)^\alpha(f(x)+f(a)) + (b-x)^\alpha(f(b)+f(x))).$$

Corollary 14: If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x) = I(x) = x$, $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators define in [15] as follows:

$$\left({}_I F^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) := {}^\alpha I^k_{a^+} f(x)$$

and

$$\left({}_I F^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) := {}^\alpha I^k_{b^-} f(x).$$

Further they satisfy the following bound for $\alpha \ge k$:

$$({}^\alpha I^k_{b^-} f)(x) + ({}^\alpha I^k_{b^-} f)(x) \le \frac{1}{k\Gamma_k(\alpha)}((x-a)^{\frac{\alpha}{k}}(f(x)+f(a))$$

$$+ (b-x)^{\frac{\alpha}{k}}(f(b)+f(x))).$$

Corollary 15: If we take $\phi(t) = t^\alpha$, $\alpha > 0$ and $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$ with $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [10], as follows:

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^\rho I^\alpha_{a^+}f)(x)$$

$$= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^\rho - t^\rho)^{\alpha-1}t^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^\rho - x^\rho)^{\alpha-1}t^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^\rho I^\alpha_{a^+}f)(x) + ({}^\rho I^\alpha_{b^-}f)(x) \leq \frac{1}{\rho^\alpha \Gamma(\alpha)}((x^\rho - a^\rho)^\alpha(f(x) + f(a))$$
$$+ (b^\rho - x^\rho)^\alpha(f(b) + f(x))).$$

Corollary 16: If we take $\phi(t) = t^\alpha, \alpha > 0$ and $g(x) = \frac{x^{s+1}}{s+1}, s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators define as follows:

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^s I^\alpha_{a^+}f)(x)$$

$$= \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{s+1} - t^{s+1})^{\alpha-1}t^s f(t)dt$$

and

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = ({}^s I^\alpha_{b^-}f)(x)$$

$$= \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{s+1} - x^{s+1})^{\alpha-1}t^s f(t)dt.$$

Further they satisfy the following bound:

$$({}^s I^\alpha_{a^+}f)(x) + ({}^s I^\alpha_{b^-}f)(x)$$
$$\leq \frac{1}{(s+1)^\alpha \Gamma(\alpha)}((x^{s+1} - a^{s+1})^\alpha(f(x) + f(a))$$
$$+ (b^{s+1} - x^{s+1})^\alpha(f(b) + f(x))).$$

Corollary 17: If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [16], as follows:

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0)$$

$$= ({}^s_k I^\alpha_{a^+}f)(x)$$
$$= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)dt$$

and

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0)$$

$$= ({}^s_k I^\alpha_{b^-}f)(x)$$
$$= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)dt.$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \geq k$ therefore, they satisfy the following bound:

$$({}^s_k I^\alpha_{a^+}f)(x) + ({}^s_k I^\alpha_{b^-}f)(x)$$
$$\leq \frac{1}{(s+1)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}((f(x) + f(a))(b^{s+1} - x^{s+1})^{\frac{\alpha}{k}}$$
$$+ (x^{s+1} - a^{s+1})^{\frac{\alpha}{k}}(f(b) + f(x))).$$

Corollary 18: If we take $\phi(t) = t^\alpha$ and $g(x) = \frac{x^{\beta+s}}{\beta+s}$, $\beta, s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [13] as follows:

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0)$$

$$= ({}^s_\beta I^\alpha_{a^+}f)(x)$$
$$= \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{\beta+s} - t^{\beta+s})^{\alpha-1}t^s f(t)dt$$

and

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0)$$

$$= ({}^s_\beta I^\alpha_{b^-}f)(x)$$
$$= \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{\beta+s} - x^{\beta+s})^{\alpha-1}t^s f(t)dt.$$

Further they satisfy the following bound:

$$({}^s_\beta I^\alpha_{a^+}f)(x) + ({}^s_\beta I^\alpha_{b^-}f)(x)$$
$$\leq \frac{1}{(\beta+s)^\alpha \Gamma(\alpha)}((x^{\beta+s} - a^{\beta+s})^\alpha(f(x) + f(a))$$
$$+ (b^{\beta+s} - x^{\beta+s})^\alpha(f(b) + f(x))).$$

Corollary 19: If we take $g(x) = \frac{(x-a)^\rho}{\rho}$, $\rho > 0$ in (9) and $g(x) = \frac{-(b-x)^\rho}{\rho}$, $\rho > 0$ in (10) with $\phi(t) = t^\alpha, \alpha > 0$, $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [12], as follows:

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0)$$

$$= ({}^\rho I^\alpha_{a^+}f)(x)$$
$$= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x ((x-a)^\rho - (t-a)^\rho)^{\alpha-1}(t-a)^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{t^\alpha,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0)$$

$$= ({}^\rho I^\alpha_{b^-}f)(x)$$
$$= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b ((b-x)^\rho - (b-t)^\rho)^{\alpha-1}(b-t)^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^\rho I^\alpha_{a^+}f)(x) + ({}^\rho I^\alpha_{b^-}f)(x)$$
$$\leq \frac{1}{\rho^\alpha \Gamma(\alpha)}((x-a)^{\rho\alpha}(f(x) + f(a))$$
$$+ (b-x)^{\rho\alpha}(f(b) + f(x))).$$

Corollary 20: If we take $g(x) = \frac{(x-a)^\rho}{\rho}$, $\rho > 0$ in (9) and $g(x) = \frac{-(b-x)^\rho}{\rho}$, $\rho > 0$ in (10) with $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$, $\alpha > k$,

$p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [11], as follows:

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0)$$

$$= ({}_k^SI_{a^+}^\alpha f)(x) = ({}_k^\rho I_{a^+}^\alpha f)(x)$$

$$= \frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x ((x-a)^\rho - (t-a)^\rho)^{\frac{\alpha}{k}-1}(t-a)^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0)$$

$$= ({}_k^SI_{a^+}^\alpha f)(x) = ({}_k^\rho I_{b^-}^\alpha f)(x)$$

$$= \frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b ((b-x)^\rho - (b-t)^\rho)^{\frac{\alpha}{k}-1}(b-t)^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}_k^\rho I_{a^+}^\alpha f)(x) + ({}_k^\rho I_{b^-}^\alpha f)(x)$$

$$\leq \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}((x-a)^{\frac{\rho\alpha}{k}}(f(x)+f(a))$$

$$+ (b-x)^{\frac{\rho\alpha}{k}}(f(b)+f(x))).$$

We will use the following lemma to get the next theorem.

Lemma 21 [9]: Let $f : [a, b] \to \mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a, b]. \tag{27}$$

The following theorem provides the Hadamard type estimation of integral operators (9) and (10).

Theorem 22: Along with statement of Theorem 8, if in addition f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b;p)\right)$$

$$\leq \left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p)\right)$$

$$\leq 2\phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$$

$$\times (f(a)+f(b)). \tag{28}$$

Proof 23: For $x \in (a, b)$, under the assumption on g and $\frac{\phi}{x}$ the following inequality holds:

$$\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)$$

$$\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p). \tag{29}$$

Using convexity of f on $[a, b]$ for $x \in (a, b)$ we have

$$f(x) \leq \frac{x-a}{b-a}f(b) + \frac{b-x}{b-a}f(a). \tag{30}$$

Multiplying (29) and (30) and then integrating with respect to x over $[a, b]$, the following inequality is obtained:

$$\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)f(x)$$

$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)dx$$

$$\leq \frac{f(b)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$$

$$\times \int_a^b (x-a)g'(x)dx$$

$$+ \frac{f(a)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$$

$$\times \int_a^b (b-x)g'(x)dx.$$

By using (9) of Definition 4 and integrating by parts we get

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$$

$$\times (\phi(g(b)-g(a)))(f(a)+f(b)). \tag{31}$$

On the other hand the following inequality holds:

$$\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^\mu;p)$$

$$\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p). \tag{32}$$

Multiplying (30) and (32) and then integrating with respect to x over $[a, b]$ and simplifying on the same pattern as we did for (29) and (30), following inequality is obtained:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)$$

$$\times (\phi(g(b)-g(a)))(f(a)+f(b)). \tag{33}$$

By adding (31) and (33), we have

$$\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p)\right)$$

$$\leq 2\phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)(f(a)+f(b)). \tag{34}$$

Multiplying both sides of (27) by $\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)$ $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)$, then integrating over $[a, b]$ we get

$$f\left(\frac{a+b}{2}\right)\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)$$

$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)dx$$

$$\leq \int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)f(x)$$

$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)f(x)dx.$$

By using (10) of Definition 4 we get

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a;p) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p). \tag{35}$$

Multiplying both sides of (27) by $\dfrac{\phi(g(b) - g(x))}{g(b) - g(x)}g'(x)$ $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(b) - g(x))^{\mu}; p)$ and integrating over $[a, b]$ we have

$$f\left(\frac{a+b}{2}\right)\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}1\right)(b; p) \leq \left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f\right)(b; p) \quad (36)$$

by adding (35) and (36), the following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\left(\left({}_{g}F_{\mu,\alpha,l,b^{-}}^{\phi,\gamma,\delta,k,c}1\right)(a; p) + \left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}1\right)(b; p)\right).$$

$$\leq \left(\left({}_{g}F_{\mu,\alpha,l,b^{-}}^{\phi,\gamma,\delta,k,c}f\right)(a; p) + \left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f\right)(b; p)\right). \quad (37)$$

Combining (34) and (37), inequality (28) can be achieved.

Remark 24: Theorem 22 can be utilized to obtain bounds of Hadamard type for fractional integral operators and conformable integrals like Corollaries $1 - 9$. We leave them for the readers.

Theorem 25: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is convex, $0 < a < b$ and $g : [a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function and $\alpha, l, \gamma, c \in \mathbb{C}, p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in (a, b)$ we have

$$\left|\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p) + \left({}_{g}F_{\mu,\alpha,l,b^{-}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p)\right|$$

$$\leq E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)\phi(g(x) - g(a))$$

$$(|f'(x)| + |f'(a)|)$$

$$+ E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(b) - g(x))^{\mu}; p)\phi(g(b) - g(x))$$

$$(|f'(x)| + |f'(b)|). \quad (38)$$

where

$$\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p)$$

$$:= \int_{a}^{x}\frac{\phi(g(x) - g(t))}{g(x) - g(t)} \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)$$

$$- g(t))^{\mu}; p)g'(t)f'(t)dt$$

$$\left({}_{g}F_{\mu,\alpha,l,b^{-}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p)$$

$$:= \int_{x}^{b}\frac{\phi(g(t) - g(x))}{g(t) - g(x)} \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)$$

$$- g(t))^{\mu}; p)g'(t)f'(t)dt.$$

Proof 26: Using the convexity of $|f'|$ over $[a, b]$ for $t \in [a, x]$ we have

$$|f'(t)| \leq \frac{x - t}{x - a}|f'(a)| + \frac{t - a}{x - a}|f'(x)|. \quad (39)$$

From which we can write

$$-\left(\frac{x - t}{x - a}|f'(a)| + \frac{t - a}{x - a}|f'(x)|\right) \leq f'(t)$$

$$\leq \left(\frac{x - t}{x - a}|f'(a)| + \frac{t - a}{x - a}|f'(x)|\right) \quad (40)$$

we consider the right hand side inequality of the above inequality i.e.

$$f'(t) \leq \left(\frac{x - t}{x - a}|f'(a)| + \frac{t - a}{x - a}|f'(x)|\right). \quad (41)$$

Further the following inequality holds true:

$$\frac{\phi(g(x) - g(t))}{g(x) - g(t)}g'(x)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(t))^{\mu}; p)$$

$$\leq \frac{\phi(g(x) - g(a))}{g(x) - g(a)}g'(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p). \quad (42)$$

Multiplying (41) and (42) and integrating with respect to t over $[a, x]$, the following inequality is obtained:

$$\int_{a}^{x}\frac{\phi(g(x) - g(t))}{g(x) - g(t)}g'(t)f'(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(t))^{\mu}; p)dt$$

$$\leq \frac{|f'(a)|}{x - a}\frac{\phi(g(x) - g(a))}{g(x) - g(a)}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)$$

$$\times \int_{a}^{x}(x - t)g'(t)dt$$

$$+ \frac{|f'(x)|}{x - a}\frac{\phi(g(x) - g(a))}{g(x) - g(a)}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)$$

$$\times \int_{a}^{x}(t - a)g'(t)dt$$

which gives

$$\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p) \leq E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)$$

$$\times \phi(g(x) - g(a))(|f'(x)| + |f'(a)|). \quad (43)$$

If we consider the left hand side inequality from the inequality (40) and proceed as we did for the right hand side inequality we have

$$\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p) \geq -E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)$$

$$\times \phi(g(x) - g(a))(|f'(x)| + |f'(a)|). \quad (44)$$

Combining (43) and (44), the following inequality is obtained:

$$\left|\left({}_{g}F_{\mu,\alpha,l,a^{+}}^{\phi,\gamma,\delta,k,c}f * g\right)(x; p)\right| \leq E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(a))^{\mu}; p)$$

$$\times \phi(g(x) - g(a))(|f'(x)| + |f'(a)|). \quad (45)$$

On the other hand using convexity of $|f'(t)|$ over $[a, b]$ for $t \in (x, b]$ we have

$$|f'(t)| \leq \frac{t - x}{b - x}|f'(b)| + \frac{b - t}{b - x}|f'(x)|. \quad (46)$$

Further the following inequality holds true:

$$\frac{\phi(g(x) - g(t))}{g(x) - g(t)} g'(x) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(t))^{\mu}; p)$$

$$\leq \frac{\phi(g(b) - g(x))}{g(b) - g(x)} g'(t) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^{\mu}; p). \quad (47)$$

By adopting the same treatment as we did for (39) and (42), one can obtain the following inequality from (46) and (47):

$$\left| \left({_g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f * g \right)(x; p) \right| \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b) - g(x))^{\mu}; p)$$

$$\times \phi(g(b) - g(x))(|f'(x)| + |f'(b)|). \quad (48)$$

Combining (45) and (48), inequality (38) can be achieved.

III. PROPOSED FRACTIONAL DIFFERENTIAL EQUATIONS

Theorem 27: Let $\mu, \alpha, l, \gamma, \nu, c \in \mathbb{C}$, $\mathfrak{R}(\mu), \mathfrak{R}(\alpha)$, $\mathfrak{R}(l) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \mathfrak{R}(\mu)$. Let $g(x) = I(x), f(x) = x^2$ and $\phi(t) = t^{\alpha}$. Then the differential equation

$$(D^{\nu}_{0+} y)(x) = \lambda_1 \left({_I}F^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,0^+} x^2 \right)(x; p) + x^2 \quad (49)$$

with initial condition $(I^{1-\nu}_{0+}) = (0+) = C$, has its solution in the $L(0, \infty)$

$$y(x) = C \frac{x^{\nu-1}}{\Gamma(\nu)} + 2\lambda_1 \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)}$$

$$\times \frac{(c)_{nk}}{\Gamma(\mu n + \alpha + \nu + 3)} \frac{\omega^n}{(l)_{n\delta}}$$

$$\times x^{\nu n + \alpha + \nu + 2} + 2 \frac{x^{\nu+2}}{\Gamma(\nu + 3)} \quad (50)$$

where C is an arbitrary constant.

Proof 28: For the function $f(x) = x^2$ the generalized fraction integral operator is calculated in [20, Thorem 3.1] as follows:

$$\left({_I}F^{t^{\alpha}\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} x^2 \right)(x; p)$$

$$= (x - a)^{\alpha} \times \left[a^2 E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega(x - a)^{\mu}; p) \right.$$

$$+ 2a(x - a) E^{\gamma,\delta,k,c}_{\mu,\alpha+2,l}(\omega(x - a)^{\mu}; p)$$

$$\left. + 2(x - a)^2 \times E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x - a)^{\mu}; p) \right]. \quad (51)$$

Now putting $a = 0$ the above equation reduces to

$$\left({_I}F^{t^{\alpha}\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,0^+} x^2 \right)(x; p) = 2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu}; p)). \quad (52)$$

Using (52) in (49) we get

$$(D^{\nu}_{0+} y)(x) = \lambda_1 2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu}; p)) + x^2. \quad (53)$$

Applying Laplace transform on both sides of (53) we have

$$L[(D^{\nu}_{0+} y)(x); s] = L[\lambda_1 2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu}; p)); s]$$

$$+ L[x^2; s]. \quad (54)$$

Laplace transform of Mittag-Leffler function is obtained as follows:

$$L[x^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x)^{\mu}; p)]$$

$$= \int_0^{\infty} x^{\alpha} e^{-sx} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x)^{\mu}; p) dx$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}}$$

$$\times \int_0^{\infty} x^{\alpha + \mu n - 1} e^{-sx} dx$$

$$= \frac{1}{s^{\alpha}} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{(l)_{n\delta}} \omega^n. \quad (55)$$

And Laplace transform of fractional derivative $D^{\nu}_{0+} f$ is calculated as follows:

$$L[D^{\nu}_{0+} f; s] = s^{\nu} F(s)$$

$$- \sum_{k=1}^{n} D^{\nu-k}_{0+} f(0+) \quad (n - 1 < \nu < n) Re(s) > 0. \quad (56)$$

Using (55) and (56) (for $n = 1$) in (54) we have

$$y(s) = Cs^{-\nu} + 2\lambda_1 s^{-(\mu n + \alpha + \nu + 3)} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)}$$

$$\times (c)_{nk} \frac{\omega^n}{(l)_{n\delta}} + 2s^{-(\nu+3)}. \quad (57)$$

Now taking the inverse Laplace transformation on both side of (57) and after some simplifications, we achieved the required result (50).

Theorem 29: Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\mathfrak{R}(\mu), \mathfrak{R}(\alpha), \mathfrak{R}(l) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \mathfrak{R}(\mu)$. Let $g(x) = I(x)$, and $\phi(t) = t^{\alpha}$. Then the differential equation

$$(D^{\alpha}_{0+} y)(x) = \lambda_1 \left({_I}F^{t^{\alpha},\gamma\delta,k,c}_{\mu,\alpha,l,0^+} 1 \right)(x; p) + \lambda_2 x^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega x^{\mu}; p) \quad (58)$$

with initial condition $(I^{1-\alpha}_{0+}) = (0+) = C$, has its solution in the $L(0, \infty)$

$$y(x) = C \frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda_1 + \lambda_2) \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)}$$

$$\times \frac{(c)_{nk}}{\Gamma(\mu n + 2\alpha + 1)} \frac{\omega^n}{(l)_{n\delta}} x^{\nu n + 2\alpha} \quad (59)$$

where C is an arbitrary constant.

Proof 30: By convenient settings of values of function $g(x) = I(x)$ and $\phi(t) = t^\alpha$ in (6), we have

$$\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} 1\right)(x;p) = \left({}_I F_{\mu,\alpha,l,\omega,a^+}^{t^\alpha \gamma,\delta,k,c} 1\right)(x;p). \quad (60)$$

By putting $a = 0$ in (60) one can obtained:

$$(D_{0^+}^\alpha y)(x) = \lambda_1 x^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^\mu; p) + x^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^\mu; p)$$

$$= (\lambda_1 + \lambda_2) x^\alpha E_{\mu,\alpha+1,l}^{\gamma,\delta,k,c}(\omega x^\mu; p). \quad (61)$$

Applying Laplace transform on both sides of (61) and after simplification one can obtained:

$$y(s) = Cs^{-\alpha} + (\lambda_1 + \lambda_2) s^{-(\mu n + 2\alpha + 1)}$$

$$\times \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} (c)_{nk} \frac{\omega^n}{(l)_{n\delta}}. \quad (62)$$

Applying inverse Laplace transform and after simplification the required result (59) can be achieved.

IV. CONCLUDING REMARKS

The findings of this research provide compact presentation of bounds for fractional integral operators and conformable integrals simultaneously. These bounds can be achieved from the bounds of unified integral operators (9) and (10) which have been established by utilizing convex functions, functions whose derivatives in absolute value are convex, symmetric convex functions, and by applying the conditions involved in definitions of unified operators.

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